Color superconductivity in a holographic model

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A holographic bottom-up model used in studying the superconducting system is applied to search for the color superconducting phase of supersymmetric Yang-Mills theory with quarks. We apply the probe analysis of this model to the supersymmetric Yang-Mills theory in both the confinement and the deconfinement phases. In this analysis, we find the color superconductivity in both phases when the baryon chemical potential exceeds a certain critical value. This result implies that, above the critical chemical potential, a color nonsinglet diquark operator, namely the Cooper pair, has its vacuum expectation value even in the confinement phase. In order to improve this peculiar situation, we proceed with the analysis by taking account of the full backreaction from the probe. As a result, the color superconducting phase, which is observed in the probe approximation, disappears in both the confinement and the deconfinement phases when parameters of the theory are set within their reasonable values.

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I. INTRODUCTION

The color superconducting (CSC) phase has been expected in QCD at the finite baryon chemical potential, but it is difficult to show it (see e.g., the review [1]). The numerical simulation is also difficult due to the complex Euclidean action problem known as the sign problem. On the one hand, the gauge/gravity duality [2–4] has been used as a powerful method to clarify many properties related to QCD in the strong coupling region. As for the CSC, however, the holographic method is not clear to be useful. The reason is that we need to introduce a color nonsinglet scalar field, which provides us the vacuum expectation value (VEV) of the color nonsinglet operator in the CSC phase. In the top-down holographic approaches, the scalar field introduced in both the bulk and the probe branes should be a color singlet.

On the other hand, from the viewpoint of the bottom-up approach, a holographic model for the superconducting phase has been studied by introducing a $U(1)$ gauge field and a charged scalar field in the bulk [5–8]. The important point in applying them as the holographic CSC model is that the charge of the $U(1)$ gauge field is regarded as the baryon number [9,10]. Namely, the $U(1)$ gauge field is dual to the baryon number current, density, and chemical potential for the quarks of the $SU(N_c)$ Yang-Mills theory. And, the charged scalar is dual to a composite field operator with a finite quark number. In the model, they are not introduced through the probe D-branes but are given by the bulk action. Although it is not known how this model is lifted up to the ten-dimensional superstring theory, we call the model considered here the holographic supersymmetric Yang-Mills (SYM) theory by respecting the basic gauge theory dual to the gravity in the AdS_5.

In Ref. [9], the matter part, the system consisting of a $U(1)$ gauge field and a charged scalar field, is treated as the probe as in Ref. [6]. In other words, any backreaction from the matter part to the bulk configuration is neglected.[1] Then the authors of Ref. [9] performed the analysis in the high temperature deconfinement phase of the SYM theory, which is dual to the five-dimensional anti–de Sitter (AdS)–Schwarzschild background, and found the CSC phase above a certain chemical potential.

In Ref. [10], on the other hand, the backreaction from the matter part to the bulk is fully taken into account. Due to the backreaction from the $U(1)$ gauge field, the high temperature deconfinement phase is described by the Reissner-Nordstrom charged black hole in the gravity side.

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[1]This approximation would be justified for the case of a large charge of the scalar; however, we do not impose this restriction here.

In Ref. [10], however, a color neutral scalar field is chosen to study the color superconductor. Furthermore, the conformal dimension of the scalar is supposed to be smaller than that of the diquark operator. As a result of these special settings, at a very low temperature, a phase transition to the CSC phase has been observed.

In this paper, taking the idea and technique of Refs. [6,9,10] into account, we proceed with the analysis of the bottom-up model in order to get more profound knowledge about the CSC phase in the SYM theory. At first, we apply a simple probe analysis, in which the matter part is treated as a probe, to the background for both the confinement and the deconfinement phases. In this analysis, the CSC phase is confirmed by the nonvanishing VEV of a color nonsinglet diquark operator, namely the Cooper pair. As a result, we find the CSC phase in both the deconfinement and the confinement backgrounds. At a glance, it seems strange that the VEV of a color nonsinglet operator exists in the confinement phase. This may indicate the breakdown of the simple probe approximation. The probe approximation is available for a large scalar charge, which is supposed as $q = 2/N_c$ in this paper, and the holographic approach is useful for large N_c. Then the two methods are not compatible. We therefore cannot trust the results obtained from the probe approximation.

In order to improve the probe approximation, it is natural to consider the vacuum solutions which are given by solving the Einstein-Maxwell equation of the system with the Einstein-Hilbert action and the $U(1)$ gauge field part. In this case, the phase diagram in the μ-T plane should be modified to the chemical potential dependent form.[2] This modification is equivalently obtained by taking account of the full backreaction from the probe action as shown in Ref. [10]. Thus, as the next step, we proceed with the analysis based on this modification.

In the next section, we set up our holographic model and make a probe analysis to find the CSC phase in SYM theory. In the resultant phase diagram obtained by the probe approximation, we find a result which is unacceptable from the viewpoint of QCD. In Sec. III, we continue the analysis by taking account of the backreaction to improve the probe approximation, and we search the CSC phase in the improved background. Summary and discussions are given in the final section.

II. A BOTTOM-UP MODEL AND A PROBE APPROACH

We consider a bottom-up holographic dual for the SYM theory. It is given by the following gravitational theory [5,6]:

$$S = \int d^{d+1}x\sqrt{-g}\mathcal{L}, \tag{2.1}$$

$$\mathcal{L} = \mathcal{L}_{\mathrm{Gravity}} + \mathcal{L}_{\mathrm{CSC}}, \tag{2.2}$$

$$\mathcal{L}_{\mathrm{Gravity}} = \mathcal{R} + \frac{d(d-1)}{L^2}, \tag{2.3}$$

$$\mathcal{L}_{\mathrm{CSC}} = -\frac{1}{4}F^2 - |D_\mu\psi|^2 - m^2|\psi|^2, \tag{2.4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu\psi = (\partial_\mu - iqA_\mu)\psi. \tag{2.5}$$

It describes $(d+1)$-dimensional gravity coupled to a $U(1)$ gauge field, A_μ, and a charged scalar field, ψ. The charge q denotes the baryon number of the scalar ψ, which is considered to be dual to the diquark operator in this paper, and for the moment it is chosen that $q = 2/3$.[3] The mass m is given to reproduce the correct conformal dimension of the diquark operator dual to the scalar field ψ. Here we put $1/2\kappa_6^2 = 1$ and consider the case of $d = 5$ hereafter.

The above holographic model, previously, has been considered to be dual to the superconductor of the electric charge [5,6] and of the R-charge [8]. And it is recently extended to a theory dual to the color superconductor in [9,10]. However, it is unknown how this theory is dual to the SYM theory and can be related to the ten-dimensional string theory. In this paper, we proceed with the analysis of this model by supposing that this is dual to the SYM theory to study the existence of the CSC phase.

Bulk and probe

Here, $\mathcal{L}_{\mathrm{CSC}}$ is considered as the probe to see the condensation of the colored operator which is expressed in terms of the scalar field ψ. Therefore, the vacuum of the dual SYM theory is given by the solution of the Einstein equation of the action,

$$S_{\mathrm{Gravity}} = \int d^{d+1}x\sqrt{-g}\left\{\mathcal{R} + \frac{20}{L^2}\right\}, \tag{2.6}$$

where $\mathcal{L}_{\mathrm{CSC}}$ is neglected. Hereafter, we put $L = 1$. After giving the vacuum of the SYM theory by solving the above action, the equations of motion of the probe action $\mathcal{L}_{\mathrm{CSC}}$ are solved without changing the background configuration given by S_{Gravity}.

In the present case, we could find two solutions of the Einstein equation (2.6). They represent a low temperature confinement phase and a high temperature deconfinement one. For each phase, we study the superconducting phase by applying the probe method mentioned above.

[2] In Ref. [8], in a context of the R-charge superconductor, an analysis has been done based on the similar background configuration.

[3] In the model, it should be taken as $q = 2/N_c$. In this section, we suppose that we are considering the dual theory as QCD with the $SU(3)$ gauge group.

In $\mathcal{L}_{\rm CSC}$ given above, the charge q is factored out by the rescaling, $qA_\mu \to A_\mu$ and $q\psi \to \psi$, as follows:

$$\mathcal{L}_{\rm CSC} \to \frac{1}{q^2}\tilde{\mathcal{L}}_{\rm CSC}, \tag{2.7}$$

where $\tilde{\mathcal{L}}_{\rm CSC}$ is independent of q. This means that the probe approximation for $\mathcal{L}_{\rm CSC}$ would be justified for the case of large q. Then we find that the probe approximation for $q = 2/3$ is not good. However, we perform the analysis in this approximation to see what kind of results are obtained.

A. High temperature deconfinement phase

First, we consider the high temperature deconfinement phase, where the temperature is given by

$$T = \frac{5r_0}{4\pi}. \tag{2.8}$$

The solution is known as the AdS–Schwarzschild solution, which is written as

$$ds^2 = r^2(-f(r)dt^2 + \sum_{i=1}^{3}(dx^i)^2 + dw^2) + \frac{dr^2}{r^2 f(r)}, \tag{2.9}$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^5, \qquad r_0 = \frac{2}{5R_w}. \tag{2.10}$$

Here the Sherk-Schwartz compactification is imposed in the direction w, and then its circle length is taken as $2\pi R_w$.

Equations of probe $\mathcal{L}_{\rm CSC}$

In this case the equations of motion are given as

$$\psi'' + \left(\frac{6}{r} + \frac{f'}{f}\right)\psi' + \frac{1}{r^2 f}\left(\frac{q^2\phi^2}{r^2 f} - m^2\right)\psi = 0, \tag{2.11}$$

$$\phi'' + \frac{4}{r}\phi' - \frac{2q^2\psi^2}{r^2 f}\phi = 0, \tag{2.12}$$

where we assumed $A = A_\mu dx^\mu = \phi(r)dt$ and $\psi = \psi(r)$.

The conformal dimension of the scalar, say Δ, is related to the mass as

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right). \tag{2.13}$$

Here we suppose that the scalar is dual to the Cooper pair, so the dimension is expected to be $\Delta = 2 \times \frac{d-1}{2} = d - 1$, which is realized for $m^2 = -(d-1)$. We notice here $d = 5$ and $m^2 = -4$. Then the asymptotic forms of ϕ and ψ are expected as

$$\phi = \mu - \frac{\bar{d}}{r^3} + \cdots, \qquad \psi = \frac{J_C}{r} + \frac{C}{r^4} + \cdots \quad (r \to \infty), \tag{2.14}$$

where μ, $\bar{d}$, J_C, and C denote the chemical potential, charge density, source, and VEV of the dual operator of ψ, respectively.

These equations are essentially equivalent to the one studied in Ref. [9]. The difference is in the dimension of the YM theory. In our case, the boundary YM theory is defined in $(4+1)$-dimensional spacetime with one compactified space (w). The effective space dimension is however three, so there is no essential difference. We adopt this model to analyze the confinement phase in a way parallel to Ref. [9], in which the vacuum solution is obtained by the double Wick rotation of (2.9) as shown below.

In order to avoid the singularity at the horizon (r_0), the boundary conditions are given as

$$\phi(r_0) = 0, \qquad \psi'(r_0) = -\frac{4}{5r_0}\psi(r_0), \tag{2.15}$$

and the temperature is given by (2.8).

In this case, we could find the color superconducting phase for $\mu > \mu_c \simeq 6.9$.[4] This is assured by the nontrivial solution of ψ for $J_C = 0$ and $C \neq 0$. As pointed out in Ref. [9], there are plural solutions (node $= 0, 1, 2, \ldots$) for large μ. The solutions for node ≥ 1 represent metastable vacua of the theory. The solution of the lowest vacuum (node $= 0$) is shown in Fig. 1.

As for the charge density, we can see that it has the singularity at the transition point. Based on Refs. [11,12], such first-order singularity (shape bend) can be understood as the singularity propagation; it means that discontinuities appearing in a particular order parameter can be propagated via the entropy density and/or the charge density. Thus, the charge density also has the same order of the singularity.

B. Low temperature confinement phase

The low temperature solution of (2.6) is known as the AdS soliton solution [13,14], and it is obtained as

$$ds^2 = r^2(\eta_{\mu\nu}dx^\mu dx^\nu + f(r)dw^2) + \frac{dr^2}{r^2 f(r)}, \tag{2.16}$$

where

[4]The transition line between the normal phase (b) and the color superconducting phase (d) for this case is shown in the phase diagram in the μ-T plane, which appears later in Fig. 3.

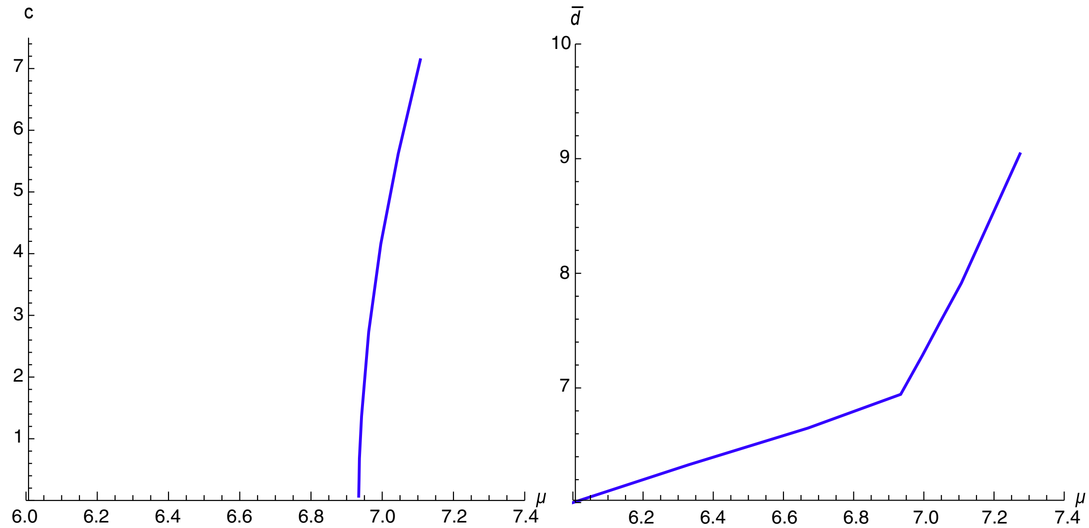

FIG. 1. The condensation (C) and the charge density ($\bar{d}$) vs μ in the deconfinement phase at $T = 5/(4\pi)$.

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^5, \qquad r_0 = \frac{2}{5R_w}, \tag{2.17}$$

and $2\pi R_w$ denotes the compactified length of w.

This configuration is realized for $T \leq \frac{5r_0}{4\pi}$, and the vacuum state is dual to the confinement phase. Namely, the line $T = \frac{5r_0}{4\pi}$ denotes the Hawking-Page transition line. Under the configuration (2.16), we find a linear potential between quark and antiquark by evaluating the Wilson loop. In this case, we suppose that the condensed scalar should be a color singlet and the charge density $\bar{d}$ is also constructed by the color singlet. This supposition would be correct when the chemical potential is not taken into account in the probe action since (2.16) is independent of μ.

In the present case, however, the chemical potential and the charge density are contained in the probe. Therefore, the chemical potential cannot affect the confinement background. This implies that it may be expected to find the phase transition to the CSC phase by solving the $\mathcal{L}_{\rm CSC}$ even in the case of (2.16). So it is very important to apply the model to the background (2.16).

Equations of probe $\mathcal{L}_{\rm CSC}$

Equations of motion of A_μ and ψ are obtained by assuming again $A = A_\mu dx^\mu = \phi(r)dt$ and $\psi = \psi(r)$:

$$\psi'' + \left(\frac{6}{r} + \frac{f'}{f}\right)\psi' + \frac{1}{r^2 f}\left(\frac{q^2\phi^2}{r^2} - m^2\right)\psi = 0, \tag{2.18}$$

$$\phi'' + \left(\frac{4}{r} + \frac{f'}{f}\right)\phi' - \frac{2q^2\psi^2}{r^2 f}\phi = 0. \tag{2.19}$$

Since $f(r)$ vanishes at $r = r_0$, Eqs. (2.18) and (2.19) should be solved under the following conditions:

$$\phi'(r_0) = \frac{2q^2\psi^2(r_0)}{5r_0}\phi(r_0),$$
$$\psi'(r_0) = -\frac{1}{5r_0}\left(\frac{q^2\phi^2(r_0)}{r_0^2} - m^2\right)\psi(r_0). \tag{2.20}$$

Here, we notice the boundary condition (2.20) allows the solution of $\phi(r_0) \neq 0$.

As expected, we could find nontrivial solutions of ψ with $J_C = 0$ and $C \neq 0$ for $\mu > \mu_c^{\rm conf} \simeq 4.7$ with $r_0 = 1$. For such nontrivial solutions, the μ dependence of $\bar{d}$ and C in the confined phase are shown in Fig. 2.

From the present analysis, we can draw the phase diagram in the μ-T plane, as shown by Fig. 3. We notice the existence of the critical line between the regions 3(a) and 3(c). As a result, there appear two superconducting phases 3(c) and 3(d). It is an interesting point how they are different from each other.

In the phase 3(c), the bulk represents the confinement phase. This phase would not be compatible with the existence of the VEV of the diquark operator and of the finite charge density. The reason why the CSC phase appears in the confinement bulk background may be reduced to the probe approximation used in this section. As explained above through Eq. (2.7), which is given by the rescaling $qA_\mu \to A_\mu$ and $q\psi \to \psi$, the probe approximation is useful for large q. However, $q(=2/N_c)$ should be very small in the present case since the holographic approach is available for large N_c.

An alternative idea to support the probe approximation is to suppress $\tilde{\mathcal{L}}_{\rm CSC}$ by a small number of the flavor branes, N_f, which would appear as the prefactor as

$$\mathcal{L} = \mathcal{L}_{\rm Gravity} + \frac{N_f}{q^2}\tilde{\mathcal{L}}_{\rm CSC}. \tag{2.21}$$

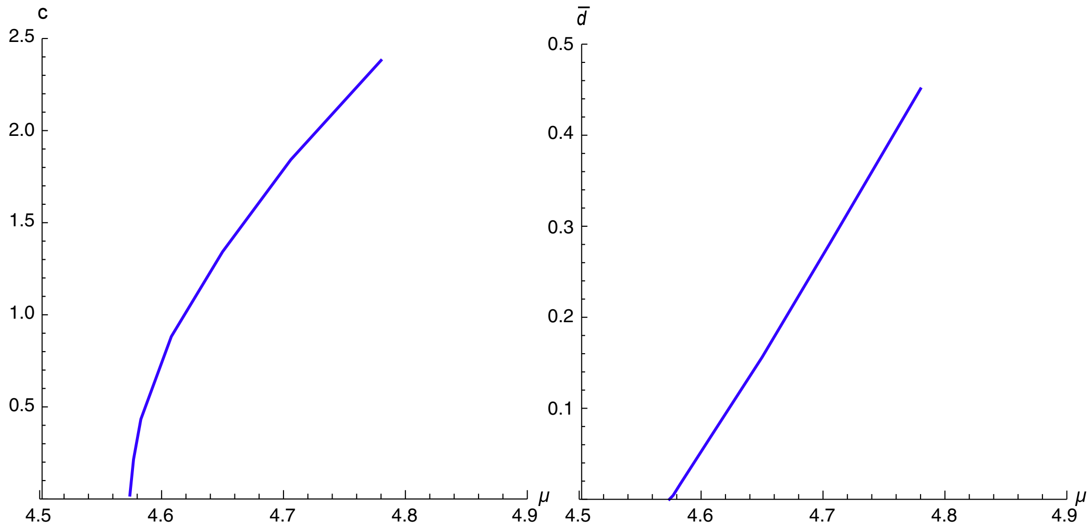

FIG. 2. The condensation (C) and the charge density ($\bar{d}$) vs μ in the confinement phase at $T = 5/(4\pi)$.

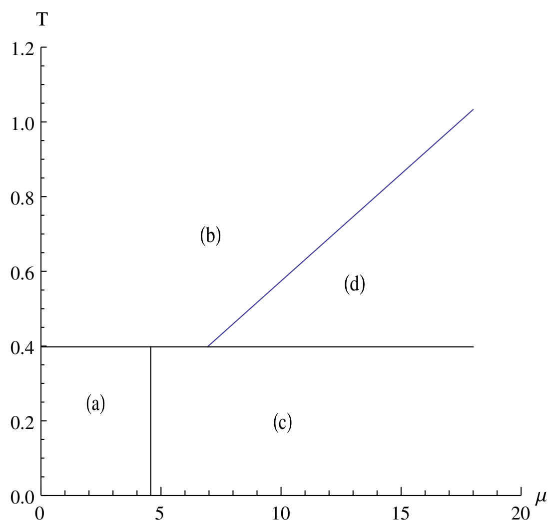

FIG. 3. Phase diagram given by the probe approximation for $q = 2/3$, $m^2 = -4$. The regions (b) and (d) represent the normal and CSC phases for the AdS–Schwarzschild deconfinement background. The critical line between (b) and (d) is given by $T = 0.058\mu$. For the AdS soliton confinement background, (a) and (c) represent the normal and CSC phases with the critical line $\mu = 4.7$ which is independent of T.

Then we can say the validity of the probe approximation for $N_f/q^2 \ll 1$. This is, however, impossible since $N_f > 1$. Therefore, it seems difficult to support the validity of the probe approximation. Then, we should improve the probe approximation by considering the backreaction from the terms of $\mathcal{L}_{\rm CSC}$ as performed in the next section.

III. SUPERCONDUCTOR IN THE BACKREACTED BACKGROUND

In the previous section, we find a phase transition to a color superconducting phase even in the confinement background of the theory. Why has such a transition been observed in the confinement vacuum?

As mentioned above, this is because of the application of the probe approximation without any backreaction from the matter field part of the model. The superconducting phase is observed in the large μ region, $\mu \geq \mu_c$. We expect that, in this region, the confinement force is suppressed by the effect of the chemical potential and the charge density so that the deconfinement phase might be realized. In order to make the situation clear, we should take account of the backreaction of the probe action to the gravity part. This is performed here according to the way given in Ref. [10].

At first, the basic background configuration is set by taking account of the $U(1)$ gauge part into the gravity part. In other words, the backreaction from the $U(1)$ gauge part to the bulk gravity is fully considered since it becomes important in the region, $\mu \geq \mu_c$.

Now, we set the bulk background configuration by solving the following action [15]:

$$S_{\rm Bulk} = \int d^6x\sqrt{-g}\left\{\mathcal{R} + \frac{20}{L^2} - \frac{1}{4}F^2\right\}. \tag{3.1}$$

The action leads to the following three solutions:

(1) AdS soliton (confinement phase)

The solution is given by the background metric (2.16), (2.17), and the constant potential,

$$A_0 = \phi = \mu. \tag{3.2}$$

(2) AdS–Schwarzschild (deconfinement phase)

The solution with the background (2.9) is compatible with the same type of a constant potential as (3.2).

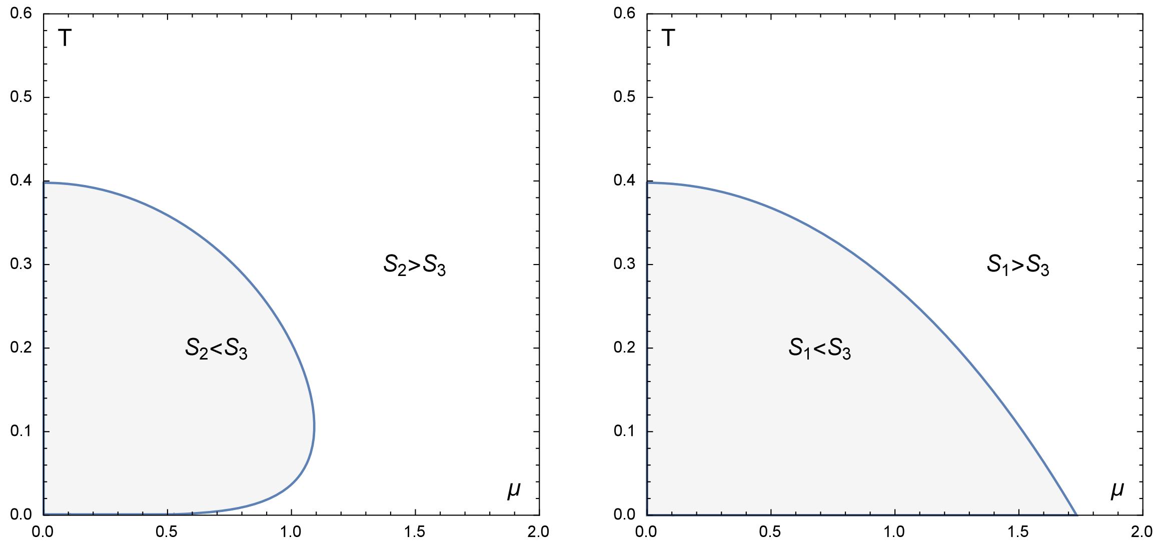

FIG. 4. Comparison among the actions (S_2 vs S_3 and S_1 vs S_3).

(3) Reissner-Nordstrom (RN) (deconfinement phase)
By considering the $U(1)$ potential A_0 with a finite charge density, a charged black hole solution (see, e.g., Refs. [15,16]) is obtained:[5]

$$ds^2 = r^2\left(-g(r)dt^2 + \sum_{i=1}^{3}(dx^i)^2 + dw^2\right) + \frac{dr^2}{r^2 g(r)}, \tag{3.3}$$

$$g(r) = 1 - \left(1 + \frac{3\mu^2}{8r_+^2}\right)\left(\frac{r_+}{r}\right)^5 + \frac{3\mu^2 r_+^6}{8r^8}, \tag{3.4}$$

$$A_0 = \phi = \mu\left(1 - \frac{r_+^3}{r^3}\right), \tag{3.5}$$

where r_+ denotes the horizon of the charged black hole, and the Hawking temperature is given by

$$T = \frac{1}{4\pi}\left(5r_+ - \frac{9\mu^2}{8r_+}\right). \tag{3.6}$$

Here we notice the setting of the parameter q. As mentioned above, by rescaling as $qA_\mu \to A_\mu$ and $q\psi \to \psi$, Eq. (2.2) is rewritten by Eq. (2.7). Then the equations of motion derived from $\tilde{\mathcal{L}}_{\rm CSC}$ for the rescaled ϕ and ψ are independent of q. However, q appears in the equations of motion when we take into account the backreaction to the background determined by gravity since q remains as the prefactor of $\tilde{\mathcal{L}}_{\rm CSC}$. Due to this fact, in this section, we solve the equations without any rescaling mentioned above.

[5]The notation is taken following Ref. [10].

A. Phase diagram before adding scalar

Before adding the scalar field, we compare the free energies of three types of vacua which arise from the background configurations, those are AdS black hole (BH), AdS soliton (Soliton), and Reissner-Nordstrom black hole (RN) given by (2.9), (2.16), and (3.3), respectively.

Action densities for the three solutions are given as

$$S_1/V_3 \equiv S_{\rm (Soliton)}/V_3 = -r_0^5 \frac{4\pi}{5r_0}\frac{1}{T}, \tag{3.7}$$

$$S_2/V_3 \equiv S_{\rm (BH)}/V_3 = -r_0^5\left(\frac{4\pi}{5r_0}\right)^2, \tag{3.8}$$

$$S_3/V_3 \equiv S_{\rm (RN)}/V_3 = -r_+^5\left(1 + \frac{3\mu^2}{8r_+^2}\right)\frac{4\pi}{5r_0}\frac{1}{T}, \tag{3.9}$$

where $V_3 = \int dxdydz$. We notice that the temperature T for RN and BH are given by (3.6) and (2.8), respectively. In the μ-T plane, we find the phase diagram by choosing the lowest action density among the above three. We find that the solution of BH is not realized when the solution of RN is added to compare.

The difference of the actions are written as

$$(S_i - S_j)/V_3 = N_{ij}(X_{ij}^5 - Y_{ij}^5), \tag{3.10}$$

where i and j run from 1 to 3, and

$$N_{23} = N_{13} = \frac{4\pi}{5r_0T}, \qquad X_{13} = X_{23} = r_+\left(1 + \frac{3\mu^2}{8r_+^2}\right)^{1/5}, \tag{3.11}$$

$$Y_{23} = r_0\left(\frac{4\pi T}{5r_0}\right)^{1/5}, \qquad Y_{13} = r_0 = 1. \tag{3.12}$$

Then, which action is larger is found by the plots of $X_{ij} - Y_{ij}$, which are shown in Fig. 4.

In the left-hand side of Fig. 4, there is a region where $S_2 < S_3$. However, we easily find $S_2 > S_1$ in this region. In fact, $S_1 < S_2$ for $T < 5r_0/4\pi$ at any μ. Then the phase diagram is constructed by the phase represented by Soliton and RN backgrounds, and it is given by the right-hand side of Fig. 4, which is separated by the phase of Soliton and RN. The critical curve in Fig. 7, which is shown in the Sec. III B, is given by

$$X_{13} = 1. \tag{3.13}$$

We notice the following point. In the previous section, the gauge field part is treated as the probe. It is not used for constructing the background metric. As a result, the phase diagram is given by comparing S_2 and S_1. In this case, the parameter space is separated into two phases, corresponding to the Soliton solution and the BH one, and the critical line is given by $T = 5r_0/4\pi$. Therefore the phase diagram in the previous section is largely changed after adding the $U(1)$ gauge part in the equations of motion to be solved.

B. Phase diagram after adding scalar

The phase diagrams shown in Fig. 4 would be changed due to the appearance of the color superconducting phase, which could be found by the scalar field condensation. We should remember that the backgrounds considered here are obtained by taking into account the backreaction of the $U(1)$ gauge part. The backreaction of the scalar is also taken into account when we solve its equation of motion. In this case, we must solve the equations of backreacted metric and gauge fields, and the equation of the scalar at the same time. Although it is straightforward but hard work to solve those simultaneous equations, we can find a phase diagram after solving them.

On the other hand, it would be possible to find the critical curve without solving the full equations. This economical method has been proposed in Ref. [10]. When a superconducting phase exists, there is a solution for the scalar with $J_C = 0$ and a finite C for $\mu > \mu_c^{(B)}$ at a finite T. And, for $\mu \to \mu_c^{(B)}$, C approaches zero.[6] Then the backreaction from the scalar to the bulk configuration becomes negligible near the critical point. This means that the backreactions to both the metric and the gauge fields disappear. This implies that the critical line can be obtained by solving the equation of motion of only the scalar which is treated as a probe in the two vacuum configurations given above. Then the task to find the critical line is to solve the equation of the scalar field in the given background of $S_{\rm Bulk}$ denoted in (3.1).

Deconfinement phase

At first we consider the equation of the motion in the RN background, (3.3)–(3.5). It is given as

$$\psi'' + \left(\frac{6}{r} + \frac{g'}{g}\right)\psi' + \frac{1}{r^2 g}\left(\frac{q^2\phi^2}{r^2 g} - m^2\right)\psi = 0. \tag{3.14}$$

The boundary condition to be imposed is

$$\psi'(r_+) = \frac{m^2}{5r_+ - \frac{9\mu^2}{8r_+}}\psi(r_+). \tag{3.15}$$

Equation (3.14) includes parameters, q, μ, and r_+. Here the temperature is expressed by r_+ and μ as (3.6). Since T should be positive, then we find the following constraint for μ:

$$0 \le \frac{\mu}{r_+} \le \frac{\sqrt{40}}{3}. \tag{3.16}$$

Under this constraint, we searched for a solution with $J_C = 0$ and $C \neq 0$. And, within the error of our numerical calculation, we get to the conclusion that there is no such solution for $q = 2/N_c \le 1$. Since our concern is in the case of $N_c = 3$ (or $N_c = 2$), we cannot say that there exists a color superconducting phase in four-dimensional (4D) Yang-Mills theories with quarks.

According to a view pointed out in Ref. [9], the reason why the solution leading to the condensation of the Cooper pair is not found in the present case is explained by considering the effective mass of the scalar, $m_{\rm eff}$. The input mass $m^2 = -4$ satisfies the Breitenlohner Freedman (BF) bound ($-25/4 < m^2$) in the $(5+1)$-dimensional AdS spacetime. However, as shown below, the mass is effectively suppressed by the coupling to the gauge potential ϕ. Noticing that Eq. (3.14) is rewritten as $(\Box_{\rm RN} - m_{\rm eff}^2)\psi = 0$, where $\Box_{\rm RN}$ denotes the Laplacian in the RN background, $m_{\rm eff}$ can be read as follows:

$$m_{\rm eff}^2 = m^2 - \Delta m^2, \qquad \Delta m^2 \equiv \frac{q^2\phi^2}{r^2 g}. \tag{3.17}$$

Then we consider that the necessary condition to destabilize the scalar field and to condense in the vacuum is to break the BF bound for $m_{\rm eff}$. It is given by $m_{\rm eff}^2 < -\frac{25}{4}$ or equivalently by

$$\Delta m^2 > \frac{9}{4}. \tag{3.18}$$

[6]Here we suppose the order parameter C has no gap at the transition point as seen in the previous section for the probe approximation.

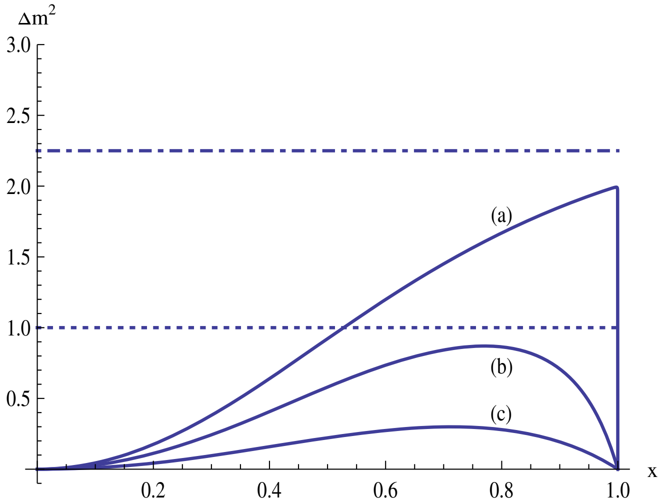

FIG. 5. The value of Δm^2 for $q = 1$ and (a) $\tilde{\mu} = \frac{\sqrt{40}}{3}$, (b) $\tilde{\mu} = 0.8 \times \frac{\sqrt{40}}{3}$, and (c) $\tilde{\mu} = 0.5 \times \frac{\sqrt{40}}{3}$. The horizontal dot-dashed and dotted lines show $9/4$ and 1.0, the bound to break the BF bound for Δm^2 in the AdS_6 and infrared AdS_2.

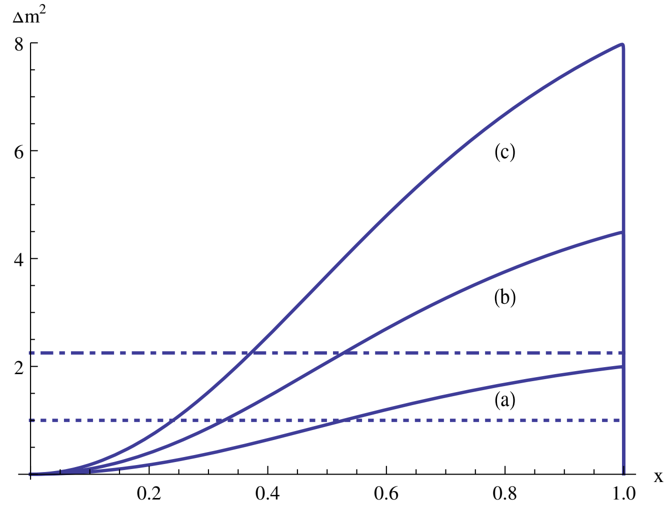

FIG. 6. The value of Δm^2 for $\tilde{\mu} = \frac{\sqrt{40}}{3}$ and (a) $q = 1$, (b) $q = 1.5$, and (c) $q = 2$. The horizontal dot-dashed and dotted lines show $9/4$ and 1.0, the bound to break the BF bound for Δm^2 in the AdS_6 and infrared AdS_2. For $q \geq 1.5$, the CSC phase has been found.

Here we notice that Δm^2 is r dependent and it is rewritten by using (3.4) and (3.5) as

$$\Delta m^2 = q^2 F(x, \tilde{\mu}), \qquad \tilde{\mu} = \frac{\mu}{r_+}, \tag{3.19}$$

$$F(x, \tilde{\mu}) = \frac{x^2(1 - x^3)^2 \tilde{\mu}^2}{1 - (1 + \frac{3\tilde{\mu}^2}{8})x^5 + \frac{3\tilde{\mu}^2}{8} x^8}, \tag{3.20}$$

where $x = r_+/r$.

The factor $F(x, \tilde{\mu})$ increases with $\tilde{\mu}$ at any x. The value of $\tilde{\mu}$ is bounded from above as shown by (3.16), so we can find the maximum form of $F(x, \tilde{\mu})$ as $F(x, \sqrt{40}/3)$ as shown by the curve (a) in Fig. 5. Further the maximum value of this function is found as $F(x, \sqrt{40}/3) < 1.993 \simeq 2$. Then we are led to

$$0 < \Delta m^2 < 2q^2 = \frac{8}{N_c^2} \tag{3.21}$$

for all regions of $r (\geq r_+)$. The above equality comes from $q = \frac{2}{N_c}$ since we assume that the scalar is dual to the diquark operator.

From (3.21) and (3.18), we have the condition that Δm^2 breaks the BF condition at some point of r, as

$$\frac{9}{4} < \Delta m^2 < \frac{8}{N_c^2}, \tag{3.22}$$

and thus we obtain

$$N_c < \frac{4\sqrt{2}}{3} \simeq 1.89. \tag{3.23}$$

This implies that it is impossible to see the scalar condensate with reasonable values of N_c, namely in the region $N_c \geq 2$, in the present holographic model. As a result, we can say that there is no CSC phase in the RN background.

In arriving at the above result, we must notice the following comments:

(C1) While a necessary condition for the condensation of the scalar is given above, we did not say anything about the sufficient condition. It is shown by a simple example studied in the previous section. For the case of the confinement phase, from Eq. (2.18) we can set it as

$$\Delta m^2 = \frac{q^2 \phi^2}{r^2} = q^2 \frac{\mu^2}{r_0^2} \left(\frac{r_0}{r}\right)^2. \tag{3.24}$$

For $r_0 = 1$, the necessary condition of BF bound breaking is given by $q\mu > 1.5$. However, we need $q\mu > 3.01$[7] to find the CSC phase. This implies that a wide enough region of r, where BF bound is broken, should be needed as a sufficient condition of the scalar condensation. This statement for the sufficient condition is available in other cases.

In Fig. 6, some examples of m_{eff} in the deconfinement phase are shown at $T = 0$. In this case, the sufficient condition is given as $q > 1.5$, and its m_{eff} is shown by the curve (b) of Fig. 6.

(C2) In the case of the deconfinement phase, the curve (a) in Fig. 5 is lower than the bound for all x. This curve is given for $q = 1$ ($N_c = 2$). On the other hand, we can find a CSC phase with the critical line $T/\mu = 0.0426$ for $q = 2$ ($N_c = 1$) [see the curve (c) of Fig. 6 and the right-hand part

[7]The value of $q\mu$ is obtained by our numerical estimation.

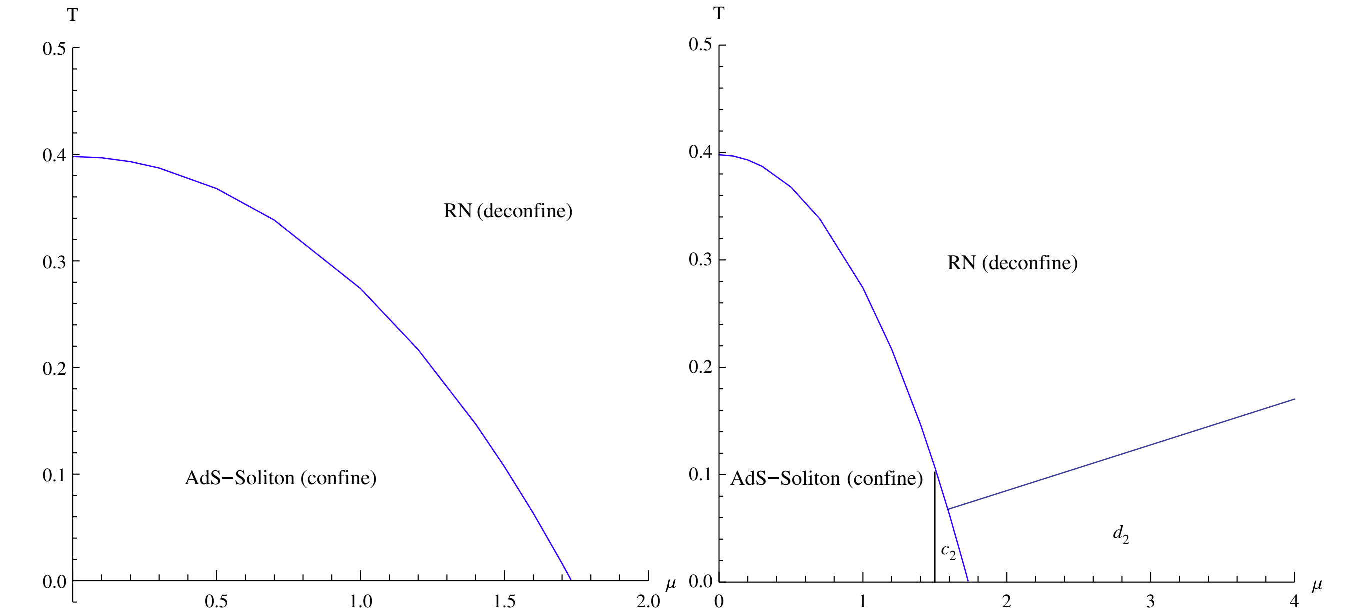

FIG. 7. Left: The phase diagram for the case of the backreacted background for $N_c \geq 2$. There is no CSC phase. The critical curve between the deconfinement and the confinement phases, which are denoted by RN (deconfine) and AdS-Soliton (confine) respectively, is given in the right-hand part of Fig. 4. Right: The phase diagram for $N_c = 1$. The regions (c_2) and (d_2) denote the CSC phases, and the critical line in the RN represents $T/\mu = 0.0426$. The vertical line in the confinement phase denotes $\mu = 1.5$.

of Fig. 7]. Then, in this case, the magnitude of the scalar charge is sufficient to generate the condensation. However, in this case, we have $N_c = 1$, which means that the dual theory is $U(1)_c$ gauge theory. So it is unrealistic in the present bottom-up model as the SYM theory and also from the holographic setup.

(C3) We notice another scalar mass bound which has been examined in [10]. Near the horizon of the RN black hole solution, we find a geometry $AdS_2 \times R^4$ [7,16] where the radius of AdS_2 is given by $L/\sqrt{20}$. Then the new BF bound near this AdS_2 geometry is given by $m^2_{\rm eff} > -5$. This bound is broken for $\Delta m^2 > 1$. From Fig. 5, the curve (a) is over the bound value in a long range of x. This implies a possibility of the solution of the scalar which indicates the Cooper pair condensation at a very small temperature as found in [10] for a neutral scalar with $m^2 < -5$. However, we cannot find such a solution in the present model. So one may consider that the curve (a) is not yet sufficient to generate the CSC phase as mentioned in the above comment (C1). An alternate reason for not finding the CSC phase may be that we are not considering the neutral scalar but a charged one. Since the operators dual to the scalars are different from each other, then our result might be compatible with the one of Ref. [10]. In any case, this point is an open problem.

Confinement phase

In the confinement region, the equations to be solved are given as

$$\psi'' + \left(\frac{6}{r} + \frac{f'}{f}\right)\psi' + \frac{1}{r^2 f}\left(\frac{q^2\phi^2}{r^2} - m^2\right)\psi = 0, \tag{3.25}$$

$$f = 1 - \left(\frac{r_0}{r}\right)^5, \qquad \phi = \mu, \tag{3.26}$$

with the following condition at $r = r_0$:

$$\psi'(r_0) = -\frac{1}{5r_0}\left(\frac{q^2\phi^2(r_0)}{r_0^2} - m^2\right)\psi(r_0). \tag{3.27}$$

In this case, the effective scalar mass has the same form with the case considered in the previous section. Hence Δm^2 is given by (3.24). Thus the instability would be seen locally for $q\mu > 1.5$ when we set it as $r_0 = 1$. However, the sufficient condition to find the CSC phase is $q\mu > 3.01$ as mentioned above. This implies $\mu > 3.01$ for $q = 1$. This region of μ is out of the confinement phase, $\mu < 1.73$. So we cannot find the CSC phase also in the confinement background. Noticing $q = 2/N_c$, the CSC phase cannot be found for $N_c \geq 2$.

We should notice that the CSC phase is realized for $q = 2$ $(N_c = 1)$ since the sufficient condition is given as $\mu > 3.01/q = 1.5$ which is within the confinement region. The phase diagram for the $q = 2$ case is given in Fig. 7.

This section can be summarized as follows:

(1) For $N_c \geq 2$, we can say that there is no CSC phase in the deconfinement and also in the confinement

phases. Then, we need not change the phase diagram even if an appropriately charged scalar is added by taking into account the backreaction to realize the color superconductivity in the SYM theory.

(2) For $N_c = 1$, a phase diagram with the CSC phase is obtained, but the setting of $N_c = 1$ is unrealistic since it is incompatible with the holography. So it is hard to accept the results obtained in this case.

We must remember that the CSC phase searched in this section is the one realized by the phase transition without the gap of the order parameter C. In this paper, we concentrated on this type of phase transition since the same type transition is found in the case of the probe approximation as shown in the Sec. II.

IV. SUMMARY AND DISCUSSION

We have studied a possibility of the CSC phase in the SYM theory by using a bottom-up holographic model which is constructed by the gravity and a simple matter action composed of a $U(1)$ gauge field and a charged scalar. The time component of the gauge field gives a finite chemical potential μ of the baryon number and its density in the vacuum of the SYM theory. The mass and the charge of the scalar give the conformal dimension and the baryon number of the composite operator of the dual SYM theory. It is chosen as a scalar which is dual to the diquark operator. According to this holographic setting, the equations of motion of the system are solved, and the CSC phase of the SYM theory is searched.

Using this model, at first, a probe analysis has been applied to the two vacuum states, the confinement and deconfinement phases. The probe action is composed of the $U(1)$ gauge field and the charged scalar. The bulk configurations of these two phases are therefore independent of μ since the chemical potential and the charge density belong to the probe. In this case, we find the CSC phase in each vacuum when the value of μ exceeds a critical point observed in each phase. In any case, this transition causes the breaking of the gauge symmetry since the VEV of the diquark operator is not a color singlet except for the case of $N_c = 2$. In this sense, it might be interesting to study the color superconductors in the holographic Higgs branch [17–20]. The approach in this direction would be given elsewhere.

In any case, this CSC transition implies the transition from the confinement to the deconfinement phase at the same time. The reason why this curious phenomenon occurs would be that the probe approximation is used in the situation where this approximation may not be useful. In fact, here the holographic approach is set for $q = 2/N_c$ with large N_c, so q is small. On the other hand, a large q is needed for the validity of the probe approximation.

In order to improve the results of the probe approximation, we consider the backreactions. At first, we reset the vacuum. It is given as a solution of the gravity with the full backreaction from the $U(1)$ gauge field. As a result, the solutions and the phase diagram, which is given by the two phases, confinement (AdS soliton) and deconfinement (Reissner-Nordstrom black hole), have been largely changed and are dependent on μ. By adding the scalar in these vacuum configurations, the superconducting phase has been searched. Here the backreaction of the scalar is also considered, and we arrive at the conclusion that there is no CSC phase in both the confinement and the deconfinement phases with the reasonable parameters of the theory.

We should, however, notice that we find a region of r where the BF bound is broken, for $N_c = 2$, in the confinement phase. But enough instability to generate the CSC phase cannot be obtained in this case.[8] There are some studies of the CSC phase via two color lattice QCD (QC_2D) since the fermion determinant is pseudoreal and then the sign problem vanishes. Through lattice simulations, actually, the CSC phase has been found in Refs. [22,23] in the deconfined regime. But the parameters used in this case lead to a result that the pion mass is rather heavier than the physical one, $m_\pi/m_\rho \sim 0.8$ where m_π and m_ρ are the π and ρ meson masses. Therefore, to conclude that our present model is correct, we will need more data of the lattice QC_2D with physical quark masses.

Another point to be noticed is that we find the CSC phase when the charge q becomes large, $q > 1.5$. In Sec. III, an example is shown for $q = 2$, where the CSC phases are found in both the deconfinement and the confinement phases with a definite critical line. However, since $q(= 2/N_c) \geq 2$ means $N_c \leq 1$, the holographic approach is not useful in this case. So we cannot trust any result for large q. The other example of CSC phase realization may be possible in the lower space dimension. For $2 + 1$ space dimension, an analysis has been given in [8] for the superconductor of R-charge (not for the baryon charge) though the equations of motion are similar. So it would be possible to study the CSC phase in such lower dimensional cases by using the holographic model used here.

As for the chiral symmetry, the phase diagram in the $\mu - T$ plane has been examined by solving the profile of probe branes based on a top-down model in the deconfinement phase [24]. And a chiral symmetry breaking phase has been found in the small μ region. However, in the present model, it is impossible to find such a broken phase in both the deconfinement and the confinement phases when the scalar is set as a field dual to the quark antiquark operator in order to investigate the chiral symmetry. This is easily understood. Since the scalar field has no baryon charge ($q = 0$), then the effective mass cannot be changed from the

[8] As a related example, another kind of phase transition in the confinement phase has been studied by using a different form of probe composed of D8$\overline{\text{D8}}$ branes, in which the baryon chemical potential is considered [21].

given mass $m^2 = -4$. As a result, the trivial solution of the scalar field is the stable one. In other words, the present bottom-up model is not suitable for studying the chiral symmetry. It is important to improve the model in this point; however, it is out of the present task.

Finally, we give a brief comment on the flavor degrees of freedom and the mass of quarks. Although, in this paper, we have not considered them, we would have a variety of phases such as two flavor CSC [25,26] and color-flavor locking [1,27]. And once this is accomplished, it might be possible to study an interesting issue in high density QCD.

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