Second-order cosmological perturbations. IV. Produced by scalar-tensor and tensor-tensor couplings during the radiation dominated stage

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We continue to study the second-order cosmological perturbations in synchronous coordinates in the framework of general relativity (GR) during the radiation dominated (RD) stage and to focus on the scalar-tensor and tensor-tensor couplings. The first-order curl velocity and the associated first-order vector metric perturbations are assumed to be vanishing. By analytically solving the second-order Einstein equation and the energy-momentum conservation equations, we obtain the second-order formal solutions (in the integral form) of all the metric perturbations, density contrast and velocity; perform the transformation between the synchronous coordinates; and identify the residual gauge modes in the second-order solutions. In addition, we present the second-order gauge transformations of the solutions from synchronous to Poisson coordinates. To apply these formal solutions to concrete cosmological study, one needs to choose proper initial conditions and carry out several numerical integrations.

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I. INTRODUCTION

The cosmological perturbations in a Friedmann-Robertson-Walker Universe are of great importance for the study of cosmology. In the past, the linear cosmological perturbations have been extensively explored [1–7] and widely applied to the large scale structure [8], the cosmic microwave background (CMB) [9–20], relic gravitational waves (RGW) [21–34], etc. Nowadays, observations with increasing accuracy [35–44] are capable of probing the nonlinear cosmological perturbations. Throughout all cosmic expansion stages, the nonlinear perturbations exist at small scales; may accumulate substantially in the course of expansion; and can leave possibly observable effects in the large scale structure, the CMB anisotropies and polarization [40,41], the non-Gaussianity of primordial perturbation [42], primordial black holes [45], etc.

The metric perturbations in GR are usually decomposed into scalars, vectors, and tensors. In regard to cosmology, the linear vector modes decrease with cosmic expansion and can be neglected in most applications, while the scalar and tensor modes are generated with comparable magnitudes during inflation. So three types of nonlinear couplings are more interesting in cosmological studies: scalar-scalar, scalar-tensor, and tensor-tensor. In the literature [46–53], the studies of the second-order perturbations mainly consider the scalar-scalar coupling, whereas the scalar-tensor and the tensor-tensor couplings have not been sufficiently investigated. The authors of Ref. [54] used the Arnowitt-Deser-Misner (ADM) method on the second-order perturbation with the tensor-tensor coupling for a zero pressure dust [55] and made comparisons between the comoving gauge and the synchronous gauge [56]. The authors of Ref. [57] presented a complete set of second-order equations for scalar, vector, and tensor perturbations in the ADM formulation for a general gauge. The authors of Ref. [58] studied the gauge dependence of the spectrum of the second-order tensor mode with scalar-scalar couplings during the matter dominated (MD) stage and showed that the second-order tensor mode could dominate over the first-order tensor mode. The authors of Refs. [59,60] studied the second-order perturbed Einstein equation during the MD stage driven by the zero pressure dust in synchronous coordinates, including only the scalar-scalar coupling, and derived the solutions of second-order scalar and tensor perturbations. In our previous works, we extended the study of Refs. [59,60], including the scalar-scalar [61], scalar-tensor, and tensor-tensor couplings [62]; obtained all the second-order solutions of the scalar, vector, tensor, and density perturbations in the MD stage; identified the residual gauge modes under synchronous-to-synchronous transformations; and also presented the second-order transformation from synchronous to Poisson coordinates. In Ref. [63], we studied the second-order perturbations in the RD stage driven by a relativistic perfect fluid, including only the scalar-scalar coupling; derived the second-order formal solutions of the scalar, vector, tensor, density, and velocity perturbations in the integral form;

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and also identified the residual gauge modes within the synchronous coordinates. In this paper as a continuation of Ref. [63], we shall include the scalar-tensor and tensor-tensor couplings. We are motivated not only by nonlinearity within one type of metric perturbation, such as the transfer of perturbation power between k-modes, etc., but also by the transfer of perturbation power between different types of perturbations, such as those between scalar and tensor modes, etc. We shall derive the second-order solutions of scalar, vector, tensor, density, and velocity perturbations in the integral form, perform the second-order synchronous-to-synchronous transformation, and identify the residual gauge modes. In addition, we shall present the second-order gauge transformations of the solutions from the synchronous to the Poisson coordinates; the latter is also commonly used in cosmological studies. Thus, together with the results in Ref. [63], a complete set of solutions of second-order cosmological perturbations of the RD stage are available.

In Sec. II, we give some basic setups of the metric perturbations and the relativistic fluid model. We also list the solutions of first-order perturbations of the RD stage for use in this paper.

In Sec. III, we decompose the second-order perturbed Einstein equation into the equations of second-order scalar, vector, tensor metric perturbations with scalar-tensor and tensor-tensor couplings as part of the effective sources, and we also derive the equations of the second-order density contrast and velocity from the covariant conservation of the stress tensor.

In Sec. IV, we derive all the second-order solutions in the integral form. We also explain how to do the time and momentum integrals that occur in the solutions by two examples.

In Sec. V, we perform the synchronous-to-synchronous transformation and identify the residual gauge modes in the second-order solutions.

In Sec. VI, we perform the transformations from synchronous coordinates to Poisson coordinates of the second-order solutions in Sec. IV and also of those with the scalar-scalar couplings that have been derived in Ref. [63].

Section VII gives the conclusions and discussions.

The Appendix gives a list of the second-order perturbed Einstein equation and covariant conservation equations of the stress tensor with the scalar-tensor and tensor-tensor couplings in a general RW spacetime. We use units with the speed of light $c = 1$.

II. BASIC SETUPS AND FIRST-ORDER SOLUTIONS

Here we give the basic setups of this paper, using the same notations as in Refs. [60–63]. A flat Robertson-Walker (RW) metric in synchronous coordinates is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)[-d\tau^2 + \gamma_{ij}dx^i dx^j], \tag{2.1}$$

$$\gamma_{ij} = \delta_{ij} + \gamma_{ij}^{(1)} + \frac{1}{2}\gamma_{ij}^{(2)}, \tag{2.2}$$

with $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ being the first- and second-order metric perturbation, respectively. Writing $g^{ij} = a^{-2}\gamma^{ij}$, one has

$$\gamma^{ij} = \delta^{ij} - \gamma^{(1)ij} - \frac{1}{2}\gamma^{(2)ij} + \gamma^{(1)ik}\gamma_k^{(1)j}. \tag{2.3}$$

Raising and lowering the three-dimensional spatial indices will be done by δ^{ij}. The metric perturbations can be further written as

$$\gamma_{ij}^{(A)} = -2\phi^{(A)}\delta_{ij} + \chi_{ij}^{(A)}, \quad \text{with} \quad A = 1,2, \tag{2.4}$$

where $\phi^{(A)}$ is the trace part of scalar perturbation, and $\chi_{ij}^{(A)}$ is traceless and can be further decomposed into the following:

$$\chi_{ij}^{(A)} = D_{ij}\chi^{\|(A)} + \chi_{ij}^{\perp(A)} + \chi_{ij}^{\top(A)}, \quad \text{with} \quad A = 1,2, \tag{2.5}$$

where $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$, $\chi^{\|(A)}$ is a scalar function, and $D_{ij}\chi^{\|(A)}$ is the traceless part of the scalar perturbation. The vector metric perturbation satisfies a condition $\partial^i\partial^j\chi_{ij}^{\perp(A)}=0$ and can be written as

$$\chi_{ij}^{\perp(A)} = \partial_i B_j^{(A)} + \partial_j B_i^{(A)}, \quad \text{with} \quad \partial^i B_i^{(A)} = 0, \quad A = 1,2, \tag{2.6}$$

where $B_i^{(A)}$ is a curl vector and has two independent modes. The tensor metric perturbation satisfies the traceless and transverse condition: $\chi^{\top(A)i}{}_i = 0$, $\partial^i\chi_{ij}^{\top(A)} = 0$, having two independent modes.

The RD stage of expansion is driven by a relativistic fluid (without a shear stress), whose energy-momentum tensor is $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + g_{\mu\nu}p$, where ρ and p are, respectively, the energy density and pressure measured by a comoving observer in the locally inertial frame, and $U^\mu = \frac{dx^\mu}{d\lambda}$ with $d\lambda^2 = -ds^2$ is the fluid 4-velocity with a normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$. We write

$$\rho = \rho^{(0)} + \rho^{(1)} + \frac{1}{2}\rho^{(2)}, \tag{2.7}$$

$$p = p^{(0)} + p^{(1)} + \frac{1}{2}p^{(2)}, \tag{2.8}$$

where $\rho^{(0)}$ is the background density and $\rho^{(1)}$, $\rho^{(2)}$ are the respective first- and second-order density perturbations. We introduce the density contrast

$$\delta^{(A)} \equiv \frac{\rho^{(A)}}{\rho^{(0)}}, \quad A = 1, 2. \tag{2.9}$$

One can define the following parameters of a fluid:

$$c_s^2 \equiv \frac{p^{(0)\prime}}{\rho^{(0)\prime}}, \qquad c_L^2 \equiv \frac{p^{(1)}}{\rho^{(1)}}, \qquad c_N^2 \equiv \frac{p^{(2)}}{\rho^{(2)}}, \tag{2.10}$$

where c_s is the sound speed. For the relativistic fluid, $c_s^2 = \frac{1}{3}$, which also equals the state of matter $\omega = \frac{p^{(0)}}{\rho^{(0)}} = \frac{1}{3}$, and $c_L^2 = \frac{1}{3}$ is taken [64]. In this paper we assume $c_N^2 = \frac{1}{3}$ for computation convenience; its actual value should be determined by future experiments. The expansion of U^μ is also expanded up to second order,

$$U^\mu \equiv U^{(0)\mu} + U^{(1)\mu} + \frac{1}{2} U^{(2)\mu}, \tag{2.11}$$

where

$$U^{(0)0} = a^{-1}, \qquad U^{(1)0} = 0, \qquad U^{(2)0} = a^{-1} v^{(1)m} v_m^{(1)}, \tag{2.12}$$

$$U^{(0)i} = 0, \qquad U^{(1)i} = a^{-1} v^{(1)i}, \qquad U^{(2)i} = a^{-1} v^{(2)i}, \tag{2.13}$$

with the 3-velocity $v^i \equiv \frac{dx^i}{d\tau} = \frac{U^i}{U^0} = v^{(1)i} + \frac{1}{2} v^{(2)i}$ [12]. $v^{(A)i}$ can be decomposed into a noncurl and a curl part as $v^{(A)i} = v^{\parallel(A),i} + v^{\perp(A)i}$, with $\partial_i v^{\perp(A)i} = 0$ for $A = 1,\ 2$. We also have

$$U_0 = -a\left(1 + \frac{1}{2} v^{(1)m} v_m^{(1)}\right),$$
$$U_i = a\left(v_i^{(1)} + \gamma_{ij}^{(1)} v^{(1)j} + \frac{1}{2} v_i^{(2)}\right). \tag{2.14}$$

The Einstein equation is expanded up to the second order of perturbations:

$$G_{\mu\nu}^{(A)} \equiv R_{\mu\nu}^{(A)} - \frac{1}{2}[g_{\mu\nu} R]^{(A)} = 8\pi G T_{\mu\nu}^{(A)}, \quad \text{with} \quad A = 0, 1, 2. \tag{2.15}$$

For each order of (2.15), the (00) component is the energy constraint, $(0i)$ components are the momentum constraints, and (ij) components are the evolution equations. The zeroth-order Einstein equation gives the Friedman equations $(\frac{a'}{a})^2 = \frac{8\pi G}{3} a^2 \rho^{(0)}$ and $-2\frac{a''}{a} + (\frac{a'}{a})^2 = 8\pi G a^2 p^{(0)}$, which have a solution for the RD stage $a(\tau) \propto \tau$ and $\rho^{(0)}(\tau) = \frac{3}{8\pi G} \frac{a'^2(\tau)}{a^4(\tau)} \propto \tau^{-4}$. The covariant conservation of the stress tensor is

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.16}$$

The dynamics of gravitational systems is determined by (2.15) and (2.16). The component $\mu = 0$ of (2.16) gives the energy conservation,

$$\begin{aligned} & g^{00} p_{,0} + \partial_0[(\rho+p)U^0U^0] + \partial_i[(\rho+p)U^0U^i] \\ & + \Gamma^0_{00}(\rho+p)U^0U^0 + \Gamma^0_{ij}(\rho+p)U^iU^j \\ & + \Gamma^0_{00}(\rho+p)U^0U^0 + \Gamma^k_{k0}(\rho+p)U^0U^0 \\ & + \Gamma^k_{km}(\rho+p)U^0U^m = 0, \end{aligned} \tag{2.17}$$

and the component $\mu = i$ gives the momentum conservation,

$$\begin{aligned} & g^{ik} p_{,k} + \partial_0[(\rho+p)U^iU^0] + \partial_m[(\rho+p)U^iU^m] \\ & + 2\Gamma^i_{m0}(\rho+p)U^mU^0 + \Gamma^i_{ml}(\rho+p)U^mU^l \\ & + \Gamma^0_{00}(\rho+p)U^iU^0 + \Gamma^k_{k0}(\rho+p)U^iU^0 \\ & + \Gamma^k_{kl}(\rho+p)U^iU^l = 0, \end{aligned} \tag{2.18}$$

where the nonvanishing Christopher symbols are listed in (A1)–(A4) of Ref. [63]. To each order of perturbation, (2.17) and (2.18) will determine ρ and U^μ.

The first-order perturbation in the RD stage is known in the literature and a complete list is given in Ref. [63]. Here we quote the first-order gauge-invariant modes for use in this paper. The transverse part of first-order velocity $U^{\perp(1)i}$ and the vector mode $\chi_{ij}^{\perp(1)}$ of the metric are decaying during the RD stage, so we take

$$U^{\perp(1)i} = \chi_{ij}^{\perp(1)} = C_{1ij} = 0, \tag{2.19}$$

which amounts to assuming that the relativistic fluid is irrotational during the RD stage. This will simplify the second-order calculation as the coupling terms involving first-order vectors vanish. The k-mode of the first-order longitudinal velocity with a constant c_L^2 is

$$\begin{aligned} v_{\mathbf{k}}^{\parallel(1)}(\tau) = {} & d_1 \left(\frac{c_L k\tau}{2}\right)^{-\frac{3c_L^2}{2}} \Gamma\left(\frac{3c_L^2}{2} + 1\right) \Big(J_{\frac{3c_L^2}{2}}(c_L k\tau) \\ & + (c_L k\tau) J_{\frac{3c_L^2}{2}+1}(c_L k\tau)\Big) \\ & + d_2 \left(\frac{c_L k\tau}{2}\right)^3 {}_1F_2\left(2; \frac{5}{2}, \frac{3c_L^2}{2} + \frac{5}{2}; -\left(\frac{c_L k\tau}{2}\right)^2\right) \\ & + d_3 \left(\frac{c_L k\tau}{2}\right)^{-3c_L^2} {}_1F_2\left(\frac{1}{2} - \frac{3c_L^2}{2}; -\frac{3c_L^2}{2} \right. \\ & \left. -\frac{1}{2}, 1 - \frac{3c_L^2}{2}; -\left(\frac{c_L k\tau}{2}\right)^2\right), \end{aligned} \tag{2.20}$$

where d_1, d_2, d_3 are $\mathbf{k}$-dependent integration constants; $J_q(x)$ is the Bessel function; $\Gamma(x)$ is the gamma function; and ${}_pF_q$ is the generalized hypergeometric function. From the solution of $v_{\mathbf{k}}^{\|(1)}$, other scalar perturbations can also be given [63]. For simplicity, we take $c_L^2 = \frac{1}{3}$ [7]; then (2.20) reduces to

$$v_{\mathbf{k}}^{\|(1)} = D_2(\mathbf{k})\left(\frac{2}{k\tau} + \frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + D_3(\mathbf{k})\left(\frac{2}{k\tau} - \frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}, \quad (2.21)$$

where $D_2 = -\frac{3\sqrt{3}}{4}d_2 + \frac{i}{2}\sqrt{3}d_1$ and $D_3 = -\frac{3\sqrt{3}}{4}d_2 + \frac{i}{2}\sqrt{3}d_1$ are $\mathbf{k}$-dependent constants. The first-order density contrast is

$$\delta_{\mathbf{k}}^{(1)} = D_2\left(\frac{8}{k\tau^2} + \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{8}{k\tau^2} - \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}. \quad (2.22)$$

The two scalar modes of the metric perturbation are

$$\phi_{\mathbf{k}}^{(1)} = D_2\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} - \frac{2k}{3}\int^{\tau}[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'}, \quad (2.23)$$

$$\chi_{\mathbf{k}}^{\|(1)} = D_2\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - D_3\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{4}{k}\int^{\tau}[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'}. \quad (2.24)$$

And the equation for the tensor mode is

$$\chi_{ij}^{\top(1)''} + \frac{2}{\tau}\chi_{ij}^{\top(1)'} - \nabla^2\chi_{ij}^{\top(1)} = 0, \quad (2.25)$$

which has the solution written in terms of Fourier modes

$$\chi_{ij}^{\top(1)}(\mathbf{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{s=+,\times}\overset{s}{\epsilon}_{ij}(\mathbf{k})\overset{s}{h}_{\mathbf{k}}(\tau), \quad \mathbf{k} = k\hat{k}, \quad (2.26)$$

with two polarization tensors satisfying

$$\overset{s}{\epsilon}_{ij}(\mathbf{k})\delta^{ij} = 0, \qquad \overset{s}{\epsilon}_{ij}(\mathbf{k})k^i = 0, \qquad \overset{s}{\epsilon}_{ij}(\mathbf{k})\overset{s'}{\epsilon}{}^{ij}(\mathbf{k}) = 2\delta_{ss'}. \quad (2.27)$$

For RGW generated during inflation [21,27,30], the two polarization modes $\overset{s}{h}_k$ with $s = +, \times$ are usually assumed to be statistically equivalent, the superscript s can be dropped. During the RD stage, the mode is given by

$$h_{\mathbf{k}}(\tau) = \frac{1}{a(\tau)}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\tau}{2}}[b_1(\mathbf{k})H_{\frac{1}{2}}^{(1)}(k\tau) + b_2(\mathbf{k})H_{\frac{1}{2}}^{(2)}(k\tau)] = \frac{1}{\tau}\frac{i}{\sqrt{2k}}[-b_1 e^{ik\tau} + b_2 e^{-ik\tau}], \quad (2.28)$$

where b_1, b_2 are $\mathbf{k}$-dependent coefficients, to be determined by the initial condition during inflation or a possible subsequent reheating stage [21,30]. There are cosmic processes occurring during the RD stage, such as the QCD transition and e^+e^- annihilation [65], which modify only slightly the amplitude of RGW and will be neglected in this study.

The scalar modes (2.23) and (2.24), the density contrast (2.22), and the longitudinal velocity (2.21) all contain a factor $e^{\pm ik\tau/\sqrt{3}}$, so that they are waves propagating at the sound speed $\frac{1}{\sqrt{3}}$ of the relativistic fluid. On the other hand, the tensor modes (2.28) are waves propagating at the speed of light. In the MD stage [60–62], the scalar and density contrast are not waves and do not propagate; only the tensor modes still propagate at the speed of light. Therefore, whether or not the scalar modes propagate actually depends on the background matter; nevertheless, the tensor modes always propagate at speed of light, regardless of the background matter. Thus, the tensor modes are radiative as dynamic degrees of freedom, differing from the scalar and vector modes.

III. THE SECOND-ORDER PERTURBED EQUATIONS

A. Equations with scalar-tensor couplings

The second-order perturbed Einstein equation is listed in the Appendix for a general RW spacetime. As said earlier in (2.19), the transverse vector mode and the curl velocity are dropped; there remain only three types of couplings: scalar-scalar, scalar-tensor, and tensor-tensor. The scalar-scalar coupling has been studied in Ref. [63]. Now we consider the scalar-tensor coupling.

The (00) component of the second-order perturbed Einstein equation is

$$G_{00}^{(2)} = 8\pi G T_{00}^{(2)}, \quad (3.1)$$

where $G_{00}^{(2)}$ and $T_{00}^{(2)}$ are given by (A16) and (A22) in Ref. [63]. For the scalar-tensor coupling, by using the Friedmann equation and moving the coupling terms to the rhs, Eq. (3.1) gives the second-order energy constraint

$$-\frac{6}{\tau}\phi_{s(t)}^{(2)'} + 2\nabla^2\phi_{s(t)}^{(2)} + \frac{1}{3}\nabla^2\nabla^2\chi_{s(t)}^{\|(2)} = \frac{3}{\tau^2}\delta_{s(t)}^{(2)} + E_{s(t)}, \quad (3.2)$$

where a subscript $s(t)$ in $\phi_{s(t)}^{(2)}$, etc., indicates the case of scalar-tensor coupling, and $E_{s(t)}$ is the scalar-tensor coupling terms as follows:

$$E_{s(t)} \equiv 2\phi^{(1),lm}\chi^{\top(1)}_{lm} + \frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\|(1)',lm} + \frac{2}{\tau}\chi^{\top(1)}_{lm}\chi^{\|(1)',lm} + \frac{2}{\tau}\chi^{\top(1)'}_{lm}\chi^{\|(1),lm} + \frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm} - \chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{2}\chi^{\top(1),l}_{mn}\chi^{\|(1),mn}_{,l}, \tag{3.3}$$

which is part of the effective source of the 2nd-order energy constraint.

The $(0i)$ component of the second-order perturbed Einstein equation is

$$G^{(2)}_{0i} = R^{(2)}_{0i} = 8\pi G T^{(2)}_{0i}, \tag{3.4}$$

where $R^{(2)}_{0i}$ and $T^{(2)}_{0i}$ are given by (A13) and (A23) in Ref. [63]. This gives the second-order momentum constraint equation for the scalar-tensor case as follows:

$$2\phi^{(2)'}_{s(t),i} + \frac{1}{2}D_{ij}\chi^{\|(2)',j}_{s(t)} + \frac{1}{2}\chi^{\perp(2)',j}_{s(t)ij} = -\frac{4}{\tau^2}v^{(2)}_{s(t)i} + M_{s(t)i}, \tag{3.5}$$

where

$$M_{s(t)i} \equiv -\frac{8}{\tau^2}\chi^{\top(1)}_{il}v^{\|(1),l} + \phi^{(1),l}\chi^{\top(1)'}_{il} - 2\phi^{(1)',l}\chi^{\top(1)}_{il} - \chi^{\top(1)'}_{lm,i}\chi^{\|(1),lm} - \frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\|(1),lm}_{,i} - \frac{1}{2}\chi^{\top(1)}_{lm,i}\chi^{\|(1)',lm} + \chi^{\top(1)'}_{il,m}\chi^{\|(1),lm} - \frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\chi^{\|(1)',l} + \frac{2}{3}\chi^{\top(1)'}_{il}\nabla^2\chi^{\|(1),l} \tag{3.6}$$

is part of the effective source. Equation (3.5) can be further decomposed into two equations. Applying $\nabla^{-2}\partial^i$ on (3.5) gives the longitudinal momentum constraint

$$2\phi^{(2)'}_{s(t)} + \frac{1}{3}\nabla^2\chi^{\|(2)'}_{s(t)} = -\frac{4}{\tau^2}v^{\|(2)}_{s(t)} + \nabla^{-2}M^{,l}_{s(t)l}, \tag{3.7}$$

where

$$M^{,l}_{s(t)l} = -\frac{8}{\tau^2}\chi^{\top(1)}_{lm}v^{\|(1),lm} + \phi^{(1),lm}\chi^{\top(1)'}_{lm} - 2\phi^{(1)',lm}\chi^{\top(1)}_{lm} - \frac{1}{2}\chi^{\top(1)'}_{lm,n}\chi^{\|(1),lmn} + \frac{1}{6}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1),lm} - \chi^{\|(1),lm}\nabla^2\chi^{\top(1)'}_{lm} - \frac{1}{2}\chi^{\top(1)}_{lm,n}\chi^{\|(1)',lmn} - \frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1)',lm} - \frac{1}{2}\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)}_{lm}. \tag{3.8}$$

A combination [(3.5)–∂_i(3.7)] gives the transverse momentum constraint

$$\frac{1}{2}\chi^{\perp(2)',j}_{s(t)ij} = -\frac{4}{\tau^2}v^{\perp(2)}_{s(t)i} + (M_{s(t)i} - \partial_i\nabla^{-2}M^{,l}_{s(t)l}), \tag{3.9}$$

where

$$\begin{aligned}(M_{s(t)i} - \partial_i\nabla^{-2}M^{,l}_{s(t)l}) = {} & -\frac{8}{\tau^2}\chi^{\top(1)}_{il}v^{\|(1),l} + \phi^{(1),l}\chi^{\top(1)'}_{il} - 2\phi^{(1)',l}\chi^{\top(1)}_{il} - \frac{1}{2}\chi^{\top(1)'}_{lm,i}\chi^{\|(1),lm} - \frac{1}{2}\chi^{\top(1)}_{lm,i}\chi^{\|(1)',lm} + \chi^{\top(1)'}_{il,m}\chi^{\|(1),lm} \\ & - \frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\chi^{\|(1)',l} + \frac{2}{3}\chi^{\top(1)'}_{il}\nabla^2\chi^{\|(1),l} + \partial_i\nabla^{-2}\Big[\frac{8}{\tau^2}\chi^{\top(1)}_{lm}v^{\|(1),lm} - \phi^{(1),lm}\chi^{\top(1)'}_{lm} \\ & + 2\phi^{(1)',lm}\chi^{\top(1)}_{lm} - \frac{1}{2}\chi^{\top(1)'}_{lm,n}\chi^{\|(1),lmn} - \frac{2}{3}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1),lm} + \frac{1}{2}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)'}_{lm} \\ & + \frac{1}{2}\chi^{\top(1)}_{lm,n}\chi^{\|(1)',lmn} + \frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1)',lm} + \frac{1}{2}\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)}_{lm}\Big].\end{aligned} \tag{3.10}$$

The (ij) component of the second-order perturbed Einstein equation is

$$G^{(2)}_{ij} = 8\pi G T^{(2)}_{ij}, \tag{3.11}$$

where $G^{(2)}_{ij}$ and $T^{(2)}_{ij}$ are given in (A18) and (A24) of Ref. [63]. For the scalar-tensor case, (3.11) gives the second-order evolution equation

$$\begin{aligned}& 2\phi^{(2)''}_{s(t)}\delta_{ij} + \frac{4}{\tau}\phi^{(2)'}_{s(t)}\delta_{ij} + \phi^{(2)}_{s(t),ij} - \nabla^2\phi^{(2)}_{s(t)}\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\|(2)''}_{s(t)} + \frac{1}{\tau}D_{ij}\chi^{\|(2)'}_{s(t)} + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(2)}_{s(t)} - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(2)}_{s(t)} \\ & + \frac{1}{2}\chi^{\perp(2)''}_{s(t)ij} + \frac{1}{\tau}\chi^{\perp(2)'}_{s(t)ij} + \frac{1}{2}\chi^{\top(2)''}_{s(t)ij} + \frac{1}{\tau}\chi^{\top(2)'}_{s(t)ij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{s(t)ij} = \frac{3c_N^2}{\tau^2}\delta^{(2)}_{s(t)}\delta_{ij} + S_{s(t)ij},\end{aligned} \tag{3.12}$$

where

$$S_{s(t)ij} \equiv \frac{6}{\tau^2} c_L^2 \chi_{ij}^{\top(1)} \delta^{(1)} - 6\phi^{(1)''} \chi_{ij}^{\top(1)} - \frac{12}{\tau} \phi^{(1)'} \chi_{ij}^{\top(1)} + 4\chi_{ij}^{\top(1)} \nabla^2 \phi^{(1)} - \phi^{(1)'} \chi_{ij}^{\top(1)'} - \phi^{(1),l} \chi_{lj,i}^{\top(1)} - \phi^{(1),l} \chi_{li,j}^{\top(1)} + 3\phi^{(1),l} \chi_{ij,l}^{\top(1)}$$
$$+ 2\phi^{(1)} \nabla^2 \chi_{ij}^{\top(1)} - 2\phi_{,i}^{(1),l} \chi_{lj}^{\top(1)} - 2\phi_{,j}^{(1),l} \chi_{li}^{\top(1)} - \chi_{lm}^{\top(1)''} \chi^{\|(1),lm} \delta_{ij} - \chi_{lm}^{\top(1)} \chi^{\|(1)'',lm} \delta_{ij} - \frac{2}{\tau} \chi_{lm}^{\top(1)'} \chi^{\|(1),lm} \delta_{ij}$$
$$- \frac{2}{\tau} \chi_{lm}^{\top(1)} \chi^{\|(1)',lm} \delta_{ij} + \chi_{li}^{\top(1)'} \chi_{,j}^{\|(1)',l} + \chi_{lj}^{\top(1)'} \chi_{,i}^{\|(1)',l} - \frac{3}{2} \chi_{lm}^{\top(1)'} \chi^{\|(1)',lm} \delta_{ij} - \frac{1}{3} \chi_{li}^{\top(1)} \nabla^2 \chi_{,j}^{\|(1),l} - \frac{1}{3} \chi_{lj}^{\top(1)} \nabla^2 \chi_{,i}^{\|(1),l}$$
$$- \chi_{lm,ij}^{\top(1)} \chi^{\|(1),lm} + \chi^{\|(1),lm} \nabla^2 \chi_{lm}^{\top(1)} \delta_{ij} - \frac{1}{2} \chi_{lm,j}^{\top(1)} \chi_{,i}^{\|(1),lm} - \frac{1}{2} \chi_{lm,i}^{\top(1)} \chi_{,j}^{\|(1),lm} + \frac{1}{2} \chi^{\|(1),nml} \chi_{nm,l}^{\top(1)} \delta_{ij} - \frac{2}{3} \chi_{ij}^{\top(1)'} \nabla^2 \chi^{\|(1)'}$$
$$+ \frac{2}{3} \chi_{ij}^{\top(1)} \nabla^2 \nabla^2 \chi^{\|(1)} + \frac{1}{3} \nabla^2 \chi_{ij}^{\top(1)} \nabla^2 \chi^{\|(1)} + \frac{1}{3} \chi_{lj,i}^{\top(1)} \nabla^2 \chi^{\|(1),l} + \frac{1}{3} \chi_{li,j}^{\top(1)} \nabla^2 \chi^{\|(1),l} + \chi_{lj,im}^{\top(1)} \chi^{\|(1),lm}$$
$$+ \chi_{li,jm}^{\top(1)} \chi^{\|(1),lm} - \chi_{ij,lm}^{\top(1)} \chi^{\|(1),lm} \tag{3.13}$$

is part of the effective source.

We also need to decompose the evolution equation (3.12) into the trace equation and the traceless equation. First, the trace equation from (3.12) is

$$2\phi_{s(t)}^{(2)''} + \frac{4}{\tau} \phi_{s(t)}^{(2)'} - \frac{2}{3} \nabla^2 \phi_{s(t)}^{(2)} - \frac{1}{9} \nabla^2 \nabla^2 \chi_{s(t)}^{\|(2)} = \frac{3}{\tau^2} c_N^2 \delta_{s(t)}^{(2)} + \frac{1}{3} S_{s(t)l}^{l}, \tag{3.14}$$

where

$$S_{s(t)l}^{l} = -4\phi^{(1),lm} \chi_{lm}^{\top(1)} - 3\chi_{lm}^{\top(1)''} \chi^{\|(1),lm} - 3\chi_{lm}^{\top(1)} \chi^{\|(1)'',lm} - \frac{6}{\tau} \chi_{lm}^{\top(1)'} \chi^{\|(1),lm} - \frac{6}{\tau} \chi_{lm}^{\top(1)} \chi^{\|(1)',lm} - \frac{5}{2} \chi_{lm}^{\top(1)'} \chi^{\|(1)',lm}$$
$$- \frac{2}{3} \chi_{lm}^{\top(1)} \nabla^2 \chi^{\|(1),lm} + 2\chi^{\|(1),lm} \nabla^2 \chi_{lm}^{\top(1)} + \frac{1}{2} \chi_{lm,n}^{\top(1)} \chi^{\|(1),lmn}. \tag{3.15}$$

The traceless equation from (3.12) is

$$D_{ij} \phi_{s(t)}^{(2)} + \frac{1}{2} D_{ij} \chi_{s(t)}^{\|(2)''} + \frac{1}{\tau} D_{ij} \chi_{s(t)}^{\|(2)'} + \frac{1}{6} \nabla^2 D_{ij} \chi_{s(t)}^{\|(2)} + \frac{1}{2} \chi_{s(t)ij}^{\perp(2)''} + \frac{1}{\tau} \chi_{s(t)ij}^{\perp(2)'} + \frac{1}{2} \chi_{s(t)ij}^{\top(2)''} + \frac{1}{\tau} \chi_{s(t)ij}^{\top(2)'} - \frac{1}{2} \nabla^2 \chi_{s(t)ij}^{\top(2)} = \bar{S}_{s(t)ij}, \tag{3.16}$$

where

$$\bar{S}_{s(t)ij} \equiv S_{s(t)ij} - \frac{1}{3} S_{s(t)l}^{l} \delta_{ij}$$
$$= \frac{6}{\tau^2} c_L^2 \chi_{ij}^{\top(1)} \delta^{(1)} - 6\phi^{(1)''} \chi_{ij}^{\top(1)} - \frac{12}{\tau} \phi^{(1)'} \chi_{ij}^{\top(1)} + 4\chi_{ij}^{\top(1)} \nabla^2 \phi^{(1)} - \phi^{(1)'} \chi_{ij}^{\top(1)'} - \phi^{(1),l} \chi_{lj,i}^{\top(1)} - \phi^{(1),l} \chi_{li,j}^{\top(1)} + 3\phi^{(1),l} \chi_{ij,l}^{\top(1)}$$
$$+ 2\phi^{(1)} \nabla^2 \chi_{ij}^{\top(1)} - 2\phi_{,i}^{(1),l} \chi_{lj}^{\top(1)} - 2\phi_{,j}^{(1),l} \chi_{li}^{\top(1)} + \frac{4}{3} \phi^{(1),lm} \chi_{lm}^{\top(1)} \delta_{ij} + \chi_{li}^{\top(1)'} \chi_{,j}^{\|(1)',l} + \chi_{lj}^{\top(1)'} \chi_{,i}^{\|(1)',l} - \frac{2}{3} \chi_{lm}^{\top(1)'} \chi^{\|(1)',lm} \delta_{ij}$$
$$- \frac{1}{3} \chi_{li}^{\top(1)} \nabla^2 \chi_{,j}^{\|(1),l} - \frac{1}{3} \chi_{lj}^{\top(1)} \nabla^2 \chi_{,i}^{\|(1),l} + \frac{2}{9} \chi_{lm}^{\top(1)} \nabla^2 \chi^{\|(1),lm} \delta_{ij} - \chi_{lm,ij}^{\top(1)} \chi^{\|(1),lm} + \frac{1}{3} \chi^{\|(1),lm} \nabla^2 \chi_{lm}^{\top(1)} \delta_{ij} - \frac{1}{2} \chi_{lm,j}^{\top(1)} \chi_{,i}^{\|(1),lm}$$
$$- \frac{1}{2} \chi_{lm,i}^{\top(1)} \chi_{,j}^{\|(1),lm} + \frac{1}{3} \chi_{lm,n}^{\top(1)} \chi^{\|(1),lmn} \delta_{ij} - \frac{2}{3} \chi_{ij}^{\top(1)'} \nabla^2 \chi^{\|(1)'} + \frac{2}{3} \chi_{ij}^{\top(1)} \nabla^2 \nabla^2 \chi^{\|(1)} + \frac{1}{3} \nabla^2 \chi_{ij}^{\top(1)} \nabla^2 \chi^{\|(1)}$$
$$+ \frac{1}{3} \chi_{lj,i}^{\top(1)} \nabla^2 \chi^{\|(1),l} + \frac{1}{3} \chi_{li,j}^{\top(1)} \nabla^2 \chi^{\|(1),l} + \chi_{lj,im}^{\top(1)} \chi^{\|(1),lm} + \chi_{li,jm}^{\top(1)} \chi^{\|(1),lm} - \chi_{ij,lm}^{\top(1)} \chi^{\|(1),lm}, \tag{3.17}$$

which still contains the scalar, vector, and tensor components. Applying $3\nabla^{-2}\nabla^{-2}\partial_i\partial_j$ on (3.16) gives the evolution equation for the scalar $\chi_{s(t)}^{\|(2)}$ as follows:

$$\chi_{s(t)}^{\|(2)''} + \frac{2}{\tau} \chi_{s(t)}^{\|(2)'} + \frac{1}{3} \nabla^2 \chi_{s(t)}^{\|(2)} + 2\phi_{s(t)}^{(2)} = 3\nabla^{-2}\nabla^{-2} \bar{S}_{s(t)lm}^{,lm}, \tag{3.18}$$

where

$$
\begin{aligned}
\bar{S}^{,lm}_{s(t)lm} = & -\frac{2}{3}\nabla^2\nabla^2[\chi^{\parallel(1),lm}\chi^{\top(1)}_{lm}] + \nabla^2\left[\frac{4}{3}\phi^{(1),lm}\chi^{\top(1)}_{lm} + \frac{1}{3}\chi^{\top(1)\prime}_{lm}\chi^{\parallel(1)\prime,lm} + \frac{8}{9}\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm} + \frac{7}{6}\chi^{\top(1)}_{lm,n}\chi^{\parallel(1),lmn}\right] \\
& + 6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lm}\delta^{(1),lm} - 6\phi^{(1)\prime\prime,lm}\chi^{\top(1)}_{lm} - \frac{12}{\tau}\phi^{(1)\prime,lm}\chi^{\top(1)}_{lm} - \phi^{(1)\prime,lm}\chi^{\top(1)\prime}_{lm} - 3\phi^{(1),lmn}\chi^{\top(1)}_{lm,n} - \chi^{\parallel(1)\prime,lm}\nabla^2\chi^{\top(1)\prime}_{lm} \\
& + \frac{1}{3}\chi^{\top(1)\prime}_{lm}\nabla^2\chi^{\parallel(1)\prime,lm} - \frac{1}{2}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\parallel(1),lmn} + \nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm} + \frac{3}{2}\chi^{\parallel(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}.
\end{aligned} \tag{3.19}
$$

By a combination $\partial^i[(3.16) - \frac{1}{2}D_{ij}(3.18)]$, one has

$$
\frac{1}{2}\chi^{\perp(2)\prime\prime,l}_{s(t)lj} + \frac{1}{\tau}\chi^{\perp(2)\prime,l}_{s(t)lj} = \bar{S}^{,l}_{s(t)lj} - \partial_j\nabla^{-2}\bar{S}^{,lm}_{s(t)lm}. \tag{3.20}
$$

By $\nabla^{-2}[\partial_i(3.20) + (i \leftrightarrow j)]$, using (2.6), one gets the evolution equation for the vector mode

$$
\frac{1}{2}\chi^{\perp(2)\prime\prime}_{s(t)ij} + \frac{1}{\tau}\chi^{\perp(2)\prime}_{s(t)ij} = V_{s(t)ij}, \tag{3.21}
$$

where

$$
\begin{aligned}
V_{s(t)ij} \equiv & \nabla^{-2}\bar{S}^{,l}_{s(t)lj,i} + \nabla^{-2}\bar{S}^{,l}_{s(t)li,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{s(t)lm,ij} \\
= & \frac{2}{3}\partial_i\partial_j[\chi^{\parallel(1),lm}\chi^{\top(1)}_{lm}] + \partial_i\partial_j\nabla^{-2}\left[-\phi^{(1),lm}\chi^{\top(1)}_{lm} - \frac{5}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm} - \frac{1}{6}\chi^{\parallel(1),lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{4}{3}\chi^{\top(1)}_{lm,n}\chi^{\parallel(1),lmn}\right] \\
& + \partial_i\nabla^{-2}\left[\frac{6}{\tau^2}c_L^2\chi^{\top(1)}_{lj}\delta^{(1),l} - 6\phi^{(1)\prime\prime,l}\chi^{\top(1)}_{lj} - \frac{12}{\tau}\phi^{(1)\prime,l}\chi^{\top(1)}_{lj} - \phi^{(1)\prime,l}\chi^{\top(1)\prime}_{lj} + 2\chi^{\top(1)}_{lj}\nabla^2\phi^{(1),l} + \phi^{(1),l}\nabla^2\chi^{\top(1)}_{lj} - \phi^{(1),lm}_{,j}\chi^{\top(1)}_{lm}\right. \\
& + \chi^{\top(1)\prime}_{lj,m}\chi^{\parallel(1)\prime,lm} - \chi^{\top(1)\prime}_{lm,j}\chi^{\parallel(1)\prime,lm} + \frac{1}{3}\chi^{\top(1)\prime}_{lj}\nabla^2\chi^{\parallel(1)\prime,l} - \frac{1}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm}_{,j} + \frac{1}{3}\chi^{\top(1)}_{lj}\nabla^2\nabla^2\chi^{\parallel(1),l} + \frac{2}{3}\nabla^2\chi^{\top(1)}_{lj}\nabla^2\chi^{\parallel(1),l} \\
& \left. + \chi^{\parallel(1),lm}\nabla^2\chi^{\top(1)}_{lj,m} - \frac{1}{2}\chi^{\parallel(1),lm}\nabla^2\chi^{\top(1)}_{lm,j}\right] + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[-\frac{6}{\tau^2}c_L^2\chi^{\top(1)}_{lm}\delta^{(1),lm} + 6\phi^{(1)\prime\prime,lm}\chi^{\top(1)}_{lm} + \frac{12}{\tau}\phi^{(1)\prime,lm}\chi^{\top(1)}_{lm}\right. \\
& + \phi^{(1)\prime,lm}\chi^{\top(1)\prime}_{lm} + 3\phi^{(1),lmn}\chi^{\top(1)}_{lm,n} + \chi^{\parallel(1)\prime,lm}\nabla^2\chi^{\top(1)\prime}_{lm} - \frac{1}{3}\chi^{\top(1)\prime}_{lm}\nabla^2\chi^{\parallel(1)\prime,lm} + \frac{1}{2}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\parallel(1),lmn} - \nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm} \\
& \left. - \frac{3}{2}\chi^{\parallel(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}\right] + (i \leftrightarrow j)
\end{aligned} \tag{3.22}
$$

is the effective source of the second-order vector. Finally, $[(3.16) - \frac{1}{2}D_{ij}(3.18) - (3.21)]$ gives the evolution equation for the second-order tensor mode

$$
\frac{1}{2}\chi^{\top(2)\prime\prime}_{s(t)ij} + \frac{1}{\tau}\chi^{\top(2)\prime}_{s(t)ij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{s(t)ij} = J_{s(t)ij}, \tag{3.23}
$$

where

$$
\begin{aligned}
J_{s(t)ij} \equiv{}& \bar{S}_{s(t)ij} - \frac{3}{2} D_{ij} \nabla^{-2}\nabla^{-2} \bar{S}^{,lm}_{s(t)lm} - \nabla^{-2} \bar{S}^{,l}_{s(t)lj,i} - \nabla^{-2} \bar{S}^{,l}_{s(t)li,j} + 2\nabla^{-2}\nabla^{-2} \bar{S}^{,lm}_{s(t)lm,ij} \\
={}& \bigg[\frac{6}{\tau^2} c_L^2 \chi^{\top(1)}_{ij} \delta^{(1)} - 6\phi^{(1)''} \chi^{\top(1)}_{ij} - \frac{12}{\tau} \phi^{(1)'} \chi^{\top(1)}_{ij} + 4\chi^{\top(1)}_{ij} \nabla^2 \phi^{(1)} - \phi^{(1)'} \chi^{\top(1)'}_{ij} \\
& - \phi^{(1),l} \chi^{\top(1)}_{lj,i} - \phi^{(1),l} \chi^{\top(1)}_{li,j} + 3\phi^{(1),l} \chi^{\top(1)}_{ij,l} + 2\phi^{(1)} \nabla^2 \chi^{\top(1)}_{ij} - 2\phi^{(1),l}_{,i} \chi^{\top(1)}_{lj} \\
& - 2\phi^{(1),l}_{,j} \chi^{\top(1)}_{li} + 2\phi^{(1),lm} \chi^{\top(1)}_{lm} \delta_{ij} + \chi^{\top(1)'}_{li} \chi^{\|(1)',l}_{,j} + \chi^{\top(1)'}_{lj} \chi^{\|(1)',l}_{,i} - \frac{1}{2} \chi^{\top(1)'}_{lm} \chi^{\|(1)',lm} \delta_{ij} \\
& - \frac{1}{3} \chi^{\top(1)}_{li} \nabla^2 \chi^{\|(1),l}_{,j} - \frac{1}{3} \chi^{\top(1)}_{lj} \nabla^2 \chi^{\|(1),l}_{,i} + \frac{1}{3} \chi^{\top(1)}_{lm} \nabla^2 \chi^{\|(1),lm} \delta_{ij} - \chi^{\top(1)}_{lm,ij} \chi^{\|(1),lm} \\
& - \frac{1}{2} \chi^{\top(1)}_{lm,j} \chi^{\|(1),lm}_{,i} - \frac{1}{2} \chi^{\top(1)}_{lm,i} \chi^{\|(1),lm}_{,j} + \frac{1}{4} \chi^{\top(1)}_{lm,n} \chi^{\|(1),lmn} \delta_{ij} - \frac{2}{3} \chi^{\top(1)'}_{ij} \nabla^2 \chi^{\|(1)'} \\
& + \frac{2}{3} \chi^{\top(1)}_{ij} \nabla^2\nabla^2 \chi^{\|(1)} + \frac{1}{3} \nabla^2 \chi^{\top(1)}_{ij} \nabla^2 \chi^{\|(1)} + \frac{1}{3} \chi^{\top(1)}_{lj,i} \nabla^2 \chi^{\|(1),l} + \frac{1}{3} \chi^{\top(1)}_{li,j} \nabla^2 \chi^{\|(1),l} \\
& + \chi^{\top(1)}_{lj,im} \chi^{\|(1),lm} + \chi^{\top(1)}_{li,jm} \chi^{\|(1),lm} - \chi^{\top(1)}_{ij,lm} \chi^{\|(1),lm} \bigg] + \delta_{ij} \nabla^{-2} \bigg[\frac{3}{\tau^2} c_L^2 \chi^{\top(1)}_{lm} \delta^{(1),lm} \\
& - 3\phi^{(1)'',lm} \chi^{\top(1)}_{lm} - \frac{6}{\tau} \phi^{(1)',lm} \chi^{\top(1)}_{lm} - \frac{1}{2} \phi^{(1)',lm} \chi^{\top(1)'}_{lm} - \frac{3}{2} \phi^{(1),lmn} \chi^{\top(1)}_{lm,n} \\
& - \frac{1}{2} \chi^{\|(1)',lm} \nabla^2 \chi^{\top(1)'}_{lm} + \frac{1}{6} \chi^{\top(1)'}_{lm} \nabla^2 \chi^{\|(1)',lm} - \frac{1}{4} \chi^{\top(1)}_{lm,n} \nabla^2 \chi^{\|(1),lmn} \\
& + \frac{1}{2} \nabla^2 \chi^{\top(1)}_{lm} \nabla^2 \chi^{\|(1),lm} + \frac{3}{4} \chi^{\|(1),lmn} \nabla^2 \chi^{\top(1)}_{lm,n} \bigg] + \partial_i \partial_j \nabla^{-2} \bigg[-\frac{1}{2} \chi^{\top(1)'}_{lm} \chi^{\|(1)',lm} \\
& + \frac{1}{4} \chi^{\top(1)}_{lm,n} \chi^{\|(1),lmn} \bigg] + \partial_i \nabla^{-2} \bigg[-\frac{6}{\tau^2} c_L^2 \chi^{\top(1)}_{lj} \delta^{(1),l} + 6\phi^{(1)'',l} \chi^{\top(1)}_{lj} + \frac{12}{\tau} \phi^{(1)',l} \chi^{\top(1)}_{lj} \\
& + \phi^{(1)',l} \chi^{\top(1)'}_{lj} - 2\chi^{\top(1)}_{lj} \nabla^2 \phi^{(1),l} - \phi^{(1),l} \nabla^2 \chi^{\top(1)}_{lj} + \phi^{(1),lm}_{,j} \chi^{\top(1)}_{lm} - \chi^{\top(1)'}_{lj,m} \chi^{\|(1)',lm} \\
& + \chi^{\top(1)'}_{lm,j} \chi^{\|(1)',lm} - \frac{1}{3} \chi^{\top(1)'}_{lj} \nabla^2 \chi^{\|(1)',l} + \frac{1}{6} \chi^{\top(1)}_{lm} \nabla^2 \chi^{\|(1),lm}_{,j} - \frac{1}{3} \chi^{\top(1)}_{lj} \nabla^2\nabla^2 \chi^{\|(1),l} \\
& - \frac{2}{3} \nabla^2 \chi^{\top(1)}_{lj} \nabla^2 \chi^{\|(1),l} - \chi^{\|(1),lm} \nabla^2 \chi^{\top(1)}_{lj,m} + \frac{1}{2} \chi^{\|(1),lm} \nabla^2 \chi^{\top(1)}_{lm,j} \bigg] \\
& + \partial_j \nabla^{-2} \bigg[-\frac{6}{\tau^2} c_L^2 \chi^{\top(1)}_{li} \delta^{(1),l} + 6\phi^{(1)'',l} \chi^{\top(1)}_{li} + \frac{12}{\tau} \phi^{(1)',l} \chi^{\top(1)}_{li} + \phi^{(1)',l} \chi^{\top(1)'}_{li} \\
& - 2\chi^{\top(1)}_{li} \nabla^2 \phi^{(1),l} - \phi^{(1),l} \nabla^2 \chi^{\top(1)}_{li} + \phi^{(1),lm}_{,i} \chi^{\top(1)}_{lm} - \chi^{\top(1)'}_{li,m} \chi^{\|(1)',lm} + \chi^{\top(1)'}_{lm,i} \chi^{\|(1)',lm} \\
& - \frac{1}{3} \chi^{\top(1)'}_{li} \nabla^2 \chi^{\|(1)',l} + \frac{1}{6} \chi^{\top(1)}_{lm} \nabla^2 \chi^{\|(1),lm}_{,i} - \frac{1}{3} \chi^{\top(1)}_{li} \nabla^2\nabla^2 \chi^{\|(1),l} - \frac{2}{3} \nabla^2 \chi^{\top(1)}_{li} \nabla^2 \chi^{\|(1),l} \\
& - \chi^{\|(1),lm} \nabla^2 \chi^{\top(1)}_{li,m} + \frac{1}{2} \chi^{\|(1),lm} \nabla^2 \chi^{\top(1)}_{lm,i} \bigg] + \partial_i \partial_j \nabla^{-2}\nabla^{-2} \bigg[\frac{3}{\tau^2} c_L^2 \chi^{\top(1)}_{lm} \delta^{(1),lm} \\
& - 3\phi^{(1)'',lm} \chi^{\top(1)}_{lm} - \frac{6}{\tau} \phi^{(1)',lm} \chi^{\top(1)}_{lm} - \frac{1}{2} \phi^{(1)',lm} \chi^{\top(1)'}_{lm} - \frac{3}{2} \phi^{(1),lmn} \chi^{\top(1)}_{lm,n} \\
& - \frac{1}{2} \chi^{\|(1)',lm} \nabla^2 \chi^{\top(1)'}_{lm} + \frac{1}{6} \chi^{\top(1)'}_{lm} \nabla^2 \chi^{\|(1)',lm} - \frac{1}{4} \chi^{\top(1)}_{lm,n} \nabla^2 \chi^{\|(1),lmn} \\
& + \frac{1}{2} \nabla^2 \chi^{\top(1)}_{lm} \nabla^2 \chi^{\|(1),lm} + \frac{3}{4} \chi^{\|(1),lmn} \nabla^2 \chi^{\top(1)}_{lm,n} \bigg],
\end{aligned}
\tag{3.24}
$$

which is the effective source of the second-order tensor. [See also (A20) for a general RW spacetime.] So far the second-order perturbed Einstein equation has been decomposed into separate equations for the second-order metric perturbations.

To solve these equations of second-order metric perturbations, we need also the second-order energy-momentum conservation. The general energy conservation is given by (2.17). Substituting $\Gamma^{\alpha}_{\mu\nu}$ in (A1)–(A4) of Ref. [63] and U^{μ} in (2.12) and (2.13) into (2.17) and keeping second order, one has the second-order energy conservation for the scalar-tensor case,

$$\delta^{(2)\prime}_{s(t)} + \frac{3}{\tau}\left(c_N^2 - \frac{1}{3}\right)\delta^{(2)}_{s(t)} = -\frac{4}{3}\nabla^2 v^{\parallel(2)}_{s(t)} + 4\phi^{(2)\prime}_{s(t)} + A_{s(t)}, \tag{3.25}$$

where

$$A_{s(t)} \equiv \frac{4}{3}\chi^{\top(1)\prime}_{lm}\chi^{\parallel(1),lm} + \frac{4}{3}\chi^{\top(1)}_{lm}\chi^{\parallel(1)\prime,lm}. \tag{3.26}$$

Equation (3.25) contains the scalar mode $\phi^{(2)}_S$ and the velocity $v^{\parallel(2)}_S$. Note that c_N defined by (2.10) appears in the second-order energy conservation equation.

The general expression of the momentum conservation has been given in (2.18). Substituting $\Gamma^\alpha_{\mu\nu}$ in (A1)–(A4) in Ref. [63] and U^μ in (2.12) and (2.13) into (2.18) to second order gives the second-order momentum conservation for the scalar-tensor case

$$c_N^2\delta^{(2)}_{s(t),i} + \frac{4}{3}v^{\parallel(2)\prime}_{s(t),i} + \frac{4}{3}v^{\perp(2)\prime}_{s(t)i} = F_{s(t)i}, \tag{3.27}$$

where

$$F_{s(t)i} \equiv 2c_L^2\delta^{(1),l}\chi^{\top(1)}_{li} - \frac{8}{3}v^{\parallel(1),l}\chi^{\top(1)\prime}_{li}. \tag{3.28}$$

Observe that (3.27) involves the second-order velocity $v^{\perp(2)i}$, yet does not involve the second-order metric perturbations, in contrast to the energy conservation (3.25). To proceed further, (3.27) is decomposed into a longitudinal part by $[\nabla^{-2}\partial^i(3.27)]$ as

$$c_N^2\delta^{(2)}_{s(t)} + \frac{4}{3}v^{\parallel(2)\prime}_{s(t)} = F^{\parallel}_{s(t)}, \tag{3.29}$$

with

$$F^{\parallel}_{s(t)} \equiv \nabla^{-2}\partial^i F_{s(t)i} = \nabla^{-2}\left[2c_L^2\delta^{(1),lm}\chi^{\top(1)}_{lm} - \frac{8}{3}v^{\parallel(1),lm}\chi^{\top(1)\prime}_{lm}\right], \tag{3.30}$$

and a transverse part by $[(3.27) - \partial_i(3.29)]$ as

$$\frac{4}{3}v^{\perp(2)\prime}_{s(t)i} = F^{\perp}_{s(t)i}, \tag{3.31}$$

with

$$\begin{aligned} F^{\perp}_{s(t)i} &\equiv F_{s(t)i} - \partial_i F^{\parallel}_{s(t)} \\ &= 2c_L^2\delta^{(1),l}\chi^{\top(1)}_{li} - \frac{8}{3}v^{\parallel(1),l}\chi^{\top(1)\prime}_{li} \\ &\quad + \partial_i\nabla^{-2}\left[-2c_L^2\delta^{(1),lm}\chi^{\top(1)}_{lm} + \frac{8}{3}v^{\parallel(1),lm}\chi^{\top(1)\prime}_{lm}\right] \end{aligned} \tag{3.32}$$

being a transverse vector function.

We find that the second-order trace evolution equation (3.14) can be given as the combination

$$(3.14) = -\frac{1}{3}(3.2) - \frac{\tau}{3}\frac{d}{d\tau}(3.2) + \frac{\tau}{3}\nabla^2(3.7) - \frac{1}{\tau}(3.25), \tag{3.33}$$

and the second-order traceless scalar evolution equation (3.18) as follows:

$$\begin{aligned} (3.18) = \nabla^{-2}\Big[&(3.2) + \tau\frac{d}{d\tau}(3.2) - \tau\nabla^2(3.7) \\ &+ 3\frac{d}{d\tau}(3.7) + \frac{6}{\tau}(3.7) + \frac{3}{\tau}(3.25) - \frac{9}{\tau^2}(3.29)\Big]. \end{aligned} \tag{3.34}$$

These mean that, for the scalars, we can solve the equations of constraints and conservations, and the solutions satisfy the evolution equations automatically. (See Sec. IV.)

B. Equations with tensor-tensor couplings

Now we consider the tensor-tensor couplings. Moving the tensor-tensor coupling terms to the rhs of the (00) component of the Einstein equation (3.1), and using the Friedmann equation, one has the second-order energy constraint equation as follows:

$$-\frac{6}{\tau}\phi^{(2)\prime}_T + 2\nabla^2\phi^{(2)}_T + \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(2)}_T = \frac{3}{\tau^2}\delta^{(2)}_T + E_T, \tag{3.35}$$

where a subscript T in $\phi^{(2)}_T$ etc. indicates the case of tensor-tensor coupling, and E_T in the above is

$$\begin{aligned} E_T \equiv& \frac{1}{4}\chi^{\top(1)\prime lm}\chi^{\top(1)\prime}_{lm} + \frac{2}{\tau}\chi^{\top(1)lm}\chi^{\top(1)\prime}_{lm} - \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm} \\ &- \frac{3}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n} + \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}. \end{aligned} \tag{3.36}$$

The $(0i)$ component of the second-order perturbed Einstein equation (3.4) gives the second-order momentum constraint equation as follows:

$$2\phi^{(2)\prime}_{T,i} + \frac{1}{2}D_{ij}\chi^{\parallel(2)\prime,j}_T + \frac{1}{2}\chi^{\perp(2)\prime,j}_{Tij} = -\frac{4}{\tau^2}v^{(2)}_{Ti} + M_{Ti}, \tag{3.37}$$

with

$$M_{Ti} \equiv \chi^{\top(1)lm}\chi^{\top(1)'}_{il,m} - \chi^{\top(1)lm}\chi^{\top(1)'}_{lm,i} - \frac{1}{2}\chi^{\top(1)lm}_{,i}\chi^{\top(1)'}_{lm}. \tag{3.38}$$

As before, $[\nabla^{-2}\partial^i(3.37)]$ gives the longitudinal momentum constraint

$$2\phi^{(2)'}_T + \frac{1}{3}\nabla^2\chi^{\|(2)'}_T = -\frac{4}{\tau^2}v^{\|(2)}_T + \nabla^{-2}M^{,l}_{Tl}, \tag{3.39}$$

where

$$\nabla^{-2}M^{,l}_{Tl} = \nabla^{-2}\left[\chi^{\top(1)lm,n}\chi^{\top(1)'}_{ln,m} - \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)'}_{lm,n} - \frac{1}{2}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)'}_{lm}\right] - \frac{1}{2}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm} \tag{3.40}$$

is the effective source term. A combination $[(3.37)-\partial_i(3.39)]$ gives the transverse momentum constraint

$$\frac{1}{2}\chi^{\perp(2)',j}_{Tij} = -\frac{4}{\tau^2}v^{\perp(2)}_{Ti} + (M_{Ti} - \partial_i\nabla^{-2}M^{,l}_{Tl}), \tag{3.41}$$

with

$$\begin{aligned}(M_{Ti} - \partial_i\nabla^{-2}M^{,l}_{Tl}) &= \chi^{\top(1)lm}\chi^{\top(1)'}_{il,m} - \frac{1}{2}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm,i} \\ &\quad + \partial_i\nabla^{-2}\left[-\chi^{\top(1)lm,n}\chi^{\top(1)'}_{ln,m} + \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)'}_{lm,n} + \frac{1}{2}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)'}_{lm}\right].\end{aligned} \tag{3.42}$$

The (ij) component of second-order perturbed Einstein equation (3.11) gives the second-order evolution equation

$$\begin{aligned}&2\phi^{(2)''}_T\delta_{ij} + \frac{4}{\tau}\phi^{(2)'}_T\delta_{ij} + \phi^{(2)}_{T,ij} - \nabla^2\phi^{(2)}_T\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\|(2)''}_T + \frac{1}{\tau}D_{ij}\chi^{\|(2)'}_T + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(2)}_T - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(2)}_T \\ &\quad + \frac{1}{2}\chi^{\perp(2)''}_{Tij} + \frac{1}{\tau}\chi^{\perp(2)'}_{Tij} + \frac{1}{2}\chi^{\top(2)''}_{Tij} + \frac{1}{\tau}\chi^{\top(2)'}_{Tij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Tij} = \frac{3c_N^2}{\tau^2}\delta^{(2)}_T\delta_{ij} + S_{Tij},\end{aligned} \tag{3.43}$$

with

$$\begin{aligned}S_{Tij} \equiv\, & -\frac{2}{\tau}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm}\delta_{ij} - \chi^{\top(1)lm}\chi^{\top(1)}_{lm,ij} + \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm}\delta_{ij} - \frac{1}{2}\chi^{\top(1)lm}_{,i}\chi^{\top(1)}_{lm,j} - \chi^{\top(1)}_{li,m}\chi^{\top(1)l,m}_{j} + \frac{3}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n}\delta_{ij} \\ & + \chi^{\top(1)}_{li,m}\chi^{\top(1)m,l}_{j} - \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\delta_{ij} + \chi^{\top(1)'l}_{i}\chi^{\top(1)'}_{lj} - \frac{3}{4}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}\delta_{ij} - \chi^{\top(1)lm}\chi^{\top(1)''}_{lm}\delta_{ij} + \chi^{\top(1)lm}\chi^{\top(1)}_{lj,im} \\ & + \chi^{\top(1)lm}\chi^{\top(1)}_{li,jm} - \chi^{\top(1)lm}\chi^{\top(1)}_{ij,lm}\end{aligned} \tag{3.44}$$

as part of the effective source term.

We also need to decompose the evolution equation (3.43) into the trace part and the traceless part. The trace part of second-order evolution equation (3.43) is

$$2\phi^{(2)''}_T + \frac{4}{\tau}\phi^{(2)'}_T - \frac{2}{3}\nabla^2\phi^{(2)}_T - \frac{1}{9}\nabla^2\nabla^2\chi^{\|(2)}_T = \frac{3}{\tau^2}c_N^2\delta^{(2)}_T + \frac{1}{3}S^l_{Tl}, \tag{3.45}$$

with

$$S^l_{Tl} = -\frac{6}{\tau}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm} + 2\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm} + \frac{3}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n} - \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m} - \frac{5}{4}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm} - 3\chi^{\top(1)lm}\chi^{\top(1)''}_{lm}. \tag{3.46}$$

The traceless part of (3.43) is

$$D_{ij}\phi^{(2)}_T + \frac{1}{2}D_{ij}\chi^{\|(2)''}_T + \frac{1}{\tau}D_{ij}\chi^{\|(2)'}_T + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(2)}_T + \frac{1}{2}\chi^{\perp(2)''}_{Tij} + \frac{1}{\tau}\chi^{\perp(2)'}_{Tij} + \frac{1}{2}\chi^{\top(2)''}_{Tij} + \frac{1}{\tau}\chi^{\top(2)'}_{Tij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Tij} = \bar{S}_{Tij}, \tag{3.47}$$

with

$$
\begin{aligned}
\bar{S}_{Tij} \equiv{}& S_{Tij} - \frac{1}{3} S^{l}_{Tl}\delta_{ij} \\
={}& -\chi^{\top(1)lm}\chi^{\top(1)}_{lm,ij} + \frac{1}{3}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm}\delta_{ij} - \frac{1}{2}\chi^{\top(1)lm}_{,i}\chi^{\top(1)}_{lm,j} - \chi^{\top(1)}_{li,m}\chi^{\top(1)l,m}_{j} \\
&+ \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n}\delta_{ij} + \chi^{\top(1)}_{li,m}\chi^{\top(1)m,l}_{j} - \frac{1}{3}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\delta_{ij} + \chi^{\top(1)'l}_{i}\chi^{\top(1)'}_{lj} \\
&- \frac{1}{3}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}\delta_{ij} + \chi^{\top(1)lm}\chi^{\top(1)}_{lj,im} + \chi^{\top(1)lm}\chi^{\top(1)}_{li,jm} - \chi^{\top(1)lm}\chi^{\top(1)}_{ij,lm}.
\end{aligned}
\tag{3.48}
$$

Equation (3.47) contains the scalar, the vector, and the tensor. Applying $3\nabla^{-2}\nabla^{-2}\partial_i\partial_j$ on (3.47) gives the evolution equation for the scalar $\chi^{\parallel(2)}_{T}$ as follows:

$$
\chi^{\parallel(2)''}_{T} + \frac{2}{\tau}\chi^{\parallel(2)'}_{T} + \frac{1}{3}\nabla^2\chi^{\parallel(2)}_{T} + 2\phi^{(2)}_{T} = 3\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm},
\tag{3.49}
$$

where

$$
\begin{aligned}
\bar{S}^{,lm}_{Tlm} ={}& -\nabla^2\nabla^2\left[\frac{1}{8}\chi^{\top(1)lm}\chi^{\top(1)}_{lm}\right] + \nabla^2\left[-\frac{1}{3}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm} + \frac{1}{12}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm} + \frac{1}{6}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\right] \\
&+ \chi^{\top(1)'lm,n}\chi^{\top(1)'}_{ln,m} - \frac{1}{2}\chi^{\top(1)lm}\nabla^2\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{2}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{lm,n} + \chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{ln,m}.
\end{aligned}
\tag{3.50}
$$

A combination $\partial^i[(3.47) - \frac{1}{2}D_{ij}(3.49)]$ gives

$$
\frac{1}{2}\chi^{\perp(2)'',l}_{Tlj} + \frac{1}{\tau}\chi^{\perp(2)',l}_{Tlj} = \bar{S}^{,l}_{Tlj} - \partial_j\nabla^{-2}\bar{S}^{,lm}_{Tlm}.
\tag{3.51}
$$

By $\nabla^{-2}[\partial_i(3.51) + (i\leftrightarrow j)]$, using (2.6), one gets the evolution equation for the vector mode

$$
\frac{1}{2}\chi^{\perp(2)''}_{Tij} + \frac{1}{\tau}\chi^{\perp(2)'}_{Tij} = V_{Tij},
\tag{3.52}
$$

where

$$
\begin{aligned}
V_{Tij} \equiv{}& \nabla^{-2}\bar{S}^{,l}_{Tlj,i} + \nabla^{-2}\bar{S}^{,l}_{Tli,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm,ij} \\
={}& \partial_i\nabla^{-2}\left[\chi^{\top(1)'lm}\chi^{\top(1)'}_{jl,m} - \frac{1}{2}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm,j} + \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{jl,m}\right] + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[-\chi^{\top(1)'lm,n}\chi^{\top(1)'}_{ln,m} + \frac{1}{2}\chi^{\top(1)lm}\nabla^2\nabla^2\chi^{\top(1)}_{lm}\right. \\
&\left. + \frac{1}{2}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{lm,n} - \chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{ln,m}\right] + (i\leftrightarrow j)
\end{aligned}
\tag{3.53}
$$

is the effective source. Finally, $[(3.47) - \frac{1}{2}D_{ij}(3.49) - (3.52)]$ gives the evolution equation for the second-order tensor mode

$$
\frac{1}{2}\chi^{\top(2)''}_{Tij} + \frac{1}{\tau}\chi^{\top(2)'}_{Tij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Tij} = J_{Tij},
\tag{3.54}
$$

where

$$
\begin{aligned}
J_{Tij} \equiv{}& \bar{S}_{Tij} - \frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm} - \nabla^{-2}\bar{S}^{,l}_{Tlj,i} - \nabla^{-2}\bar{S}^{,l}_{Tli,j} + 2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm,ij} \\
={}& \Big[-\frac{5}{8}\chi^{\top(1)lm}\chi^{\top(1)}_{lm,ij} + \frac{1}{4}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm}\delta_{ij} - \frac{1}{8}\chi^{\top(1)lm}_{,i}\chi^{\top(1)}_{lm,j} - \chi^{\top(1)}_{li,m}\chi^{\top(1)l,m}_{j} + \frac{3}{8}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n}\delta_{ij} + \chi^{\top(1)}_{li,m}\chi^{\top(1)m,l}_{j} \\
& - \frac{1}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\delta_{ij} + \chi^{\top(1)'l}_{i}\chi^{\top(1)'}_{lj} - \frac{1}{2}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}\delta_{ij} + \chi^{\top(1)lm}\chi^{\top(1)}_{lj,im} + \chi^{\top(1)lm}\chi^{\top(1)}_{li,jm} - \chi^{\top(1)lm}\chi^{\top(1)}_{ij,lm}\Big] \\
& + \delta_{ij}\nabla^{-2}\Big[\frac{1}{2}\chi^{\top(1)'lm,n}\chi^{\top(1)'}_{ln,m} - \frac{1}{4}\chi^{\top(1)lm}\nabla^2\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{4}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{lm,n} + \frac{1}{2}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{ln,m}\Big] \\
& + \partial_i\nabla^{-2}\Big[-\chi^{\top(1)'lm}\chi^{\top(1)'}_{jl,m} + \frac{1}{2}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm,j} - \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{jl,m}\Big] \\
& + \partial_j\nabla^{-2}\Big[-\chi^{\top(1)'lm}\chi^{\top(1)'}_{il,m} + \frac{1}{2}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm,i} - \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{il,m}\Big] \\
& + \partial_i\partial_j\nabla^{-2}\Big[\frac{1}{2}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm} - \frac{1}{8}\chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\Big] \\
& + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big[\frac{1}{2}\chi^{\top(1)'lm,n}\chi^{\top(1)'}_{ln,m} - \frac{1}{4}\chi^{\top(1)lm}\nabla^2\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{4}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{lm,n} + \frac{1}{2}\chi^{\top(1)lm,n}\nabla^2\chi^{\top(1)}_{ln,m}\Big]
\end{aligned} \tag{3.55}
$$

is the effective source. So far the second-order perturbed Einstein equation has been decomposed into the separate equations for the second-order metric perturbations.

Next, we derive the second-order energy-momentum conservation with tensor-tensor couplings. Expanding (2.17) to second order, one has the second-order energy conservation for the tensor-tensor case as

$$\delta^{(2)'}_T + \frac{3}{\tau}\left(c_N^2 - \frac{1}{3}\right)\delta^{(2)}_T = -\frac{4}{3}\nabla^2 v^{\|(2)}_T + 4\phi^{(2)'}_T + A_T, \tag{3.56}$$

where

$$A_T \equiv \frac{4}{3}\chi^{\top(1)'}_{lm}\chi^{\top(1)lm}. \tag{3.57}$$

Expanding (2.18) to second order, one has the second-order momentum conservation for the tensor-tensor case as

$$c_N^2\delta^{(2)}_{T,i} + \frac{4}{3}v^{\|(2)'}_{T,i} + \frac{4}{3}v^{\perp(2)'}_{Ti} = 0, \tag{3.58}$$

which is homogeneous, involving no tensor-tensor coupling terms. Observe that (3.58) involves the second-order velocity $v^{\perp(2)}_{Ti}$, yet does not involve the second-order metric perturbations, in contrast to the energy conservation (3.25). To proceed further, (3.58) can be decomposed into a longitudinal part by $[\nabla^{-2}\partial^i(3.58)]$ as

$$c_N^2\delta^{(2)}_T + \frac{4}{3}v^{\|(2)'}_T = 0 \tag{3.59}$$

and a transverse part by $[(3.58) - \partial_i(3.59)]$ as

$$\frac{4}{3}v^{\perp(2)'}_{Ti} = 0. \tag{3.60}$$

Similar to the relations (3.33) and (3.34), the trace of the second-order evolution equation (3.45) can be given by a combination as

$$(3.45) = -\frac{1}{3}(3.35) - \frac{\tau}{3}\frac{d}{d\tau}(3.35) + \frac{\tau}{3}\nabla^2(3.39) - \frac{1}{\tau}(3.56), \tag{3.61}$$

and the scalar part of the second-order traceless evolution equation (3.49) is given by

$$
\begin{aligned}
(3.45) = \nabla^{-2}\Big[&(3.35) + \tau\frac{d}{d\tau}(3.35) - \tau\nabla^2(3.39) + 3\frac{d}{d\tau}(3.39) \\
&+ \frac{6}{\tau}(3.39) + \frac{3}{\tau}(3.56) - \frac{9}{\tau^2}(3.59)\Big].
\end{aligned} \tag{3.62}
$$

So we can use the equations of constraints and conservations to solve the scalars, and the solutions satisfy the evolution equations automatically.

IV. SOLUTION OF THE SECOND-ORDER PERTURBATIONS

A. Solution for scalar-tensor couplings

Given the second-order perturbed equations, we shall solve for the second-order perturbations. This subsection is for the scalar-tensor case. First, the solution of tensor equation (3.23) is

$$
\begin{aligned}
\chi^{\top(2)}_{s(t)ij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot x}\Big(& \bar{I}_{s(t)ij}(\mathbf{k},\tau) \\
& + \sum_{s=+,\times} \overset{s}{\epsilon}_{ij}(\mathbf{k})\Big[-a^s_1\sqrt{\frac{2}{\pi}}\frac{ie^{ik\tau}}{k\tau} + a^s_2\sqrt{\frac{2}{\pi}}\frac{ie^{-ik\tau}}{k\tau}\Big]\Big),
\end{aligned} \tag{4.1}
$$

where $a_1^s(\mathbf{k})$ and $a_2^s(\mathbf{k})$ are polarization-dependent and $\mathbf{k}$-dependent coefficients, to be determined by initial conditions; their associated term is the homogeneous solution of (3.23) and has the same form as (2.28). In certain applications, they can be absorbed into (2.28). The integrand of the inhomogeneous solution in (4.1) is given by

$$\bar{I}_{s(t)ij}(\mathbf{k},\tau) \equiv \frac{ie^{-ik\tau}}{k\tau}\int^\tau \tau' e^{ik\tau'}\bar{J}_{s(t)ij}(\mathbf{k},\tau')d\tau' - \frac{ie^{ik\tau}}{k\tau}\int^\tau \tau' e^{-ik\tau'}\bar{J}_{s(t)ij}(\mathbf{k},\tau')d\tau', \quad (4.2)$$

with $\bar{J}_{s(t)ij}$ being the Fourier transform of the source $J_{s(t)ij}$ in (3.24) that contains many terms of products of first-order solutions.

Next, the vector solution of evolution equation (3.21) is

$$\chi^{\perp(2)}_{s(t)ij}(\mathbf{x},\tau) = q_{1ij}(\mathbf{x}) + \frac{q_{2ij}(\mathbf{x})}{\tau} + \int^\tau \frac{d\tau'}{\tau'^2}\int^{\tau'} 2\tau''^2 V_{s(t)ij}(\mathbf{x},\tau'')d\tau'', \quad (4.3)$$

where $q_{1ij}(\mathbf{x})$ and $q_{2ij}(\mathbf{x})$ are two time-independent functions determined by initial values and $V_{s(t)ij}$ is given by (3.22). Note that $q_{1ij}(\mathbf{x})$ is a gauge mode as shall be seen in Sec. V. Plugging the solution (4.3) into (3.9) yields the transverse part of the second-order velocity

$$v^{\perp(2)}_{s(t)i} = \frac{q^{,j}_{2ij}(\mathbf{x})}{8} + \frac{\tau^2}{4}(M_{s(t)i} - \partial_i\nabla^{-2}M^{,k}_{s(t)k}) - \frac{1}{4}\int^\tau \tau'^2 V^{,j}_{s(t)ij}(\mathbf{x},\tau')d\tau', \quad (4.4)$$

where $(M_{s(t)i} - \partial_i\nabla^{-2}M^{,k}_{s(t)k})$ is in (3.10). This solution can also be derived from integration of the transverse momentum conservation (3.31), as we have checked. Thus, although the first-order curl vector $v_i^{\perp(1)}$ is vanishing by assumption, nevertheless, the second-order curl vector $v^{\perp(2)}_{s(t)i}$ is generated by the coupling according to (4.4).

Next, we solve the scalars. From the longitudinal momentum conservation (3.29), one has

$$\delta^{(2)}_{s(t)} = -\frac{4}{3c_N^2}v^{\|(2)'}_{s(t)} + \frac{1}{c_N^2}F^{\|}_{s(t)}. \quad (4.5)$$

Plugging the above $\delta^{(2)}_{s(t)}$ into the energy conservation (3.25) gives $\phi^{(2)'}_{s(t)}$ in terms of $v^{\|(2)}_{s(t)}$, $v^{\|(2)'}_{s(t)}$, $v^{\|(2)''}_{s(t)}$ as

$$\phi^{(2)'}_{s(t)} = -\frac{1}{3c_N^2}v^{\|(2)''}_{s(t)} - \frac{1}{\tau}\frac{c_N^2-\frac{1}{3}}{c_N^2}v^{\|(2)'}_{s(t)} + \frac{1}{3}\nabla^2 v^{\|(2)}_{s(t)} + \frac{1}{4c_N^2}F^{\|'}_{s(t)} + \frac{3}{4\tau}\frac{c_N^2-\frac{1}{3}}{c_N^2}F^{\|}_{s(t)} - \frac{1}{4}A_{s(t)}. \quad (4.6)$$

To use the energy constraint (3.2), taking $[\frac{d}{d\tau}(3.2)]$ gives

$$-\frac{6}{\tau}\phi^{(2)''}_{s(t)} + \frac{6}{\tau^2}\phi^{(2)'}_{s(t)} + \nabla^2\left[2\phi^{(2)'}_{s(t)} + \frac{1}{3}\nabla^2\chi^{\|(2)'}_{s(t)}\right] = \frac{3}{\tau^2}\delta^{(2)'}_{s(t)} - \frac{6}{\tau^3}\delta^{(2)}_{s(t)} + E'_{s(t)}. \quad (4.7)$$

Plugging the momentum constraint (3.7) into the above to eliminate the $\nabla^2\nabla^2\chi^{\|(2)'}_{s(t)}$ term leads to

$$-\frac{6}{\tau}\phi^{(2)''}_{s(t)} + \frac{6}{\tau^2}\phi^{(2)'}_{s(t)} - \frac{4}{\tau^2}\nabla^2 v^{\|(2)}_{s(t)} + M^{,k}_{s(t)k} = \frac{3}{\tau^2}\delta^{(2)'}_{s(t)} - \frac{6}{\tau^3}\delta^{(2)}_{s(t)} + E'_{s(t)}. \quad (4.8)$$

Then, plugging $\delta^{(2)}_{s(t)}$ of (4.5) and $\phi^{(2)'}_{s(t)}$ of (4.6) into the above yields a third-order differential equation of $v^{\|(2)}_{s(t)}$ as follows:

$$v^{\|(2)'''}_{s(t)} + \frac{3c_N^2}{\tau}v^{\|(2)''}_{s(t)} - \frac{6c_N^2+2}{\tau^2}v^{\|(2)'}_{s(t)} - \frac{c_N^2}{\tau}\nabla^2 v^{\|(2)}_{s(t)} - c_N^2\nabla^2 v^{\|(2)'}_{s(t)} = Z_{s(t)}, \quad (4.9)$$

with

$$\begin{aligned}
Z_{s(t)} \equiv{}& \frac{3}{4}F^{\|''}_{s(t)} + \frac{9c_N^2}{4\tau}F^{\|'}_{s(t)} - \frac{9c_N^2+3}{2\tau^2}F^{\|}_{s(t)} - \frac{3c_N^2}{4}A'_{s(t)} + \frac{3c_N^2}{4\tau}A_{s(t)} - \frac{\tau}{2}c_N^2 M^{,l}_{s(t)l} + \frac{\tau}{2}c_N^2 E'_{s(t)} \\
={}& \left[\frac{4}{3\tau}\chi^{\top(1)}_{lm}v^{\|(1),lm} + \frac{2\tau}{3}\phi^{(1)',lm}\chi^{\top(1)}_{lm} + \frac{\tau}{6}\phi^{(1),lm}\chi^{\top(1)'}_{lm} + \frac{\tau}{12}\chi^{\top(1)''}_{lm}\chi^{\|(1)',lm} + \frac{\tau}{12}\chi^{\top(1)'}_{lm}\chi^{\|(1)'',lm} + \frac{\tau}{36}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1),lm}\right. \\
&\left. + \frac{\tau}{9}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1)',lm} - \frac{\tau}{12}\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)}_{lm}\right] + \nabla^{-2}\left[\frac{3}{2}c_L^2\delta^{(1)'',lm}\chi^{\top(1)}_{lm} + \frac{3}{2}c_L^2\delta^{(1),lm}\chi^{\top(1)''}_{lm} + \frac{3}{2\tau}c_L^2\delta^{(1)',lm}\chi^{\top(1)}_{lm}\right. \\
& + \frac{3}{2\tau}c_L^2\delta^{(1),lm}\chi^{\top(1)'}_{lm} - \frac{6}{\tau^2}c_L^2\delta^{(1),lm}\chi^{\top(1)}_{lm} + 3c_L^2\delta^{(1)',lm}\chi^{\top(1)'}_{lm} - 2v^{\|(1)'',lm}\chi^{\top(1)'}_{lm} - 2v^{\|(1),lm}\chi^{\top(1)'''}_{lm} - 4v^{\|(1)',lm}\chi^{\top(1)''}_{lm} \\
&\left. - \frac{2}{\tau}v^{\|(1),lm}\chi^{\top(1)''}_{lm} - \frac{2}{\tau}v^{\|(1)',lm}\chi^{\top(1)'}_{lm} + \frac{8}{\tau^2}v^{\|(1),lm}\chi^{\top(1)'}_{lm}\right],
\end{aligned} \quad (4.10)$$

being the effective source, formed from the products of first-order scalar and tensor modes. Written in the $\mathbf{k}$-space, (4.9) is

$$v^{\|(2)'''}_{s(t)\mathbf{k}} + \frac{3c_N^2}{\tau} v^{\|(2)''}_{s(t)\mathbf{k}} + \left(c_N^2 k^2 - \frac{6c_N^2+2}{\tau^2}\right) v^{\|(2)'}_{s(t)\mathbf{k}} + \frac{c_N^2}{\tau} k^2 v^{\|(2)}_{s(t)\mathbf{k}} = Z_{s(t)\mathbf{k}}(\tau), \tag{4.11}$$

where

$$Z_{s(t)\mathbf{k}}(\tau) = \frac{1}{(2\pi)^3} \int d^3k_2 \left\{ \left[k^l k^m \sum_{s=+,\times} \overset{s}{\epsilon}_{lm}(\mathbf{k}_2) \right] \tilde{Z}(\tau; \mathbf{k}, \mathbf{k}_2) \right\} \tag{4.12}$$

is the Fourier transform of (4.10), with

$$\begin{aligned}
\tilde{Z}(\tau; \mathbf{k}, \mathbf{k}_2) = {} & -\frac{4}{3\tau} h_{\mathbf{k}_2} v^{\|(1)}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{2\tau}{3} h_{\mathbf{k}_2} \phi^{(1)'}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{\tau}{6} h'_{\mathbf{k}_2} \phi^{(1)}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{\tau}{12} h''_{\mathbf{k}_2} \chi^{\|(1)'}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{\tau}{12} h'_{\mathbf{k}_2} \chi^{\|(1)''}_{(\mathbf{k}-\mathbf{k}_2)} + \frac{\tau}{36} |\mathbf{k}-\mathbf{k}_2|^2 h'_{\mathbf{k}_2} \chi^{\|(1)}_{(\mathbf{k}-\mathbf{k}_2)} \\
& + \frac{\tau}{9} |\mathbf{k}-\mathbf{k}_2|^2 h_{\mathbf{k}_2} \chi^{\|(1)'}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{\tau}{12} (k_2)^2 h_{\mathbf{k}_2} \chi^{\|(1)'}_{(\mathbf{k}-\mathbf{k}_2)} + \frac{3}{2} (k)^{-2} c_L^2 h_{\mathbf{k}_2} \delta^{(1)''}_{(\mathbf{k}-\mathbf{k}_2)} + \frac{3}{2} (k)^{-2} c_L^2 h''_{\mathbf{k}_2} \delta^{(1)}_{(\mathbf{k}-\mathbf{k}_2)} \\
& + \frac{3}{2\tau} (k)^{-2} c_L^2 h_{\mathbf{k}_2} \delta^{(1)'}_{(\mathbf{k}-\mathbf{k}_2)} + \frac{3}{2\tau} (k)^{-2} c_L^2 h'_{\mathbf{k}_2} \delta^{(1)}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{6}{\tau^2} (k)^{-2} c_L^2 h_{\mathbf{k}_2} \delta^{(1)}_{(\mathbf{k}-\mathbf{k}_2)} + 3 (k)^{-2} c_L^2 h'_{\mathbf{k}_2} \delta^{(1)'}_{(\mathbf{k}-\mathbf{k}_2)} \\
& - 2 (k)^{-2} h'_{\mathbf{k}_2} v^{\|(1)''}_{(\mathbf{k}-\mathbf{k}_2)} - 2 (k)^{-2} h'''_{\mathbf{k}_2} v^{\|(1)}_{(\mathbf{k}-\mathbf{k}_2)} - 4 (k)^{-2} h''_{\mathbf{k}_2} v^{\|(1)'}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{2}{\tau} (k)^{-2} h''_{\mathbf{k}_2} v^{\|(1)}_{(\mathbf{k}-\mathbf{k}_2)} - \frac{2}{\tau} (k)^{-2} h'_{\mathbf{k}_2} v^{\|(1)'}_{(\mathbf{k}-\mathbf{k}_2)} \\
& + \frac{8}{\tau^2} (k)^{-2} h'_{\mathbf{k}_2} v^{\|(1)}_{(\mathbf{k}-\mathbf{k}_2)},
\end{aligned} \tag{4.13}$$

where $\overset{s}{\epsilon}_{ij}(\mathbf{k}) k^i = 0$ has been used. Plugging the first-order solutions (2.21)–(2.28) into the above yields

$$\begin{aligned}
\tilde{Z}(\tau; \mathbf{k}, \mathbf{k}_2) = {} & b_1(\mathbf{k}_2) D_2(\mathbf{k}-\mathbf{k}_2) \frac{i}{\sqrt{2k_2}} e^{i(k_2 - |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau} \left[-\frac{40}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^5} - \frac{40i}{\sqrt{3}(k)^2\tau^4} + \frac{40ik_2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^4} - \frac{4i}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^4} \right. \\
& + \frac{20|\mathbf{k}-\mathbf{k}_2|}{3(k)^2\tau^3} + \frac{20(k_2)^2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^3} - \frac{4k_2}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^3} - \frac{40k_2}{\sqrt{3}(k)^2\tau^3} - \frac{4i(k_2)^3}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^2} + \frac{16i(k_2)^2}{\sqrt{3}(k)^2\tau^2} + \frac{8i|\mathbf{k}-\mathbf{k}_2|^2}{3\sqrt{3}(k)^2\tau^2} \\
& \left. - \frac{20i|\mathbf{k}-\mathbf{k}_2|k_2}{3(k)^2\tau^2} - \frac{2|\mathbf{k}-\mathbf{k}_2|^3}{9(k)^2\tau} + \frac{2|\mathbf{k}-\mathbf{k}_2|^2 k_2}{\sqrt{3}(k)^2\tau} - \frac{2|\mathbf{k}-\mathbf{k}_2|(k_2)^2}{(k)^2\tau} + \frac{2(k_2)^3}{\sqrt{3}(k)^2\tau} \right] + b_1(\mathbf{k}_2) D_3(\mathbf{k}-\mathbf{k}_2) \\
& \times \frac{i}{\sqrt{2k_2}} e^{i(k_2 + |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau} \left[-\frac{40}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^5} + \frac{40i}{\sqrt{3}(k)^2\tau^4} + \frac{40ik_2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^4} + \frac{4i}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^4} + \frac{20|\mathbf{k}-\mathbf{k}_2|}{3(k)^2\tau^3} \right. \\
& + \frac{20(k_2)^2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^3} + \frac{4k_2}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^3} + \frac{40k_2}{\sqrt{3}(k)^2\tau^3} - \frac{4i(k_2)^3}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^2} - \frac{16i(k_2)^2}{\sqrt{3}(k)^2\tau^2} - \frac{8i|\mathbf{k}-\mathbf{k}_2|^2}{3\sqrt{3}(k)^2\tau^2} \\
& \left. - \frac{20i|\mathbf{k}-\mathbf{k}_2|k_2}{3(k)^2\tau^2} - \frac{2|\mathbf{k}-\mathbf{k}_2|^3}{9(k)^2\tau} - \frac{2|\mathbf{k}-\mathbf{k}_2|^2 k_2}{\sqrt{3}(k)^2\tau} - \frac{2|\mathbf{k}-\mathbf{k}_2|(k_2)^2}{(k)^2\tau} - \frac{2(k_2)^3}{\sqrt{3}(k)^2\tau} \right] + b_2(\mathbf{k}_2) D_2(\mathbf{k}-\mathbf{k}_2) \\
& \times \frac{i}{\sqrt{2k_2}} e^{-i(k_2 + |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau} \left[\frac{40}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^5} + \frac{40i}{\sqrt{3}(k)^2\tau^4} + \frac{40ik_2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^4} + \frac{4i}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^4} - \frac{20|\mathbf{k}-\mathbf{k}_2|}{3(k)^2\tau^3} \right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{20(k_2)^2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^3}-\frac{4k_2}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^3}-\frac{40k_2}{\sqrt{3}(k)^2\tau^3}-\frac{4i(k_2)^3}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^2}-\frac{16i(k_2)^2}{\sqrt{3}(k)^2\tau^2}-\frac{8i|\mathbf{k}-\mathbf{k}_2|^2}{3\sqrt{3}(k)^2\tau^2}-\frac{20i|\mathbf{k}-\mathbf{k}_2|k_2}{3(k)^2\tau^2}+\frac{2|\mathbf{k}-\mathbf{k}_2|^3}{9(k)^2\tau}\\
&+\frac{2|\mathbf{k}-\mathbf{k}_2|^2k_2}{\sqrt{3}(k)^2\tau}+\frac{2|\mathbf{k}-\mathbf{k}_2|(k_2)^2}{(k)^2\tau}+\frac{2(k_2)^3}{\sqrt{3}(k)^2\tau}\bigg]+b_2(\mathbf{k}_2)D_3(\mathbf{k}-\mathbf{k}_2)\frac{i}{\sqrt{2k_2}}e^{-i(k_2-|\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}\bigg[\frac{40}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^5}\\
&-\frac{40i}{\sqrt{3}(k)^2\tau^4}+\frac{40ik_2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^4}-\frac{4i}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^4}-\frac{20|\mathbf{k}-\mathbf{k}_2|}{3(k)^2\tau^3}-\frac{20(k_2)^2}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^3}+\frac{4k_2}{\sqrt{3}|\mathbf{k}-\mathbf{k}_2|^2\tau^3}+\frac{40k_2}{\sqrt{3}(k)^2\tau^3}\\
&-\frac{4i(k_2)^3}{|\mathbf{k}-\mathbf{k}_2|(k)^2\tau^2}+\frac{16i(k_2)^2}{\sqrt{3}(k)^2\tau^2}+\frac{8i|\mathbf{k}-\mathbf{k}_2|^2}{3\sqrt{3}(k)^2\tau^2}-\frac{20i|\mathbf{k}-\mathbf{k}_2|k_2}{3(k)^2\tau^2}+\frac{2|\mathbf{k}-\mathbf{k}_2|^3}{9(k)^2\tau}-\frac{2|\mathbf{k}-\mathbf{k}_2|^2k_2}{\sqrt{3}(k)^2\tau}+\frac{2|\mathbf{k}-\mathbf{k}_2|(k_2)^2}{(k)^2\tau}-\frac{2(k_2)^3}{\sqrt{3}(k)^2\tau}\bigg].
\end{aligned}
\tag{4.14}
$$

Notice that (4.14) contains no time-integration terms.

The homogeneous solution of (4.11) for a general value of c_N is similar to Eq. (2.20) just by a replacement of $c_L \to c_N$, and the inhomogeneous solution of (4.11) is complicated. For the case $c_N^2 = \frac{1}{3}$ and a general c_L, (4.11) becomes

$$
v^{\|(2)'''}_{s(t)\mathbf{k}} + \frac{1}{\tau}v^{\|(2)''}_{s(t)\mathbf{k}} + \left(\frac{k^2}{3}-\frac{4}{\tau^2}\right)v^{\|(2)'}_{s(t)\mathbf{k}} + \frac{k^2}{3\tau}v^{\|(2)}_{s(t)\mathbf{k}} = Z_{s(t)\mathbf{k}}(\tau). \tag{4.15}
$$

The solution of (4.15) is

$$
\begin{aligned}
v^{\|(2)}_{s(t)\mathbf{k}} = {}& \frac{P_1(\mathbf{k})}{k\tau} + P_2(\mathbf{k})\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + P_3(\mathbf{k})\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}\\
&-\left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&-\left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\frac{1}{k\tau}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau',
\end{aligned}
\tag{4.16}
$$

where $P_1(\mathbf{k})$, $P_2(\mathbf{k})$, $P_3(\mathbf{k})$ are arbitrary time-independent functions, determined by initial conditions, and their associated terms are the homogeneous solution. The $P_1(\mathbf{k})$ terms are gauge modes as shall be seen in Sec. V.

The solution of $\delta^{(2)}_{s(t)\mathbf{k}}$ is directly given by (4.5) in $\mathbf{k}$-space as

$$
\begin{aligned}
\delta^{(2)}_{s(t)\mathbf{k}} &= -4v^{\|(2)'}_{s(t)\mathbf{k}} + 3F^{\|}_{s(t)\mathbf{k}}\\
&= \frac{4P_1(\mathbf{k})}{k\tau^2} + P_2(\mathbf{k})\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}} + P_3(\mathbf{k})\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}} + 3F^{\|}_{s(t)\mathbf{k}}\\
&\quad+\left(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right.\\
&\quad\left.+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \left(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right.\\
&\quad\left.+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&\quad+\frac{1}{k\tau^2}\int^\tau\frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau'.
\end{aligned}
\tag{4.17}
$$

Integrating the k-mode equation of (4.6) yields the scalar solution

$$
\begin{aligned}
\phi^{(2)}_{s(t)\mathbf{k}} &= -v^{\parallel(2)\prime}_{s(t)\mathbf{k}} - \int^{\tau} \frac{k^2}{3} v^{\parallel(2)}_{s(t)\mathbf{k}} d\tau' + \frac{3}{4} F^{\parallel}_{s(t)\mathbf{k}} - \frac{1}{4}\int^{\tau} A_{s(t)\mathbf{k}} d\tau' + P_4(\mathbf{k})(\tau) \\
&= P_1(\mathbf{k})\left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right) + P_2(\mathbf{k})\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right) e^{-ik\tau/\sqrt{3}} + P_3(\mathbf{k})\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right) e^{ik\tau/\sqrt{3}} + P_4(\mathbf{k}) \\
&\quad - \frac{2k}{3}\int^{\tau} [P_2(\mathbf{k}) e^{-ik\tau'/\sqrt{3}} + P_3(\mathbf{k}) e^{ik\tau'/\sqrt{3}}] \frac{d\tau'}{\tau'} + \frac{3}{4} F^{\parallel}_{s(t)\mathbf{k}}(\tau) - \frac{1}{4}\int^{\tau} A_{s(t)\mathbf{k}}(\tau') d\tau' \\
&\quad + \int^{\tau} \frac{(k^2\tau'^2 + 6)\ln\tau' + 3}{k^2\tau'} Z_{s(t)\mathbf{k}}(\tau') d\tau' + \left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right)\int^{\tau} \frac{3(k^2\tau'^2 + 6)}{k^3\tau'} Z_{s(t)\mathbf{k}}(\tau') d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'} d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'} d\tau' \\
&\quad + \int^{\tau}\left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'} d\tau'\right] d\tau'' \\
&\quad + \int^{\tau}\left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'} d\tau'\right] d\tau'', \qquad (4.18)
\end{aligned}
$$

where $A_{s(t)} \equiv \frac{d}{d\tau}[\frac{4}{3}\chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}]$ by (3.26), and

$$
\begin{aligned}
\int^{\tau} A_{s(t)\mathbf{k}} d\tau' &= \frac{1}{(2\pi)^3}\int d^3k_2 \left\{\left[k^l k^m \sum_{s=+,\times} \overset{s}{\epsilon}_{lm}(\mathbf{k}_2)\right]\left[-\frac{4}{3} h_{\mathbf{k}_2}\chi^{\parallel(1)}_{(\mathbf{k}-\mathbf{k}_2)}\right]\right\} \\
&= \frac{1}{(2\pi)^3}\int d^3k_2 \left\{\left[k^l k^m \sum_{s=+,\times} \overset{s}{\epsilon}_{lm}(\mathbf{k}_2)\right]\frac{16i}{3\sqrt{2k_2}|\mathbf{k}-\mathbf{k}_2|\tau}\right. \\
&\quad \times \left[b_1(\mathbf{k}_2) D_2(\mathbf{k}-\mathbf{k}_2)\left(\frac{\sqrt{3}ie^{i(k_2 - |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}}{|\mathbf{k}-\mathbf{k}_2|\tau} - e^{ik_2\tau}\int^{\tau}\frac{e^{-i|\mathbf{k}-\mathbf{k}_2|\tau'/\sqrt{3}}}{\tau'} d\tau'\right)\right. \\
&\quad + b_1(\mathbf{k}_2) D_3(\mathbf{k}-\mathbf{k}_2)\left(-\frac{\sqrt{3}ie^{i(k_2 + |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}}{|\mathbf{k}-\mathbf{k}_2|\tau} - e^{ik_2\tau}\int^{\tau}\frac{e^{i|\mathbf{k}-\mathbf{k}_2|\tau'/\sqrt{3}}}{\tau'} d\tau'\right) \\
&\quad + b_2(\mathbf{k}_2) D_2(\mathbf{k}-\mathbf{k}_2)\left(-\frac{\sqrt{3}ie^{-i(k_2 + |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}}{|\mathbf{k}-\mathbf{k}_2|\tau} + e^{-ik_2\tau}\int^{\tau}\frac{e^{-i|\mathbf{k}-\mathbf{k}_2|\tau'/\sqrt{3}}}{\tau'} d\tau'\right) \\
&\quad \left.\left. + b_2(\mathbf{k}_2) D_3(\mathbf{k}-\mathbf{k}_2)\left(\frac{\sqrt{3}ie^{-i(k_2 - |\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}}{|\mathbf{k}-\mathbf{k}_2|\tau} + e^{-ik_2\tau}\int^{\tau}\frac{e^{i|\mathbf{k}-\mathbf{k}_2|\tau'/\sqrt{3}}}{\tau'} d\tau'\right)\right]\right\}. \qquad (4.19)
\end{aligned}
$$

Notice that in the above, $P_4(\mathbf{k})$ terms in (4.18) are gauge modes, as shall be seen in Sec. V.

Finally, plugging (4.17) and (4.18) into (3.2) in $\mathbf{k}$-space gives the scalar solution

$$\begin{aligned}
\chi^{\parallel(2)}_{s(t)\mathbf{k}} &= \frac{18}{k^4\tau}\phi^{(2)\prime}_{s(t)\mathbf{k}} + \frac{6}{k^2}\phi^{(2)}_{s(t)\mathbf{k}} + \frac{9}{k^4\tau^2}\delta^{(2)}_{s(t)\mathbf{k}} + \frac{3}{k^4}E_{s(t)\mathbf{k}} \\
&= -P_1(\mathbf{k})\frac{2\ln\tau}{k} + P_2(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - P_3(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} + \frac{6P_4(\mathbf{k})}{k^2} - \frac{4}{k}\int^\tau [P_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + P_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} \\
&\quad - \frac{2\ln\tau}{k}\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' + \int^\tau \frac{6(k^2\tau'^2+6)\ln\tau' + 18}{k^4\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' - \frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right. \\
&\quad \left. + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \int^\tau\left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \int^\tau\left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \frac{3}{k^4}E_{s(t)\mathbf{k}} + \frac{27}{2k^4\tau}F^{\parallel\prime}_{s(t)\mathbf{k}}(\tau) + \frac{27}{k^4\tau^2}F^{\parallel}_{s(t)\mathbf{k}} + \frac{9}{2k^2}F^{\parallel}_{s(t)\mathbf{k}}(\tau) - \frac{9}{2k^4\tau}A_{s(t)\mathbf{k}}(\tau) - \frac{3}{2k^2}\int^\tau A_{s(t)\mathbf{k}}(\tau')d\tau'. \qquad (4.20)
\end{aligned}$$

We have checked that the scalar solutions (4.16)–(4.18) and (4.20) satisfy the scalar parts of the evolution equation, (3.14) and (3.18).

The above second-order solutions involve many integrals $\int d^3k$ or $\int d\tau$ of the scalar-tensor coupling terms. In the $\mathbf{k}$-integrations, the four functions $b_1(\mathbf{k})$, $b_2(\mathbf{k})$, $D_2(\mathbf{k})$, and $D_3(\mathbf{k})$ depend upon the concrete initial conditions and inflation models. Moreover, in actually doing integration, one should avoid IR and UV divergences which may arise from the lower and upper limits of $\int d^3k$ [66,67]. As an illustration, suppose $b_1(\mathbf{k}) \propto k^{N_1}$, $b_2(\mathbf{k}) \propto k^{N_2}$, $D_2(\mathbf{k}) \propto k^{N_3}$, and $D_3(\mathbf{k}) \propto k^{N_4}$. Then we shall have the following typical integration terms:

$$\int d^3k_2\left[e^{-i(k_2+|\mathbf{k}-\mathbf{k}_2|/\sqrt{3})\tau}(k_2)^{n_1}|\mathbf{k}-\mathbf{k}_2|^{n_2}k^l k^m\sum_{s=+,\times}\overset{s}{\epsilon}_{lm}(\mathbf{k}_2)\right], \qquad (4.21)$$

where n_1 and n_2 are linearly related to N_1, N_2, N_3, and N_4. Let $\mathbf{k}$ be along the z-axis and θ be the angle between $\mathbf{k}_2$ and $\mathbf{k}$. The unit vector along $\mathbf{k}_2$ is

$$\hat{k}_2 = \cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z}, \qquad (4.22)$$

the orthogonal unit vectors normal to $\mathbf{k}_2$ are

$$\hat{u} = \sin\phi\hat{x} - \cos\phi\hat{y}, \qquad (4.23)$$

$$\hat{v} = \cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z}, \qquad (4.24)$$

and the corresponding polarization tensors of the first-order tensor mode are as follows:

$$\overset{\times}{\epsilon}_{ij}(\mathbf{k}_2) = \hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j = \begin{pmatrix} 2\sin\phi\cos\phi\cos\theta & \sin^2\phi\cos\theta - \cos^2\phi\cos\theta & -\sin\phi\sin\theta \\ \sin^2\phi\cos\theta - \cos^2\phi\cos\theta & -2\sin\phi\cos\phi\cos\theta & \cos\phi\sin\phi \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & 0 \end{pmatrix}, \qquad (4.25)$$

$$\overset{+}{\epsilon}_{ij}(\mathbf{k}_2) = \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j = \begin{pmatrix} \sin^2\phi - \cos^2\phi\cos^2\theta & -\sin\phi\cos\phi(1+\cos^2\theta) & \cos\phi\cos\theta\sin\theta \\ -\sin\phi\cos\phi(1+\cos^2\theta) & \cos^2\phi - \sin^2\phi\cos^2\theta & \sin\phi\cos\theta\sin\theta \\ \cos\phi\cos\theta\sin\theta & \sin\phi\cos\theta\sin\theta & -\sin^2\theta \end{pmatrix}. \qquad (4.26)$$

So one has $k^l k^m \sum_{s=+,\times} \overset{s}{\epsilon}_{lm}(\mathbf{k}_2) = -(k_2)^2 \sin^2\theta$. The angular integration in (4.21) can be carried out, yielding

$$
\begin{aligned}
&-2\pi \int_{K_1}^{K_2} dk_2 \int_0^{\pi} d\theta [e^{-i(k_2+\sqrt{(k)^2+(k_2)^2-2k_2 k\cos\theta}/\sqrt{3})\tau}(k_2)^{n_1+2}((k)^2+(k_2)^2-2k_2 k\cos\theta)^{\frac{n_2}{2}}(k)^2\sin^3\theta] \\
&= -\pi \int_{K_1}^{K_2} dk_2 \left\{ \frac{3^{\frac{n_2}{2}+1}}{2k\tau^6}(k_2)^{n_1-1} e^{-ik_2\tau}\tau^{-n_2}(-i)^{n_2}\left[\tau^4((k)^2-(k_2)^2)^2\Gamma\left(n_2+2, \frac{i|k-k_2|\tau}{\sqrt{3}}\right)\right.\right. \\
&\quad + 6\tau^2((k)^2+(k_2)^2)\Gamma\left(n_2+4, \frac{i|k-k_2|\tau}{\sqrt{3}}\right) + 9\Gamma\left(n_2+6, \frac{i|k-k_2|\tau}{\sqrt{3}}\right) \\
&\quad - \tau^4((k)^2-(k_2)^2)^2\Gamma\left(n_2+2, \frac{i(k+k_2)\tau}{\sqrt{3}}\right) - 6\tau^2((k)^2+(k_2)^2)\Gamma\left(n_2+4, \frac{i(k+k_2)\tau}{\sqrt{3}}\right) \\
&\quad \left.\left. - 9\Gamma\left(n_2+6, \frac{i(k+k_2)\tau}{\sqrt{3}}\right)\right]\right\},
\end{aligned}
$$

where the integration limits K_1 and K_2 are cutoffs to ensure IR and UV convergence, and the remaining integration over k_2 can be done numerically. All other $\int d^3k$ terms can be treated similarly.

The time integrations can also be carried out. $Z_{s(t)\mathbf{k}}$ in (4.12) and (4.14) only has one type of term: $\frac{1}{\tau^N} e^{-i\frac{k\tau}{\sqrt{3}}}$, so that the single time integrations of $Z_{s(t)\mathbf{k}}$ have the nontrivial terms,

$$
\int^{\tau} \frac{d\tau'}{\tau'^n} \exp\left[-i\frac{k_1\tau'}{\sqrt{3}}\right] \propto k^{n-1}\Gamma\left(1-n, \frac{ik\tau}{\sqrt{3}}\right), \tag{4.27}
$$

$$
\begin{aligned}
&\int^{\tau} \frac{d\tau'}{\tau'^n} \ln\tau' \exp\left[-i\frac{k_1\tau'}{\sqrt{3}}\right] \\
&\propto 3^{\frac{1-n}{2}}(ik)^{n-1}\ln\tau\Gamma\left(1-n, 0, \frac{ik\tau}{\sqrt{3}}\right) \\
&\quad - \frac{\tau^{1-n}}{(1-n)^2}{}_2F_2\left(1-n, 1-n; 2-n, 2-n; -\frac{ik\tau}{\sqrt{3}}\right),
\end{aligned} \tag{4.28}
$$

and the double time integrations of $Z_{s(t)\mathbf{k}}$ have the following nontrivial term:

$$
\int^{\tau} \frac{d\tau'}{\tau'} \exp\left[i\frac{k_1\tau'}{\sqrt{3}}\right] \int^{\tau'} \frac{d\tau''}{\tau''^n} \exp\left[i\frac{k_2\tau''}{\sqrt{3}}\right] \equiv z_5(\tau; n; k_1, k_2), \tag{4.29}
$$

which, in actual computing, can be defined as a function and recalled [63]. As an illustration, we plot the real part of z_5 in Fig. 1.

B. Solution for tensor-tensor couplings

Now we solve the second-order perturbations with tensor-tensor couplings. The tensor part of the traceless part of evolution equation (3.54) has the general solution

$$
\begin{aligned}
\chi^{\top(2)}_{Tij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \Bigg(\bar{I}_{Tij}(\mathbf{k},\tau) + \sum_{s=+,\times} \overset{s}{\epsilon}_{ij}(\mathbf{k}) \\
\times \left[-f_1^s \sqrt{\frac{2}{\pi}} \frac{ie^{ik\tau}}{k\tau} + f_2^s \sqrt{\frac{2}{\pi}} \frac{ie^{-ik\tau}}{k\tau} \right] \Bigg),
\end{aligned} \tag{4.30}
$$

where $f_1^s(\mathbf{k})$ and $f_2^s(\mathbf{k})$ terms represent a homogeneous solution, similar to (4.1). The integrand of the inhomogeneous solution in (4.30) is given by

$$
\begin{aligned}
\bar{I}_{Tij}(\mathbf{k},\tau) \equiv &\frac{ie^{-ik\tau}}{k\tau} \int^{\tau} \tau' e^{ik\tau'} \bar{J}_{Tij}(\mathbf{k},\tau') d\tau' \\
&- \frac{ie^{ik\tau}}{k\tau} \int^{\tau} \tau' e^{-ik\tau'} \bar{J}_{Tij}(\mathbf{k},\tau') d\tau',
\end{aligned} \tag{4.31}
$$

with $\bar{J}_{Tij}$ being the Fourier transform of the source J_{Tij} in (3.55) that contains many terms of products of first-order solutions. The solution (4.30) has a similar structure to the

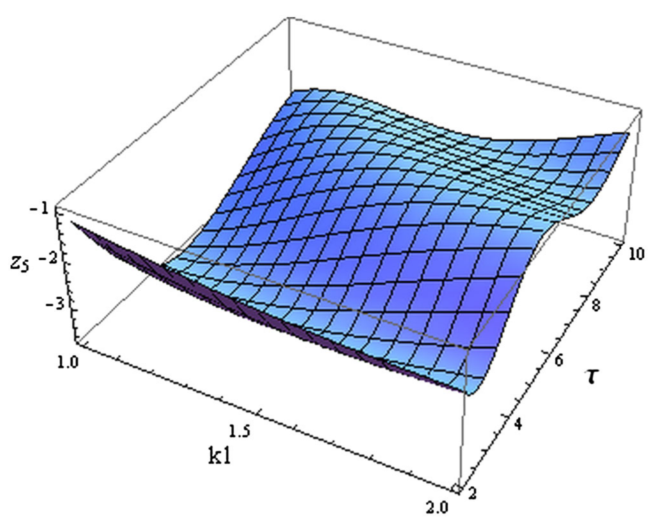


FIG. 1. Real part of $z_5(\tau; 1; k_1, k_2)$ at fixed k_2.

solution (4.1) except for the inhomogeneous part with a different source term.

The vector solution of Eq. (3.52) is

$$\chi^{\perp(2)}_{Tij}(\mathbf{x},\tau) = q_{3ij}(\mathbf{x}) + \frac{q_{4ij}(\mathbf{x})}{\tau} + \int^{\tau}\frac{d\tau'}{\tau'^2}\int^{\tau'} 2\tau''^2 V_{Tij}(\mathbf{x},\tau'')d\tau'', \quad (4.32)$$

with V_{Tij} given by (3.53) and $q_{3ij}(\mathbf{x})$ and $q_{4ij}(\mathbf{x})$ are two coefficients to be determined by initial conditions. Note that $q_{3ij}(\mathbf{x})$ is a gauge mode as shall be seen in the next section. Also the solution (4.32) has a similar structure to the solution (4.3). Plugging the solution (4.32) into (3.41), one has the transverse part of the second-order velocity as

$$v^{\perp(2)}_{Ti} = \frac{q^{,j}_{4ij}(\mathbf{x})}{8} + \frac{\tau^2}{4}(M_{Ti} - \partial_i\nabla^{-2}M^{,k}_{Tk}) - \frac{1}{4}\int^{\tau}\tau'^2 V^{,j}_{Tij}(\mathbf{x},\tau')d\tau', \quad (4.33)$$

where $(M_{Ti} - \partial_i\nabla^{-2}M^{,k}_{Tk})$ is in (3.42). This solution satisfies the transverse momentum conservation (3.60), i.e., $v^{\perp(2)'}_{Ti} = 0$, as we have checked. According to (4.33), the second-order curl vector $v^{\perp(2)}_{i}$ is generated by the coupling, although the first-order curl vector $v^{\perp(1)}_{i}$ is zero by assumption.

Next, we solve the scalars. From the longitudinal momentum conservation (3.59),

$$\delta^{(2)}_{T} = -\frac{4}{3c_N^2}v^{\parallel(2)'}_{T}. \quad (4.34)$$

Plugging the above $\delta^{(2)}_{T}$ into the energy conservation (3.56) gives $\phi^{(2)'}_{T}$ in terms of $v^{\parallel(2)}_{T}$, $v^{\parallel(2)'}_{T}$, $v^{\parallel(2)''}_{T}$ as

$$\phi^{(2)'}_{T} = -\frac{1}{3c_N^2}v^{\parallel(2)''}_{T} - \frac{1}{\tau}\frac{c_N^2-\frac{1}{3}}{c_N^2}v^{\parallel(2)'}_{T} + \frac{1}{3}\nabla^2 v^{\parallel(2)}_{T} - \frac{1}{4}A_T. \quad (4.35)$$

To use the energy constraint (3.35), taking $[\frac{d}{d\tau}(3.35)]$ gives

$$-\frac{6}{\tau}\phi^{(2)''}_{T} + \frac{6}{\tau^2}\phi^{(2)'}_{T} + \nabla^2\left[2\phi^{(2)'}_{T} + \frac{1}{3}\nabla^2\chi^{\parallel(2)'}_{T}\right] = \frac{3}{\tau^2}\delta^{(2)'}_{T} - \frac{6}{\tau^3}\delta^{(2)}_{T} + E'_T. \quad (4.36)$$

Plugging the momentum constraint (3.39) into the above to eliminate $\nabla^2\nabla^2\chi^{\parallel(2)'}_{T}$, one has

$$-\frac{6}{\tau}\phi^{(2)''}_{T} + \frac{6}{\tau^2}\phi^{(2)'}_{T} - \frac{4}{\tau^2}\nabla^2 v^{\parallel(2)}_{T} + M^{,k}_{Tk} = \frac{3}{\tau^2}\delta^{(2)'}_{T} - \frac{6}{\tau^3}\delta^{(2)}_{T} + E'_T. \quad (4.37)$$

Then, plugging $\delta^{(2)}_{T}$ of (4.34) and $\phi^{(2)'}_{T}$ of (4.35) into the above yields a third-order differential equation of $v^{\parallel(2)}_{T}$ as

$$v^{\parallel(2)'''}_{T} + \frac{3c_N^2}{\tau}v^{\parallel(2)''}_{T} - \frac{6c_N^2+2}{\tau^2}v^{\parallel(2)'}_{T} - \frac{c_N^2}{\tau}\nabla^2 v^{\parallel(2)}_{T} - c_N^2\nabla^2 v^{\parallel(2)'}_{T} = Z_T, \quad (4.38)$$

with

$$Z_T \equiv -\frac{3c_N^2}{4}A'_T + \frac{3c_N^2}{4\tau}A_T - \frac{\tau}{2}c_N^2 M^{,l}_{Tl} + \frac{\tau}{2}c_N^2 E'_T, \quad (4.39)$$

which can also be written as

$$Z_T = -\frac{1}{6}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}, \quad (4.40)$$

where the first-order GW equation (2.25) is used. Equation (4.38) in the $\mathbf{k}$-space is

$$v^{\parallel(2)'''}_{T\mathbf{k}} + \frac{3c_N^2}{\tau}v^{\parallel(2)''}_{T\mathbf{k}} + \left(c_N^2k^2 - \frac{6c_N^2+2}{\tau^2}\right)v^{\parallel(2)'}_{T\mathbf{k}} + \frac{c_N^2}{\tau}k^2 v^{\parallel(2)}_{T\mathbf{k}} = Z_{T\mathbf{k}}(\tau), \quad (4.41)$$

where the source

$$Z_{T\mathbf{k}} = \frac{1}{(2\pi)^{3/2}}\int d^3x\left[-\frac{1}{6}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}\right]e^{-i\mathbf{k}\cdot\mathbf{x}} = -\frac{1}{6(2\pi)^3}\int d^3k_2\left[\sum_{s_1=+,\times}\sum_{s_2=+,\times}\overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)\right] \times h'_{\mathbf{k}-\mathbf{k}_2}h'_{\mathbf{k}_2} \quad (4.42)$$

is the Fourier transform of $Z_T(\mathbf{x},\tau)$ in (4.39). Plugging (2.28) into the above, one has

$$
\begin{aligned}
Z_{T\mathbf{k}} = \frac{1}{12(2\pi)^3}\int d^3k_2\Bigg[\sum_{s_1=+,\times}\sum_{s_2=+,\times}\overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)\frac{1}{\sqrt{k_2|\mathbf{k}-\mathbf{k}_2|}} \\
\times\Bigg\{-b_1(\mathbf{k}_2)b_2(\mathbf{k}-\mathbf{k}_2)e^{i(k_2-|\mathbf{k}-\mathbf{k}_2|)\tau}\left(\frac{1}{\tau^4}+\frac{i(|\mathbf{k}-\mathbf{k}_2|-k_2)}{\tau^3}+\frac{k_2|\mathbf{k}-\mathbf{k}_2|}{\tau^2}\right) \\
+b_1(\mathbf{k}_2)b_1(\mathbf{k}-\mathbf{k}_2)e^{i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau}\left(\frac{1}{\tau^4}-\frac{i(|\mathbf{k}-\mathbf{k}_2|+k_2)}{\tau^3}-\frac{k_2|\mathbf{k}-\mathbf{k}_2|}{\tau^2}\right) \\
+b_2(\mathbf{k}_2)b_2(\mathbf{k}-\mathbf{k}_2)e^{-i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau}\left(\frac{1}{\tau^4}+\frac{i(|\mathbf{k}-\mathbf{k}_2|+k_2)}{\tau^3}-\frac{k_2|\mathbf{k}-\mathbf{k}_2|}{\tau^2}\right) \\
-b_2(\mathbf{k}_2)b_1(\mathbf{k}-\mathbf{k}_2)e^{-i(k_2-|\mathbf{k}-\mathbf{k}_2|)\tau}\left(\frac{1}{\tau^4}-\frac{i(|\mathbf{k}-\mathbf{k}_2|-k_2)}{\tau^3}+\frac{k_2|\mathbf{k}-\mathbf{k}_2|}{\tau^2}\right)\Bigg\}\Bigg]. \qquad (4.43)
\end{aligned}
$$

The homogeneous solution of (4.41) for a general value of c_N is similar to the first-order solution (2.20) just by a replacement of $c_L \to c_N$ and a new set of integration constants d_1, d_2, d_3. For simplicity, we take $c_N^2 = \frac{1}{3}$, and c_L can be a general value in the following calculations. Then, (4.41) becomes

$$
v_{T\mathbf{k}}^{\parallel(2)'''}+\frac{1}{\tau}v_{T\mathbf{k}}^{\parallel(2)''}+\left(\frac{k^2}{3}-\frac{4}{\tau^2}\right)v_{T\mathbf{k}}^{\parallel(2)'}+\frac{k^2}{3\tau}v_{T\mathbf{k}}^{\parallel(2)} = Z_{T\mathbf{k}}(\tau). \qquad (4.44)
$$

The solution of (4.44) is

$$
\begin{aligned}
v_{T\mathbf{k}}^{\parallel(2)} = \frac{Q_1(\mathbf{k})}{k\tau}+Q_2(\mathbf{k})\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}}+Q_3(\mathbf{k})\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}} \\
-\left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
-\left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
+\frac{1}{k\tau}\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau', \qquad (4.45)
\end{aligned}
$$

where $Q_1(\mathbf{k})$, $Q_2(\mathbf{k})$, $Q_3(\mathbf{k})$ are time-independent coefficients, determined by initial conditions, corresponding to the homogeneous terms of the solution. Note that the $Q_1(\mathbf{k})$ term is a gauge mode as shall be seen in Sec. V. The solution of $\delta_{T\mathbf{k}}^{(2)}$ is directly given by (4.34) in $\mathbf{k}$-space as

$$
\begin{aligned}
\delta_{T\mathbf{k}}^{(2)} &= -4v_{T\mathbf{k}}^{\parallel(2)'} \\
&= \frac{4Q_1(\mathbf{k})}{k\tau^2}+Q_2(\mathbf{k})\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}+Q_3(\mathbf{k})\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}} \\
&\quad+\left(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad+\left(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right. \\
&\quad\left.-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'+\frac{1}{k\tau^2}\int^{\tau}\frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau'. \qquad (4.46)
\end{aligned}
$$

Integrating (4.35) yields the solution of $\phi_{T\mathbf{k}}^{(2)}$ as

$$
\begin{aligned}
\phi^{(2)}_{T\mathbf{k}} &= -v^{\|(2)\prime}_{T\mathbf{k}} - \int^\tau \frac{k^2}{3} v^{\|(2)}_{T\mathbf{k}} d\tau' - \frac{1}{4}\int^\tau A_{T\mathbf{k}} d\tau' + Q_4(\mathbf{k})(\tau) \\
&= Q_1(\mathbf{k})\left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right) + Q_2(\mathbf{k})\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + Q_3(\mathbf{k})\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} + Q_4(\mathbf{k}) \\
&\quad - \frac{2k}{3}\int^\tau [Q_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + Q_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} - \frac{1}{4}\int^\tau A_{T\mathbf{k}}(\tau')d\tau' \\
&\quad + \int^\tau \frac{(k^2\tau'^2+6)\ln\tau' + 3}{k^2\tau'} Z_{T\mathbf{k}}(\tau')d\tau' + \left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right)\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'} Z_{T\mathbf{k}}(\tau')d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \int^\tau \left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \int^\tau \left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'',
\end{aligned} \tag{4.47}
$$

where [by $A_T = \frac{d}{d\tau}[\frac{2}{3}\chi^{\top(1)}_{lm}\chi^{\top(1)lm}]$ in (3.57)]

$$
\begin{aligned}
\int^\tau A_{T\mathbf{k}} d\tau' &= \frac{2}{3(2\pi)^3}\int d^3k_2 \left[\sum_{s_1=+,\times}\sum_{s_2=+,\times} \overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2) h_{\mathbf{k}-\mathbf{k}_2} h_{\mathbf{k}_2}\right] \\
&= \frac{1}{3(2\pi)^3}\int d^3k_2 \left[\frac{1}{\tau^2\sqrt{k_2|\mathbf{k}-\mathbf{k}_2|}}\sum_{s_1=+,\times}\sum_{s_2=+,\times} \overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)\right. \\
&\quad \times (b_1(\mathbf{k}_2)b_2(\mathbf{k}-\mathbf{k}_2)e^{i(k_2-|\mathbf{k}-\mathbf{k}_2|)\tau} - b_1(\mathbf{k}_2)b_1(\mathbf{k}-\mathbf{k}_2)e^{i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau} \\
&\quad \left. - b_2(\mathbf{k}_2)b_2(\mathbf{k}-\mathbf{k}_2)e^{-i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau} + b_2(\mathbf{k}_2)b_1(\mathbf{k}-\mathbf{k}_2)e^{-i(k_2-|\mathbf{k}-\mathbf{k}_2|)\tau})\right].
\end{aligned} \tag{4.48}
$$

In the above and the following solutions, $Q_4(\mathbf{k})$ terms are gauge modes, as shall be seen in Sec. V.

Finally, plugging (4.46) and (4.47) into (3.35) in $\mathbf{k}$-space, one has the scalar solution

$$
\begin{aligned}
\chi^{\|(2)}_{T\mathbf{k}} &= \frac{18}{k^4\tau}\phi^{(2)\prime}_{T\mathbf{k}} + \frac{6}{k^2}\phi^{(2)}_{T\mathbf{k}} + \frac{9}{k^4\tau^2}\delta^{(2)}_{T\mathbf{k}} + \frac{3}{k^4}E_{T\mathbf{k}} \\
&= -Q_1(\mathbf{k})\frac{2\ln\tau}{k} + Q_2(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - Q_3(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} + \frac{6Q_4(\mathbf{k})}{k^2} - \frac{4}{k}\int^\tau [Q_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + Q_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} \\
&\quad - \frac{2\ln\tau}{k}\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau' + \int^\tau \frac{6(k^2\tau'^2+6)\ln\tau' + 18}{k^4\tau'}Z_{T\mathbf{k}}(\tau')d\tau' - \frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right. \\
&\quad \left. + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \int^\tau \left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \int^\tau \left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \frac{3}{k^4}E_{T\mathbf{k}} - \frac{9}{2k^4\tau}A_{T\mathbf{k}}(\tau) - \frac{3}{2k^2}\int^\tau A_{T\mathbf{k}}(\tau')d\tau'.
\end{aligned} \tag{4.49}
$$

We have checked that the scalar solutions (4.45)–(4.47) and (4.49) satisfy the scalar parts of the evolution equation, (3.45) and (3.49). So far, the solutions for the second-order perturbations have all been given. The expressions of the second-order solutions for the RD stage generally contain many more terms than those for the MD stage [61,62].

The second-order solutions contain several terms of integrals of the tensor-tensor coupling. The $\int d^3k$ integral can be treated in a similar way to the paragraph below (4.20). As an illustration, suppose $b_1(\mathbf{k}) \propto k^{N_1}$ and $b_2(\mathbf{k}) \propto k^{N_2}$. Then we shall have the following typical integration,

$$\int d^3k_2 \left[e^{-i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau}(k_2)^{n_3}|\mathbf{k}-\mathbf{k}_2|^{n_4} \sum_{s_1=+,\times}\sum_{s_2=+,\times} \overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)\right]$$
$$= \int_{K_3}^{K_4} dk_2 \int_0^{\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left[e^{-i(k_2+|\mathbf{k}-\mathbf{k}_2|)\tau}(k_2)^{n_3+2}|\mathbf{k}-\mathbf{k}_2|^{n_4} \sum_{s_1=+,\times}\sum_{s_2=+,\times} \overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)\right], \quad (4.50)$$

where n_3 and n_4 are linearly related to N_1 and N_2, and K_3 and K_4 are cutoffs to ensure IR and UV convergence. Take $\mathbf{k}$ as the z-axis and θ as the angle between $\mathbf{k}_2$ and $\mathbf{k}$. The polarization tensors $\overset{s}{\epsilon}_{ij}(\mathbf{k}_2)$ are the same as (4.25) and (4.26), and $\overset{s}{\epsilon}_{ij}(\mathbf{k}-\mathbf{k}_2)$ are given by a replacement of $\theta \to \arctan(\frac{k_2\sin\theta}{k-k_2\cos\theta})$ and $\phi \to \phi + \pi$ in (4.25) and (4.26). With these relations, the integration over $\mathbf{k}_2$ can be treated similarly to the paragraph below (4.20). As for the time integrations $\int d\tau$, since $Z_{T\mathbf{k}}(\tau)$ in (4.43) only has one type of term, $\frac{1}{\tau^{N'}}e^{-ik\tau}$, the time integrations in the tensor-tensor coupling case have similar forms as in (4.27)–(4.29) and can be done numerically.

V. SECOND-ORDER RESIDUAL GAUGE MODES IN SYNCHRONOUS COORDINATES

The second-order perturbation solutions in the last section contain the gauge modes. In this section, we shall give the second-order residual gauge transformations for scalar-tensor and tensor-tensor coupling cases and eliminate the second-order residual gauge modes in the solutions.

Consider the coordinate transformation up to second order [60,61,63]:

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^{(1)\mu} + \frac{1}{2}\xi^{(1)\mu}_{,\alpha}\xi^{(1)\alpha} + \frac{1}{2}\xi^{(2)\mu}, \quad (5.1)$$

where $\xi^{(1)\mu}$ is a first-order vector field and $\xi^{(2)\mu}$ is a second-order vector field, and they can be written in terms of their respective parameters,

$$\xi^{(A)0} = \alpha^{(A)}, \quad \text{with} \quad A = 1, 2, \quad (5.2)$$

$$\xi^{(A)i} = \partial^i\beta^{(A)} + d^{(A)i} \quad \text{with} \quad \partial_i d^{(A)i} = 0. \quad (5.3)$$

For the RD stage with $a(\tau) \propto \tau$,

$$\xi^{(1)0}(\tau,\mathbf{x}) = \frac{A^{(1)}(\mathbf{x})}{\tau}, \quad (5.4)$$

$$\xi^{(1)i}(\tau,\mathbf{x}) = A^{(1),i}\ln\tau + C^{\parallel(1),i}(\mathbf{x}) + C^{\perp(1)i}(\mathbf{x}), \quad (5.5)$$

where $A^{(1)}$ and $C^{\parallel(1)}$ are τ-independent scalar functions, $C_i^{\perp(1)}$ is a τ-independent curl vector, and all of them are of first order. [See (C12) and (C13) of Ref. [63].] The first-order gauge transformations of metric perturbations between two synchronous coordinate systems are given in (3.37)–(3.50) of Ref. [63].

The general second-order synchronous-to-synchronous gauge transformations in a general RW spacetime are given in Appendix C in Ref. [63]. Here we apply them to the case of $a(\tau) \propto \tau$. First we shall give the second-order gauge transformations for the scalar-tensor coupling. From (C27), (C29), and (C30) of Ref. [63], keeping only the $\chi_{ij}^{\top(1)}$-linear-dependent terms, the second-order vector field $\xi^{(2)\mu}$ is given as follows:

$$\alpha^{(2)} = \frac{A^{(2)}(\mathbf{x})}{\tau}, \quad (5.6)$$

$$\beta^{(2)} = \nabla^{-2}\left[-A^{(1),lm}\int^\tau \frac{2\chi_{lm}^{\top(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\right] + A^{(2)}(\mathbf{x})\ln\tau + C^{\parallel(2)}(\mathbf{x}), \quad (5.7)$$

$$d_i^{(2)} = \partial_i\nabla^{-2}\left[2A^{(1),lm}\int^\tau \frac{\chi_{lm}^{\top(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\right] - 2A^{(1),l}\int^\tau \frac{\chi_{li}^{\top(1)}(\tau',\mathbf{x})}{\tau'}d\tau' + C_i^{\perp(2)}(\mathbf{x}), \quad (5.8)$$

where $A^{(2)}$ and $C^{\parallel(2)}$ are τ-independent scalar functions, $C_i^{\perp(2)}$ is a τ-independent curl vector, and all of them are of second order. From (C31)–(C34) of Ref. [63] applied to the RD stage and for the scalar-tensor coupling, we have the second-order transformation of metric perturbations

$$\bar{\phi}^{(2)}_{s(t)} = \phi^{(2)}_{s(t)} + \frac{2\ln\tau}{3}\chi^{\top(1)}_{lm}A^{(1),lm} - A^{(1),lm}\int^{\tau}\frac{2\chi^{\top(1)}_{lm}(\tau',\mathbf{x})}{3\tau'}d\tau' + \frac{1}{\tau^2}A^{(2)} + \frac{\ln\tau}{3}\nabla^2A^{(2)} + \frac{1}{3}\nabla^2C^{\|(2)}, \tag{5.9}$$

$$\begin{aligned}\bar{\chi}^{\|(2)} = {}& \chi^{\|(2)} - \frac{6}{\tau^2}\nabla^{-2}\nabla^{-2}(\chi^{\top(1)}_{lm}A^{(1),lm}) - \frac{3}{\tau}\nabla^{-2}\nabla^{-2}(\chi^{\top(1)\prime}_{lm}A^{(1),lm})\\ &+ [\ln\tau]\{2\nabla^{-2}(\chi^{\top(1)}_{lm}A^{(1),lm}) - 3\nabla^{-2}\nabla^{-2}(3\chi^{\top(1)}_{lm,n}A^{(1),lmn} + 2\chi^{\top(1)}_{lm}\nabla^2A^{(1),lm})\}\\ &+ \{2\nabla^{-2}(\chi^{\top(1)}_{lm}C^{\|(1),lm}) - 3\nabla^{-2}\nabla^{-2}(3\chi^{\top(1)}_{lm,n}C^{\|(1),lmn} + 2\chi^{\top(1)}_{lm}\nabla^2C^{\|(1),lm})\}\\ &- 4\nabla^{-2}\left\{-A^{(1),lm}\int^{\tau}\frac{\chi^{\top(1)}_{lm}(\tau',\mathbf{x})}{\tau'}d\tau'\right\} - 2[\ln\tau]A^{(2)} - 2C^{\|(2)},\end{aligned} \tag{5.10}$$

$$\begin{aligned}\bar{\chi}^{\perp(2)}_{ij} = {}& \chi^{\perp(2)}_{ij} + \frac{2}{\tau^2}\{-2\partial_i\nabla^{-2}(\chi^{\top(1)}_{lj}A^{(1),l}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{\top(1)}_{lm}A^{(1),lm})\} - \frac{1}{\tau}\{2\partial_i\nabla^{-2}(\chi^{\top(1)\prime}_{lj}A^{(1),l})\\ &- 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{\top(1)\prime}_{lm}A^{(1),lm})\} - [\ln\tau]\{2\partial_i\nabla^{-2}(2\chi^{\top(1)}_{lj,m}A^{(1),lm} + \chi^{\top(1)}_{lm}A^{(1),lm}_{,j} + \chi^{\top(1)}_{lj}\nabla^2A^{(1),l})\\ &- 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(3\chi^{\top(1)}_{lm,n}A^{(1),lmn} + 2\chi^{\top(1)}_{lm}\nabla^2A^{(1),lm})\} + \{-2\partial_i\nabla^{-2}(2\chi^{\top(1)}_{lj,m}C^{\|(1),lm} + \chi^{\top(1)}_{lm}C^{\|(1),lm}_{,j} + \chi^{\top(1)}_{lj}\nabla^2C^{\|(1),l})\\ &+ 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(+3\chi^{\top(1)}_{lm,n}C^{\|(1),lmn} + 2\chi^{\top(1)}_{lm}\nabla^2C^{\|(1),lm})\} + \partial_i\left\{A^{(1),l}\int^{\tau}\frac{2\chi^{\top(1)}_{lj}(\tau',\mathbf{x})}{\tau'}d\tau'\right\}\\ &+ \partial_i\partial_j\nabla^{-2}\left\{-A^{(1),lm}\int^{\tau}\frac{2\chi^{\top(1)}_{lm}(\tau',\mathbf{x})}{\tau'}d\tau\right\} - C^{\perp(2)}_{i,j} + (i\leftrightarrow j),\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\bar{\chi}^{\top(2)}_{ij} = {}& \chi^{\top(2)}_{ij} + \frac{2}{\tau^2}\{-\delta_{ij}\nabla^{-2}(\chi^{\top(1)}_{lm}A^{(1),lm}) - 2\chi^{\top(1)}_{ij}A^{(1)} + 2\partial_i\nabla^{-2}(\chi^{\top(1)}_{lj}A^{(1),l}) + 2\partial_j\nabla^{-2}(\chi^{\top(1)}_{li}A^{(1),l})\\ &- \partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{\top(1)}_{lm}A^{(1),lm})\} + \frac{1}{\tau}\{-\delta_{ij}\nabla^{-2}(\chi^{\top(1)\prime}_{lm}A^{(1),lm}) - 2\chi^{\top(1)\prime}_{ij}A^{(1)} + 2\partial_i\nabla^{-2}(\chi^{\top(1)\prime}_{lj}A^{(1),l})\\ &+ 2\partial_j\nabla^{-2}(\chi^{\top(1)\prime}_{li}A^{(1),l}) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{\top(1)\prime}_{lm}A^{(1),lm})\} + [\ln\tau]\{\delta_{ij}\nabla^{-2}(\chi^{\top(1)}_{lm,n}A^{(1),lmn} + 2A^{(1),lm}\nabla^2\chi^{\top(1)}_{lm})\\ &- 2\chi^{\top(1)}_{ij,l}A^{(1),l} - 2\chi^{\top(1)}_{li}A^{(1),l}_{,j} - 2\chi^{\top(1)}_{lj}A^{(1),l}_{,i} + 2\partial_i\nabla^{-2}(2\chi^{\top(1)}_{lj,m}A^{(1),lm} + \chi^{\top(1)}_{lm}A^{(1),lm}_{,j} + \chi^{\top(1)}_{lj}\nabla^2A^{(1),l})\\ &+ 2\partial_j\nabla^{-2}(2\chi^{\top(1)}_{li,m}A^{(1),lm} + \chi^{\top(1)}_{lm}A^{(1),lm}_{,i} + \chi^{\top(1)}_{li}\nabla^2A^{(1),l}) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}(7\chi^{\top(1)}_{lm,n}A^{(1),lmn} + 4\chi^{\top(1)}_{lm}\nabla^2A^{(1),lm}\\ &+ 2A^{(1),lm}\nabla^2\chi^{\top(1)}_{lm})\} + \{\delta_{ij}\nabla^{-2}(\chi^{\top(1)}_{lm,n}C^{\|(1),lmn} + 2C^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm}) - 2\chi^{\top(1)}_{ij,l}C^{\|(1),l} - 2\chi^{\top(1)}_{li}C^{\|(1),l}_{,j} - 2\chi^{\top(1)}_{lj}C^{\|(1),l}_{,i}\\ &+ 2\partial_i\nabla^{-2}(2\chi^{\top(1)}_{lj,m}C^{\|(1),lm} + \chi^{\top(1)}_{lm}C^{\|(1),lm}_{,j} + \chi^{\top(1)}_{lj}\nabla^2C^{\|(1),l}) + 2\partial_j\nabla^{-2}(2\chi^{\top(1)}_{li,m}C^{\|(1),lm} + \chi^{\top(1)}_{lm}C^{\|(1),lm}_{,i}\\ &+ \chi^{\top(1)}_{li}\nabla^2C^{\|(1),l}) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}(7\chi^{\top(1)}_{lm,n}C^{\|(1),lmn} + 4\chi^{\top(1)}_{lm}\nabla^2C^{\|(1),lm} + 2C^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm})\}.\end{aligned} \tag{5.12}$$

Equation (5.12) tells that the transformation of the second-order tensor involves only $\xi^{(1)\mu}$, independent of the second-order vector field $\xi^{(2)\mu}$. So the second-order tensor is effectively gauge invariant under the second-order gauge transformations. This is similar to the case of the MD stage in Ref. [61].

From (C38), (C39), and (C42)–(C44) in Ref. [63], applied to the RD stage and for the scalar-tensor coupling, we have the second-order transformation for the density contrast and the velocity as follows:

$$\bar{\delta}^{(2)}_{s(t)} = \delta^{(2)}_{s(t)} + \frac{4}{\tau^2}A^{(2)}(\mathbf{x}), \tag{5.13}$$

$$\bar{v}^{\|(2)}_{s(t)} = v^{\|(2)}_{s(t)} + \frac{1}{\tau}\nabla^{-2}(-2A^{(1),lm}\chi^{\top(1)}_{lm}) + \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{5.14}$$

$$\bar{v}^{\perp(2)}_{s(t)i} = v^{\perp(2)}_{s(t)i} + \frac{1}{\tau}[-2A^{(1),l}\chi^{\top(1)}_{li} + \partial^i\nabla^{-2}(2A^{(1),lm}\chi^{\top(1)}_{lm})]. \tag{5.15}$$

The above general synchronous-to-synchronous transformations contain two vector fields $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$. However, in applications, distinctions should be made between $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$. If one sets $\xi^{(2)\mu} = 0$ [68,69], only $\xi^{(1)\mu}$ remains, which ensures $\bar{g}^{(1)}_{00} = 0$, $\bar{g}^{(1)}_{0i} = 0$ and keeps the obtained first-order solutions as gauge invariant; one has no freedom to make $\bar{g}^{(2)}_{00} = 0$ and $\bar{g}^{(2)}_{0i} = 0$ anymore, because $\xi^{(1)\mu}$ has been already fixed. Thus, this kind of second-order transformation is not effective when $\xi^{(2)\mu} = 0$. The more interesting case of second-order gauge transformation is when the first-order solutions are held fixed and only the second-order $\xi^{(2)\mu}$ is allowed [70]. In this case one simply sets $\xi^{(1)\mu} = 0$ but $\xi^{(2)\mu} \neq 0$ in (5.6)–(5.8), which reduce to

$$\alpha^{(2)}(\tau, \mathbf{x}) = \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{5.16}$$

$$\beta^{(2)}(\tau, \mathbf{x}) = A^{(2)}(\mathbf{x}) \ln\tau + C^{\parallel(2)}(\mathbf{x}), \tag{5.17}$$

$$d_i^{(2)}(\mathbf{x}) = C_i^{\perp(2)}(\mathbf{x}), \tag{5.18}$$

and (5.9)–(5.15) reduce to

$$\bar{\phi}^{(2)}_{s(t)}(\tau, \mathbf{x}) = \phi^{(2)}_{s(t)}(\tau, \mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau^2} + \frac{\ln\tau}{3}\nabla^2 A^{(2)}(\mathbf{x}) + \frac{1}{3}\nabla^2 C^{\parallel(2)}(\mathbf{x}), \tag{5.19}$$

$$\bar{\chi}^{\parallel(2)}_{s(t)}(\tau, \mathbf{x}) = \chi^{\parallel(2)}_{s(t)}(\tau, \mathbf{x}) - 2A^{(2)}(\mathbf{x})\ln\tau - 2C^{\parallel(2)}(\mathbf{x}), \tag{5.20}$$

$$\bar{\chi}^{\perp(2)}_{s(t)ij}(\tau, \mathbf{x}) = \chi^{\perp(2)}_{s(t)ij}(\tau, \mathbf{x}) - \partial_j C_i^{\perp(2)}(\mathbf{x}) - \partial_i C_j^{\perp(2)}(\mathbf{x}), \tag{5.21}$$

$$\bar{\chi}^{\top(2)}_{s(t)ij}(\tau, \mathbf{x}) = \chi^{\top(2)}_{s(t)ij}(\tau, \mathbf{x}), \tag{5.22}$$

$$\bar{\delta}^{(2)}_{s(t)}(\tau, \mathbf{x}) = \delta^{(2)}_{s(t)}(\tau, \mathbf{x}) + \frac{4}{\tau^2}A^{(2)}(\mathbf{x}), \tag{5.23}$$

$$\bar{v}^{\parallel(2)}_{s(t)}(\tau, \mathbf{x}) = v^{\parallel(2)}_{s(t)}(\tau, \mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{5.24}$$

$$\bar{v}^{\perp(2)}_{s(t)i}(\tau, \mathbf{x}) = v^{\perp(2)}_{s(t)i}(\tau, \mathbf{x}), \tag{5.25}$$

which has the same structure as the first-order residual transforms [63]. It is seen that $\chi^{\top(2)}_{s(t)ij}$ and $v^{\perp(2)i}_{s(t)}$ are invariant within synchronous coordinates. Equation (5.21) tells that the τ-independent term $q_{1ij}(\mathbf{x})$ in the vector solution (4.3) is a gauge term and can be eliminated by a choice of $C_j^{\perp(2)}$, so that the gauge-invariant vector solution is

$$\chi^{\perp(2)}_{s(t)ij}(\mathbf{x}, \tau) = \frac{q_{2ij}(\mathbf{x})}{\tau} + \int^{\tau}\frac{d\tau'}{\tau'^2}\int^{\tau'} 2\tau''^2 V_{s(t)ij}(\mathbf{x}, \tau'')d\tau''. \tag{5.26}$$

To identify the residual gauge modes in the second-order scalar solutions, we write (5.19), (5.20), (5.23), and (5.24) in $\mathbf{k}$-space:

$$\bar{\phi}^{(2)}_{s(t)\mathbf{k}}(\tau) = \phi^{(2)}_{s(t)\mathbf{k}}(\tau) + A^{(2)}_{\mathbf{k}}\left(\frac{1}{\tau^2} - \frac{k^2}{3}\ln\tau\right) - \frac{k^2}{3}C^{\parallel(2)}_{\mathbf{k}}, \tag{5.27}$$

$$\bar{\chi}^{\parallel(2)}_{s(t)\mathbf{k}}(\tau) = \chi^{\parallel(2)}_{s(t)\mathbf{k}}(\tau) - 2A^{(2)}_{\mathbf{k}}\ln\tau - 2C^{\parallel(2)}_{\mathbf{k}}, \tag{5.28}$$

$$\bar{\delta}^{(2)}_{s(t)\mathbf{k}}(\tau) = \delta^{(2)}_{s(t)\mathbf{k}}(\tau) + \frac{4}{\tau^2}A^{(2)}_{\mathbf{k}}, \tag{5.29}$$

$$\bar{v}^{\parallel(2)}_{s(t)\mathbf{k}}(\tau) = v^{\parallel(2)}_{s(t)\mathbf{k}}(\tau) + \frac{A^{(2)}_{\mathbf{k}}}{\tau}. \tag{5.30}$$

Comparing (5.27)–(5.30) with the solutions (4.18), (4.20), (4.17), and (4.16), respectively, tells us that the $P_1(\mathbf{k})$ and $P_4(\mathbf{k})$ terms in (4.18), (4.20), (4.17), and (4.16) are gauge terms, which can be eliminated simultaneously by choosing

$$A^{(2)}_{\mathbf{k}} = -\frac{P_1(\mathbf{k})}{k}, \tag{5.31}$$

$$C^{\parallel(2)}_{\mathbf{k}} = \frac{3P_4(\mathbf{k})}{k^2}. \tag{5.32}$$

Thus, the gauge-invariant modes of the second-order scalar perturbations are

$$\begin{aligned}
\phi^{(2)}_{s(t)\mathbf{k}} = {} & P_2(\mathbf{k})\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + P_3(\mathbf{k})\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} \\
& - \frac{2k}{3}\int^{\tau}[P_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + P_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} + \frac{3}{4}F^{\parallel}_{s(t)\mathbf{k}}(\tau) - \frac{1}{4}\int^{\tau}A_{s(t)\mathbf{k}}(\tau')d\tau' \\
& + \int^{\tau}\frac{(k^2\tau'^2 + 6)\ln\tau' + 3}{k^2\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' + \left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right)\int^{\tau}\frac{3(k^2\tau'^2 + 6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' \\
& - \left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\int^\tau\left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\int^\tau\left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'', \qquad (5.33)
\end{aligned}
$$

$$
\begin{aligned}
\chi^{\parallel(2)}_{s(t)\mathbf{k}}=&\,P_2(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}}-P_3(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}}-\frac{4}{k}\int^\tau[P_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}}+P_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'}\\
&-\frac{2\ln\tau}{k}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau'+\int^\tau\frac{6(k^2\tau'^2+6)\ln\tau'+18}{k^4\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau'\\
&-\frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\int^\tau\left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\int^\tau\left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\frac{3}{k^4}E_{s(t)\mathbf{k}}+\frac{27}{2k^4\tau}F^{\parallel'}_{s(t)\mathbf{k}}(\tau)+\frac{27}{k^4\tau^2}F^{\parallel}_{s(t)\mathbf{k}}+\frac{9}{2k^2}F^{\parallel}_{s(t)\mathbf{k}}(\tau)\\
&-\frac{9}{2k^4\tau}A_{s(t)\mathbf{k}}(\tau)-\frac{3}{2k^2}\int^\tau A_{s(t)\mathbf{k}}(\tau')d\tau', \qquad (5.34)
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}_{s(t)\mathbf{k}}=&\,P_2(\mathbf{k})\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}+P_3(\mathbf{k})\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}+3F^{\parallel}_{s(t)\mathbf{k}}\\
&+\left(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\left(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\frac{1}{k\tau^2}\int^\tau\frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau', \qquad (5.35)
\end{aligned}
$$

$$
\begin{aligned}
v^{\parallel(2)}_{s(t)\mathbf{k}}=&\,P_2(\mathbf{k})\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}}+P_3(\mathbf{k})\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}\\
&-\left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&-\left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\frac{1}{k\tau}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau', \qquad (5.36)
\end{aligned}
$$

where $Z_{s(t)}$, $E_{s(t)}$, $A_{s(t)}$, and $F^{\parallel}_{s(t)}$ are in (4.10), (3.3), (3.26), and (3.30).

Next is the case of tensor-tensor coupling. The analysis is similar to the above paragraphs. In particular, the second-order residual gauge transformation is effectively implemented only by the second-order vector field $\xi^{(2)\mu}$, and the gauge transformations are

$$\bar{\phi}_T^{(2)}(\tau,\mathbf{x}) = \phi_T^{(2)}(\tau,\mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau^2} + \frac{\ln\tau}{3}\nabla^2 A^{(2)}(\mathbf{x}) + \frac{1}{3}\nabla^2 C^{\parallel(2)}(\mathbf{x}), \tag{5.37}$$

$$\bar{\chi}_T^{\parallel(2)}(\tau,\mathbf{x}) = \chi_T^{\parallel(2)}(\tau,\mathbf{x}) - 2A^{(2)}(\mathbf{x})\ln\tau - 2C^{\parallel(2)}(\mathbf{x}), \tag{5.38}$$

$$\bar{\chi}_{Tij}^{\perp(2)}(\tau,\mathbf{x}) = \chi_{Tij}^{\perp(2)}(\tau,\mathbf{x}) - \partial_j C_i^{\perp(2)}(\mathbf{x}) - \partial_i C_j^{\perp(2)}(\mathbf{x}), \tag{5.39}$$

$$\bar{\chi}_{Tij}^{\top(2)}(\tau,\mathbf{x}) = \chi_{Tij}^{\top(2)}(\tau,\mathbf{x}), \tag{5.40}$$

$$\bar{\delta}_T^{(2)}(\tau,\mathbf{x}) = \delta_T^{(2)}(\tau,\mathbf{x}) + \frac{4}{\tau^2}A^{(2)}(\mathbf{x}), \tag{5.41}$$

$$\bar{v}_T^{\parallel(2)}(\tau,\mathbf{x}) = v_T^{\parallel(2)}(\tau,\mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{5.42}$$

$$\bar{v}_{Ti}^{\perp(2)}(\tau,\mathbf{x}) = v_{Ti}^{\perp(2)}(\tau,\mathbf{x}). \tag{5.43}$$

Again, $\chi_{Tij}^{\top(2)}$ in (4.30) and $v_{Ti}^{\perp(2)}$ in (4.33) are invariant within synchronous coordinates. By (5.39), the $q_{3ij}(\mathbf{x})$ term in the solution (4.32) is a gauge term and can be eliminated, so that the gauge-invariant vector solution is

$$\chi_{Tij}^{\perp(2)}(\mathbf{x},\tau) = \frac{q_{4ij}(\mathbf{x})}{\tau} + \int^\tau \frac{d\tau'}{\tau'^2}\int^{\tau'} 2\tau''^2 V_{Tij}(\mathbf{x},\tau'')d\tau''. \tag{5.44}$$

To identify the residual gauge modes in the second-order scalar solutions, we write (5.37), (5.38), (5.41), and (5.42) in $\mathbf{k}$-space:

$$\bar{\phi}_{T\mathbf{k}}^{(2)}(\tau) = \phi_{T\mathbf{k}}^{(2)}(\tau) + A_{\mathbf{k}}^{(2)}\left(\frac{1}{\tau^2} - \frac{k^2}{3}\ln\tau\right) - \frac{k^2}{3}C_{\mathbf{k}}^{\parallel(2)}, \tag{5.45}$$

$$\bar{\chi}_{T\mathbf{k}}^{\parallel(2)}(\tau) = \chi_{T\mathbf{k}}^{\parallel(2)}(\tau) - 2A_{\mathbf{k}}^{(2)}\ln\tau - 2C_{\mathbf{k}}^{\parallel(2)}, \tag{5.46}$$

$$\bar{\delta}_{T\mathbf{k}}^{(2)}(\tau) = \delta_{T\mathbf{k}}^{(2)}(\tau) + \frac{4}{\tau^2}A_{\mathbf{k}}^{(2)}, \tag{5.47}$$

$$\bar{v}_{T\mathbf{k}}^{\parallel(2)}(\tau) = v_{T\mathbf{k}}^{\parallel(2)}(\tau) + \frac{A_{\mathbf{k}}^{(2)}}{\tau}. \tag{5.48}$$

Comparing (5.45)–(5.48) with the solutions (4.47), (4.49), (4.46), and (4.45), respectively, tells us that the $Q_1(\mathbf{k})$ and $Q_4(\mathbf{k})$ terms in (4.47), (4.49), (4.46), and (4.45) are gauge terms, which can be removed simultaneously by choosing

$$A_{\mathbf{k}}^{(2)} = -\frac{Q_1(\mathbf{k})}{k}, \tag{5.49}$$

$$C_{\mathbf{k}}^{\parallel(2)} = \frac{3Q_4(\mathbf{k})}{k^2}. \tag{5.50}$$

Thus, the gauge-invariant solutions of the second-order scalar perturbations are

$$\begin{aligned}
\phi_{T\mathbf{k}}^{(2)} = {}& Q_2(\mathbf{k})\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + Q_3(\mathbf{k})\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} - \frac{2k}{3}\int^\tau [Q_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + Q_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} \\
& - \frac{1}{4}\int^\tau A_{T\mathbf{k}}(\tau')d\tau' + \int^\tau \frac{(k^2\tau'^2+6)\ln\tau' + 3}{k^2\tau'}Z_{T\mathbf{k}}(\tau')d\tau' + \left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right)\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau' \\
& - \left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& - \left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \int^\tau \left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
& + \int^\tau \left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'',
\end{aligned} \tag{5.51}$$

$$
\begin{aligned}
\chi^{\parallel(2)}_{T\mathbf{k}} =& \, Q_2(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - Q_3(\mathbf{k})\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{4}{k}\int^\tau [Q_2(\mathbf{k})e^{-ik\tau'/\sqrt{3}} + Q_3(\mathbf{k})e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} \\
& - \frac{2\ln\tau}{k}\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau' + \int^\tau \frac{6(k^2\tau'^2+6)\ln\tau' + 18}{k^4\tau'}Z_{T\mathbf{k}}(\tau')d\tau' \\
& - \frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \int^\tau \left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
& + \int^\tau \left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
& + \frac{3}{k^4}E_{T\mathbf{k}} - \frac{9}{2k^4\tau}A_{T\mathbf{k}}(\tau) - \frac{3}{2k^2}\int^\tau A_{T\mathbf{k}}(\tau')d\tau', \qquad (5.52)
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}_{T\mathbf{k}} =& \, Q_2(\mathbf{k})\left(\frac{8}{k\tau^2} + \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}} + Q_3(\mathbf{k})\left(\frac{8}{k\tau^2} - \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{ik\tau/\sqrt{3}} + \left(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\
& \left. + \frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \left(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\
& \left. + \frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{1}{k\tau^2}\int^\tau \frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau', \qquad (5.53)
\end{aligned}
$$

$$
\begin{aligned}
v^{\parallel(2)}_{T\mathbf{k}} =& \, Q_2(\mathbf{k})\left(\frac{2}{k\tau} + \frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + Q_3(\mathbf{k})\left(\frac{2}{k\tau} - \frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}} \\
& - \left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& - \left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \frac{1}{k\tau}\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau', \qquad (5.54)
\end{aligned}
$$

where Z_T is in (4.40), E_T is in (3.36), and A_T is in (3.57).

Thus, we have obtained all the gauge-invariant solutions of the second-order perturbations with scalar-tensor couplings in Eqs. (5.33)–(5.36), (5.26), (4.4), and (4.1) and with tensor-tensor couplings in Eqs. (5.51)–(5.54), (5.44), (4.33), and (4.30).

To illustrate the physical picture of the solved second-order perturbations, we now calculate the power spectrum of the second-order scalar perturbation $\phi^{(2)}_{T\mathbf{k}}$ in (4.47) generated by the tensor-tensor coupling. Equation (4.47) contains many terms. For an illustration, here we consider only the term $-\frac{1}{4}\int^\tau A_{T\mathbf{k}}(\tau')d\tau'$ in the first line in (4.47) as follows:

$$
\begin{aligned}
\phi^{(2)}_{T\mathbf{k}} &= -\frac{1}{4}\int^\tau A_{T\mathbf{k}}(\tau')d\tau' \\
&= -\frac{1}{6(2\pi)^3}\int d^3k_2 \\
&\quad \times \left[\sum_{s_1,s_2=+,\times} \overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2}{\epsilon}{}^{lm}(\mathbf{k}_2)h_{\mathbf{k}-\mathbf{k}_2}h_{\mathbf{k}_2}\right], \qquad (5.55)
\end{aligned}
$$

where $h_{\mathbf{k}}$ is the first-order tensor mode given by (2.28). As a complex random variable, $h_{\mathbf{k}}$ has a zero mean

$$\langle h_{\mathbf{k}}\rangle = 0 \quad \text{for each } \mathbf{k}, \qquad (5.56)$$

and its real and imaginary parts have the same Gaussian probability distribution, with no correlation between them; i.e., the phase of $h_{\mathbf{k}}$ is random with a uniform probability distribution [71]. Furthermore, $h_{\mathbf{k}}$ are statistically independent for different $\mathbf{k}$, and the power spectrum of the first-order tensor is defined in terms of the ensemble average as follows [30,72]:

$$\langle h^*_{\mathbf{k}} h_{\mathbf{k}'}\rangle \equiv \frac{2\pi^2}{k^3}\delta(\mathbf{k}-\mathbf{k}')\Delta_t^2(k,\tau). \tag{5.57}$$

Also,

$$\langle h_{\mathbf{k}} h_{\mathbf{k}}\rangle = 0 \tag{5.58}$$

holds since the phase of $h_{\mathbf{k}}$ is random [71,73]. By these, the statistical properties of $h_{\mathbf{k}}$ are completely determined. [In parallel, one can write down the properties similar to (5.56)–(5.58), for the first-order scalar and vector metric perturbations, as well as for the first-order density and velocity perturbations.] For inflation models described by a power law $a(\tau) \propto |\tau|^{1+\beta}$, the first-order power spectrum is known for the RD stage resulting from the cosmic evolution from inflation [30]. We define the second-order power spectrum in terms of the covariance as follows:

$$\begin{aligned}
&\langle(\phi^{(2)*}_{T\mathbf{k}} - \langle\phi^{(2)*}_{T\mathbf{k}}\rangle)(\phi^{(2)}_{T\mathbf{k}'} - \langle\phi^{(2)}_{T\mathbf{k}'}\rangle)\rangle \\
&= \langle\phi^{(2)*}_{T\mathbf{k}}\phi^{(2)}_{T\mathbf{k}'}\rangle - \langle\phi^{(2)*}_{T\mathbf{k}}\rangle\langle\phi^{(2)}_{T\mathbf{k}'}\rangle \\
&\equiv \frac{2\pi^2}{k^3}\delta(\mathbf{k}-\mathbf{k}')P_{\phi_T}(k,\tau),
\end{aligned} \tag{5.59}$$

where the mean of the second-order scalar perturbation is given by

$$\begin{aligned}
\langle\phi^{(2)}_{T\mathbf{k}}\rangle = -\frac{1}{6(2\pi)^3}\int d^3k_2\Bigg[\sum_{s_1,s_2=+,\times}\overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2 lm}{\epsilon} \\
\times(\mathbf{k}_2)\langle h_{\mathbf{k}-\mathbf{k}_2}h_{\mathbf{k}_2}\rangle\Bigg] = 0,
\end{aligned} \tag{5.60}$$

where the second equality follows from (5.58). In fact, a second-order perturbation, either scalar, vector, or tensor, always has a vanishing mean since it is formed from the products of the first-order perturbations with the statistical properties (5.57) and (5.58). The covariance (5.59) is always proportional to the Dirac delta function as we have checked. This is consistent with Refs. [52,58] in other coordinates. (We remark that the second-order perturbations, such as $\phi^{(2)}_{T\mathbf{k}}$, are statistically independent for different $\mathbf{k}$ since their covariances are zero for $\mathbf{k} \neq \mathbf{k}'$. However, $\phi^{(2)}_{T\mathbf{k}}$ for different $\mathbf{k}$ are dynamically related, since it contains products of the first-order perturbations with different $\mathbf{k}$.) By using the "factorization" property [52,72,74]

$$\langle x_1x_2x_3x_4\rangle = \langle x_1x_2\rangle\langle x_3x_4\rangle + \langle x_1x_3\rangle\langle x_2x_4\rangle + \langle x_1x_4\rangle\langle x_2x_3\rangle, \tag{5.61}$$

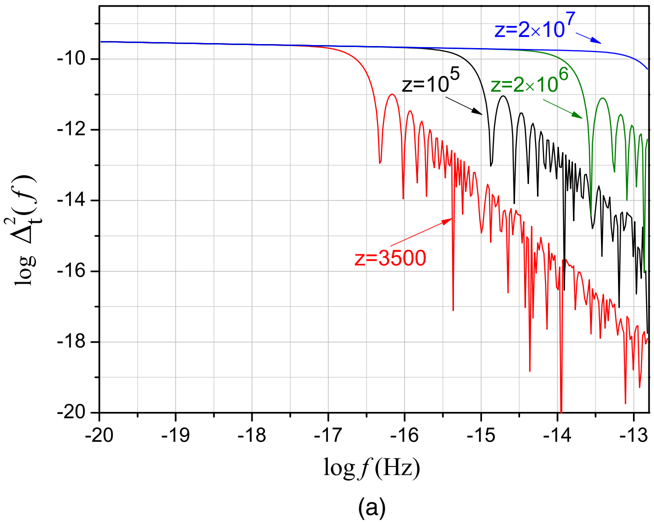
which is valid for generic Gaussian random variables x_1, x_2, x_3, x_4, each having zero mean; using the property (2.27) of the polarization tensors; and using (5.55), (5.57), and (5.58), the covariance is written as

$$\begin{aligned}
\langle\phi^{(2)*}_{T\mathbf{k}}\phi^{(2)}_{T\mathbf{k}'}\rangle &= \delta(\mathbf{k}-\mathbf{k}')\frac{(2\pi^2)^2}{36(2\pi)^6}\int d^3k_2\frac{2}{(k_2)^3|\mathbf{k}-\mathbf{k}_2|^3} \\
&\times\left(\sum_{s_1,s_2=+,\times}\overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}_2)\overset{s_2 lm}{\epsilon}(\mathbf{k}_2)\right)^2 \\
&\times\Delta_t^2(|\mathbf{k}-\mathbf{k}_2|,\tau)\Delta_t^2(k_2,\tau).
\end{aligned} \tag{5.62}$$

So by the definition (5.59), we read off the second-order power spectrum

$$\begin{aligned}
P_{\phi_T}(k,\tau) = \frac{k^3}{36(2\pi)^4}\int\frac{d^3k'}{k'^3} \\
\times\Bigg[\frac{1}{|\mathbf{k}-\mathbf{k}'|^3}\left(\sum_{s_1,s_2=+,\times}\overset{s_1}{\epsilon}_{lm}(\mathbf{k}-\mathbf{k}')\overset{s_2 lm}{\epsilon}(\mathbf{k}')\right)^2 \\
\times\Delta_t^2(|\mathbf{k}-\mathbf{k}'|,\tau)\Delta_t^2(k',\tau)\Bigg].
\end{aligned} \tag{5.63}$$

Given $\Delta_t^2(k,\tau)$ in a cosmological model [30], the power spectrum $P_{\phi_T}(k,\tau)$ is determined by the integration (5.63). The frequency at a conformal time τ is related to k via $f(\tau) = k/2\pi a(\tau)$. By the normalization of $a(\tau)$ in Ref. [66], the present frequency is related to k as $f \simeq 1.7\times10^{-19}\,k\,\mathrm{Hz}$. We plot $\Delta_t^2(f)$ and $P_{\phi_T}(f)$ in Fig. 2 for different values of cosmic redshift z in the low frequency range, which may affect the CMB anisotropies and polarization. It is seen that, at low f, $\Delta_t^2(f) \sim 10^{-10}$ in magnitude, and $P_{\phi_T}(f) \sim [\Delta_t^2(f)]^2 \sim 10^{-20}$, so that the amplitude of the second-order spectrum is extremely small in this case. As expected, the amplitudes of the power spectra are decreasing as z becomes smaller, since the amplitude $h_{\mathbf{k}}$ of RGW is decreasing inside the horizon [30]. So far in the above, we present only the contribution of one term $-\frac{1}{4}\int^\tau A_{T\mathbf{k}}d\tau'$, as an illustration. The whole contribution to the second-order spectrum of $\phi^{(2)}_{T\mathbf{k}}$ will involve hundreds of terms like this. Besides, there are the second-order scalars $\phi^{(2)}_{s(t)\mathbf{k}}$, $\chi^{\|(2)}_{s(t)\mathbf{k}}$, generated by the scalar-tensor coupling, etc. All these require many more efforts of computation and will be left for future study.



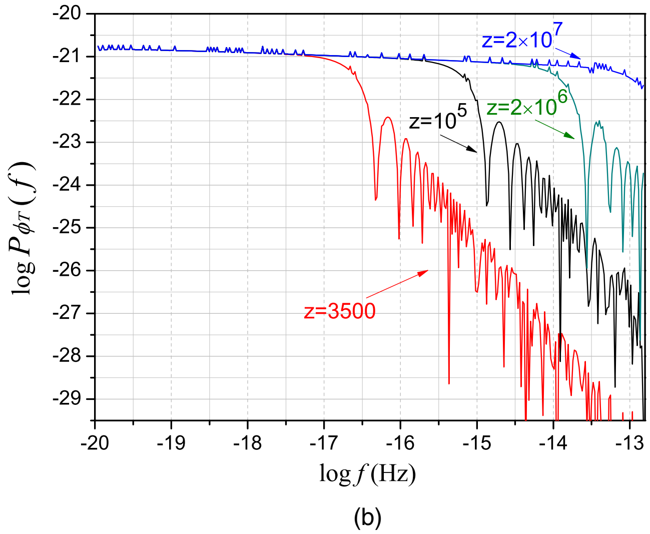


FIG. 2. (a) The power spectrum of first-order tensor. (b) The power spectrum (5.63) of second-order scalar $\phi^{(2)}_{T\mathbf{k}}$ generated by the first-order tensor-tensor coupling term (5.55).

VI. TRANSFORMATION OF SECOND-ORDER SOLUTIONS FROM SYNCHRONOUS TO POISSON COORDINATES

The Poisson coordinates are also commonly used in cosmological study, so in this section we perform transformation of the second-order solutions from synchronous coordinates into Poisson coordinates. A perturbed metric up to second order in the Poisson coordinates is generally written as [11,12,60]

$$g_{00} = -a^2\left[1+\psi^{(1)}_P+\frac{1}{2}\psi^{(2)}_P\right], \tag{6.1}$$

$$g_{0i} = a^2\left[w^{(1)}_{Pi}+\frac{1}{2}w^{(2)}_{Pi}\right], \tag{6.2}$$

$$g_{ij} = a^2\left[\delta_{ij}-2\left(\phi^{(1)}_P+\frac{1}{2}\phi^{(2)}_P\right)\delta_{ij}+\chi^{\top(1)}_{Pij}+\frac{1}{2}\chi^{\top(2)}_{Pij}\right], \tag{6.3}$$

with the vectors satisfying

$$\partial^i w^{(A)}_{Pi} = 0, \qquad A = 1,2, \tag{6.4}$$

and the tensor satisfying

$$\chi^{\top(A)i}_{Pi} = 0, \qquad \partial^i\chi^{\top(A)}_{Pij} = 0. \tag{6.5}$$

[When the vector and tensor modes are zero, the coordinate system (6.1)–(6.3) is also called the longitudinal (conformal-Newtonian) coordinate.] Consider a transformation of the solutions of metric perturbations from a synchronous coordinate to a Poisson coordinate. The first- and second-order transformation vectors $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ are introduced, respectively, as (5.2) and (5.3). Let the first-order metric perturbations, $\phi^{(1)}$, $D_{ij}\chi^{\|(1)}$, $\chi^{\top(1)}_{ij}$ be given as (2.23), (2.24), and (2.26) in a synchronous coordinate without a vector mode. By the coordinate transformation (C4) in Ref. [63], one gets the first-order metric perturbation in Poisson coordinates as follows [60]:

$$\psi^{(1)}_P = -\alpha^{(1)\prime}-\frac{a'}{a}\alpha^{(1)}, \tag{6.6}$$

$$w^{(1)}_{Pi} = \alpha^{(1)}_{,i}-\beta^{(1)\prime}_{,i}-d^{(1)\prime}_i, \tag{6.7}$$

$$\phi^{(1)}_P = \phi^{(1)}+\frac{1}{3}\nabla^2\beta^{(1)}+\frac{a'}{a}\alpha^{(1)}, \tag{6.8}$$

$$\chi^{\top(1)}_{Pij} = D_{ij}\chi^{\|(1)}+\chi^{\top(1)}_{ij}-2D_{ij}\beta^{(1)}-d^{(1)}_{i,j}-d^{(1)}_{j,i}. \tag{6.9}$$

By the constraints (6.4) and (6.5) in the Poisson coordinate, the transformation parameters satisfy the following constraints:

$$\alpha^{(1)} = \beta^{(1)\prime}, \tag{6.10}$$

$$D_{ij}(\chi^{\|(1)}-2\beta^{(1)}) = 0, \tag{6.11}$$

$$d^{(1)}_i = 0, \tag{6.12}$$

in the absence of the first-order vector mode. Also, their solutions are given by $\alpha^{(1)} = \frac{1}{2}\chi^{\|(1)\prime}$, $\beta^{(1)} = \frac{1}{2}\chi^{\|(1)}$. In $\mathbf{k}$-space, the solution $\chi^{\|(1)}_{\mathbf{k}}$ is known in (2.24), by which one obtains the first-order transformation parameters

$$\alpha^{(1)}_{\mathbf{k}} = \frac{1}{2}\chi^{\|(1)\prime}_{\mathbf{k}} = -D_2\frac{2\sqrt{3}i}{k^2\tau^2}e^{-ik\tau/\sqrt{3}} + D_3\frac{2\sqrt{3}i}{k^2\tau^2}e^{ik\tau/\sqrt{3}}, \tag{6.13}$$

$$\begin{aligned}\beta^{(1)}_{\mathbf{k}} &= \frac{1}{2}\chi^{\|(1)}_{\mathbf{k}} \\ &= D_2\frac{2\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - D_3\frac{2\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} \\ &\quad -\frac{2}{k}\int^{\tau}[D_2e^{-ik\tau'/\sqrt{3}} + D_3e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'}.\end{aligned} \tag{6.14}$$

Plugging these and (2.23), (2.24) into (6.6)–(6.9) yields the metric perturbations in Poisson coordinates

$$\begin{aligned}\psi^{(1)}_{P\mathbf{k}} &= \phi^{(1)}_{P\mathbf{k}} \\ &= D_2\left(\frac{2}{k\tau^2} - \frac{2\sqrt{3}i}{k^2\tau^3}\right)e^{-ik\tau/\sqrt{3}} \\ &\quad + D_3\left(\frac{2}{k\tau^2} + \frac{2\sqrt{3}i}{k^2\tau^3}\right)e^{ik\tau/\sqrt{3}},\end{aligned} \tag{6.15}$$

$$w^{(1)}_{Pi} = 0, \tag{6.16}$$

$$\chi^{\top(1)}_{Pij} = \chi^{\top(1)}_{ij}. \tag{6.17}$$

Our result (6.15) is the general first-order scalar solution. By taking $D_2 = (-A_r + iB_r)/(4\sqrt{3}k)$ and $D_3 = (-A_r - iB_r)/(4\sqrt{3}k)$, Eq. (6.15) is the same as Eq. (43) of Ref. [52]. The first-order scalar mode Φ in Ref. [52] is related to our $\chi^{\|(1)}$ as $\Phi = -\frac{1}{2}\chi^{\|(1)\prime\prime} - \frac{1}{2\tau}\chi^{\|(1)\prime}$. Also note that Ref. [52] ignored certain terms as decaying modes in (6.15) in their actual subsequent computing. We see that in the Poisson coordinate the first-order vector mode is still zero and the first-order tensor mode is the same as in synchronous coordinates, and the two first-order scalar modes are equal, a fact that can be checked using (6.7), (6.8), (6.13), and (6.14),

$$\phi^{(1)}_P - \psi^{(1)}_P = \frac{1}{2}\chi^{\|(1)\prime\prime} + \frac{1}{\tau}\chi^{\|(1)\prime} + \frac{1}{6}\nabla^2\chi^{\|(1)} + \phi^{(1)} = 0,$$

where in the last step the evolution equation (3.11) in Ref. [63] is used. That is why the Poisson coordinate is often used when only the scalar metric perturbation is present.

Next, we turn to the second-order metric perturbations with scalar-tensor couplings in the Poisson coordinates. By (C5) in Ref. [63] with the scalar-tensor coupling, one has

$$\psi^{(2)}_{P_{s(t)}} = -\alpha^{(2)\prime}_{s(t)} - \frac{a'}{a}\alpha^{(2)}_{s(t)}, \tag{6.18}$$

$$w^{(2)}_{P_{s(t)}i} = W_{s(t)i} + \alpha^{(2)}_{s(t),i} - \beta^{(2)\prime}_{s(t),i} - d^{(2)\prime}_{s(t)i}, \tag{6.19}$$

$$\phi^{(2)}_{P_{s(t)}} = \phi^{(2)}_{s(t)} + \frac{1}{3}\chi^{\top(1)}_{lm}\chi^{\|,lm} + \frac{a'}{a}\alpha^{(2)}_{s(t)} + \frac{1}{3}\nabla^2\beta^{(2)}_{s(t)}, \tag{6.20}$$

$$\begin{aligned}\chi^{\top(2)}_{P_{s(t)}ij} &= D_{ij}\chi^{\|(2)}_{s(t)} + \chi^{\perp(2)}_{s(t)ij} + \chi^{\top(2)}_{s(t)ij} + Y_{s(t)ij} \\ &\quad - (d^{(2)}_{s(t)i,j} + d^{(2)}_{s(t)j,i} + 2D_{ij}\beta^{(2)}_{s(t)}),\end{aligned} \tag{6.21}$$

where

$$W_{s(t)i} \equiv -\chi^{\top(1)}_{il}\chi^{\|(1)\prime,l}, \tag{6.22}$$

$$\begin{aligned}Y_{s(t)ij} \equiv& -\frac{2a'}{a}\chi^{\top(1)}_{ij}\chi^{\|(1)\prime} - \chi^{\top(1)\prime}_{ij}\chi^{\|(1)\prime} - \chi^{\top(1)}_{ij,l}\chi^{\|(1),l} \\ &-\chi^{\top(1)}_{il}\chi^{\|(1),l}_{,j} - \chi^{\top(1)}_{jl}\chi^{\|(1),l}_{,i} + \frac{2}{3}\chi^{\top(1)}_{lm}\chi^{\|(1),lm}\delta_{ij}.\end{aligned} \tag{6.23}$$

In order to get perturbations in the Poisson coordinates, one needs to solve for the parameters $\alpha^{(2)}_{s(t)}$, $\beta^{(2)}_{s(t)}$, and $d^{(2)}_{s(t)i}$. We decompose $W_{s(t)i}$ in (6.19) into two parts, $W_{s(t)i} = W^{\|}_{s(t),i} + W^{\perp}_{s(t)i}$, with

$$W^{\|}_{s(t)} = \nabla^{-2}\partial^i W_{s(t)i} = \nabla^{-2}[-\chi^{\top(1)}_{lm}\chi^{\|(1)\prime,lm}], \tag{6.24}$$

$$\begin{aligned}W^{\perp}_{s(t)i} &= W_{s(t)i} - \partial_i W^{\|}_{s(t)} \\ &= -\chi^{\top(1)}_{il}\chi^{\|(1)\prime,l} + \partial_i\nabla^{-2}[\chi^{\top(1)}_{lm}\chi^{\|(1)\prime,lm}].\end{aligned} \tag{6.25}$$

Since $w^{(2)}_{P_{s(t)}i}$ contains only the curl part, the gradient part of Eq. (6.19) gives an equation

$$W^{\|}_{s(t)} + \alpha^{(2)}_{s(t)} - \beta^{(2)\prime}_{s(t)} = 0. \tag{6.26}$$

Also $\chi^{\top(2)}_{P_{s(t)}ij}$ is traceless and transverse; i.e., the rhs of (6.21) should contain no scalar, vector modes. We decompose $Y_{s(t)ij}$ into scalar, vector, and tensor modes as

$$Y_{s(t)ij} = D_{ij}Y^{\|}_{s(t)} + 2Y^{\perp}_{s(t)(i,j)} + Y^{\top}_{s(t)ij}, \tag{6.27}$$

where $Y^{\perp}_{s(t)(i,j)} = \frac{1}{2}(Y^{\perp}_{s(t)i,j} + Y^{\perp}_{s(t)j,i})$, $Y^{\perp,i}_{s(t)i} = 0$, and $Y^{\top,i}_{s(t)ij} = 0$. We get

$$\begin{aligned}Y^{\|}_{s(t)} =& \frac{3}{2}\nabla^{-2}\nabla^{-2}Y^{,lm}_{s(t)lm} \\ =& \nabla^{-2}[\chi^{\top(1)}_{lm}\chi^{\|(1),lm}] + \nabla^{-2}\nabla^{-2}\Big[-\frac{3a'}{a}\chi^{\top(1)}_{lm}\chi^{\|(1)\prime,lm} \\ &-\frac{3}{2}\chi^{\top(1)\prime}_{lm}\chi^{\|(1)\prime,lm} - \frac{9}{2}\chi^{\top(1)}_{lm,n}\chi^{\|(1),lmn} - 3\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}\Big],\end{aligned} \tag{6.28}$$

$$
\begin{aligned}
Y^{\perp}_{s(t)j} &= \nabla^{-2} Y^{,l}_{s(t)lj} - \frac{2}{3} Y^{\parallel}_{s(t),j} \\
&= \nabla^{-2}\left[-\frac{2a'}{a}\chi^{\top(1)}_{lj}\chi^{\parallel(1)',l} - \chi^{\top(1)'}_{lj}\chi^{\parallel(1)',l} - 2\chi^{\top(1)}_{jl,m}\chi^{\parallel(1),lm} - \chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}_{,j} - \chi^{\top(1)}_{lj}\nabla^2\chi^{\parallel(1),l}\right] \\
&\quad + \partial_j \nabla^{-2}\nabla^{-2}\left[\frac{2a'}{a}\chi^{\top(1)}_{lm}\chi^{\parallel(1)',lm} + \chi^{\top(1)'}_{lm}\chi^{\parallel(1)',lm} + 3\chi^{\top(1)}_{lm,n}\chi^{\parallel(1),lmn} + 2\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm}\right],
\end{aligned} \tag{6.29}
$$

$$
\begin{aligned}
Y^{\top}_{s(t)ij} &= Y_{s(t)ij} - D_{ij} Y^{\parallel}_{s(t)} - 2Y^{\perp}_{s(t)(i,j)} \\
&= -\frac{2a'}{a}\chi^{\top(1)}_{ij}\chi^{\parallel(1)'} - \chi^{\top(1)'}_{ij}\chi^{\parallel(1)'} - \chi^{\top(1)}_{ij,l}\chi^{\parallel(1),l} - \chi^{\top(1)}_{il}\chi^{\parallel(1),l}_{,j} - \chi^{\top(1)}_{jl}\chi^{\parallel(1),l}_{,i} + \chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}\delta_{ij} \\
&\quad + \delta_{ij}\nabla^{-2}\left[-\frac{a'}{a}\chi^{\top(1)}_{lm}\chi^{\parallel(1)',lm} - \frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\parallel(1)',lm} - \frac{3}{2}\chi^{\top(1)}_{lm,n}\chi^{\parallel(1),lmn} - \chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm}\right] \\
&\quad + \partial_i\nabla^{-2}\left[\frac{2a'}{a}\chi^{\top(1)}_{lj}\chi^{\parallel(1)',l} + \chi^{\top(1)'}_{lj}\chi^{\parallel(1)',l} + 2\chi^{\top(1)}_{jl,m}\chi^{\parallel(1),lm} + \chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}_{,j} + \chi^{\top(1)}_{lj}\nabla^2\chi^{\parallel(1),l}\right] \\
&\quad + \partial_j\nabla^{-2}\left[\frac{2a'}{a}\chi^{\top(1)}_{li}\chi^{\parallel(1)',l} + \chi^{\top(1)'}_{li}\chi^{\parallel(1)',l} + 2\chi^{\top(1)}_{il,m}\chi^{\parallel(1),lm} + \chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}_{,i} + \chi^{\top(1)}_{li}\nabla^2\chi^{\parallel(1),l}\right] \\
&\quad - \partial_i\partial_j\nabla^{-2}[\chi^{\top(1)}_{lm}\chi^{\parallel(1),lm}] + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[-\frac{a'}{a}\chi^{\top(1)}_{lm}\chi^{\parallel(1)',lm} - \frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\parallel(1)',lm}\right. \\
&\quad \left. - \frac{3}{2}\chi^{\top(1)}_{lm,n}\chi^{\parallel(1),lmn} - \chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm}\right].
\end{aligned} \tag{6.30}
$$

Requiring the scalar and vector modes in Eq. (6.21) to be zero leads to the two equations

$$\chi^{\parallel(2)}_{s(t)} + Y^{\parallel}_{s(t)} - 2\beta^{(2)}_{s(t)} = 0, \tag{6.31}$$

$$\nabla^{-2}\partial^j\chi^{\perp(2)}_{s(t)ij} + Y^{\perp}_{s(t)i} - d^{(2)}_{s(t)i} = 0. \tag{6.32}$$

The solutions of (6.26), (6.31), and (6.32) are as follows:

$$\alpha^{(2)}_{s(t)} = \frac{1}{2}\chi^{\parallel(2)'}_{s(t)} + \frac{1}{2}Y^{\parallel'}_{s(t)} - W^{\parallel}_{s(t)}, \tag{6.33}$$

$$\beta^{(2)}_{s(t)} = \frac{1}{2}\chi^{\parallel(2)}_{s(t)} + \frac{1}{2}Y^{\parallel}_{s(t)}, \tag{6.34}$$

$$d^{(2)}_{s(t)i} = \nabla^{-2}\chi^{\perp(2),j}_{s(t)ij} + Y^{\perp}_{s(t)i}. \tag{6.35}$$

Plugging the solutions $\chi^{\perp(2)}_{s(t)ij}$ (5.26) and $\chi^{\parallel(2)}_{s(t)}$ (5.34) into the above equations, one obtains the parameters

$$
\begin{aligned}
\alpha^{(2)}_{s(t)\mathbf{k}} &= -P_2\frac{2\sqrt{3}i}{k^2\tau^2}e^{-ik\tau/\sqrt{3}} + P_3\frac{2\sqrt{3}i}{k^2\tau^2}e^{ik\tau/\sqrt{3}} - \frac{1}{k\tau}\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' - \frac{9}{k^4\tau}Z_{s(t)\mathbf{k}} \\
&\quad + \frac{2\sqrt{3}}{k^2\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad - \frac{2\sqrt{3}}{k^2\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \frac{3}{2k^4}E'_{s(t)\mathbf{k}} + \frac{27}{4k^4\tau}F^{\parallel''}_{s(t)\mathbf{k}} + \frac{27}{4k^4\tau^2}F^{\parallel'}_{s(t)\mathbf{k}} - \frac{27}{k^4\tau^3}F^{\parallel}_{s(t)\mathbf{k}} + \frac{9}{4k^2}F^{\parallel'}_{s(t)\mathbf{k}} \\
&\quad - \frac{9}{4k^4\tau}A'_{s(t)\mathbf{k}} + \frac{9}{4k^4\tau^2}A_{s(t)\mathbf{k}} - \frac{3}{4k^2}A_{s(t)\mathbf{k}} + \frac{1}{2}Y^{\parallel'}_{s(t)\mathbf{k}} - W^{\parallel}_{s(t)\mathbf{k}},
\end{aligned} \tag{6.36}
$$

$$
\begin{aligned}
\beta^{(2)}_{s(t)\mathbf{k}} = {} & P_2\frac{2\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - P_3\frac{2\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{2}{k}\int^\tau [P_2e^{-ik\tau'/\sqrt{3}} + P_3e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} - \frac{\ln\tau}{k}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' \\
& + \int^\tau\frac{3(k^2\tau'^2+6)\ln\tau'+9}{k^4\tau'}Z_{s(t)\mathbf{k}}(\tau')d\tau' - \frac{2\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \frac{2\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \int^\tau\left[\frac{6}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
& + \int^\tau\left[\frac{6}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' + \frac{3}{2k^4}E_{s(t)\mathbf{k}} + \frac{27}{4k^4\tau}F^{\parallel'}_{s(t)\mathbf{k}}(\tau) \\
& + \frac{27}{2k^4\tau^2}F^{\parallel}_{s(t)\mathbf{k}} + \frac{9}{4k^2}F^{\parallel}_{s(t)\mathbf{k}}(\tau) - \frac{9}{4k^4\tau}A_{s(t)\mathbf{k}}(\tau) - \frac{3}{4k^2}\int^\tau A_{s(t)\mathbf{k}}(\tau')d\tau' + \frac{1}{2}Y^{\parallel}_{s(t)\mathbf{k}},
\end{aligned} \tag{6.37}
$$

$$
d^{(2)}_{s(t)i} = \frac{\nabla^{-2}q^{,j}_{2ij}}{\tau} + 2\int^\tau\frac{d\tau'}{\tau'^2}\int^{\tau'}\tau''^2\nabla^{-2}V^{,j}_{s(t)ij}(\mathbf{x},\tau'')d\tau'' + Y^{\perp}_{s(t)i}, \tag{6.38}
$$

where $E_{s(t)}$, $F_{s(t)}$, $A_{s(t)}$, $Z_{s(t)}$, and $V_{s(t)ij}$ are given by Eqs. (3.3), (3.30), (3.26), (4.10), and (3.22).

Plugging Eqs. (5.33) and (6.36)–(6.38) into (6.18)–(6.20) with $a(\tau)\propto\tau$, one obtains the second-order metric perturbations in the Poisson coordinate as

$$
\psi^{(2)}_{P_{s(t)}} = \int\frac{d^3k}{(2\pi)^{3/2}}\left[-\alpha^{(2)'}_{s(t)\mathbf{k}} - \frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.39}
$$

where

$$
\begin{aligned}
-\alpha^{(2)'}_{s(t)\mathbf{k}} - \frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}} = {} & P_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3} + \frac{2}{k\tau^2}\right)e^{-ik\tau/\sqrt{3}} + P_3\left(-\frac{2\sqrt{3}i}{k^2\tau^3} + \frac{2}{k\tau^2}\right)e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\
& \left. - \frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& + \frac{3}{k^2}Z_{s(t)\mathbf{k}} + \frac{9}{k^4\tau}Z'_{s(t)\mathbf{k}} - \frac{3}{2k^4\tau}E'_{s(t)\mathbf{k}} - \frac{3}{2k^4}E''_{s(t)\mathbf{k}} - \frac{27}{4k^4\tau}F^{\parallel'''}_{s(t)\mathbf{k}} + \frac{135}{4k^4\tau^3}F^{\parallel'}_{s(t)\mathbf{k}} \\
& - \frac{27}{4k^4\tau^2}F^{\parallel''}_{s(t)\mathbf{k}} - \frac{54}{k^4\tau^4}F^{\parallel}_{s(t)\mathbf{k}} - \frac{9}{4k^2\tau}F^{\parallel'}_{s(t)\mathbf{k}} - \frac{9}{4k^2}F^{\parallel''}_{s(t)\mathbf{k}} + \frac{9}{4k^4\tau}A''_{s(t)\mathbf{k}} + \frac{9}{4k^4\tau^3}A_{s(t)\mathbf{k}} \\
& - \frac{9}{4k^4\tau^2}A'_{s(t)\mathbf{k}} + \frac{3}{4k^2\tau}A_{s(t)\mathbf{k}} + \frac{3}{4k^2}A'_{s(t)\mathbf{k}} - \frac{1}{2\tau}Y^{\parallel'}_{s(t)\mathbf{k}} - \frac{1}{2}Y^{\parallel''}_{s(t)\mathbf{k}} + \frac{1}{\tau}W^{\parallel}_{s(t)\mathbf{k}} + W^{\parallel'}_{s(t)\mathbf{k}},
\end{aligned} \tag{6.40}
$$

and

$$
\phi^{(2)}_{P_{s(t)}} = \int\frac{d^3k}{(2\pi)^{3/2}}\left[\phi^{(2)}_{s(t)\mathbf{k}} + \frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}} - \frac{k^2}{3}\beta^{(2)}_{s(t)\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{1}{3}\chi^{\top(1)}_{lm}\chi^{\parallel,lm}, \tag{6.41}
$$

where

$$\begin{aligned}\phi^{(2)}_{s(t)\mathbf{k}}+\frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}}-\frac{k^2}{3}\beta^{(2)}_{s(t)\mathbf{k}}&=P_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{-ik\tau/\sqrt{3}}+P_3\left(\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{ik\tau/\sqrt{3}}\\&+\left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\&+\left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\&-\frac{9}{k^4\tau^2}Z_{s(t)\mathbf{k}}-\frac{9}{4k^4\tau^2}A'_{s(t)\mathbf{k}}+\frac{9}{4k^4\tau^3}A_{s(t)\mathbf{k}}+\frac{3}{2k^4\tau}E'_{s(t)\mathbf{k}}+\frac{27}{4k^4\tau^2}F^{\parallel''}_{s(t)\mathbf{k}}+\frac{27}{4k^4\tau^3}F^{\parallel'}_{s(t)\mathbf{k}}\\&-\frac{27}{k^4\tau^4}F^{\parallel}_{s(t)\mathbf{k}}-\frac{1}{2k^2}E_{s(t)\mathbf{k}}-\frac{9}{2k^2\tau^2}F^{\parallel}_{s(t)\mathbf{k}}-\frac{k^2}{6}Y^{\parallel}_{s(t)\mathbf{k}}+\frac{1}{2\tau}Y^{\parallel'}_{s(t)\mathbf{k}}-\frac{1}{\tau}W^{\parallel}_{s(t)\mathbf{k}},\end{aligned}\tag{6.42}$$

and

$$\begin{aligned}w^{(2)}_{P_{s(t)}i}&=\frac{\nabla^{-2}q^{,j}_{2ij}}{\tau^2}\\&-\frac{2}{\tau^2}\int^\tau\tau'^2\nabla^{-2}V^{,j}_{s(t)ij}(\mathbf{x},\tau')d\tau'-Y^{\perp'}_{s(t)i}+W^{\perp}_{s(t)i},\end{aligned}\tag{6.43}$$

with $W^{\parallel}_{s(t)}$, $Y^{\parallel}_{s(t)}$, $Y^{\perp}_{s(t)i}$, $W^{\perp}_{s(t)i}$ given in (6.24)–(6.29). Also, the tensor $\chi^{\top(2)}_{P_{s(t)}ij}$ is given from (6.21) as

$$\chi^{\top(2)}_{P_{s(t)}ij}=\chi^{\top(2)}_{s(t)ij}+Y^{\top}_{s(t)ij},\tag{6.44}$$

with $\chi^{\top(2)}_{s(t)ij}$ in (4.1) and $Y^{\top}_{s(t)ij}$ in (6.30). From the above, it is seen that in the Poisson coordinate the two second-order scalar modes are no longer equal, that the second-order vector mode is nonzero and is contributed by the scalar-tensor couplings, and that the second-order tensor mode is different from the one in synchronous coordinates by the coupling term $Y^{\top}_{s(t)ij}$.

Next, we calculate the density in the Poisson coordinate. Equation (C6) in Ref. [63] gives that the zeroth-order mass-energy density as

$$\rho^{(0)}_P=\rho^{(0)}=\frac{3}{8\pi G}\frac{a'^2(\tau)}{a^4(\tau)}\propto\tau^{-4}.\tag{6.45}$$

By Eq. (C7) in Ref. [63], the first-order $\rho^{(1)}_P$ is given by

$$\rho^{(1)}_P=\rho^{(1)}-\rho^{(0)}_{,\mu}\xi^{(1)\mu}.\tag{6.46}$$

For the RD stage, from $\delta^{(1)}_{\mathbf{k}}$ in (2.22) and $\alpha^{(1)}_{\mathbf{k}}$ in (6.13), the above gives

$$\begin{aligned}\delta^{(1)}_{P\mathbf{k}}&=D_2\left(-\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}\\&+D_3\left(\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}.\end{aligned}\tag{6.47}$$

To second order, by using (C8) in Ref. [63], one has

$$\begin{aligned}\rho^{(2)}_P&=\rho^{(2)}-2\rho^{(1)}_{,0}\xi^{(1)0}-2\rho^{(1)}_{,k}\xi^{(1)k}+\rho^{(0)}_{,0}\xi^{(1)0}_{,0}\xi^{(1)0}\\&+\rho^{(0)}_{,0}\xi^{(1)0}_{,k}\xi^{(1)k}+\rho^{(0)}_{,00}\xi^{(1)0}\xi^{(1)0}-\rho^{(0)}_{,0}\xi^{(2)0},\end{aligned}\tag{6.48}$$

which can be written in terms of the second-order density contrast contributed by scalar-tensor couplings as

$$\delta^{(2)}_{P_{s(t)}}=\delta^{(2)}_{s(t)}+\left[-2\frac{a''(\tau)}{a'(\tau)}+4\frac{a'(\tau)}{a(\tau)}\right]\alpha^{(2)}_{s(t)}.\tag{6.49}$$

For the RD stage, using the solution $\delta^{(2)}_{s(t)\mathbf{k}}$ in (5.35) and $\alpha^{(2)}_{s(t)\mathbf{k}}$ in (6.36), the above becomes

$$\delta^{(2)}_{P_{s(t)}}=\int\frac{d^3k}{(2\pi)^{3/2}}[\delta^{(2)}_{s(t)\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}}]e^{i\mathbf{k}\cdot\mathbf{x}},\tag{6.50}$$

where

$$
\begin{aligned}
\delta^{(2)}_{s(t)\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}} = {}& P_2\left(-\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}+P_3\left(\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}\\
&+\left(\frac{8\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.\\
&\left.+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\left(-\frac{8\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right.\\
&\left.+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&-\frac{36}{k^4\tau^2}Z_{s(t)\mathbf{k}}+\frac{6}{k^4\tau}E'_{s(t)\mathbf{k}}+\frac{27}{k^4\tau^2}F^{\|''}_{s(t)\mathbf{k}}+\frac{27}{k^4\tau^3}F^{\|'}_{s(t)\mathbf{k}}-\frac{108}{k^4\tau^4}F^{\|}_{s(t)\mathbf{k}}+\frac{9}{k^2\tau}F^{\|'}_{s(t)\mathbf{k}}\\
&+3F^{\|}_{s(t)\mathbf{k}}-\frac{9}{k^4\tau^2}A'_{s(t)\mathbf{k}}+\frac{9}{k^4\tau^3}A_{s(t)\mathbf{k}}-\frac{3}{k^2\tau}A_{s(t)\mathbf{k}}+\frac{2}{\tau}Y^{\|'}_{s(t)\mathbf{k}}-\frac{4}{\tau}W^{\|}_{s(t)\mathbf{k}}. \qquad (6.51)
\end{aligned}
$$

It is seen that only $\xi^{(2)\mu}$ contributes to $\delta^{(2)}_{P_{s(t)}}$, yet $\xi^{(1)\mu}$ does not contribute.

Next, we solve for the 4-velocity in the Poisson coordinate. By (C9) in Ref. [63], the zeroth-order 4-velocity is $U_P^{(0)0}=U^{(0)0}=a^{-1}$ and $U_P^{(0)i}=U^{(0)i}=0$. By (C10) in Ref. [63], the first-order velocity transforms as

$$U_P^{(1)0}=U^{(1)0}+\frac{a'}{a^2}\alpha^{(1)}+\frac{1}{a}\alpha^{(1)\prime}, \qquad (6.52)$$

$$v_P^{(1)i}=v^{(1)i}+\beta^{(1)\prime,i}+d^{(1)i\prime}, \qquad (6.53)$$

where the definition $U_P^{(1)i}=a^{-1}v_P^{(1)i}$ is used. From $\alpha^{(1)}_{\mathbf{k}}$ in (6.13) for the RD stage, and using $U^{(1)0}=0$ in (2.12), Eq. (6.52) becomes

$$
\begin{aligned}
U_P^{(1)0}={}&D_2\left(\frac{2\sqrt{3}i}{k^2\tau^4}-\frac{2}{k\tau^3}\right)e^{-ik\tau/\sqrt{3}}\\
&+D_3\left(-\frac{2\sqrt{3}i}{k^2\tau^4}-\frac{2}{k\tau^3}\right)e^{ik\tau/\sqrt{3}}, \qquad (6.54)
\end{aligned}
$$

which is nonzero, even in the synchronous coordinate $U^{(1)0}=0$. By using $v^{\|(1)}$ in (2.21) and $\beta^{(1)}$ in (6.14), and taking $v^{\perp(1)i}=0$ and $d^{(1)i}=0$, Eq. (6.53) gives

$$v_P^{\perp(1)i}=0, \qquad (6.55)$$

$$
\begin{aligned}
v^{\|(1)}_{P\mathbf{k}}={}&D_2\left(-\frac{2\sqrt{3}i}{k^2\tau^2}+\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}}\\
&+D_3\left(\frac{2\sqrt{3}i}{k^2\tau^2}+\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}. \qquad (6.56)
\end{aligned}
$$

Thus the first-order velocity remains noncurl in the Poisson coordinate.

To second order, from (C11) in Ref. [63] and (2.12), (2.13), (6.13), and (6.14), one has the 0-component of the 4-velocity as

$$U^{(2)0}_{P_{s(t)}}=-\frac{1}{\tau}\int\frac{d^3k}{(2\pi)^{3/2}}\left[-\alpha^{(2)\prime}_{s(t)\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (6.57)$$

where $-\alpha^{(2)\prime}_{s(t)\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{s(t)\mathbf{k}}$ is given in (6.40). By a definition $U^{(2)i}_{P_{s(t)}}=a^{-1}v^{(2)i}_{P_{s(t)}}$, one has the i-component as

$$v^{(2)}_{P_{s(t)}i}=v^{(2)}_{s(t)i}+\beta^{(2)\prime}_{s(t),i}+d^{(2)\prime}_{s(t)i}, \qquad (6.58)$$

where only $\xi^{(2)\mu}$ contributes to $U^{(2)\mu}_{P_{s(t)}}$. By writing $v^{(2)}_{s(t)i}=v^{\|(2)}_{s(t),i}+v^{\perp(2)}_{s(t)i}$ with $v^{\perp(2),i}_{s(t)i}=0$, $[\nabla^{-2}\partial^i(6.58)]$ gives the noncurl part of $v^{(2)}_{P_{s(t)}i}$ as

$$v^{\|(2)}_{P_{s(t)}}=\int\frac{d^3k}{(2\pi)^{3/2}}[v^{\|(2)}_{s(t)\mathbf{k}}+\beta^{(2)\prime}_{s(t)\mathbf{k}}]e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (6.59)$$

where

$$
\begin{aligned}
v^{\|(2)}_{s(t)\mathbf{k}} + \beta^{(2)'}_{s(t)\mathbf{k}} = {} & P_2\left(-\frac{2\sqrt{3}i}{k^2\tau^2}+\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + P_3\left(\frac{2\sqrt{3}i}{k^2\tau^2}+\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\
& \left.-\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
& +\left(-\frac{2\sqrt{3}}{k^2\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right. \\
& \left.-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{s(t)\mathbf{k}}(\tau')}{k^3\tau'}d\tau' - \frac{9}{k^4\tau}Z_{s(t)\mathbf{k}} + \frac{3}{2k^4}E'_{s(t)\mathbf{k}} + \frac{27}{4k^4\tau}F^{\|''}_{s(t)\mathbf{k}} + \frac{27}{4k^4\tau^2}F^{\|'}_{s(t)\mathbf{k}} - \frac{27}{k^4\tau^3}F^{\|}_{s(t)\mathbf{k}} \\
& +\frac{9}{4k^2}F^{\|'}_{s(t)\mathbf{k}} - \frac{9}{4k^4\tau}A'_{s(t)\mathbf{k}} + \frac{9}{4k^4\tau^2}A_{s(t)\mathbf{k}} - \frac{3}{4k^2}A_{s(t)\mathbf{k}} + \frac{1}{2}Y^{\|'}_{s(t)\mathbf{k}}, \qquad (6.60)
\end{aligned}
$$

with $E_{s(t)}$, $F_{s(t)}$, $A_{s(t)}$, $Z_{s(t)}$, and $Y^{\|}_{s(t)}$ given by Eqs. (3.3), (3.30), (3.26), (4.10), and (6.28).

The curl part of $v^{(2)}_{P_{s(t)}i}$ is given by [(6.58)–∂_i(6.59)] as follows:

$$
v^{\perp(2)}_{P_{s(t)}i} = \frac{q^{\cdot j}_{2ij}(\mathbf{x})}{8} - \frac{\nabla^{-2}q^{\cdot j}_{2ij}}{\tau^2} - \frac{1}{4}\int^\tau \tau'^2 V^{\cdot j}_{s(t)ij}(\mathbf{x},\tau')d\tau' + \frac{2}{\tau^2}\int^\tau \tau'^2\nabla^{-2}V^{\cdot j}_{s(t)ij}(\mathbf{x},\tau')d\tau' + \frac{\tau^2}{4}(M_{s(t)i} - \partial_i\nabla^{-2}M^{\cdot k}_{s(t)k}) + Y^{\perp'}_{s(t)i}, \qquad (6.61)
$$

where $(M_{s(t)i} - \partial_i\nabla^{-2}M^{\cdot k}_{s(t)k})$, $V_{s(t)ij}$, and $Y^{\perp}_{s(t)i}$ are given in (3.10), (3.22), and (6.29), respectively. Thus the second-order curl part of the 3-velocity is nonzero even if $v^{\perp(1)}_{Pi} = 0$.

Next, we will calculate the second-order perturbations with tensor-tensor couplings in the Poisson coordinate, by similar procedures to the above. By (C5) with the coupling of tensor-tensor in Ref. [63], one has

$$
\psi^{(2)}_{P_T} = -\alpha^{(2)'}_T - \frac{a'}{a}\alpha^{(2)}_T, \qquad (6.62)
$$

$$
w^{(2)}_{P_Ti} = \alpha^{(2)}_{T,i} - \beta^{(2)'}_{T,i} - d^{(2)'}_{Ti}, \qquad (6.63)
$$

$$
\phi^{(2)}_{P_T} = \phi^{(2)}_T + \frac{a'}{a}\alpha^{(2)}_T + \frac{1}{3}\nabla^2\beta^{(2)}_T, \qquad (6.64)
$$

$$
\begin{aligned}
\chi^{\top(2)}_{P_Tij} = {} & D_{ij}\chi^{\|(2)}_T + \chi^{\perp(2)}_{Tij} + \chi^{\top(2)}_{Tij} \\
& - (d^{(2)}_{Ti,j} + d^{(2)}_{Tj,i} + 2D_{ij}\beta^{(2)}_T), \qquad (6.65)
\end{aligned}
$$

where only $\xi^{(2)\mu}$ contributes to the above transformations, as $\xi^{(1)\mu}$ does not contain the tensor mode.

We need to solve for $\alpha^{(2)}_T$, $\beta^{(2)}_T$, and $d^{(2)}_{Ti}$. To do this, we use the transverse condition of $w^{(2)}_{P_Ti}$ and the traceless and transverse condition of $\chi^{\top(2)}_{P_Tij}$, which lead to

$$
\alpha^{(2)}_T - \beta^{(2)'}_T = 0, \qquad (6.66)
$$

$$
\chi^{\|(2)}_T - 2\beta^{(2)}_T = 0, \qquad (6.67)
$$

$$
\nabla^{-2}\partial^j\chi^{\perp(2)}_{Tij} - d^{(2)}_{Ti} = 0, \qquad (6.68)
$$

and the solutions

$$
\alpha^{(2)}_T = \frac{1}{2}\chi^{\|(2)'}_T, \qquad (6.69)
$$

$$
\beta^{(2)}_T = \frac{1}{2}\chi^{\|(2)}_T, \qquad (6.70)
$$

$$
d^{(2)}_{Ti} = \nabla^{-2}\chi^{\perp(2),j}_{Tij}. \qquad (6.71)
$$

Plugging the solutions $\chi^{\perp(2)}_{Tij}$ (5.44) and $\chi^{\|(2)}_T$ (5.52) into the above equations, one has

$$\begin{aligned}\alpha^{(2)}_{T\mathbf{k}} =& -Q_2\frac{2\sqrt{3}i}{k^2\tau^2}e^{-ik\tau/\sqrt{3}} + Q_3\frac{2\sqrt{3}i}{k^2\tau^2}e^{ik\tau/\sqrt{3}} - \frac{1}{k\tau}\int^\tau \frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau' - \frac{9}{k^4\tau}Z_{T\mathbf{k}}\\ &+\frac{2\sqrt{3}}{k^2\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\ &-\frac{2\sqrt{3}}{k^2\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\ &+\frac{3}{2k^4}E'_{T\mathbf{k}} - \frac{9}{4k^4\tau}A'_{T\mathbf{k}} + \frac{9}{4k^4\tau^2}A_{T\mathbf{k}} - \frac{3}{4k^2}A_{T\mathbf{k}},\end{aligned} \tag{6.72}$$

$$\begin{aligned}\beta^{(2)}_{T\mathbf{k}} =& Q_2\frac{2\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - Q_3\frac{2\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{2}{k}\int^\tau[Q_2e^{-ik\tau'/\sqrt{3}}+Q_3e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} - \frac{\ln\tau}{k}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{T\mathbf{k}}(\tau')d\tau'\\ &+\int^\tau\frac{3(k^2\tau'^2+6)\ln\tau'+9}{k^4\tau'}Z_{T\mathbf{k}}(\tau')d\tau' - \frac{2\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\ &+\frac{2\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \int^\tau\left[\frac{6}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right.\right.\\ &\left.\left.+\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' + \int^\tau\left[\frac{6}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\ &+\frac{3}{2k^4}E_{T\mathbf{k}} - \frac{9}{4k^4\tau}A_{T\mathbf{k}}(\tau) - \frac{3}{4k^2}\int^\tau A_{T\mathbf{k}}(\tau')d\tau',\end{aligned} \tag{6.73}$$

$$d^{(2)}_{Ti} = \frac{\nabla^{-2}q^{,j}_{4ij}}{\tau} + 2\int^\tau\frac{d\tau'}{\tau'^2}\int^{\tau'}\tau''^2\nabla^{-2}V^{,j}_{Tij}(\mathbf{x},\tau'')d\tau'', \tag{6.74}$$

where E_T, A_T, Z_T, and V_{Tij} are given by Eqs. (3.36), (3.57), (4.40), and (3.53).

Plugging Eqs. (5.51), (6.72), and (6.73) into (6.62) and (6.64) with $a(\tau)\propto\tau$, one obtains the second-order metric perturbations in the Poisson gauge as

$$\psi^{(2)}_{P_T} = \int\frac{d^3k}{(2\pi)^{3/2}}\left[-\alpha^{(2)\prime}_{T\mathbf{k}} - \frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.75}$$

where

$$\begin{aligned}-\alpha^{(2)\prime}_{T\mathbf{k}} - \frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}} =& Q_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{-ik\tau/\sqrt{3}} + Q_3\left(-\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.\\ &\left.-\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right.\\ &\left.-\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{3}{k^2}Z_{T\mathbf{k}} + \frac{9}{k^4\tau}Z'_{T\mathbf{k}} - \frac{3}{2k^4\tau}E'_{T\mathbf{k}}\\ &-\frac{3}{2k^4}E''_{T\mathbf{k}} + \frac{9}{4k^4\tau}A''_{T\mathbf{k}} + \frac{9}{4k^4\tau^3}A_{T\mathbf{k}} - \frac{9}{4k^4\tau^2}A'_{T\mathbf{k}} + \frac{3}{4k^2\tau}A_{T\mathbf{k}} + \frac{3}{4k^2}A'_{T\mathbf{k}},\end{aligned} \tag{6.76}$$

and

$$\phi^{(2)}_{P_T} = \int\frac{d^3k}{(2\pi)^{3/2}}\left[\phi^{(2)}_{T\mathbf{k}} + \frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}} - \frac{k^2}{3}\beta^{(2)}_{T\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.77}$$

where

$$\phi^{(2)}_{T\mathbf{k}}+\frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}}-\frac{k^2}{3}\beta^{(2)}_{T\mathbf{k}}=Q_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{-ik\tau/\sqrt{3}}+Q_3\left(\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{ik\tau/\sqrt{3}}+\left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.-\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'+\left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.-\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'$$
$$-\frac{9}{k^4\tau^2}Z_{T\mathbf{k}}-\frac{9}{4k^4\tau^2}A'_{T\mathbf{k}}+\frac{9}{4k^4\tau^3}A_{T\mathbf{k}}+\frac{3}{2k^4\tau}E'_{T\mathbf{k}}-\frac{1}{2k^2}E_{T\mathbf{k}}. \tag{6.78}$$

Thus, we see that in the Poisson gauge, the two second-order scalar modes are not equal. This can be checked by using (6.62), (6.64), (6.69), and (6.70),

$$\phi^{(2)}_{P_T}-\psi^{(2)}_{P_T}=\frac{1}{2}\chi^{\parallel(2)''}_{T}+\frac{1}{\tau}\chi^{\parallel(2)'}_{T}+\frac{1}{6}\nabla^2\chi^{\parallel(2)}_{T}+\phi^{(2)}_{T}$$
$$=\frac{3}{2}\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm},$$

according to the evolution equation (3.49). A similar relation holds for the scalar-tensor and scalar-scalar cases.

The vector mode is given by (6.63) and (6.72)–(6.74) as

$$w^{(2)}_{P_Ti}=\frac{\nabla^{-2}q^{,j}_{4ij}}{\tau^2}-\frac{2}{\tau^2}\int^\tau\tau'^2\nabla^{-2}V^{,j}_{Tij}(\mathbf{x},\tau')d\tau', \tag{6.79}$$

which is nonzero and sourced by the first-order couplings V_{Tij} in (3.53).

The tensor $\chi^{\top(2)}_{P_Tij}$ is given from (6.65) as

$$\chi^{\top(2)}_{P_Tij}=\chi^{\top(2)}_{Tij}, \tag{6.80}$$

which is the same as in the synchronous coordinate.

Next, we calculate the second-order density in the Poisson coordinate. We write (6.48) in terms of the second-order density contrast contributed by tensor-tensor couplings as

$$\delta^{(2)}_{P_T}=\delta^{(2)}_{T}+\left[-2\frac{a''(\tau)}{a'(\tau)}+4\frac{a'(\tau)}{a(\tau)}\right]\alpha^{(2)}_{T}. \tag{6.81}$$

For the RD stage, using the solution $\delta^{(2)}_{T\mathbf{k}}$ in (5.53) and $\alpha^{(2)}_{T\mathbf{k}}$ in (6.72), the above becomes

$$\delta^{(2)}_{P_T}=\int\frac{d^3k}{(2\pi)^{3/2}}\left[\delta^{(2)}_{T\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{T\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.82}$$

where

$$\delta^{(2)}_{T\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{T\mathbf{k}}=Q_2\left(-\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}+Q_3\left(\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}+\left(\frac{8\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'$$
$$+\left(-\frac{8\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right.$$
$$\left.-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'}d\tau'-\frac{36}{k^4\tau^2}Z_{T\mathbf{k}}+\frac{6}{k^4\tau}E'_{T\mathbf{k}}-\frac{9}{k^4\tau^2}A'_{T\mathbf{k}}+\frac{9}{k^4\tau^3}A_{T\mathbf{k}}-\frac{3}{k^2\tau}A_{T\mathbf{k}}. \tag{6.83}$$

It is seen that only $\xi^{(2)\mu}$ contributes to $\delta^{(2)}_{P_T}$, yet $\xi^{(1)\mu}$ does not contribute.

The second-order 4-velocity in the Poisson coordinate is derived from (C11) in Ref. [63] and from (2.12), (2.13), (6.13), and (6.14) as

$$U^{(2)0}_{P_T}=-\frac{1}{\tau}\int\frac{d^3k}{(2\pi)^{3/2}}\left[-\alpha^{(2)'}_{T\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.84}$$

with $-\alpha^{(2)'}_{T\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{T\mathbf{k}}$ given in (6.76), and

$$v^{(2)}_{P_Ti}=v^{(2)}_{Ti}+\beta^{(2)'}_{T,i}+d^{(2)'}_{Ti}, \tag{6.85}$$

with $U^{(2)i}_{P_T}=a^{-1}v^{(2)i}_{P_T}$. Only $\xi^{(2)\mu}$ contributes to $U^{(2)\mu}_{P_T}$. By writing $v^{(2)}_{Ti}=v^{\parallel(2)}_{T,i}+v^{\perp(2)}_{Ti}$ with $v^{\perp(2),i}_{Ti}=0$, $[\nabla^{-2}\partial^i(6.85)]$ gives the noncurl part of $v^{(2)}_{P_Ti}$ as

$$v_{P_T}^{\|(2)} = \int \frac{d^3k}{(2\pi)^{3/2}} [v_{T\mathbf{k}}^{\|(2)} + \beta_{T\mathbf{k}}^{(2)\prime}] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.86}$$

where

$$\begin{aligned} v_{T\mathbf{k}}^{\|(2)} + \beta_{T\mathbf{k}}^{(2)\prime} = {} & Q_2\left(-\frac{2\sqrt{3}i}{k^2\tau^2} + \frac{2}{k\tau} + \frac{i}{\sqrt{3}}\right) e^{-ik\tau/\sqrt{3}} + Q_3\left(\frac{2\sqrt{3}i}{k^2\tau^2} + \frac{2}{k\tau} - \frac{i}{\sqrt{3}}\right) e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\ & \left. - \frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right) \int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right) \frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'} d\tau' + \left(-\frac{2\sqrt{3}}{k^2\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\ & \left. - \frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right) \int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right) \frac{Z_{T\mathbf{k}}(\tau')}{k^3\tau'} d\tau' \\ & - \frac{9}{k^4\tau} Z_{T\mathbf{k}} + \frac{3}{2k^4} E'_{T\mathbf{k}} - \frac{9}{4k^4\tau} A'_{T\mathbf{k}} + \frac{9}{4k^4\tau^2} A_{T\mathbf{k}} - \frac{3}{4k^2} A_{T\mathbf{k}}, \end{aligned} \tag{6.87}$$

with E_T, A_T, and Z_T given by Eqs. (3.36), (3.57), and (4.40). The curl part of $v_{P_T i}^{(2)}$ is given by [(6.85)–∂_i(6.86)] as follows:

$$\begin{aligned} v_{P_T i}^{\perp(2)} = {} & \frac{q_{4ij}^{,j}(\mathbf{x})}{8} - \frac{\nabla^{-2} q_{4ij}^{,j}}{\tau^2} - \frac{1}{4}\int^\tau \tau'^2 V_{Tij}^{,j}(\mathbf{x},\tau') d\tau' \\ & + \frac{2}{\tau^2}\int^\tau \tau'^2 \nabla^{-2} V_{Tij}^{,j}(\mathbf{x},\tau') d\tau' \\ & + \frac{\tau^2}{4}(M_{Ti} - \partial_i \nabla^{-2} M_{Tk}^{,k}), \end{aligned} \tag{6.88}$$

where $(M_{Ti} - \partial_i \nabla^{-2} M_{Tk}^{,k})$ and V_{Tij} are given in (3.42) and (3.53), respectively. The second-order curl part of the 3-velocity is nonzero even if $v_{Pi}^{\perp(1)} = 0$.

The second-order perturbations with scalar-scalar couplings in Poisson coordinates have not been given in Ref. [63]. Using the solutions in synchronous coordinates in Ref. [63], by similar gauge transformation procedures to the above we obtain the results as follows (the computation details are skipped).

With the source terms E_S, $F_S^{\|}$, A_S, Z_S, and V_{Sij} given in Eqs. (4.2), (4.28), (4.24), (5.14), and (4.19) of Ref. [63], the second-order metric perturbations in the Poisson coordinates are

$$\begin{aligned} \psi_{P_S}^{(2)} = {} & \int \frac{d^3k}{(2\pi)^{3/2}} \left[-\alpha_{S\mathbf{k}}^{(2)\prime} - \frac{1}{\tau}\alpha_{S\mathbf{k}}^{(2)}\right] e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{1}{4}\chi^{\|(1)\prime\prime\prime}\chi^{\|(1)\prime} \\ & + \frac{5}{4\tau}\chi^{\|(1)\prime\prime}\chi^{\|(1)\prime} + \frac{1}{4\tau^2}\chi^{\|(1)\prime}\chi^{\|(1)\prime} + \frac{1}{4}\chi_{,l}^{\|(1)\prime\prime}\chi^{\|(1),l} \\ & + \frac{1}{4\tau}\chi_{,l}^{\|(1)\prime}\chi^{\|(1),l} + \frac{1}{2}\chi^{\|(1)\prime\prime}\chi^{\|(1)\prime\prime}, \end{aligned} \tag{6.89}$$

where

$$\begin{aligned} -\alpha_{S\mathbf{k}}^{(2)\prime} - \frac{1}{\tau}\alpha_{S\mathbf{k}}^{(2)} = {} & G_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3} + \frac{2}{k\tau^2}\right) e^{-ik\tau/\sqrt{3}} + G_3\left(-\frac{2\sqrt{3}i}{k^2\tau^3} + \frac{2}{k\tau^2}\right) e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\ & \left. - \frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right) \int^\tau \left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right) \frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'} d\tau' + \left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right. \\ & \left. - \frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right) \int^\tau \left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right) \frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'} d\tau' + \frac{3}{k^2} Z_{S\mathbf{k}} + \frac{9}{k^4\tau} Z'_{S\mathbf{k}} - \frac{3}{2k^4\tau} E'_{S\mathbf{k}} \\ & - \frac{3}{2k^4} E''_{S\mathbf{k}} - \frac{27}{4k^4\tau} F_{S\mathbf{k}}^{\|\prime\prime\prime} + \frac{135}{4k^4\tau^3} F_{S\mathbf{k}}^{\|\prime} - \frac{27}{4k^4\tau^2} F_{S\mathbf{k}}^{\|\prime\prime} - \frac{54}{k^4\tau^4} F_{S\mathbf{k}}^{\|} - \frac{9}{4k^2\tau} F_{S\mathbf{k}}^{\|\prime} - \frac{9}{4k^2} F_{S\mathbf{k}}^{\|\prime\prime} + \frac{9}{4k^4\tau} A''_{S\mathbf{k}} \\ & + \frac{9}{4k^4\tau^3} A_{S\mathbf{k}} - \frac{9}{4k^4\tau^2} A'_{S\mathbf{k}} + \frac{3}{4k^2\tau} A_{S\mathbf{k}} + \frac{3}{4k^2} A'_{S\mathbf{k}} - \frac{1}{2\tau} Y_{S\mathbf{k}}^{\|\prime} - \frac{1}{2} Y_{S\mathbf{k}}^{\|\prime\prime} + \frac{1}{\tau} W_{S\mathbf{k}}^{\|} + W_{S\mathbf{k}}^{\|\prime}, \end{aligned} \tag{6.90}$$

and

$$\phi^{(2)}_{P_S}=\int\frac{d^3k}{(2\pi)^{3/2}}\left[\phi^{(2)}_{S\mathbf{k}}+\frac{1}{\tau}\alpha^{(2)}_{S\mathbf{k}}-\frac{k^2}{3}\beta^{(2)}_{S\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}-\phi^{(1)\prime}\chi^{\|(1)\prime}-\frac{2}{\tau}\phi^{(1)}\chi^{\|(1)\prime}-\frac{1}{4\tau^2}\chi^{\|(1)\prime}\chi^{\|(1)\prime}-\frac{1}{4\tau}\chi^{\|(1)\prime\prime}\chi^{\|(1)\prime}-\frac{2}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}$$
$$-\frac{1}{12}\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}-\frac{1}{12}\chi^{\|(1),l}\nabla^2\chi^{\|(1)}_{,l}-\frac{1}{3\tau}\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)}-\frac{1}{9}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)}-\chi^{\|(1),l}\phi^{(1)}_{,l}-\frac{1}{4\tau}\chi^{\|(1),l}\chi^{\|(1)\prime}_{,l}$$
$$+\frac{1}{6}\chi^{\|(1)}_{,lm}\chi^{\|(1),lm}, \tag{6.91}$$

where

$$\phi^{(2)}_{S\mathbf{k}}+\frac{1}{\tau}\alpha^{(2)}_{S\mathbf{k}}-\frac{k^2}{3}\beta^{(2)}_{S\mathbf{k}}=G_2\left(-\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{-ik\tau/\sqrt{3}}+G_3\left(\frac{2\sqrt{3}i}{k^2\tau^3}+\frac{2}{k\tau^2}\right)e^{ik\tau/\sqrt{3}}$$
$$+\left(\frac{2\sqrt{3}}{k^2\tau^3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'$$
$$+\left(-\frac{2\sqrt{3}}{k^2\tau^3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'$$
$$-\frac{9}{k^4\tau^2}Z_{S\mathbf{k}}-\frac{9}{4k^4\tau^2}A'_{S\mathbf{k}}+\frac{9}{4k^4\tau^3}A_{S\mathbf{k}}+\frac{3}{2k^4\tau}E'_{S\mathbf{k}}+\frac{27}{4k^4\tau^2}F^{\|\prime\prime}_{S\mathbf{k}}+\frac{27}{4k^4\tau^3}F^{\|\prime}_{S\mathbf{k}}$$
$$-\frac{27}{k^4\tau^4}F^{\|}_{S\mathbf{k}}-\frac{1}{2k^2}E_{S\mathbf{k}}-\frac{9}{2k^2\tau^2}F^{\|}_{S\mathbf{k}}-\frac{k^2}{6}Y^{\|}_{S\mathbf{k}}+\frac{1}{2\tau}Y^{\|\prime}_{S\mathbf{k}}-\frac{1}{\tau}W^{\|}_{S\mathbf{k}}, \tag{6.92}$$

and

$$w^{(2)}_{P_Si}=\frac{\nabla^{-2}c^{;j}_{2ij}}{\tau^2}-\frac{2}{\tau^2}\int^\tau\tau'^2\nabla^{-2}V^{;j}_{Sij}(\mathbf{x},\tau')d\tau'-Y^{\perp\prime}_{Si}+W^{\perp}_{Si}, \tag{6.93}$$

and the tensor $\chi^{\top(2)}_{Pij}$ is

$$\chi^{\top(2)}_{P_Sij}=\chi^{\top(2)}_{Sij}+Y^{\top}_{Sij}, \tag{6.94}$$

with $\chi^{\top(2)}_{Sij}$ in (5.1) of Ref. [63] and $W^{\|}_S$, $Y^{\|}_S$, $W^{\perp}_{Si}$, $Y^{\perp}_{Si}$, $Y^{\top}_{Sij}$ as

$$\begin{aligned}W^{\|}_S&=\nabla^{-2}\partial^iW_i\\&=\nabla^{-2}\Big[2\phi^{(1),l}\chi^{\|(1)\prime}_{,l}+2\phi^{(1)}\nabla^2\chi^{\|(1)\prime}-\frac{1}{2}\chi^{\|(1)\prime\prime,l}\chi^{\|(1)\prime}_{,l}-\frac{1}{2}\chi^{\|(1)\prime\prime}\nabla^2\chi^{\|(1)\prime}\\&\quad-\frac{1}{6}\chi^{\|(1)\prime}_{,l}\nabla^2\chi^{\|(1),l}+\frac{1}{3}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)}-\frac{1}{2}\chi^{\|(1)\prime}_{,lm}\chi^{\|(1),lm}\Big].\end{aligned} \tag{6.95}$$

The curl part of W_{Si} is

$$\begin{aligned}W^{\perp}_{Si}=W_{Si}-\partial_iW^{\|}_S&=2\phi^{(1)}\chi^{\|(1)\prime}_{,i}-\frac{1}{2}\chi^{\|(1)\prime\prime}\chi^{\|(1)\prime}_{,i}+\frac{1}{3}\chi^{\|(1)\prime}_{,i}\nabla^2\chi^{\|(1)}-\frac{1}{2}\chi^{\|(1)\prime}_{,l}\chi^{\|(1),l}_{,i}\\&\quad+\partial_i\nabla^{-2}\Big[-2\phi^{(1),l}\chi^{\|(1)\prime}_{,l}-2\phi^{(1)}\nabla^2\chi^{\|(1)\prime}+\frac{1}{2}\chi^{\|(1)\prime\prime,l}\chi^{\|(1)\prime}_{,l}+\frac{1}{2}\chi^{\|(1)\prime\prime}\nabla^2\chi^{\|(1)\prime}\\&\quad+\frac{1}{6}\chi^{\|(1)\prime}_{,l}\nabla^2\chi^{\|(1),l}-\frac{1}{3}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)}+\frac{1}{2}\chi^{\|(1)\prime}_{,lm}\chi^{\|(1),lm}\Big].\end{aligned} \tag{6.96}$$

The scalar, vector, and tensor modes of Y_{Sij} are

$$
\begin{aligned}
Y_S^{\|} = & \frac{3}{2}\nabla^{-2}\nabla^{-2}Y_{Slm}^{,lm} \\
= & \frac{1}{8}\chi^{\|(1)\prime}\chi^{\|(1)\prime} - \frac{1}{8}\chi^{\|(1),l}\chi_{,l}^{\|(1)} + \nabla^{-2}\left[-2\phi^{(1)}\nabla^2\chi^{\|(1)} + \frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)} - \frac{5}{8}\chi^{\|(1)\prime,l}\chi_{,l}^{\|(1)\prime} - \frac{3}{8}\chi^{\|(1),lm}\chi_{,lm}^{\|(1)}\right] \\
& + \nabla^{-2}\nabla^{-2}\Bigg[6\phi^{(1),lm}\chi_{,lm}^{\|(1)} + 12\phi^{(1),l}\nabla^2\chi_{,l}^{\|(1)} + 6\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)} - \frac{3}{4}\chi^{\|(1)\prime,l}\nabla^2\chi_{,l}^{\|(1)\prime} \\
& - \frac{3}{4}\chi^{\|(1)\prime}\nabla^2\nabla^2\chi^{\|(1)\prime} - \frac{1}{4}\chi^{\|(1),lm}\nabla^2\chi_{,lm}^{\|(1)} - \frac{1}{4}\chi^{\|(1),l}\nabla^2\nabla^2\chi_{,l}^{\|(1)}\Bigg],
\end{aligned} \tag{6.97}
$$

$$
\begin{aligned}
Y_{Sj}^{\perp} = & \nabla^{-2}Y_{Slj}^{,l} - \frac{2}{3}Y_{S,j}^{\|} \\
= & \nabla^{-2}\left[4\phi^{(1),l}\chi_{,lj}^{\|(1)} + 4\phi^{(1)}\nabla^2\chi_{,j}^{\|(1)} - \frac{1}{2}\chi^{\|(1)\prime}\nabla^2\chi_{,j}^{\|(1)\prime} - \frac{1}{6}\chi^{\|(1),l}\nabla^2\chi_{,lj}^{\|(1)}\right] \\
& + \partial_j\nabla^{-2}\nabla^{-2}\Bigg[-4\phi^{(1),lm}\chi_{,lm}^{\|(1)} - 8\phi^{(1),l}\nabla^2\chi_{,l}^{\|(1)} - 4\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)} + \frac{1}{2}\chi^{\|(1)\prime,l}\nabla^2\chi_{,l}^{\|(1)\prime} \\
& + \frac{1}{2}\chi^{\|(1)\prime}\nabla^2\nabla^2\chi^{\|(1)\prime} + \frac{1}{6}\chi^{\|(1),lm}\nabla^2\chi_{,lm}^{\|(1)} + \frac{1}{6}\chi^{\|(1),l}\nabla^2\nabla^2\chi_{,l}^{\|(1)}\Bigg],
\end{aligned} \tag{6.98}
$$

$$
\begin{aligned}
Y_{Sij}^{\top} = & Y_{Sij} - D_{ij}Y_S^{\|} - 2Y_{S(i,j)}^{\perp} \\
= & 4\phi^{(1)}\chi_{,ij}^{\|(1)} - 2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij} + \frac{2}{3}\chi_{,ij}^{\|(1)}\nabla^2\chi^{\|(1)} - \frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)}\delta_{ij} \\
& - \frac{3}{4}\chi^{\|(1)\prime}\chi_{,ij}^{\|(1)\prime} + \frac{1}{4}\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\delta_{ij} - \frac{1}{4}\chi_{,i}^{\|(1)\prime}\chi_{,j}^{\|(1)\prime} - \frac{1}{8}\chi^{\|(1)\prime,l}\chi_{,l}^{\|(1)\prime}\delta_{ij} \\
& - \frac{1}{4}\chi^{\|(1),l}\chi_{,lij}^{\|(1)} + \frac{1}{12}\chi^{\|(1),l}\nabla^2\chi_{,l}^{\|(1)}\delta_{ij} - \frac{3}{4}\chi_{,il}^{\|(1)}\chi_{,j}^{\|(1),l} + \frac{1}{8}\chi^{\|(1),lm}\chi_{,lm}^{\|(1)}\delta_{ij} \\
& + \delta_{ij}\nabla^{-2}\Bigg[2\phi^{(1),lm}\chi_{,lm}^{\|(1)} + 4\phi^{(1),l}\nabla^2\chi_{,l}^{\|(1)} + 2\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)} - \frac{1}{4}\chi^{\|(1)\prime,l}\nabla^2\chi_{,l}^{\|(1)\prime} \\
& - \frac{1}{4}\chi^{\|(1)\prime}\nabla^2\nabla^2\chi^{\|(1)\prime} - \frac{1}{12}\chi^{\|(1),lm}\nabla^2\chi_{,lm}^{\|(1)} - \frac{1}{12}\chi^{\|(1),l}\nabla^2\nabla^2\chi_{,l}^{\|(1)}\Bigg] \\
& + \partial_i\nabla^{-2}\left[-4\phi^{(1),l}\chi_{,lj}^{\|(1)} - 4\phi^{(1)}\nabla^2\chi_{,j}^{\|(1)} + \frac{1}{2}\chi^{\|(1)\prime}\nabla^2\chi_{,j}^{\|(1)\prime} + \frac{1}{6}\chi^{\|(1),l}\nabla^2\chi_{,lj}^{\|(1)}\right] \\
& + \partial_j\nabla^{-2}\left[-4\phi^{(1),l}\chi_{,li}^{\|(1)} - 4\phi^{(1)}\nabla^2\chi_{,i}^{\|(1)} + \frac{1}{2}\chi^{\|(1)\prime}\nabla^2\chi_{,i}^{\|(1)\prime} + \frac{1}{6}\chi^{\|(1),l}\nabla^2\chi_{,li}^{\|(1)}\right] \\
& + \partial_i\partial_j\nabla^{-2}\left[2\phi^{(1)}\nabla^2\chi^{\|(1)} - \frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)} + \frac{5}{8}\chi^{\|(1)\prime,l}\chi_{,l}^{\|(1)\prime} + \frac{3}{8}\chi^{\|(1),lm}\chi_{,lm}^{\|(1)}\right] \\
& + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\Bigg[2\phi^{(1),lm}\chi_{,lm}^{\|(1)} + 4\phi^{(1),l}\nabla^2\chi_{,l}^{\|(1)} + 2\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)} - \frac{1}{4}\chi^{\|(1)\prime,l}\nabla^2\chi_{,l}^{\|(1)\prime} \\
& - \frac{1}{4}\chi^{\|(1)\prime}\nabla^2\nabla^2\chi^{\|(1)\prime} - \frac{1}{12}\chi^{\|(1),lm}\nabla^2\chi_{,lm}^{\|(1)} - \frac{1}{12}\chi^{\|(1),l}\nabla^2\nabla^2\chi_{,l}^{\|(1)}\Bigg].
\end{aligned} \tag{6.99}
$$

Note that Ref. [52] also gave the second-order vector mode of the metric perturbation in Poisson coordinates. However, by detailed checking of Eq. (17) of Ref. [52], we find that in their source $\sum_{lm}$ the asymmetric, last term

$$2\mathcal{H}(\partial_l\Phi)(\partial_m\Phi')$$

should be replaced by the symmetrized term

$$\mathcal{H}(\partial_l\Phi)(\partial_m\Phi') + \mathcal{H}(\partial_l\Phi')(\partial_m\Phi)$$

because $\sum_{lm}$ as the source in the Einstein equation must be symmetric, and the solution $S(\mathbf{k},\tau)$ of (20) in Ref. [52] should also be revised accordingly. We project the inhomogeneous part of $w_{P_S i}$ of (6.93) into the following:

$$e^j w^{(2)}_{P_S j} = e^j\Big\{-\frac{2}{\tau^2}\int^\tau \tau'^2\nabla^{-2}\Big[\frac{\tau'^2}{2}\chi^{\|(1)'''}_{,j}\nabla^2\chi^{\|(1)'''}+\tau'\chi^{\|(1)''}_{,j}\nabla^2\chi^{\|(1)'''}+\tau'\chi^{\|(1)'''}_{,j}\nabla^2\chi^{\|(1)''}+\frac{1}{\tau'^2}\chi^{\|(1)'}_{,j}\nabla^2\chi^{\|(1)'}+\frac{1}{\tau'}\chi^{\|(1)'}_{,j}\nabla^2\chi^{\|(1)''}$$
$$+\frac{1}{\tau'}\chi^{\|(1)''}_{,j}\nabla^2\chi^{\|(1)'}+3\chi^{\|(1)''}_{,j}\nabla^2\chi^{\|(1)''}\Big]d\tau'\Big\}, \tag{6.100}$$

where e_i is a polarization vector orthogonal to the wave vector. The Fourier mode of our result (6.100) agrees with the corrected $S(\mathbf{k},\tau)$ in Eq. (20) of Ref. [52], where the relation $\Phi=-\frac{1}{2}\chi^{\|(1)''}-\frac{1}{2\tau}\chi^{\|(1)'}$ is used.

The second-order density contrast is

$$\delta^{(2)}_{P_S}=\int\frac{d^3k}{(2\pi)^{3/2}}\Big[\delta^{(2)}_{S\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{S\mathbf{k}}\Big]e^{i\mathbf{k}\cdot\mathbf{x}}+\frac{4}{\tau}\delta^{(1)}\chi^{\|(1)'}-\delta^{(1)'}\chi^{\|(1)'}-\delta^{(1)}_{,l}\chi^{\|(1),l}-\frac{1}{\tau}\chi^{\|(1)''}\chi^{\|(1)'}-\frac{1}{\tau}\chi^{\|(1)'}_{,l}\chi^{\|(1),l}+\frac{5}{\tau^2}\chi^{\|(1)'}\chi^{\|(1)'}, \tag{6.101}$$

where

$$\begin{aligned}\delta^{(2)}_{S\mathbf{k}}+\frac{4}{\tau}\alpha^{(2)}_{S\mathbf{k}} &= G_2\Big(-\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\Big)e^{-ik\tau/\sqrt{3}}+G_3\Big(\frac{8\sqrt{3}i}{k^2\tau^3}+\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\Big)e^{ik\tau/\sqrt{3}}\\
&+\Big(\frac{8\sqrt{3}}{k^2\tau^3}\sin\Big(\frac{k\tau}{\sqrt{3}}\Big)-\frac{8}{k\tau^2}\cos\Big(\frac{k\tau}{\sqrt{3}}\Big)-\frac{8}{\sqrt{3}\tau}\sin\Big(\frac{k\tau}{\sqrt{3}}\Big)\\
&+\frac{4k}{3}\cos\Big(\frac{k\tau}{\sqrt{3}}\Big)\Big)\int^\tau\Big(9\cos\Big(\frac{k\tau'}{\sqrt{3}}\Big)+3\sqrt{3}k\tau'\sin\Big(\frac{k\tau'}{\sqrt{3}}\Big)\Big)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\Big(-\frac{8\sqrt{3}}{k^2\tau^3}\cos\Big(\frac{k\tau}{\sqrt{3}}\Big)-\frac{8}{k\tau^2}\sin\Big(\frac{k\tau}{\sqrt{3}}\Big)+\frac{8}{\sqrt{3}\tau}\cos\Big(\frac{k\tau}{\sqrt{3}}\Big)\\
&+\frac{4k}{3}\sin\Big(\frac{k\tau}{\sqrt{3}}\Big)\Big)\int^\tau\Big(9\sin\Big(\frac{k\tau'}{\sqrt{3}}\Big)-3\sqrt{3}k\tau'\cos\Big(\frac{k\tau'}{\sqrt{3}}\Big)\Big)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&-\frac{36}{k^4\tau^2}Z_{S\mathbf{k}}+\frac{6}{k^4\tau}E'_{S\mathbf{k}}+\frac{27}{k^4\tau^2}F^{\|''}_{S\mathbf{k}}+\frac{27}{k^4\tau^3}F^{\|'}_{S\mathbf{k}}-\frac{108}{k^4\tau^4}F^{\|}_{S\mathbf{k}}+\frac{9}{k^2\tau}F^{\|'}_{S\mathbf{k}}\\
&+3F^{\|}_{S\mathbf{k}}-\frac{9}{k^4\tau^2}A'_{S\mathbf{k}}+\frac{9}{k^4\tau^3}A_{S\mathbf{k}}-\frac{3}{k^2\tau}A_{S\mathbf{k}}+\frac{2}{\tau}Y^{\|'}_{S\mathbf{k}}-\frac{4}{\tau}W^{\|}_{S\mathbf{k}}.\end{aligned} \tag{6.102}$$

The 0-component of the 4-velocity is

$$U^{(2)0}_{P_S}=-\frac{1}{\tau}\int\frac{d^3k}{(2\pi)^{3/2}}\Big[-\alpha^{(2)'}_{S\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{S\mathbf{k}}\Big]e^{i\mathbf{k}\cdot\mathbf{x}}+\frac{1}{\tau}v^{\|(1),l}v^{\|(1)}_{,l}+\frac{1}{\tau}\chi^{\|(1)'}_{,l}v^{\|(1),l}+\frac{1}{2\tau^3}\chi^{\|(1)'}\chi^{\|(1)'}+\frac{1}{4\tau^2}\chi^{\|(1)''}\chi^{\|(1)'}$$
$$-\frac{1}{4\tau}\chi^{\|(1)'''}\chi^{\|(1)'}+\frac{1}{4\tau}\chi^{\|(1)''}\chi^{\|(1)''}-\frac{1}{4\tau^2}\chi^{\|(1)'}_{,l}\chi^{\|(1),l}-\frac{1}{4\tau}\chi^{\|(1)''}_{,l}\chi^{\|(1),l}+\frac{1}{4\tau}\chi^{\|(1)',l}\chi^{\|(1)'}_{,l}, \tag{6.103}$$

where $-\alpha^{(2)'}_{S\mathbf{k}}-\frac{1}{\tau}\alpha^{(2)}_{S\mathbf{k}}$ is given in (6.90). By the definition $U^{(2)i}_{P_S}=a^{-1}v^{(2)i}_{P_S}$, one has the i-component as

$$v^{(2)}_{P_S i}=v^{(2)}_{Si}+\beta^{(2)'}_{S,i}+d^{(2)'}_{Si}+\frac{1}{\tau}v^{\|(1)}_{,i}\chi^{\|(1)'}-v^{\|(1)'}_{,i}\chi^{\|(1)'}-v^{\|(1)}_{,li}\chi^{\|(1),l}+v^{\|(1),l}\chi^{\|(1)}_{,li}+\frac{1}{2\tau}\chi^{\|(1)'}_{,i}\chi^{\|(1)'}-\frac{1}{4}\chi^{\|(1)''}_{,i}\chi^{\|(1)'}$$
$$+\frac{1}{4}\chi^{\|(1)'}_{,i}\chi^{\|(1)''}-\frac{1}{4}\chi^{\|(1)'}_{,li}\chi^{\|(1),l}+\frac{1}{4}\chi^{\|(1)}_{,li}\chi^{\|(1)',l}. \tag{6.104}$$

By writing $v^{(2)}_{Si}=v^{\|(2)}_{S,i}+v^{\perp(2)}_{Si}$ with $v^{\perp(2),i}_{Si}=0$, $[\nabla^{-2}\partial^i(6.104)]$ gives the noncurl part of $v^{(2)}_{P_S i}$ as

$$v_{P_S}^{\parallel(2)} = \int \frac{d^3k}{(2\pi)^{3/2}}[v_{S\mathbf{k}}^{\parallel(2)} + \beta_{S\mathbf{k}}^{(2)\prime}]e^{i\mathbf{k}\cdot\mathbf{x}} - v_{,l}^{\parallel(1)}\chi^{\parallel(1),l} + \frac{1}{4\tau}\chi^{\parallel(1)\prime}\chi^{\parallel(1)\prime} - \frac{1}{4}\chi^{\parallel(1)\prime\prime}\chi^{\parallel(1)\prime} - \frac{1}{4}\chi_{,l}^{\parallel(1)\prime}\chi^{\parallel(1),l}$$
$$+ \nabla^{-2}\Big[\frac{1}{\tau}v_{,l}^{\parallel(1)}\chi^{\parallel(1)\prime,l} + \frac{1}{\tau}\chi^{\parallel(1)\prime}\nabla^2 v^{\parallel(1)} - \chi^{\parallel(1)\prime}\nabla^2 v^{\parallel(1)\prime} - v_{,l}^{\parallel(1)\prime}\chi^{\parallel(1)\prime,l} + 2v^{\parallel(1),l}\nabla^2\chi_{,l}^{\parallel(1)} + 2v^{\parallel(1),lm}\chi_{,lm}^{\parallel(1)}$$
$$+ \frac{1}{2}\chi_{,l}^{\parallel(1)\prime}\chi^{\parallel(1)\prime\prime,l} + \frac{1}{2}\chi^{\parallel(1)\prime\prime}\nabla^2\chi^{\parallel(1)\prime} + \frac{1}{2}\chi^{\parallel(1)\prime,l}\nabla^2\chi_{,l}^{\parallel(1)} + \frac{1}{2}\chi_{,lm}^{\parallel(1)}\chi^{\parallel(1)\prime,lm}\Big], \tag{6.105}$$

where

$$v_{S\mathbf{k}}^{\parallel(2)} + \beta_{S\mathbf{k}}^{(2)\prime} = G_2\left(-\frac{2\sqrt{3}i}{k^2\tau^2} + \frac{2}{k\tau} + \frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + G_3\left(\frac{2\sqrt{3}i}{k^2\tau^2} + \frac{2}{k\tau} - \frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}} + \left(\frac{2\sqrt{3}}{k^2\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.- \frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'$$
$$+ \left(-\frac{2\sqrt{3}}{k^2\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right.$$
$$\left.- 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' - \frac{9}{k^4\tau}Z_{S\mathbf{k}} + \frac{3}{2k^4}E'_{S\mathbf{k}} + \frac{27}{4k^4\tau}F_{S\mathbf{k}}^{\parallel\prime\prime} + \frac{27}{4k^4\tau^2}F_{S\mathbf{k}}^{\parallel\prime} - \frac{27}{k^4\tau^3}F_{S\mathbf{k}}^{\parallel}$$
$$+ \frac{9}{4k^2}F_{S\mathbf{k}}^{\parallel\prime} - \frac{9}{4k^4\tau}A'_{S\mathbf{k}} + \frac{9}{4k^4\tau^2}A_{S\mathbf{k}} - \frac{3}{4k^2}A_{S\mathbf{k}} + \frac{1}{2}Y_{S\mathbf{k}}^{\parallel\prime}. \tag{6.106}$$

The curl part of $v_{P_Si}^{(2)}$ is given by [(6.104)–∂_i(6.105)] as

$$v_{P_Si}^{\perp(2)} = \frac{c_{2ij}^{,j}(\mathbf{x})}{8} - \frac{\nabla^{-2}c_{2ij}^{,j}}{\tau^2} - \frac{1}{4}\int^\tau \tau'^2 V_{Sij}^{,j}(\mathbf{x},\tau')d\tau' + \frac{2}{\tau^2}\int^\tau \tau'^2\nabla^{-2}V_{Sij}^{,j}(\mathbf{x},\tau')d\tau' + \frac{\tau^2}{4}(M_{Si} - \partial_i\nabla^{-2}M_{Sk}^{,k}) + Y_{Si}^{\perp\prime}$$
$$+ \frac{1}{\tau}v_{,i}^{\parallel(1)}\chi^{\parallel(1)\prime} - v_{,i}^{\parallel(1)\prime}\chi^{\parallel(1)\prime} + 2v^{\parallel(1),l}\chi_{,li}^{\parallel(1)} + \frac{1}{2}\chi_{,i}^{\parallel(1)\prime}\chi^{\parallel(1)\prime\prime} + \frac{1}{2}\chi_{,li}^{\parallel(1)}\chi^{\parallel(1)\prime,l} + \partial_i\nabla^{-2}\Big[-\frac{1}{\tau}v_{,l}^{\parallel(1)}\chi^{\parallel(1)\prime,l} - \frac{1}{\tau}\chi^{\parallel(1)\prime}\nabla^2 v^{\parallel(1)}$$
$$+ \chi^{\parallel(1)\prime}\nabla^2 v^{\parallel(1)\prime} + v_{,l}^{\parallel(1)\prime}\chi^{\parallel(1)\prime,l} - 2v^{\parallel(1),l}\nabla^2\chi_{,l}^{\parallel(1)} - 2v^{\parallel(1),lm}\chi_{,lm}^{\parallel(1)} - \frac{1}{2}\chi_{,l}^{\parallel(1)\prime}\chi^{\parallel(1)\prime\prime,l} - \frac{1}{2}\chi^{\parallel(1)\prime\prime}\nabla^2\chi^{\parallel(1)\prime}$$
$$- \frac{1}{2}\chi^{\parallel(1)\prime,l}\nabla^2\chi_{,l}^{\parallel(1)} - \frac{1}{2}\chi_{,lm}^{\parallel(1)}\chi^{\parallel(1)\prime,lm}\Big], \tag{6.107}$$

where $(M_{Si} - \partial_i\nabla^{-2}M_{Sk}^{,k})$ is given in (4.8) in Ref. [63]. The curl part of the 3-velocity is nonzero in the second order, even if $v_{P_Si}^{\perp(1)} = 0$.

VII. CONCLUSION

As part of a series study [63] of the second-order cosmological perturbations in the RD stage in synchronous coordinates, this paper presents the second-order formal solutions for the scalar-tensor and tensor-tensor couplings in the integral form. The cosmic matter content is represented by a relativistic fluid. The first-order vector perturbations and the first-order transverse velocity are assumed to be zero.

From the second-order solutions we find that in the RD stage the scalar modes, density contrast, and longitudinal velocity all propagate as a wave at the sound speed $\frac{1}{\sqrt{3}}$, while the tensor modes are waves at the speed of light. These wave features are similar to the first-order solutions in the RD stage. This is in contrast to the MD stage, in which only tensor modes are waves at the speed of light, and other types of perturbations evolve as a power law of the comoving time τ [61,62].

A general second-order gauge transformation, from synchronous to synchronous coordinates, is implemented by first-order and second-order transformation vector fields. When the gauge-invariant first-order solutions are used in the couplings, only the second-order transformation vector is effective at carrying out the second-order transformations. With this, we obtain the gauge-invariant second-order formal solutions in the integral form within a chosen synchronous coordinate. In addition, we also perform the second-order gauge transformations of the solutions from synchronous to Poisson coordinates and obtain the second-order formal solutions in Poisson coordinates. In the Poisson coordinates, for the second-order vector mode with scalar-scalar couplings, we find that the last term in Eq. (17) of Ref. [52] should be symmetrized. After correcting this mistake in Eq. (17) of Ref. [52], the revised solution Eq. (20) in Ref. [52] will agree with our Eq. (6.100).

From these second-order solutions, one will be able to investigate the properties of nonlinearity evolution not only within one type of metric perturbation, such as the transfer of perturbation power among various k-modes, the influence by the growing and decaying modes, etc., but also the transfer of perturbation power between different types of perturbations, such as those between scalar and tensor modes, etc. After accumulations during cosmic expansion, these nonlinear effects might possibly lead to changes of the tensor/scalar ratio of metric perturbations observed in CMB anisotropies from that of the primordial metric perturbations generated during inflation. A detailed investigation of these issues would require the initial conditions pertinent to the cosmic evolution and the proper adjoining of perturbations from the precedent expansion stages such as inflation and a possible period of reheating.

To use these solutions, one needs to do three types of $\int d\tau$ integrals and one type of $\int d^3\mathbf{k}$ integral that involve the functions $Z_{s(t)\mathbf{k}}$, $A_{s(t)\mathbf{k}}$, $Z_{T\mathbf{k}}$, $A_{T\mathbf{k}}$, etc., which can be done numerically. To apply these in the cosmological study, one needs to specify the initial conditions which are presented by $D_2(\mathbf{k})$, $D_3(\mathbf{k})$, $b_1(\mathbf{k})$, $b_2(\mathbf{k})$, $P_2(\mathbf{k})$, $P_3(\mathbf{k})$, $Q_2(\mathbf{k})$, $Q_3(\mathbf{k})$, etc.

The results of scalar-tensor and tensor-tensor in this paper, together with the results of scalar-scalar couplings in Ref. [63], constitute the full solution of the second-order cosmological perturbations in the integral form in the RD stage in synchronous coordinates. These second-order results of the RD stage, in conjunction with the second-order results of the MD stage [61,62], can be used to study the nonlinear effects of cosmological perturbations.

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APPENDIX: SECOND-ORDER PERTURBED EINSTEIN EQUATION AND CONSERVATION EQUATIONS

In this appendix, we shall list the second-order perturbed Einstein equation with the scalar-tensor and tensor-tensor couplings, as well as the second-order covariant conservation equations of the stress tensor in a general RW spacetime.

First, we present the second-order perturbed equations with scalar-tensor couplings. The (00) component of second-order perturbed Einstein equation, i.e., the second-order energy constraint, is

$$-\frac{6a'}{a}\phi^{(2)\prime}_{s(t)} + 2\nabla^2\phi^{(2)}_{s(t)} + \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(2)}_{s(t)} = 3\left(\frac{a'}{a}\right)^2\delta^{(2)}_{s(t)} + E_{s(t)}, \tag{A1}$$

where

$$E_{s(t)} \equiv 2\phi^{(1),lm}\chi^{\top(1)}_{lm} + \frac{1}{2}\chi^{\top(1)\prime}_{lm}\chi^{\parallel(1)\prime,lm} + \frac{2a'}{a}\chi^{\top(1)}_{lm}\chi^{\parallel(1)\prime,lm} + \frac{2a'}{a}\chi^{\top(1)\prime}_{lm}\chi^{\parallel(1),lm} + \frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\parallel(1),lm} - \chi^{\parallel(1),lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{1}{2}\chi^{\top(1),l}_{mn}\chi^{\parallel(1),mn}_{,l}. \tag{A2}$$

The $(0i)$ component of the second-order perturbed Einstein equation, i.e., the second-order momentum constraint, is

$$2\phi^{(2)\prime}_{s(t),i} + \frac{1}{2}D_{ij}\chi^{\parallel(2)\prime,j}_{s(t)} + \frac{1}{2}\chi^{\perp(2)\prime,j}_{s(t)ij} = -3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{(2)}_{s(t)i} + M_{s(t)i}, \tag{A3}$$

where

$$M_{s(t)i} \equiv -6\left(\frac{a'}{a}\right)^2(1+c_s^2)\chi^{\top(1)}_{il}v^{\parallel(1),l} + \phi^{(1),l}\chi^{\top(1)\prime}_{il} - 2\phi^{(1)\prime,l}\chi^{\top(1)}_{il} - \chi^{\top(1)\prime}_{lm,i}\chi^{\parallel(1),lm} - \frac{1}{2}\chi^{\top(1)\prime}_{lm}\chi^{\parallel(1),lm}_{,i} - \frac{1}{2}\chi^{\top(1)}_{lm,i}\chi^{\parallel(1)\prime,lm} + \chi^{\top(1)\prime}_{il,m}\chi^{\parallel(1),lm} - \frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\chi^{\parallel(1)\prime,l} + \frac{2}{3}\chi^{\top(1)\prime}_{il}\nabla^2\chi^{\parallel(1),l}. \tag{A4}$$

The longitudinal part of the momentum constraint (A3) is

$$2\phi^{(2)\prime}_{s(t)}+\frac{1}{3}\nabla^2\chi^{\|(2)\prime}_{s(t)}=-3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(2)}_{s(t)}+\nabla^{-2}M^{,l}_{s(t)l}, \tag{A5}$$

and the transverse part is

$$\frac{1}{2}\chi^{\perp(2)\prime,j}_{s(t)ij}=-3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\perp(2)}_{s(t)i}+(M_{s(t)i}-\partial_i\nabla^{-2}M^{,l}_{s(t)l}), \tag{A6}$$

where

$$\begin{aligned}M^{,l}_{s(t)l}={}&-6\left(\frac{a'}{a}\right)^2(1+c_s^2)\chi^{\top(1)}_{lm}v^{\|(1),lm}+\phi^{(1),lm}\chi^{\top(1)\prime}_{lm}-2\phi^{(1)\prime,lm}\chi^{\top(1)}_{lm}-\frac{1}{2}\chi^{\top(1)\prime}_{lm,n}\chi^{\|(1),lmn}+\frac{1}{6}\chi^{\top(1)\prime}_{lm}\nabla^2\chi^{\|(1),lm}\\&-\chi^{\|(1),lm}\nabla^2\chi^{\top(1)\prime}_{lm}-\frac{1}{2}\chi^{\top(1)}_{lm,n}\chi^{\|(1)\prime,lmn}-\frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1)\prime,lm}-\frac{1}{2}\chi^{\|(1)\prime,lm}\nabla^2\chi^{\top(1)}_{lm},\end{aligned} \tag{A7}$$

and

$$\begin{aligned}(M_{s(t)i}-\partial_i\nabla^{-2}M^{,l}_{s(t)l})={}&-6\left(\frac{a'}{a}\right)^2(1+c_s^2)\chi^{\top(1)}_{il}v^{\|(1),l}+\phi^{(1),l}\chi^{\top(1)\prime}_{il}-2\phi^{(1)\prime,l}\chi^{\top(1)}_{il}-\frac{1}{2}\chi^{\top(1)\prime}_{lm,i}\chi^{\|(1),lm}\\&-\frac{1}{2}\chi^{\top(1)}_{lm,i}\chi^{\|(1)\prime,lm}+\chi^{\top(1)\prime}_{il,m}\chi^{\|(1),lm}-\frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\chi^{\|(1)\prime,l}+\frac{2}{3}\chi^{\top(1)\prime}_{il}\nabla^2\chi^{\|(1),l}\\&+\partial_i\nabla^{-2}\bigg[6\left(\frac{a'}{a}\right)^2(1+c_s^2)\chi^{\top(1)}_{lm}v^{\|(1),lm}-\phi^{(1),lm}\chi^{\top(1)\prime}_{lm}+2\phi^{(1)\prime,lm}\chi^{\top(1)}_{lm}\\&-\frac{1}{2}\chi^{\top(1)\prime}_{lm,n}\chi^{\|(1),lmn}-\frac{2}{3}\chi^{\top(1)\prime}_{lm}\nabla^2\chi^{\|(1),lm}+\frac{1}{2}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)\prime}_{lm}\\&+\frac{1}{2}\chi^{\top(1)}_{lm,n}\chi^{\|(1)\prime,lmn}+\frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1)\prime,lm}+\frac{1}{2}\chi^{\|(1)\prime,lm}\nabla^2\chi^{\top(1)}_{lm}\bigg].\end{aligned} \tag{A8}$$

The (ij) component of the second-order perturbed Einstein equation, i.e., the second-order evolution equation, is

$$\begin{aligned}&2\phi^{(2)\prime\prime}_{s(t)}\delta_{ij}+4\frac{a'}{a}\phi^{(2)\prime}_{s(t)}\delta_{ij}+\phi^{(2)}_{s(t),ij}-\nabla^2\phi^{(2)}_{s(t)}\delta_{ij}+\left[4\frac{a''}{a}+6\left(c_s^2-\frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi^{(2)}_{s(t)}\delta_{ij}\\&+\frac{1}{2}D_{ij}\chi^{\|(2)\prime\prime}_{s(t)}+\frac{a'}{a}D_{ij}\chi^{\|(2)\prime}_{s(t)}+\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]D_{ij}\chi^{\|(2)}_{s(t)}\\&+\frac{1}{6}\nabla^2D_{ij}\chi^{\|(2)}_{s(t)}-\frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(2)}_{s(t)}+\frac{1}{2}\chi^{\perp(2)\prime\prime}_{s(t)ij}+\frac{a'}{a}\chi^{\perp(2)\prime}_{s(t)ij}+\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi^{\perp(2)}_{s(t)ij}\\&+\frac{1}{2}\chi^{\top(2)\prime\prime}_{s(t)ij}+\frac{a'}{a}\chi^{\top(2)\prime}_{s(t)ij}+\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi^{\top(2)}_{s(t)ij}-\frac{1}{2}\nabla^2\chi^{\top(2)}_{s(t)ij}\\&=3c_N^2\left(\frac{a'}{a}\right)^2\delta^{(2)}_{s(t)}\delta_{ij}+S_{s(t)ij},\end{aligned} \tag{A9}$$

where

$$
\begin{aligned}
S_{s(t)ij} \equiv\; & 6c_L^2\left(\frac{a'}{a}\right)^2\chi_{ij}^{\top(1)}\delta^{(1)} - 6\phi^{(1)''}\chi_{ij}^{\top(1)} - 12\frac{a'}{a}\phi^{(1)'}\chi_{ij}^{\top(1)} + 4\chi_{ij}^{\top(1)}\nabla^2\phi^{(1)} - \phi^{(1)'}\chi_{ij}^{\top(1)'} \\
& - \phi^{(1),l}\chi_{lj,i}^{\top(1)} - \phi^{(1),l}\chi_{li,j}^{\top(1)} + 3\phi^{(1),l}\chi_{ij,l}^{\top(1)} + 2\phi^{(1)}\nabla^2\chi_{ij}^{\top(1)} - 2\phi_{,i}^{(1),l}\chi_{lj}^{\top(1)} \\
& - 2\phi_{,j}^{(1),l}\chi_{li}^{\top(1)} - \chi_{lm}^{\top(1)''}\chi^{\|(1),lm}\delta_{ij} - \chi_{lm}^{\top(1)}\chi^{\|(1)'',lm}\delta_{ij} - \frac{2a'}{a}\chi_{lm}^{\top(1)'}\chi^{\|(1),lm}\delta_{ij} \\
& - \frac{2a'}{a}\chi_{lm}^{\top(1)}\chi^{\|(1)',lm}\delta_{ij} + \chi_{li}^{\top(1)'}\chi_{,j}^{\|(1)',l} + \chi_{lj}^{\top(1)'}\chi_{,i}^{\|(1)',l} - \frac{3}{2}\chi_{lm}^{\top(1)'}\chi^{\|(1)',lm}\delta_{ij} \\
& - \frac{1}{3}\chi_{li}^{\top(1)}\nabla^2\chi_{,j}^{\|(1),l} - \frac{1}{3}\chi_{lj}^{\top(1)}\nabla^2\chi_{,i}^{\|(1),l} - \chi_{lm,ij}^{\top(1)}\chi^{\|(1),lm} + \chi^{\|(1),lm}\nabla^2\chi_{lm}^{\top(1)}\delta_{ij} \\
& - \frac{1}{2}\chi_{lm,j}^{\top(1)}\chi_{,i}^{\|(1),lm} - \frac{1}{2}\chi_{lm,i}^{\top(1)}\chi_{,j}^{\|(1),lm} + \frac{1}{2}\chi^{\|(1),nml}\chi_{nm,l}^{\top(1)}\delta_{ij} - \frac{2}{3}\chi_{ij}^{\top(1)'}\nabla^2\chi^{\|(1)'} \\
& + \frac{2}{3}\chi_{ij}^{\top(1)}\nabla^2\nabla^2\chi^{\|(1)} + \frac{1}{3}\nabla^2\chi_{ij}^{\top(1)}\nabla^2\chi^{\|(1)} + \frac{1}{3}\chi_{lj,i}^{\top(1)}\nabla^2\chi^{\|(1),l} + \frac{1}{3}\chi_{li,j}^{\top(1)}\nabla^2\chi^{\|(1),l} \\
& + \chi_{lj,im}^{\top(1)}\chi^{\|(1),lm} + \chi_{li,jm}^{\top(1)}\chi^{\|(1),lm} - \chi_{ij,lm}^{\top(1)}\chi^{\|(1),lm}.
\end{aligned} \tag{A10}
$$

The trace part of the second-order evolution equation (A9) is

$$
2\phi_{s(t)}^{(2)''} + 4\frac{a'}{a}\phi_{s(t)}^{(2)'} - \frac{2}{3}\nabla^2\phi_{s(t)}^{(2)} + \left[4\frac{a''}{a} + 6\left(c_s^2 - \frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi_{s(t)}^{(2)} - \frac{1}{9}\nabla^2\nabla^2\chi_{s(t)}^{\|(2)} = 3c_N^2\left(\frac{a'}{a}\right)^2\delta_{s(t)}^{(2)} + \frac{1}{3}S_{s(t)l}^{l}, \tag{A11}
$$

where

$$
\begin{aligned}
S_{s(t)l}^{l} =\; & -4\phi^{(1),lm}\chi_{lm}^{\top(1)} - 3\chi_{lm}^{\top(1)''}\chi^{\|(1),lm} - 3\chi_{lm}^{\top(1)}\chi^{\|(1)'',lm} - \frac{6a'}{a}\chi_{lm}^{\top(1)'}\chi^{\|(1),lm} \\
& - \frac{6a'}{a}\chi_{lm}^{\top(1)}\chi^{\|(1)',lm} - \frac{5}{2}\chi_{lm}^{\top(1)'}\chi^{\|(1)',lm} - \frac{2}{3}\chi_{lm}^{\top(1)}\nabla^2\chi^{\|(1),lm} + 2\chi^{\|(1),lm}\nabla^2\chi_{lm}^{\top(1)} + \frac{1}{2}\chi_{lm,n}^{\top(1)}\chi^{\|(1),lmn}.
\end{aligned} \tag{A12}
$$

The traceless part of the second-order evolution equation (A9) is

$$
\begin{aligned}
& D_{ij}\phi_{s(t)}^{(2)} + \frac{1}{2}D_{ij}\chi_{s(t)}^{\|(2)''} + \frac{a'}{a}D_{ij}\chi_{s(t)}^{\|(2)'} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi_{s(t)}^{\|(2)} + \frac{1}{6}\nabla^2 D_{ij}\chi_{s(t)}^{\|(2)} + \frac{1}{2}\chi_{s(t)ij}^{\perp(2)''} + \frac{a'}{a}\chi_{s(t)ij}^{\perp(2)'} \\
& + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{s(t)ij}^{\perp(2)} + \frac{1}{2}\chi_{s(t)ij}^{\top(2)''} + \frac{a'}{a}\chi_{s(t)ij}^{\top(2)'} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{s(t)ij}^{\top(2)} - \frac{1}{2}\nabla^2\chi_{s(t)ij}^{\top(2)} = \bar{S}_{s(t)ij},
\end{aligned} \tag{A13}
$$

where

$$
\begin{aligned}
\bar{S}_{s(t)ij} \equiv\; & S_{s(t)ij} - \frac{1}{3}S_{s(t)k}^{k}\delta_{ij} \\
=\; & 6c_L^2\left(\frac{a'}{a}\right)^2\chi_{ij}^{\top(1)}\delta^{(1)} - 6\phi^{(1)''}\chi_{ij}^{\top(1)} - 12\frac{a'}{a}\phi^{(1)'}\chi_{ij}^{\top(1)} + 4\chi_{ij}^{\top(1)}\nabla^2\phi^{(1)} - \phi^{(1)'}\chi_{ij}^{\top(1)'} - \phi^{(1),l}\chi_{lj,i}^{\top(1)} - \phi^{(1),l}\chi_{li,j}^{\top(1)} \\
& + 3\phi^{(1),l}\chi_{ij,l}^{\top(1)} + 2\phi^{(1)}\nabla^2\chi_{ij}^{\top(1)} - 2\phi_{,i}^{(1),l}\chi_{lj}^{\top(1)} - 2\phi_{,j}^{(1),l}\chi_{li}^{\top(1)} + \frac{4}{3}\phi^{(1),lm}\chi_{lm}^{\top(1)}\delta_{ij} + \chi_{li}^{\top(1)'}\chi_{,j}^{\|(1)',l} + \chi_{lj}^{\top(1)'}\chi_{,i}^{\|(1)',l} \\
& - \frac{2}{3}\chi_{lm}^{\top(1)'}\chi^{\|(1)',lm}\delta_{ij} - \frac{1}{3}\chi_{li}^{\top(1)}\nabla^2\chi_{,j}^{\|(1),l} - \frac{1}{3}\chi_{lj}^{\top(1)}\nabla^2\chi_{,i}^{\|(1),l} + \frac{2}{9}\chi_{lm}^{\top(1)}\nabla^2\chi^{\|(1),lm}\delta_{ij} - \chi_{lm,ij}^{\top(1)}\chi^{\|(1),lm} \\
& + \frac{1}{3}\chi^{\|(1),lm}\nabla^2\chi_{lm}^{\top(1)}\delta_{ij} - \frac{1}{2}\chi_{lm,j}^{\top(1)}\chi_{,i}^{\|(1),lm} - \frac{1}{2}\chi_{lm,i}^{\top(1)}\chi_{,j}^{\|(1),lm} + \frac{1}{3}\chi_{lm,n}^{\top(1)}\chi^{\|(1),lmn}\delta_{ij} - \frac{2}{3}\chi_{ij}^{\top(1)'}\nabla^2\chi^{\|(1)'} \\
& + \frac{2}{3}\chi_{ij}^{\top(1)}\nabla^2\nabla^2\chi^{\|(1)} + \frac{1}{3}\nabla^2\chi_{ij}^{\top(1)}\nabla^2\chi^{\|(1)} + \frac{1}{3}\chi_{lj,i}^{\top(1)}\nabla^2\chi^{\|(1),l} + \frac{1}{3}\chi_{li,j}^{\top(1)}\nabla^2\chi^{\|(1),l} + \chi_{lj,im}^{\top(1)}\chi^{\|(1),lm} + \chi_{li,jm}^{\top(1)}\chi^{\|(1),lm} \\
& - \chi_{ij,lm}^{\top(1)}\chi^{\|(1),lm}.
\end{aligned} \tag{A14}
$$

The scalar part of the traceless part of the second-order evolution equation (A13) is

$$\phi^{(2)}_{s(t)} + \frac{1}{2}\chi^{\|(2)''}_{s(t)} + \frac{a'}{a}\chi^{\|(2)'}_{s(t)} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\|(2)}_{s(t)} + \frac{1}{6}\nabla^2\chi^{\|(2)}_{s(t)} = \frac{3}{2}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{s(t)kl}, \tag{A15}$$

where

$$\begin{aligned}\bar{S}^{,lm}_{s(t)lm} =& -\frac{2}{3}\nabla^2\nabla^2[\chi^{\|(1),lm}\chi^{\top(1)}_{lm}] + \nabla^2\left[\frac{4}{3}\phi^{(1),lm}\chi^{\top(1)}_{lm} + \frac{1}{3}\chi^{\top(1)'}_{lm}\chi^{\|(1)',lm} + \frac{8}{9}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm} + \frac{7}{6}\chi^{\top(1)}_{lm,n}\chi^{\|(1),lmn}\right]\\ &+ 6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lm}\delta^{(1),lm} - 6\phi^{(1)'',lm}\chi^{\top(1)}_{lm} - 12\frac{a'}{a}\phi^{(1)',lm}\chi^{\top(1)}_{lm} - \phi^{(1)',lm}\chi^{\top(1)'}_{lm} - 3\phi^{(1),lmn}\chi^{\top(1)}_{lm,n}\\ &-\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)'}_{lm} + \frac{1}{3}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1)',lm} - \frac{1}{2}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\|(1),lmn} + \nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm} + \frac{3}{2}\chi^{\|(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}.\end{aligned} \tag{A16}$$

The vector part of the traceless part of the second-order evolution equation (A13) is

$$\frac{1}{2}\chi^{\perp(2)''}_{s(t)ij} + \frac{a'}{a}\chi^{\perp(2)'}_{s(t)ij} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\perp(2)}_{s(t)ij} = \nabla^{-2}\bar{S}^{,k}_{s(t)kj,i} + \nabla^{-2}\bar{S}^{,k}_{s(t)ki,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{s(t)kl,ij}, \tag{A17}$$

where the rhs of the above is

$$\begin{aligned}&\nabla^{-2}\bar{S}^{,l}_{s(t)lj,i} + \nabla^{-2}\bar{S}^{,l}_{s(t)li,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{s(t)lm,ij}\\ &= \frac{2}{3}\partial_i\partial_j[\chi^{\|(1),lm}\chi^{\top(1)}_{lm}] + \partial_i\partial_j\nabla^{-2}\left[-\phi^{(1),lm}\chi^{\top(1)}_{lm} - \frac{5}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}\right.\\ &\quad\left. - \frac{1}{6}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{4}{3}\chi^{\top(1)}_{lm,n}\chi^{\|(1),lmn}\right] + \partial_i\nabla^{-2}\left[6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lj}\delta^{(1),l} - 6\phi^{(1)'',l}\chi^{\top(1)}_{lj}\right.\\ &\quad - 12\frac{a'}{a}\phi^{(1)',l}\chi^{\top(1)}_{lj} - \phi^{(1)',l}\chi^{\top(1)'}_{lj} + 2\chi^{\top(1)}_{lj}\nabla^2\phi^{(1),l} + \phi^{(1),l}\nabla^2\chi^{\top(1)}_{lj} - \phi^{(1),lm}_{,j}\chi^{\top(1)}_{lm}\\ &\quad + \chi^{\top(1)'}_{lj,m}\chi^{\|(1)',lm} - \chi^{\top(1)'}_{lm,j}\chi^{\|(1)',lm} + \frac{1}{3}\chi^{\top(1)'}_{lj}\nabla^2\chi^{\|(1)',l} - \frac{1}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}_{,j}\\ &\quad\left. + \frac{1}{3}\chi^{\top(1)}_{lj}\nabla^2\nabla^2\chi^{\|(1),l} + \frac{2}{3}\nabla^2\chi^{\top(1)}_{lj}\nabla^2\chi^{\|(1),l} + \chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lj,m} - \frac{1}{2}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm,j}\right]\\ &\quad + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[-6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lm}\delta^{(1),lm} + 6\phi^{(1)'',lm}\chi^{\top(1)}_{lm} + 12\frac{a'}{a}\phi^{(1)',lm}\chi^{\top(1)}_{lm}\right.\\ &\quad + \phi^{(1)',lm}\chi^{\top(1)'}_{lm} + 3\phi^{(1),lmn}\chi^{\top(1)}_{lm,n} + \chi^{\|(1)',lm}\nabla^2\chi^{\top(1)'}_{lm} - \frac{1}{3}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1)',lm}\\ &\quad\left. + \frac{1}{2}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\|(1),lmn} - \nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm} - \frac{3}{2}\chi^{\|(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}\right]\\ &\quad + (i\leftrightarrow j).\end{aligned} \tag{A18}$$

The tensor part (GW) of the second-order evolution equation (A13) is

$$\begin{aligned}&\frac{1}{2}\chi^{\top(2)''}_{s(t)ij} + \frac{a'}{a}\chi^{\top(2)'}_{s(t)ij} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\top(2)}_{s(t)ij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{s(t)ij}\\ &= \bar{S}_{s(t)ij} - \frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{s(t)kl} - \nabla^{-2}\bar{S}^{,k}_{s(t)kj,i} - \nabla^{-2}\bar{S}^{,k}_{s(t)ki,j} + 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{s(t)kl,ij},\end{aligned} \tag{A19}$$

where the rhs of the above is

$$\begin{aligned}
&\bar{S}_{s(t)ij}-\frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{s(t)lm}-\nabla^{-2}\bar{S}^{,l}_{s(t)lj,i}-\nabla^{-2}\bar{S}^{,l}_{s(t)li,j}+2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{s(t)lm,ij}\\
&=\bigg[6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{ij}\delta^{(1)}-6\phi^{(1)''}\chi^{\top(1)}_{ij}-12\frac{a'}{a}\phi^{(1)'}\chi^{\top(1)}_{ij}+4\chi^{\top(1)}_{ij}\nabla^2\phi^{(1)}-\phi^{(1)'}\chi^{\top(1)'}_{ij}\\
&\quad-\phi^{(1),l}\chi^{\top(1)}_{lj,i}-\phi^{(1),l}\chi^{\top(1)}_{li,j}+3\phi^{(1),l}\chi^{\top(1)}_{ij,l}+2\phi^{(1)}\nabla^2\chi^{\top(1)}_{ij}-2\phi^{(1),l}_{,i}\chi^{\top(1)}_{lj}-2\phi^{(1),l}_{,j}\chi^{\top(1)}_{li}\\
&\quad+2\phi^{(1),lm}\chi^{\top(1)}_{lm}\delta_{ij}+\chi^{\top(1)'}_{li}\chi^{\|(1)',l}_{,j}+\chi^{\top(1)'}_{lj}\chi^{\|(1)',l}_{,i}-\frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\|(1)',lm}\delta_{ij}\\
&\quad-\frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\chi^{\|(1),l}_{,j}-\frac{1}{3}\chi^{\top(1)}_{lj}\nabla^2\chi^{\|(1),l}_{,i}+\frac{1}{3}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}\delta_{ij}-\chi^{\top(1)}_{lm,ij}\chi^{\|(1),lm}\\
&\quad-\frac{1}{2}\chi^{\top(1)}_{lm,j}\chi^{\|(1),lm}_{,i}-\frac{1}{2}\chi^{\top(1)}_{lm,i}\chi^{\|(1),lm}_{,j}+\frac{1}{4}\chi^{\top(1)}_{lm,n}\chi^{\|(1),lmn}\delta_{ij}-\frac{2}{3}\chi^{\top(1)'}_{ij}\nabla^2\chi^{\|(1)'}\\
&\quad+\frac{2}{3}\chi^{\top(1)}_{ij}\nabla^2\nabla^2\chi^{\|(1)}+\frac{1}{3}\nabla^2\chi^{\top(1)}_{ij}\nabla^2\chi^{\|(1)}+\frac{1}{3}\chi^{\top(1)}_{lj,i}\nabla^2\chi^{\|(1),l}+\frac{1}{3}\chi^{\top(1)}_{li,j}\nabla^2\chi^{\|(1),l}\\
&\quad+\chi^{\top(1)}_{lj,im}\chi^{\|(1),lm}+\chi^{\top(1)}_{li,jm}\chi^{\|(1),lm}-\chi^{\top(1)}_{ij,lm}\chi^{\|(1),lm}\bigg]+\delta_{ij}\nabla^{-2}\bigg[3c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lm}\delta^{(1),lm}\\
&\quad-3\phi^{(1)'',lm}\chi^{\top(1)}_{lm}-6\frac{a'}{a}\phi^{(1)',lm}\chi^{\top(1)}_{lm}-\frac{1}{2}\phi^{(1)',lm}\chi^{\top(1)'}_{lm}-\frac{3}{2}\phi^{(1),lmn}\chi^{\top(1)}_{lm,n}\\
&\quad-\frac{1}{2}\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)'}_{lm}+\frac{1}{6}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1)',lm}-\frac{1}{4}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\|(1),lmn}+\frac{1}{2}\nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}\\
&\quad+\frac{3}{4}\chi^{\|(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}\bigg]+\partial_i\partial_j\nabla^{-2}\bigg[-\frac{1}{2}\chi^{\top(1)'}_{lm}\chi^{\|(1)',lm}+\frac{1}{4}\chi^{\top(1)}_{lm,n}\chi^{\|(1),lmn}\bigg]\\
&\quad+\partial_i\nabla^{-2}\bigg[-6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lj}\delta^{(1),l}+6\phi^{(1)'',l}\chi^{\top(1)}_{lj}+12\frac{a'}{a}\phi^{(1)',l}\chi^{\top(1)}_{lj}+\phi^{(1)',l}\chi^{\top(1)'}_{lj}\\
&\quad-2\chi^{\top(1)}_{lj}\nabla^2\phi^{(1),l}-\phi^{(1),l}\nabla^2\chi^{\top(1)}_{lj}+\phi^{(1),lm}_{,j}\chi^{\top(1)}_{lm}-\chi^{\top(1)'}_{lj,m}\chi^{\|(1)',lm}+\chi^{\top(1)'}_{lm,j}\chi^{\|(1)',lm}\\
&\quad-\frac{1}{3}\chi^{\top(1)'}_{lj}\nabla^2\chi^{\|(1)',l}+\frac{1}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}_{,j}-\frac{1}{3}\chi^{\top(1)}_{lj}\nabla^2\nabla^2\chi^{\|(1),l}-\frac{2}{3}\nabla^2\chi^{\top(1)}_{lj}\nabla^2\chi^{\|(1),l}\\
&\quad-\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lj,m}+\frac{1}{2}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm,j}\bigg]+\partial_j\nabla^{-2}\bigg[-6c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{li}\delta^{(1),l}\\
&\quad+6\phi^{(1)'',l}\chi^{\top(1)}_{li}+12\frac{a'}{a}\phi^{(1)',l}\chi^{\top(1)}_{li}+\phi^{(1)',l}\chi^{\top(1)'}_{li}-2\chi^{\top(1)}_{li}\nabla^2\phi^{(1),l}-\phi^{(1),l}\nabla^2\chi^{\top(1)}_{li}\\
&\quad+\phi^{(1),lm}_{,i}\chi^{\top(1)}_{lm}-\chi^{\top(1)'}_{li,m}\chi^{\|(1)',lm}+\chi^{\top(1)'}_{lm,i}\chi^{\|(1)',lm}-\frac{1}{3}\chi^{\top(1)'}_{li}\nabla^2\chi^{\|(1)',l}+\frac{1}{6}\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}_{,i}\\
&\quad-\frac{1}{3}\chi^{\top(1)}_{li}\nabla^2\nabla^2\chi^{\|(1),l}-\frac{2}{3}\nabla^2\chi^{\top(1)}_{li}\nabla^2\chi^{\|(1),l}-\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{li,m}+\frac{1}{2}\chi^{\|(1),lm}\nabla^2\chi^{\top(1)}_{lm,i}\bigg]\\
&\quad+\partial_i\partial_j\nabla^{-2}\nabla^{-2}\bigg[3c_L^2\left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{lm}\delta^{(1),lm}-3\phi^{(1)'',lm}\chi^{\top(1)}_{lm}-6\frac{a'}{a}\phi^{(1)',lm}\chi^{\top(1)}_{lm}\\
&\quad-\frac{1}{2}\phi^{(1)',lm}\chi^{\top(1)'}_{lm}-\frac{3}{2}\phi^{(1),lmn}\chi^{\top(1)}_{lm,n}-\frac{1}{2}\chi^{\|(1)',lm}\nabla^2\chi^{\top(1)'}_{lm}+\frac{1}{6}\chi^{\top(1)'}_{lm}\nabla^2\chi^{\|(1)',lm}\\
&\quad-\frac{1}{4}\chi^{\top(1)}_{lm,n}\nabla^2\chi^{\|(1),lmn}+\frac{1}{2}\nabla^2\chi^{\top(1)}_{lm}\nabla^2\chi^{\|(1),lm}+\frac{3}{4}\chi^{\|(1),lmn}\nabla^2\chi^{\top(1)}_{lm,n}\bigg].
\end{aligned}\tag{A20}$$

Next, we present the second-order perturbed Einstein equation with tensor-tensor couplings. The (00) component of the second-order perturbed Einstein equation, i.e., the second-order energy constraint, is

$$-\frac{6a'}{a}\phi^{(2)'}_T+2\nabla^2\phi^{(2)}_T+\frac{1}{3}\nabla^2\nabla^2\chi^{\|(2)}_T=3\left(\frac{a'}{a}\right)^2\delta^{(2)}_T+E_T,\tag{A21}$$

where

$$E_T \equiv \frac{1}{4}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm} + \frac{2a'}{a}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm} - \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm} - \frac{3}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n} + \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}. \tag{A22}$$

The $(0i)$ component of the second-order perturbed Einstein equation, i.e., the second-order momentum constraint, is

$$2\phi^{(2)'}_{T,i} + \frac{1}{2}D_{ij}\chi^{\|(2)',j}_T + \frac{1}{2}\chi^{\perp(2)',j}_{Tij} = -3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{(2)}_{Ti} + M_{Ti}, \tag{A23}$$

where M_{Ti} is the same as (3.38). The longitudinal part of the momentum constraint (A23) is

$$2\phi^{(2)'}_T + \frac{1}{3}\nabla^2\chi^{\|(2)'}_T = -3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(2)}_T + \nabla^{-2}M^{,l}_{Tl}, \tag{A24}$$

and the transverse part is

$$\frac{1}{2}\chi^{\perp(2)',j}_{Tij} = -3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\perp(2)}_{Ti} + (M_{Ti} - \partial_i\nabla^{-2}M^{,l}_{Tl}), \tag{A25}$$

where $\nabla^{-2}M^{,l}_{Tl}$ and $(M_{Ti} - \partial_i\nabla^{-2}M^{,l}_{Tl})$ are the same as (3.40) and (3.42).

The (ij) component of second-order perturbed Einstein equation, i.e., the second-order evolution equation, is

$$\begin{aligned}
&2\phi^{(2)''}_T\delta_{ij} + 4\frac{a'}{a}\phi^{(2)'}_T\delta_{ij} + \phi^{(2)}_{T,ij} - \nabla^2\phi^{(2)}_T\delta_{ij} + \left[4\frac{a''}{a} + 6\left(c_s^2 - \frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi^{(2)}_T\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\|(2)''}_T + \frac{a'}{a}D_{ij}\chi^{\|(2)'}_T \\
&\quad + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi^{\|(2)}_T + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(2)}_T - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(2)}_T + \frac{1}{2}\chi^{\perp(2)''}_{Tij} + \frac{a'}{a}\chi^{\perp(2)'}_{Tij} \\
&\quad + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\perp(2)}_{Tij} + \frac{1}{2}\chi^{\top(2)''}_{Tij} + \frac{a'}{a}\chi^{\top(2)'}_{Tij} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\top(2)}_{Tij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Tij} \\
&= 3c_N^2\left(\frac{a'}{a}\right)^2\delta^{(2)}_T\delta_{ij} + S_{Tij},
\end{aligned} \tag{A26}$$

where

$$\begin{aligned}
S_{Tij} \equiv{}& -2\frac{a'}{a}\chi^{\top(1)lm}\chi^{\top(1)'}_{lm}\delta_{ij} - \chi^{\top(1)lm}\chi^{\top(1)}_{lm,ij} + \chi^{\top(1)lm}\nabla^2\chi^{\top(1)}_{lm}\delta_{ij} - \frac{1}{2}\chi^{\top(1)lm}_{,i}\chi^{\top(1)}_{lm,j} - \chi^{\top(1)}_{li,m}\chi^{\top(1)l,m}_j + \frac{3}{4}\chi^{\top(1)lm,n}\chi^{\top(1)}_{lm,n}\delta_{ij} \\
&+ \chi^{\top(1)}_{li,m}\chi^{\top(1)m,l}_j - \frac{1}{2}\chi^{\top(1)lm,n}\chi^{\top(1)}_{ln,m}\delta_{ij} + \chi^{\top(1)'l}_i\chi^{\top(1)'}_{lj} - \frac{3}{4}\chi^{\top(1)'lm}\chi^{\top(1)'}_{lm}\delta_{ij} - \chi^{\top(1)lm}\chi^{\top(1)''}_{lm}\delta_{ij} + \chi^{\top(1)lm}\chi^{\top(1)}_{lj,im} \\
&+ \chi^{\top(1)lm}\chi^{\top(1)}_{li,jm} - \chi^{\top(1)lm}\chi^{\top(1)}_{ij,lm}.
\end{aligned} \tag{A27}$$

The trace part of the second-order evolution equation (A26) is

$$2\phi^{(2)''}_T + 4\frac{a'}{a}\phi^{(2)'}_T - \frac{2}{3}\nabla^2\phi^{(2)}_T + \left[4\frac{a''}{a} + 6\left(c_s^2 - \frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi^{(2)}_T - \frac{1}{9}\nabla^2\nabla^2\chi^{\|(2)}_T = 3c_N^2\left(\frac{a'}{a}\right)^2\delta^{(2)}_T + \frac{1}{3}S^l_{Tl}, \tag{A28}$$

where S^l_{Tl} is the same as (3.46). The traceless part of the second-order evolution equation (A26) is

$$\begin{aligned}
&D_{ij}\phi^{(2)}_T + \frac{1}{2}D_{ij}\chi^{\|(2)''}_T + \frac{a'}{a}D_{ij}\chi^{\|(2)'}_T + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi^{\|(2)}_T + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(2)}_T \\
&\quad + \frac{1}{2}\chi^{\perp(2)''}_{Tij} + \frac{a'}{a}\chi^{\perp(2)'}_{Tij} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\perp(2)}_{Tij} \\
&\quad + \frac{1}{2}\chi^{\top(2)''}_{Tij} + \frac{a'}{a}\chi^{\top(2)'}_{Tij} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\top(2)}_{Tij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Tij} = \bar{S}_{Tij},
\end{aligned} \tag{A29}$$

where $\bar{S}_{Tij}$ is the same as (3.48). The scalar part of the traceless part of second-order evolution equation (A29) is

$$\chi_T^{\parallel(2)''} + \frac{2a'}{a}\chi_T^{\parallel(2)'} + 2\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_T^{\parallel(2)} + \frac{1}{3}\nabla^2\chi_T^{\parallel(2)} + 2\phi_T^{(2)} = 3\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm}, \tag{A30}$$

where $\bar{S}^{,lm}_{Tlm}$ is the same as (3.50). The vector part of the traceless part (A29) is

$$\frac{1}{2}\chi_{Tij}^{\perp(2)''} + \frac{a'}{a}\chi_{Tij}^{\perp(2)'} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Tij}^{\perp(2)} = \nabla^{-2}\bar{S}^{,k}_{Tkj,i} + \nabla^{-2}\bar{S}^{,k}_{Tki,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Tkl,ij}, \tag{A31}$$

where $\nabla^{-2}\bar{S}^{,l}_{Tlj,i} + \nabla^{-2}\bar{S}^{,l}_{Tli,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,lm}_{Tlm,ij}$ is the same as (3.53). The tensor part of (A29) is

$$\begin{aligned}&\frac{1}{2}\chi_{Tij}^{\top(2)''} + \frac{a'}{a}\chi_{Tij}^{\top(2)'} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Tij}^{\top(2)} - \frac{1}{2}\nabla^2\chi_{Tij}^{\top(2)}\\ &\quad = \bar{S}_{Tij} - \frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Tkl} - \nabla^{-2}\bar{S}^{,k}_{Tkj,i} - \nabla^{-2}\bar{S}^{,k}_{Tki,j} + 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Tkl,ij},\end{aligned} \tag{A32}$$

where the rhs of the above is the same as (3.55).

Finally, we present the second-order equations of the covariant conservation of the energy-momentum tensor, which are also needed for solving the second-order perturbations. The conservation equations with all the couplings for a general RW spacetime are given in (2.17) and (2.18). Keeping only the scalar-tensor terms, we have

$$\begin{aligned}&\delta_{s(t)}^{(2)'} + 2a''(a')^{-1}\delta_{s(t)}^{(2)} + (-1+3c_N^2)a'a^{-1}\delta_{s(t)}^{(2)} + (1+c_s^2)\nabla^2 v_{s(t)}^{\parallel(2)} - 3(1+c_s^2)\phi_{s(t)}^{(2)'}\\ &\quad - (1+c_s^2)\chi_{lm}^{\top(1)'}\chi^{\parallel(1),lm} - (1+c_s^2)\chi_{lm}^{\top(1)}\chi^{\parallel(1)',lm} = 0,\end{aligned} \tag{A33}$$

$$\begin{aligned}&c_N^2\delta_{s(t),i}^{(2)} + 2(1+c_s^2)a''(a')^{-1}v_{s(t),i}^{\parallel(2)} + (1+c_s^2)v_{s(t),i}^{\parallel(2)'} + 2(1+c_s^2)a''(a')^{-1}v_{s(t)i}^{\perp(2)}\\ &\quad + (1+c_s^2)v_{s(t)i}^{\perp(2)'} - 2c_L^2\delta^{(1),l}\chi_{li}^{\top(1)} + 2(1+c_s^2)v^{\parallel(1),l}\chi_{li}^{\top(1)'} = 0.\end{aligned} \tag{A34}$$

Keeping only the tensor-tensor coupling terms, we have

$$\delta_T^{(2)'} + 2a''(a')^{-1}\delta_T^{(2)} + (-1+3c_N^2)a'a^{-1}\delta_T^{(2)} + (1+c_s^2)\nabla^2 v_T^{\parallel(2)} - 3(1+c_s^2)\phi_T^{(2)'} - (1+c_s^2)\chi_{lm}^{\top(1)'}\chi^{\top(1)lm} = 0, \tag{A35}$$

$$c_N^2\delta_{T,i}^{(2)} + 2(1+c_s^2)a''(a')^{-1}v_{T,i}^{\parallel(2)} + (1+c_s^2)v_{T,i}^{\parallel(2)'} + 2(1+c_s^2)a''(a')^{-1}v_{Ti}^{\perp(2)} + (1+c_s^2)v_{Ti}^{\perp(2)'} = 0. \tag{A36}$$

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