Comment on "Hadamard states for a scalar field in anti–de Sitter spacetime with arbitrary boundary conditions"

J. P. M. Pitelli*
Departamento de Matemática Aplicada, Universidade Estadual de Campinas, 13083-859, Campinas, São Paulo, Brazil

(Received 22 April 2019; published 28 May 2019)

In a recent paper [C. Dappiaggi and H. C. R. Ferreira, Phys. Rev. D **94**, 125016 (2016)], the authors argued that the singularities of the two-point functions on the Poincaré domain of the n-dimensional anti–de Sitter spacetime (PAdS_n) have the Hadamard form, regardless of which (Robin) boundary condition is chosen at the conformal boundary. However, the argument used to prove this statement was based on an incorrect expression for the two-point function $G^+(x,x')$, which was obtained by demanding AdS invariance for the vacuum state. In this comment I show that their argument works only for Dirichlet and Neumann boundary conditions and that the full AdS symmetry cannot be respected by nontrivial Robin conditions (i.e., those which are neither Dirichlet nor Neumann). By studying the conformal scalar field on PAdS_2, I find the correct expression for $G^+(x,x')$ and show that, notwithstanding this problem, it still has the Hadamard form.

DOI: 10.1103/PhysRevD.99.108701

In a seminal paper [1], Allen and Jacobson presented a method of finding two-point functions in maximally symmetric spacetimes. Their method was based on the assumption that the state $|\psi\rangle$ is maximally symmetric. Within this assumption, the two-point functions $G(x,x')$ constructed using $|\psi\rangle$ depend on x and x' only upon the geodesic distance $\sigma(x,x')$. The wave equation $(\Box - m^2 - \xi R)\varphi(x) = 0$ then implies that $G(\sigma)$ satisfies the differential equation (my notation agrees with that of Ref. [2])

$$u(1-u)\frac{d^2G(\sigma)}{du^2} + [c-(a+b+1)u]\frac{dG(\sigma)}{du} - abG(\sigma) = 0, \tag{1}$$

where u is related to the geodesic distance σ by $u = \cosh^2(\frac{\sqrt{2\sigma}}{2})$ [for anti–de Sitter (AdS) spacetime] and

$$\begin{aligned} a &= \frac{n-1}{2} - \nu, \\ b &= \frac{n-1}{2} + \nu, \\ c &= n/2. \end{aligned} \tag{2}$$

In Eq. (2), n is the spacetime dimension and $\nu = \frac{1}{2}\sqrt{1+4\tilde{m}^2}$, with $\tilde{m}^2 \equiv m^2 + (\xi - \frac{n-2}{4(n-1)})R$, where m represents the mass parameter and ξ is the scalar-curvature coupling constant. A convenient pair of linear independent solutions of Eq. (1) is given by

$$\begin{aligned} &(1/u)^a {}_2F_1(a, a-c+1; a-b+1; 1/u), \\ &(1/u)^b {}_2F_1(b, b-c+1; b-a+1; 1/u), \end{aligned} \tag{3}$$

with ${}_2F_1$ being the Gauss' hypergeometric function. Clearly, any linear combination of the solutions in Eq. (3) will be AdS invariant. In Ref. [2], it was argued that the Green's function $G^+(x,x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$, constructed from a field φ satisfying a general Robin boundary condition, can be represented by such a linear combination. My claim in this comment is that this assumption is in general incorrect, being true only for Dirichlet and Neumann boundary conditions. In this way, except for these two particular boundary conditions, $G(x,x')$ will not be maximally symmetric.

To illustrate my previous observations, let me focus on conformal fields on the Poincaré domain of the n-dimensional anti–de Sitter spacetime (PAdS_2), since a closed form for the two-point function can be easily derived in this case. The metric on PAdS_2 has the form

$$ds^2 = \frac{-dt^2 + dz^2}{z^2}, \qquad z > 0, \tag{4}$$

with the conformal boundary located at $z = 0$. The geodesic distance satisfies the relation $u = 1 + \frac{\sigma_M}{2zz'}$, where

$$\sigma_M = \frac{1}{2}[-(t-t')^2 + (z-z')^2]. \tag{5}$$

For conformal fields in two dimensions we must have $\tilde{m}^2 = 0$, so that the general solution for $G(x,x')$ is given by

$$\begin{aligned} G(x,x') = {} & A_2F_1(0,0;0;1/u) \\ & + B(1/u)_2F_1(1,1;2;1/u). \end{aligned} \tag{6}$$

*Also at The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois.
pitelli@ime.unicamp.br

At the conformal boundary we have $u \to \infty$. Therefore, in this limit we have

$$G(x,x') \sim A + B(1/u) = A + \frac{4Bzz'}{-(t-t')^2 + (z+z')^2}. \quad (7)$$

Notice that if $A = 0$, then $G(x,x')$ satisfies the Dirichlet boundary condition

$$G(t, z=0; t', z') = G(t, z; t', z'=0) = 0, \quad (8)$$

while if $B = 0$, $G(x,x')$ satisfies the Neumann boundary condition

$$\frac{\partial G(t,z=0;t',z')}{\partial z} = \frac{\partial G(t,z;t',z'=0)}{\partial z'} = 0. \quad (9)$$

If we try to impose that $G(x,x')$ satisfy the Robin boundary condition at the conformal boundary, i.e., that

$$G(t,z=0;t'z) - \beta \frac{\partial G(t,z=0;t',z')}{\partial z} = 0,$$
$$G(t,z;t',z'=0) - \beta \frac{\partial G(t,z;t',z'=0)}{\partial z} = 0, \quad (10)$$

then A and B would satisfy

$$A - \frac{B\beta z'}{-(t-t')^2 + z'^2} = A - \frac{B\beta z}{-(t-t')^2 + z^2} = 0. \quad (11)$$

This does not make sense since they are constant. We hence conclude that for the general Robin boundary condition ($\beta \neq 0$), the two-point function G does not satisfy Eq. (1). Therefore the vacuum $|0\rangle$ cannot be maximally symmetric. Notice that a length scale is introduced in the semiclassical theory when β is finite and nonzero. This extra length scale is responsible for the break of AdS invariance. Clearly, this is not the case for Dirichlet and Neumann boundary conditions.

In Ref. [2], it was correctly proved that $G(x,x')$ has the Hadamard form for Dirichlet and Neumann boundary conditions. The argument was then extended to generic Robin boundary conditions by simply taking the linear combination of the fundamental solutions above, as can be seen in Eq. (4.17) in [2]. However, this is not correct as I showed above.[1]

The only thing left to do in the conformal case is to find the correct expression for $G(x,x')$ in the case $\beta \neq 0$. In order to do so, I use the mode sum method, which is correct with or without additional symmetries.[2] It can be easily checked that the complete set of solutions $\{u_\omega^{(\beta)}(x)\}$ of the wave equation

$$\Box\varphi(x) = -\frac{\partial^2\varphi(x)}{\partial t^2} + \frac{\partial^2\varphi(x)}{\partial z^2} = 0, \quad (12)$$

which satisfies Robin boundary conditions, and orthogonal in the Klein-Gordon inner product, is given by

$$u_\omega^{(\beta)}(t,z) = \frac{1}{\sqrt{\pi\omega}} \frac{\sin\omega z + \beta\omega\cos\omega z}{\sqrt{1+\beta^2\omega^2}} e^{-i\omega t}. \quad (13)$$

As in Ref. [2], I choose to work with the Green's function $G^+(x,x')$. As a sum of modes, it is given by

$$G^+(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle = \int_0^\infty \frac{d\omega}{\pi\omega} \frac{(\sin\omega z + \beta\omega\cos\omega z)(\sin\omega z' + \beta\omega\cos\omega z')}{1+\beta^2\omega^2} \times e^{-i\omega(t-t')-\epsilon\omega}. \quad (14)$$

Notice that the Robin boundary condition for z and z' is trivially satisfied in this case.

The integral (14) can be exactly calculated, and is found to be

$$G^{(+)}(x,x') = \left\{ \frac{1}{2\pi} e^{\frac{-\Delta t+z+z'+i\epsilon}{\beta}} \mathrm{E}_1\left(-\frac{(1-i\omega\beta)(\Delta t - z - z' - i\epsilon)}{\beta}\right) + \frac{1}{2\pi} e^{\frac{\Delta t+z+z'-i\epsilon}{\beta}} \mathrm{E}_1\left(\frac{i(-i+\omega\beta)(\Delta t + z + z' - i\epsilon)}{\beta}\right) + \frac{1}{4\pi}[-\mathrm{E}_1(-i\omega(-\Delta t + z + z' + i\epsilon)) + \mathrm{E}_1(i\omega(\Delta t + z - z' - i\epsilon)) + \mathrm{E}_1(i\omega(\Delta t - z + z' - i\epsilon)) - \mathrm{E}_1(i\omega(\Delta t + z + z' - i\epsilon))] \right\}_0^\infty. \quad (15)$$

[1] Although the two-dimensional case was not considered in [2], I emphasize that Eq. (4.17) in that reference represents the more general AdS invariant solution regardless the spacetime dimension. Moreover, by generalizing my previous argument, it can be easily seen that it is also impossible to fulfil the nontrivial Robin boundary condition with such an invariant expression even for $n \geq 3$. The reason for choosing the simplest case $n = 2$ is that a closed expression for the Green's function can be found, which makes my analysis clearer.

[2] The construction of the Green's function in terms of modes is correct in Ref. [2]. This microlocal analysis is sufficient to ensure that the vacuum state is of the Hadamard form [3]. However, in [2], it was assumed that such construction leads to an AdS invariant two-point function, with closed form given by Eq. (4.17). As I have shown, this is not the case. Nevertheless, I will show precisely how the Hadamard form is preserved even with the breaking of AdS invariance.

In the above expression, $E_1(z)$ is the exponential integral defined by

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt. \qquad (16)$$

By using the asymptotic expansions for E_1 given by [4],

$$\mathrm{E}_1(z) \sim -\gamma - \log z, \qquad |z| \ll 1$$
$$\mathrm{E}_1(z) \sim \frac{e^{-z}}{z}, \qquad |z| \to \infty, \qquad (17)$$

we arrive at

$$G^{(+)}(x,x') = \frac{1}{2\pi} e^{\frac{\Delta t + z + z' - i\epsilon}{\beta}} E_1\left(\frac{\Delta t + z + z' - i\epsilon}{\beta}\right) + \frac{1}{2\pi} e^{\frac{-\Delta t + z + z' + i\epsilon}{\beta}} E_1\left(-\frac{\Delta t - z - z' - i\epsilon}{\beta}\right)$$
$$+ \frac{1}{4\pi}[\log(\Delta t - z - z' - i\epsilon) - \log(\Delta t + z - z' - i\epsilon) - \log(\Delta t - z + z' - i\epsilon) + \log(\Delta t + z + z' - i\epsilon)]. \qquad (18)$$

Notice that the first two terms in Eq. (18) are regular in the limit $\Delta t \to 0$ and $z' \to z$. Moreover, the last term satisfies the wave equation and the Dirichlet boundary condition. Let us concentrate on the second term: a simple calculation shows that

$$G^{(+)}_{\text{Dirichlet}} = \frac{1}{4\pi} \log\left(1 - \frac{1}{\cosh^2(\frac{\sqrt{2\sigma}}{2})}\right), \qquad (19)$$

so that in the limit $\sigma \to 0$ we have

$$G^{(+)}_{\text{Dirichlet}} \sim \frac{1}{2\pi}(\log\sigma - \log 2), \qquad (20)$$

which has the expected Hadamard form.

In summary: although the AdS spacetime is maximally symmetric, the vacuum state does not respect its symmetries, except for fields satisfying Dirichlet or Neumann boundary conditions. In spite of that, the two-point function $G^+(x,x)$ thus has the expected Hadamard form for all Robin boundary conditions. In the above example, this happened because the Green’s function could be separated into one term respecting the Dirichlet boundary condition and one term depending on the boundary condition parameter β with the last term being completely regular in the coincidence limit. For more general situations—possibly nonconformal fields on PAdS_n—we could, in principle, expand the mode sum in terms of powers of the boundary condition parameter β. The zeroth order contribution will satisfy the Dirichlet boundary condition and respect AdS symmetries. Therefore, it will certainly have the required Hadamard form. We then expect that the remaining terms are regular when $\sigma \to 0$. This is the subject of work in progress [5].

I am indebted to Professor R. M. Wald, G. Satishchandran, V. S. Barroso and Professor R. A. Mosna for clarifying several questions during the development of this comment and also thank the Enrico Fermi Institute for the kind hospitality. Finally, I thank Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (Grants No. 2018/01558-9 and No. 2013/09357-9) for financial support.

[1] B. Allen and T. Jacobson, Vector two-point functions in maximally symmetric spaces, Commun. Math. Phys. **103**, 669 (1986).
[2] C. Dappiaggi and H. C. R. Ferreira, Hadamard states for a scalar field in anti–de Sitter spacetime with arbitrary boundary conditions, Phys. Rev. D **94**, 125016 (2016).
[3] H. Sahlmann and R. Verch, Passivity and microlocal spectrum condition, Commun. Math. Phys. **214**, 705 (2000).
[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 10th ed. (U.S. GPO, Washington, DC, 1972).
[5] J. P. M. Pitelli, V. S. Barroso, and R. A. Mosna, Boundary conditions and renormalized stress-tensor on PAdS_2, arXiv:1904.10806.