Pressure effects in the weak-field limit of $f(R) = R + \alpha R^2$ gravity

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We investigate the linear regime of $f(R) = R + \alpha R^2$ gravity for static, spherically symmetric and asymptotically flat configurations of matter. We show that, in vacuum and deep inside the range of the extra scalar degree of freedom, the post-Newtonian parameter γ is not equal to $1/2$, as established in the literature, but it assumes larger values depending on the pressure of the star. We provide an explicit expression for γ in terms of the mass, of the integrated pressure of the star and of the ratio between the star's radius and the range of the extra degree of freedom. We corroborate our results by providing numerical solutions for the case of a neutron star.

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I. INTRODUCTION

Among the numerous extensions of the theory of General Relativity (GR), in the past two decades considerable attention has been given to $f(R)$ gravity (where R is the Ricci scalar) as a possible alternative to dark energy, and to explore modifications of gravity in the strong gravity regime. The action describing this modification of GR is obtained from the Einstein-Hilbert action by simply trading R for a generic nonlinear function of it [1–4], that is,

$$S = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{-g} f(R) + S_M, \qquad (1.1)$$

where G_N is Newton's constant and S_M is the action for the matter fields.

In the metric formalism of $f(R)$ [5], which we adopt here, this apparently innocent procedure spoils the feature of GR of having field equations of second order. In metric $f(R)$ the evolution is in fact described by fourth-order differential equations, which is in general worrying from the point of view of Ostrogradsky instability [6]. It can be however shown that one of the assumptions of Ostrogradsky's theorem (nondegeneracy) is violated [7], and that the theory is safe. A generic prescription on the form of f, at least for small modifications of GR, is that $f_{RR} > 0$, in order to avoid the Dolgov-Kawasaki instability [8].[1]

A well-known property of R-regular $f(R)$ theories is that they can be mapped into a scalar-tensor theory (in fact, a Brans-Dicke theory with parameter $\omega = 0$ and a nontrivial potential), where the derivative f_R corresponds to the scalar component.[2] In comparison to GR, this component therefore acts effectively as an extra degree of freedom. As usual, it is possible to pass from the Jordan to the Einstein frame by means of a conformal transformation. The coupling of the scalar degree of freedom to the matter sector, which is induced by the transformation, is at the basis of the chameleon mechanism [9,10].

In this paper we focus on the linear regime of $f(R)$ gravity in a static, spherically symmetric and asymptotically flat configuration, providing a simple model for a star. This regime has been extensively considered, although at times with cosmological asymptotics, especially in connection with Solar System tests (see e.g., [11–51]). A well-established conclusion in this regard is that, outside the star and well within the range of the extra scalar degree of freedom, the metric potentials display the Newtonian $1/r$ behavior but have different magnitude. This difference is encoded in the post-Newtonian parameter (PPN) γ, whose value is found to be equal to $1/2$ and therefore in clear disagreement with observation (which confirms the GR value $\gamma = 1$ within few parts in 10^5 [52]). An analysis of the behavior of the PPN parameter γ in nonminimally coupled models of gravity [which have a $f(R)$ limit] can be found in [53]. The behavior of the post-Newtonian parameters β

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[1] We indicate derivatives with respect to R with a subscript $_R$.

[2] A $f(R)$ theory is called R-regular if the second derivative f_{RR} never vanishes.

and γ in a generic scalar-tensor theory, although with very crude assumptions on the structure of the star, is studied in [54,55].

The $\gamma = 1/2$ result is based on the assumption of negligible pressure of the spherically symmetric energy-momentum configuration. Here we show that γ is in general larger than $1/2$ when the pressure is taken into account. Our analysis is limited to the model $f(R) = R + \alpha R^2$, also known as Starobinsky model [56], which is one of the most successful models in inflationary cosmology [57]. Nevertheless, we expect our analysis to be relevant for any $f(R)$ theory such that f is analytic in $R = 0$. We also comment on a relativistic limit, i.e., when the spherical configuration of energy momentum has a relativistic equation of state, in which case the $\gamma = 1$ limit is recovered. Furthermore, we explore the role of mild nonlinear effects on the behavior of gravity.

The paper is organized as follows: in Sec. II we set up our notation and derive the equations of motions for our system. In Sec. III we implement the linear approximation of the latter equations and derive the external and internal solutions. In Sec. IV we discuss the consistency of our approximation and draw some conclusions about the behavior of gravity. In Sec. V we support our analysis by solving numerically the nonlinear equations of motion. We present our conclusions and comments in Sec. VI.

We adopt the "mostly plus" signature $(-,+,+,+)$ and, unless stated otherwise, we use units of measure where $c = 1$.

II. FIELD EQUATIONS

We consider the $f(R)$ gravity model given by

$$f(R) = R + \alpha R^2, \tag{2.1}$$

in the metric formalism. The field equations resulting from the variation of the action with respect to the metric are the following:

$$(1+2\alpha R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R+\alpha R^2) - 2\alpha(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)R = 8\pi G_N T_{\mu\nu}, \tag{2.2}$$

where the connection is the Levi-Civita one, so the Ricci scalar is a functional of the metric $R = R[g_{\mu\nu}]$, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the (curved space) d'Alembert operator and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.3}$$

Defining $\zeta = (f_R - 1)/2$, the fourth-order equation (2.2) can be shown to be equivalent to the second-order system for the metric $g_{\mu\nu}$ and the scalar field ζ [58]:

$$(1+2\zeta)G^\mu{}_\nu = -3m^2\zeta^2\delta^\mu{}_\nu + 2(g^{\mu\lambda}\nabla_\nu\partial_\lambda - \delta^\mu{}_\nu\Box)\zeta + 8\pi G_N T^\mu{}_\nu, \tag{2.4}$$

$$\Box\zeta - m^2\zeta = \frac{4\pi G_N}{3}T, \tag{2.5}$$

where T is the trace of the energy-momentum tensor and we defined

$$m^2 \equiv \frac{1}{6\alpha} \tag{2.6}$$

as the mass associated to the scalar degree of freedom. The metric and the scalar field are independent degrees of freedom, as far as the initial value problem of the system (2.4) and (2.5) is concerned, however when the solutions of (2.4) and (2.5) are considered the scalar field is related to the scalar curvature by $\zeta = \alpha R$.

A. Static and spherically symmetric case

As a model for a nonrotating star, we assume a static, spherically symmetric metric:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{2.7}$$

and a perfect fluid-type energy-momentum tensor:

$$T^0{}_0 = -\rho(r), \qquad T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = p(r), \tag{2.8}$$

where ρ and p are the density and pressure of the star, respectively, and they depend only on the radial coordinate.

The field equations can be straightforwardly computed and read:

$$(1+2\zeta+r\zeta')\frac{1}{Br}\frac{A'}{A} = \frac{1+2\zeta}{r^2}\left(1-\frac{1}{B}\right) - 3m^2\zeta^2 - \frac{4}{Br}\zeta' + 8\pi G_N p, \tag{2.9a}$$

$$(1+2\zeta+r\zeta')\frac{1}{Br}\frac{B'}{B} = \frac{1+2\zeta}{r^2}\left(\frac{1}{B}-1\right) + 3m^2\zeta^2 + \frac{2}{B}\left(\zeta'' + \frac{2}{r}\zeta'\right) + 8\pi G_N\rho, \tag{2.9b}$$

$$\frac{1}{B}\left[\zeta'' + \left(\frac{2}{r}+\frac{A'}{2A}-\frac{B'}{2B}\right)\zeta'\right] = m^2\zeta + \frac{4\pi G_N}{3}(3p-\rho), \tag{2.9c}$$

where the prime denotes derivation with respect to r. Using Eqs. (2.9a) and (2.9c), the Eq. (2.9b) can be cast in the form

$$\frac{1+2\zeta}{r}\frac{B'}{B^2} = \left(1+\frac{r\zeta'}{1+2\zeta+r\zeta'}\right)\left[\frac{1+2\zeta}{r^2}\left(\frac{1}{B}-1\right)+3m^2\zeta^2\right] + 2m^2\zeta + \frac{4}{B}\frac{\zeta'^2}{1+2\zeta+r\zeta'} + 8\pi G_N\left(\frac{1+2\zeta}{1+2\zeta+r\zeta'}p+\frac{2}{3}\rho\right). \quad (2.10)$$

III. LINEARIZATION OF THE FIELD EQUATIONS

We now want to study analytically the above system of equations, in the linear regime. To this aim it is useful to introduce the gravitational potentials Φ and Ψ:

$$A(r) = 1 + 2\Phi(r), \qquad B^{-1}(r) = 1 + 2\Psi(r). \quad (3.1)$$

Since we consider asymptotically flat solutions, we look for (approximated) solutions of the equations above such that ζ, Φ and Ψ decay to zero when $r \to \infty$.

A. The scalar degree of freedom

Assuming that the following conditions hold:

$$|\Phi| \ll 1, \qquad |\Psi| \ll 1, \qquad r|\Phi' + \Psi'| \ll 1, \quad (3.2)$$

the evolution equation for the scalar degree of freedom, Eq. (2.9c), can be greatly simplified as follows:

$$\zeta'' + \frac{2}{r}\zeta' - m^2\zeta = \frac{4\pi G_N}{3}(3p - \rho), \quad (3.3)$$

and can be readily solved, since it does not contain the gravitational potentials. The (asymptotically decaying) Green's function associated to the differential operator acting on ζ is the following:

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} e^{-m|\mathbf{r}-\mathbf{r}'|}, \quad (3.4)$$

so that the complete solution of Eq. (3.3) which decays at infinity reads

$$\zeta(\mathbf{r}) = \frac{G_N}{3}\int_{V_\star} \frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}(\rho(\mathbf{r}') - 3p(\mathbf{r}'))d^3\mathbf{r}', \quad (3.5)$$

where the integration is performed on $V_\star$, the spherical volume occupied by the star.

Since ρ and p depend only on the radial coordinate, the angular integration in (3.5) can be performed exactly. This is achieved by making explicit the modulus:

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\beta}, \quad (3.6)$$

where β is the angle between the directions of $\mathbf{r}$ and $\mathbf{r}'$, and then using the integral:

$$\int_{-1}^{1} d(\cos\beta)\frac{e^{-m\sqrt{r^2+r'^2-2rr'\cos\beta}}}{\sqrt{r^2+r'^2-2rr'\cos\beta}} = \frac{e^{-m|r-r'|} - e^{-m(r+r')}}{mrr'}. \quad (3.7)$$

1. External and internal solutions

Let us consider the region outside the star. The integration of Eq. (3.5) then gives

$$\zeta_{\text{ext}}(r) = \frac{G_N}{3}(\tilde{M}_\star - 3\tilde{P}_\star)\frac{e^{-mr}}{r}, \quad (3.8)$$

where we defined

$$\tilde{M}_\star \equiv 4\pi\int_0^{r_\star}\frac{\sinh mr}{mr}\rho r^2 dr, \qquad \tilde{P}_\star \equiv 4\pi\int_0^{r_\star}\frac{\sinh mr}{mr}p r^2 dr, \quad (3.9)$$

and indicated with $r_\star$ the radius of the star. The quantities $\tilde{M}_\star$ and $\tilde{P}_\star$ are related respectively to the integrated density and pressure of the star, and in fact reduce to them when $mr_\star \ll 1$:

$$\tilde{M}_\star \underset{mr_\star \ll 1}{\longrightarrow} M_\star \equiv 4\pi\int_0^{r_\star}\rho r^2 dr, \qquad \tilde{P}_\star \underset{mr_\star \ll 1}{\longrightarrow} P_\star \equiv 4\pi\int_0^{r_\star}p r^2 dr. \quad (3.10)$$

In this case, which corresponds to the range of the scalar degree of freedom being much larger than the radius of the star, outside of the star there exists a region where the exterior solution becomes[3]

$$r_\star \le r \ll m^{-1} \Rightarrow \zeta_{\text{ext}}(r) = \frac{G_N}{3r}(M_\star - 3P_\star). \quad (3.11)$$

It is suggestive to introduce a characteristic radius associated with the extra scalar degree of freedom:

$$r_\zeta \equiv 2G_N(\tilde{M}_\star - 3\tilde{P}_\star), \quad (3.12)$$

which may be considered as the "Schwarzschild radius" of ζ. In terms of r_ζ the external solution (3.8) reads

$$\zeta_{\text{ext}}(r) = \frac{r_\zeta}{6r}e^{-mr}. \quad (3.13)$$

Let us now consider the behavior of ζ inside the star. Because of the absolute value $|r - r'|$ appearing in (3.7), it is convenient to split the radial integration into two parts.

[3] This result is equivalent to Eq. (13) of [48] specialized for the $R + \alpha R^2$ model, apart from the fact that here the contribution due to the pressure is taken into account.

We then obtain

$$\zeta_{\rm int}(r) = \frac{G_N}{3}\left[\frac{e^{-mr}}{r}(\tilde{M}(r) - 3\tilde{P}(r)) + \frac{\sinh mr}{mr}\tilde{S}(r)\right], \tag{3.14}$$

where we defined

$$\tilde{M}(r) \equiv 4\pi \int_0^r \frac{\sinh my}{my}\rho y^2 dy, \\ \tilde{P}(r) \equiv 4\pi \int_0^r \frac{\sinh my}{my} p y^2 dy, \tag{3.15}$$

and

$$\tilde{S}(r) \equiv 4\pi \int_r^{r_\star} e^{-my}(\rho(y) - 3p(y))y dy. \tag{3.16}$$

Note that

$$\tilde{M}(r_\star) = \tilde{M}_\star, \qquad \tilde{P}(r_\star) = \tilde{P}_\star, \qquad \tilde{S}(r_\star) = 0, \tag{3.17}$$

so the expression (3.14) actually holds also outside the star provided we define $\tilde{M}(r)$, $\tilde{P}(r)$ and $\tilde{S}(r)$ to be continuous, and constant outside the star. Furthermore, the limit of (3.14) for $r \to 0$ is well defined, since $\tilde{M}(r)/r \to 0$ and $\tilde{P}(r)/r \to 0$, and we have

$$\zeta_{\rm int}(0) = \frac{G_N}{3}\tilde{S}(0). \tag{3.18}$$

A word on the case of relativistic matter. It is apparent that when $p = \rho/3$ the functions $\tilde{M}(r) - 3\tilde{P}(r)$ and $\tilde{S}(r)$ vanish identically, and therefore the scalar degree of freedom vanishes identically as well (both inside and outside the star). In this case, therefore, the scalar degree of freedom is "masked." We further comment on this point in Sec. IV B 3.

B. The gravitational potentials

Let us consider the extra conditions

$$|\zeta| \ll 1, \qquad r|\zeta'| \ll 1, \qquad rm^2\zeta^2 \ll |\zeta'|, \tag{3.19}$$

and suppose that the term containing ζ'^2 can be neglected in Eq. (2.10). All these assumptions are to be discussed more in detail below. Under this approximation the field equations (2.9a) and (2.10) simplify to

$$r\Phi' = -\Psi - 2r\zeta' + 4\pi G_N r^2 p, \tag{3.20a}$$

$$(r\Psi)' = -r^2 m^2 \zeta - \frac{4\pi G_N}{3} r^2 (2\rho + 3p). \tag{3.20b}$$

Knowing the explicit solution for ζ, we can straightforwardly obtain those for the gravitational potentials Ψ and Φ.

1. The potential Ψ

Let us start considering the internal solution for the potential Ψ. Integrating Eq. (3.20b) from $r = 0$ to a certain radius $r \le r_\star$ we obtain

$$r\Psi(r) = -m^2 \int_0^r \zeta(y) y^2 dy - \frac{G_N}{3}(2M(r) + 3P(r)), \tag{3.21}$$

where we assumed that $r\Psi(r) \to 0$ for $r \to 0$ (which should be safe, since we expect the gravitational field to vanish at the center of the star). Let us now define

$$\Xi(r) \equiv \frac{m^2}{G_N}\int_0^r \zeta(y) y^2 dy, \tag{3.22}$$

and adopt the notation[4]:

$$\Xi_\star \equiv \Xi(r_\star). \tag{3.23}$$

Once the configuration of ρ and p is known, the function $\Xi(r)$ can be computed (in principle exactly, but most likely performing the integrations numerically) by employing Eqs. (3.14)–(3.16). The solution for Ψ inside the star can thus be written as

$$\Psi_{\rm int}(r) = -\frac{G_N}{r}\left(\frac{2}{3}M(r) + P(r) + \Xi(r)\right). \tag{3.24}$$

The external ($r \ge r_\star$) solution for Ψ can be obtained by finding a primitive of the right-hand side of (3.20b) in vacuum [since we know explicitly the external solution (3.8) for ζ] and matching the resulting function with the internal solution (3.24) at the star's surface. We get

$$r\Psi_{\rm ext}(r) = \frac{r_\zeta}{6}[e^{-mr}(1 + mr) - e^{-mr_\star}(1 + mr_\star)] \\ - G_N\left(\frac{2}{3}M_\star + P_\star + \Xi_\star\right). \tag{3.25}$$

When $r \to \infty$, the right-hand side of the above equation tends to a finite value C:

[4]Note that $\Xi_\star$ is the amount of scalar curvature enclosed by the star, divided by $6G_N$.

$$C\equiv -G_N\left[\left(\frac{\tilde{M}_\star}{3}-\tilde{P}_\star\right)e^{-mr_\star}(1+mr_\star)+\frac{2M_\star}{3}+P_\star+\Xi_\star\right], \tag{3.26}$$

which defines a geometric mass $M_g = -C/G_N$ distinct from $M_\star$ or, equivalently, a new characteristic radius $r_g \equiv -2C$ for the gravitational potential. We can then write in a more compact form the external solution for $\Psi(r)$:

$$\Psi_{\rm ext}(r) = (1+mr)\frac{r_\zeta}{6r}e^{-mr} - \frac{r_g}{2r}, \tag{3.27}$$

from which it is evident that the external solution correctly decays to zero asymptotically.

2. The potential Φ

Let us now consider the gravitational potential Φ. Regarding the external solution, also in this case it can be found by looking for a primitive of the Eq. (3.20a) in vacuum [using the explicit external solutions (3.8) and (3.27) respectively for ζ and Ψ], and imposing it to decay to zero asymptotically. We obtain

$$\Phi_{\rm ext}(r) = -\frac{r_\zeta}{6r}e^{-mr} - \frac{r_g}{2r}. \tag{3.28}$$

The internal solution can be expressed in terms of Ψ, ζ and p (and so, ultimately, in terms of ρ and p) by integrating Eq. (3.20a) and using Eqs. (3.14) and (3.24). We get

$$\Phi_{\rm int}(r) = \Phi(0) - \int_0^r \frac{\Psi(y)}{y}dy - 2\zeta(r) + \frac{2G_N}{3}\tilde{S}(0) + 4\pi G_N\int_0^r p(y)y\,dy, \tag{3.29}$$

where the integration constant $\Phi(0)$ has to be chosen so as to match the external solution (3.28) at the star's surface. Again, although $\Phi_{\rm int}$ is expressed exactly in terms of (multiple integrations of) ρ and p, in practice it will be computed by performing the integrations numerically.

IV. GRAVITY IN THE LINEAR REGIME

A. Consistency of the approximation

Let us comment now on the meaning and consistency of the assumptions (3.2) and (3.19), which we may group as follows:

$$|\Phi| \ll 1, \qquad |\Psi| \ll 1, \qquad |\zeta| \ll 1, \tag{4.1}$$

$$r|\Phi' + \Psi'| \ll 1, \qquad r|\zeta'| \ll 1, \qquad rm^2\zeta^2 \ll |\zeta'|, \tag{4.2}$$

and on the assumption that the term containing ζ'^2 can be neglected in Eq. (2.10).

1. External solutions

Let us consider the external solutions. From the expressions (3.13), (3.27) and (3.28) it is apparent that the first group of assumptions (4.1) is essentially equivalent (unless $mr_\star$ is much bigger than one) to the condition of the characteristic radii r_ζ and r_g being much smaller than the star's radius,

$$r_\zeta \ll r_\star, \qquad r_g \ll r_\star. \tag{4.3}$$

It is not difficult to see that, in this case, the assumptions (4.2) are automatically satisfied. Note in fact that

$$r|\Phi' + \Psi'| = \left|\frac{r_\zeta}{6r}f_1(mr) - \frac{r_g}{r}\right|, \qquad r|\zeta'| = \frac{r_\zeta}{6r}f_2(mr),$$
$$\frac{rm^2\zeta^2}{|\zeta'|} = \frac{r_\zeta}{6r}f_3(mr),$$

where we introduced the functions

$$f_1(x) = x^2e^{-x}, \qquad f_2(x) = (1+x)e^{-x},$$
$$f_3(x) = \frac{x^2}{1+x}e^{-x}.$$

Since f_1, f_2 and f_3 are smaller than 1 on $[0, +\infty)$, the thesis follows.

When $mr_\star \gg 1$, instead, the condition $r_\zeta \ll r_\star$ becomes superfluous, since the exponential e^{-mr} suppresses all the terms containing r_ζ. Therefore, in this case the assumptions (4.1) and (4.2) are equivalent to the condition $r_g \ll r_\star$ only. This is consistent with GR being recovered in the $m \to \infty$ limit, since in GR the nonlinear terms in the equations of motion are negligible whenever the characteristic radius (i.e., the Schwarzschild one) is much smaller than the star's radius.

Regarding the role of the ζ'^2 term in Eq. (2.10), let us consider the latter equation imposing the conditions (4.1) and (4.2) but without neglecting ζ'^2. Instead of (3.20b), we obtain

$$(r\Psi)' = -r^2m^2\zeta - 2r^2\zeta'^2 - \frac{4\pi G_N}{3}r^2(2\rho + 3p), \tag{4.4}$$

which can be readily integrated (in vacuum) to give

$$\Psi_{\rm ext}(r) = (1+mr)\frac{r_\zeta}{6r}e^{-mr} - \frac{r_g}{2r} + (2+mr)\left(\frac{r_\zeta}{6r}e^{-mr}\right)^2. \tag{4.5}$$

By comparing (4.5) and (3.27), it is apparent that the term in (2.10) containing ζ'^2 just produces a second order correction to $\Psi_{\rm ext}$, which can be consistently neglected whenever the conditions (4.1) hold.

We conclude that, as far as the external solutions are concerned, only the weak-field conditions (4.1) need to be assumed, since the conditions (4.2) and the negligibility of the ζ'^2 term follow from them. In particular this implies that the nonlinear terms in the equations of motion are negligible whenever the characteristic radii r_ζ and r_g are much smaller than the star's radius, which nicely generalizes the analogous result which holds in GR.

2. Internal solutions

Regarding the internal solutions, the relationship between the assumptions (4.1) and (4.2) is less clear. This is so because the internal solutions depend on the *functions* $\rho(r)$ and $p(r)$, instead of on the numbers r_ζ and r_g, and do so in a fairly complicated way.

On general grounds, we expect the scalar curvature ζ/α to increase when we move inwards, towards the center of the star. This expectation can be substantiated as follows. Let us divide $\zeta_{\rm int}$ into a "density" and a "pressure" part,

$$\zeta_{\rm int}(r) = \zeta^{\rho}_{\rm int}(r) - 3\zeta^{p}_{\rm int}(r), \qquad (4.6)$$

where

$$\zeta^{\rho}_{\rm int}(r) = \frac{4\pi G_N}{3}\left(\frac{e^{-mr}}{r}\int_0^r \frac{\sinh my}{my}\rho y^2 dy + \frac{\sinh mr}{mr}\int_r^{r_\star} e^{-my}\rho(y) y dy\right), \qquad (4.7)$$

and $\zeta^{p}_{\rm int}$ is obtained by substituting $\rho \to p$ in the expression above. Let us consider the case $mr_\star \ll 1$. It is then easy to check that, if ρ and p are positive, then $\zeta^{\rho}_{\rm int}$ and $\zeta^{p}_{\rm int}$ are positive and the derivatives $\zeta^{\rho}_{\rm int}{}'$ and $\zeta^{p}_{\rm int}{}'$ are negative, so it follows that $|\zeta_{\rm int}|$ is decreasing in the direction of increasing r. In the case $mr_\star \gg 1$, on the other hand, ζ/α is approximately given by the GR relation $\zeta/\alpha = R = 8\pi G_N(\rho - 3p)$. It is then evident that, if ρ and p are positive and decreasing, then $|\zeta_{\rm int}|$ again is decreasing (in the direction of increasing r). Since it is natural to expect ρ and p to have the mentioned properties, the analysis of these two limits indeed suggests $|\zeta_{\rm int}|$ to be decreasing. Were this the case, the assumption $|\zeta| \ll 1$ (inside *and* outside the star) would be implied by the condition

$$|\zeta_{\rm int}(0)| = \frac{4\pi G_N}{3}\int_0^{r_\star} e^{-my}(\rho(y) - 3p(y)) y dy \ll 1. \qquad (4.8)$$

Overall it is likely that, to thoroughly understand the relationship between the assumptions (4.1) and (4.2) inside the star, a detailed discussion about the properties of ρ and p is needed. Here we prefer not to embark in this discussion, and simply limit our analysis to those configurations where (4.1) and (4.2) hold, and where the term containing ζ'^2 can be neglected in Eq. (2.10). We check *a posteriori*, by numerical means, that these assumptions are justified for the configurations analyzed in Sec. V.

B. The behavior of gravity

We are now in the position of drawing three interesting conclusions.

1. Relationship between the masses

The first is a relationship between the geometric mass $M_g = -C/G_N$ and the masses $M_\star$ and $\tilde{M}_\star$ computed from the density of the star. From Eq. (3.26) we have in fact

$$M_g = \left(\frac{\tilde{M}_\star}{3} - \tilde{P}_\star\right)e^{-mr_\star}(1 + mr_\star) + \frac{2M_\star}{3} + P_\star + \Xi_\star, \qquad (4.9)$$

so the geometrical mass takes into account the amount of curvature inside the star (through $\Xi_\star$), its integrated density and pressure, and depends explicitly on the range of the extra scalar degree of freedom. The difference between the two masses can be expressed as follows:

$$M_g - M_\star = \left(\frac{\tilde{M}_\star}{3} - \tilde{P}_\star\right)e^{-mr_\star}(1 + mr_\star) - \frac{M_\star}{3} + P_\star + \Xi_\star. \qquad (4.10)$$

It is not immediately evident which mass is larger, since both $\tilde{M}_\star/3 - \tilde{P}_\star$ and $M_\star/3 - P_\star$ are non-negative (for nonexotic matter at least).

In the $m \to \infty$ limit, which corresponds to the GR limit $\alpha \to 0$, from Eq. (3.3) we recover the GR result $\zeta/\alpha \to 8\pi G_N(\rho - 3p)$ and, thus,

$$\Xi_\star \underset{m\to\infty}{\to} \frac{M_\star}{3} - P_\star \quad \text{and} \quad M_g \underset{m\to\infty}{\to} M_\star, \qquad (4.11)$$

as expected.[5] In the opposite limit $m \to 0$, which means that the range of ζ tends to infinity, taking into account (3.10) we get

[5]The result $\zeta/\alpha \to 8\pi G_N(\rho - 3p)$ when $m \to \infty$ can also be obtained from the expression (3.5), by recognizing an appropriate realization of the Dirac delta inside the integral.

$$M_g - M_\star \underset{m\to 0}{\longrightarrow} \Xi_\star. \tag{4.12}$$

Since the expression (3.14) implies that ζ remains bounded when $m \to 0$, from the definition (3.22) it follows that in this limit $\Xi_\star \to 0$ and therefore

$$M_g - M_\star \underset{m\to 0}{\longrightarrow} 0. \tag{4.13}$$

In the forthcoming paper [59] we discuss the magnitude and physical meaning of the difference between M_g and $M_\star$, focusing for the sake of concreteness on neutron stars and their mass-radius relation.

2. The PPN gamma parameter

The second interesting conclusion that we draw from our analysis is a prediction on the post-Newtonian parameter γ,

$$\gamma = \frac{\Psi}{\Phi}, \tag{4.14}$$

which from Eqs. (3.27) and (3.28) can be expressed as follows:

$$\gamma = \frac{3r_g - r_\zeta e^{-mr}(1+mr)}{3r_g + r_\zeta e^{-mr}}, \tag{4.15}$$

and explicitly

$$\gamma = \frac{(\tilde{M}_\star - 3\tilde{P}_\star)e^{-mr_\star}(1+mr_\star) + 2M_\star + 3P_\star + 3\Xi_\star - (\tilde{M}_\star - 3\tilde{P}_\star)e^{-mr}(1+mr)}{(\tilde{M}_\star - 3\tilde{P}_\star)e^{-mr_\star}(1+mr_\star) + 2M_\star + 3P_\star + 3\Xi_\star + (\tilde{M}_\star - 3\tilde{P}_\star)e^{-mr}}. \tag{4.16}$$

Well outside the range of the scalar degree of freedom, that is for radii $r \gg m^{-1}$, we recover $\gamma = 1$ as expected (the spacetime being asymptotically flat).

Let us however suppose that $mr_\star \ll 1$, and consider the region well inside the range of the scalar degree of freedom (that is, $r_\star \le r \ll m^{-1}$). Recalling Eq. (3.10) and the fact that $\Xi \to 0$ when $m \to 0$, we get

$$\gamma \simeq \frac{2M_\star + 3P_\star}{4M_\star - 3P_\star} \quad \text{for } r_\star \le r \ll m^{-1}, \tag{4.17}$$

which in general is smaller than 1 but larger than 1/2, unless the pressure vanishes. This relation generalizes the results of various papers in the literature, for example [48–50], which find $\gamma = 1/2$ by neglecting the pressure. It is worthwhile to recall that the condition of asymptotic flatness is consistent for models where f is analytic in $R = 0$, while this is not true in models like, for example, $f(R) = R - \mu^4/R$. Therefore, our result should be properly seen as a generalization of the results concerning the former class of $f(R)$ models.

3. Relativistic (conformal) matter

Note that, for a hypothetical star composed exclusively of relativistic matter and radiation, we obtain $\gamma = 1$ independently from the range of the extra scalar degree of freedom. This agrees with our previous comment at the end of Sec. III A 1, about the extra degree of freedom being masked. Interestingly, we may expect the central part of a neutron star to be described by the equation of state $p = \rho/3$ when the central density is high enough, although with certainty it will not be so for the outer layers [60]. While this means that the masking of the scalar degrees of freedom (DOF) cannot be relevant for ordinary stars, it leaves open the possibility of this behavior being relevant for exotic stars. In particular, the bag model for quark stars [61–64] predicts an equation of state of the form $p = \rho/3 + B$, where B is the bag model constant, whose value is taken to be about $B \sim 100\ \text{MeV}^4$. In that case

$$\gamma = \frac{M_\star + B_\star}{M_\star - B_\star}, \tag{4.18}$$

where $B_\star \equiv \int_{V_\star} B dV$. Considering a star with the same mass of the sun $M_\star \sim M_\odot$, we get

$$\gamma \sim 1.2. \tag{4.19}$$

Both the conclusions $\gamma = 1$ and $\zeta = 0$ could have been foretold directly from the fourth order equation (2.2). Let us consider the case of conformal matter (i.e., matter whose energy-momentum tensor is trace-free). Taking first of all the trace of the Eq. (2.2), we get the following second order equation for R:

$$\Box R - m^2 R = 0. \tag{4.20}$$

Considering static configurations, the only regular and asymptotically decaying solution of the latter equation is $R = 0$. This conclusion relies crucially on the condition $T = 0$ being valid everywhere (i.e., also at the origin). Inserting $R = 0$ back into the Eq. (2.2) we get

$$R_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{4.21}$$

which is Einstein's equation in the presence of conformal matter. This shows that, at least for static, regular and asymptotically flat configurations, Starobinsky $f(R)$

gravity gives the same predictions as GR when only conformal matter is present.

V. NUMERICAL SOLUTIONS

In this section we support our analytic study in the linear approximation with fully numerical solutions of the nonlinear system of Eq. (2.9). The part which especially calls for confirmation is Sec. IVA 2, which concerns our assumptions about the behavior of the fields Φ, Ψ and ζ inside the star. While we showed that, outside the star, our approximations follow from the weak-field conditions $|\Phi| \ll 1$, $|\Psi| \ll 1$, and $|\zeta| \ll 1$, regarding the interior of the star we just assumed this to be the case. It is important to understand whether this indeed happens, and whether it is a quite general property or not. To investigate this, and in particular probe the generality of our assumptions, it is convenient to consider a star configuration which is close to the limit of validity of the linear approximation. For this reason we decide to use neutron stars as a testbench for our analysis.

This choice may seem questionable, since neutron stars are naively associated with the idea of the gravitational field being "strong." Recall however that, for what concerns static and spherically symmetric solutions in GR, the nonlinear terms in the equations of motion become of the same order of the linear ones at the Schwarzschild radius, which means that outside a (static and spherically symmetric) neutron star the former are subdominant (although in general not negligible). To compromise with the request of the linear approximation being valid, we consider neutron stars with suitably small central density. We model our star with the polytropic equation of state:

$$p = k\rho^2, \tag{5.1}$$

with

$$k = 4.012 \times 10^{-4}\ \mathrm{fm}^3/\mathrm{MeV}. \tag{5.2}$$

The choice of this equation of state, which admittedly provides only an approximated description of a neutron star [65], is justified because our main aim here is testing the validity of the analysis of Secs. III and IV. For this reason we do not feel it is necessary to use a more realistic Sly equation of state [66].

A. Numerical strategy

As a preparation to the numerical integration, we recast the system (2.9) in terms of dimensionless quantities. To this aim, we choose to normalize the quantities under consideration to the Sun's half Schwarzschild radius:

$$r_0 \equiv \frac{G_N M_\odot}{c^2} \simeq 1.5\ \mathrm{km}, \tag{5.3}$$

and to provide a transparent dimensional analysis we temporarily reinstate the constant c. We therefore introduce the dimensionless radial coordinate $x = r/r_0$, and work with the following dimensionless quantities:

$$\hat{\rho}(x) = \frac{G_N r_0^2}{c^4}\rho(r_0 x), \qquad \hat{p}(x) = \frac{G_N r_0^2}{c^4}p(r_0 x),$$
$$\hat{\alpha} = \frac{\alpha}{r_0^2}, \qquad \hat{k} = \frac{c^4}{G_N r_0^2}k \simeq 139, \tag{5.4}$$

along with $\hat{\zeta}(x) = \zeta(r_0 x)$, $\hat{A}(x) = A(r_0 x)$ and $\hat{B}(x) = B(r_0 x)$.

To find the numerical solutions we employ a shooting method, imposing the initial conditions at the center of the star. Note on this respect that, although the Eqs. (2.9) are formally singular at $r = 0$, they admit solutions which are regular at the origin. It is not difficult to see that, asking the pressure and the density at the center of the star to be finite, solutions which are analytic in $r = 0$ can exist only provided

$$A'(0) = 0 \quad B(0) = 1 \quad B'(0) = 0 \quad \zeta'(0) = 0. \tag{5.5}$$

Furthermore, the Eqs. (2.9) are invariant under the rescaling of A by an arbitrary constant. It follows that changing the initial condition $A(0) = A_0$ results simply in a rescaling of the function $A(r)$, while leaving $B(r)$ and $\zeta(r)$ unchanged. We are therefore free to choose the initial condition $A_0 = 1$, and *a posteriori* rescale the function $A(r)$ such that it tends to 1 far from the star. This leads us to consider the conditions

$$\hat{A}(0) = 1 \quad \hat{B}(0) = 1 \quad \hat{\zeta}(0) = \hat{\zeta}_0 \quad \frac{d\hat{\zeta}}{dx}(0) = 0 \quad \hat{\rho}(0) = \hat{\rho}_0, \tag{5.6}$$

where the pressure $\hat{p}$ has not been included since it is determined by $\hat{\rho}$ via the equation of state. The central density $\hat{\rho}_0$ and central curvature $\hat{\zeta}_0$ are free parameters. We choose the former *a priori* and then determine $\hat{\zeta}_0$ via the shooting, selecting the solution for $\hat{\zeta}$ which decays exponentially to zero far from the star [meaning at $x \gg \sqrt{\hat{\alpha}} \sim (m r_0)^{-1}$] where the external solution Eq. (3.8) is valid. The star's radius is determined to be that for which $\hat{\rho}$ becomes negative.

There is a technical subtlety regarding the initial conditions. Since the formal singularity at $r = 0$ is problematic for the numerical integrator, it is not possible to set the initial conditions for the numerical code exactly at $x = 0$. We therefore use $x_i = 10^{-5}$ as the initial radius for the numerical code. Plugging into the Eqs. (2.9) the Taylor expansion of $\hat{A}$, $\hat{B}$, $\hat{\zeta}$, $\hat{\rho}$ and $\hat{p}$ around $x = 0$, and working at first order in x_i, it can be seen that the conditions (5.6) imply the following initial conditions:

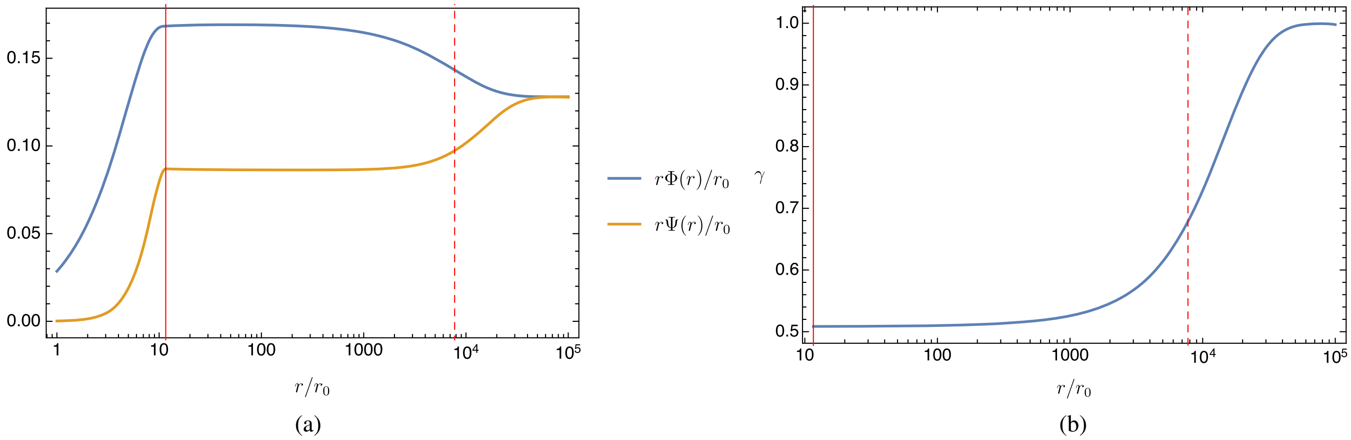

FIG. 1. The behavior of the gravitational field. The continuous vertical line marks the star's surface while the dashed line marks the range of the scalar DOF. (a) The gravitational potentials times r/r_0. (b) The PPN parameter γ.

$$\hat{A}(x_i) = 1, \qquad \hat{B}(x_i) = 1, \qquad \hat{\rho}(x_i) = \hat{\rho}_0, \qquad \hat{\zeta}(x_i) = \hat{\zeta}_0, \tag{5.7}$$

and

$$\frac{d\hat{\zeta}}{dx}(x_i) = \frac{1}{6\times 10^5}\left[\frac{\hat{\zeta}_0}{\hat{\alpha}} + 8\pi\hat{\rho}_0(3\hat{k}\hat{\rho}_0 - 1)\right]. \tag{5.8}$$

B. Numerical results

We choose for definiteness the model with $\alpha = 10^7 r_0^2$ and consider a neutron star with central density $\rho_0 = 4\times 10^{13}$ g/cm^3. The star's radius inferred from the shooting is $r_\star \simeq 11.60 r_0$, while the range of the scalar degree of freedom is $m^{-1} \simeq 7.746\times 10^3 r_0$, so the ratio of the former to the latter is $m r_\star \simeq 1.498\times 10^{-3}$. This implies that we are in the regime where (4.17) should hold, that is the regime where the range of the scalar degree of freedom is much bigger than the star's radius.

The behavior of the gravitational field is clearly illustrated in the Figs. 1(a) and 1(b). In Fig. 1(a) the gravitational potentials, multiplied by r/r_0, are plotted. In agreement with the analysis of the previous sections, it is apparent that there are two Newtonian regions, one for $r_\star \leq r \ll m^{-1}$ and one for $r \gg m^{-1}$, with a non-Newtonian transition region in between. This behavior is confirmed by the plot of the PPN parameter γ, displayed in Fig. 1(b), which tends asymptotically to unity as expected. Noteworthy, the value of the γ parameter outside the star is $\gamma_\star \simeq 0.5085$, which is higher than 1/2 at the 1% level.

In Figs. 2(a) and 2(b) the gravitational potentials (this time not multiplied by r/r_0) and the scalar degree of freedom ζ are plotted. It is easy to see that they are everywhere of the order 10^{-2} or smaller, so they satisfy the weak field conditions $|\Phi| \ll 1$, $|\Psi| \ll 1$ and $|\zeta| \ll 1$. In relation to the discussion of Sec. IV A, the numerically inferred gravitational radii are $r_\zeta = 0.2503 r_0$ and $r_g = 0.2561 r_0$, a factor 10^2 smaller than the star's radius. According to our analysis, this should imply the conditions (4.2) to be satisfied outside

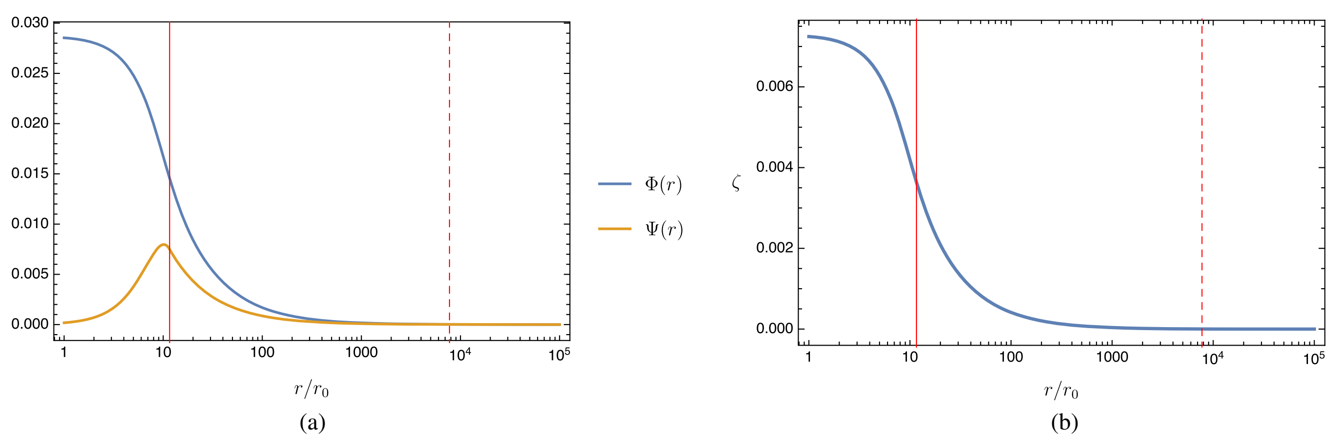

FIG. 2. The amplitude of the gravitational potentials and the scalar DOF ζ. The meaning of the vertical lines is the same as above. (a) The gravitational potentials. (b) The scalar DOF ζ.

the star. Indeed, we verified numerically that all the terms appearing in (4.2) are of order 10^{-2} or smaller. Most importantly, this holds also *inside* the star, which confirms our assumption about the internal behavior of the solutions (see Sec. IV A 2). In particular ζ is indeed monotonically decreasing, as suggested and motivated in the aforementioned section. Overall, these numerical results strongly support the consistency of our approximations and the validity of the analysis of Secs. III and IV.

Let us comment on the relation (4.17). The numerical analysis gives the following estimates:

$$M_\star = 0.128682\ M_\odot \quad P_\star = 4.7918 \times 10^{-4}\ M_\odot$$
$$\Xi_\star = 4.11868 \times 10^{-8}\ M_\odot \tag{5.9}$$

and indicates that $\tilde{M}_\star$ is equal to $M_\star$ to the fifth significant digit (the same happens for $\tilde{P}_\star$ and $P_\star$). The smallness of $\Xi_\star$, $\tilde{M}_\star - M_\star$ and $\tilde{P}_\star - P_\star$ is expected from (3.10) and from the analysis of Sec. IV B 1, and implies that approximating (4.16) with (4.17) is justified in the context of the linear approximation.

VI. CONCLUSIONS AND COMMENTS

In this paper we studied static, spherically symmetric and asymptotically flat configurations of Starobinsky $f(R)$ gravity in the metric formalism. We focused on the behavior of gravity in the linear regime in the presence of a static star, without *a priori* assuming the pressure of the latter to be negligible.

We found that the PPN parameter γ outside the star and well inside the range of the scalar degree of freedom can be larger than $1/2$. We provided a relation which explicitly expresses γ in terms of the mass, of the integrated pressure of the star and of the ratio between the star's radius and the range of the extra degree of freedom. We showed explicitly that at least two different, sensible notions of mass (frequently used in the literature for analyzing e.g., the stability of neutron stars) are possible in the context of the $f(R)$ model under consideration (and possibly also for more general models). We derived an analytic formula relating the two, showing, as expected, that they coincide in the GR limit. Though aware that this might be an unrealistic configuration, we also showed that in the limit where the matter content forming the star tends to pure radiation, then no effect from the extra scalar degree of freedom appears, as if it were masked.

Our analysis and results are based on a set of assumptions, which include the usual weak-field conditions, whose consistency was analytically proved only for the exterior solutions of the fields. The interior ones are described by semianalytic formulas and depend on integrals which can be computed, in practice, only numerically. Therefore, in order to assess the viability of our assumptions regarding the interior of the star, we performed a numerical analysis considering for concreteness the case of a neutron star. We found that the conditions we assumed to hold inside the star are indeed satisfied, thus providing firm ground to our analytic study and results also in that range of radii.

It is worthwhile at this point to discuss how our analysis, and in particular our choice of considering asymptotically flat configurations, relates to that of [48], where a (homogeneous) cosmological background, with associated cosmological energy-momentum tensor, is included. Note that in [48], whenever the gravitational field outside a star is studied, the time dependence of the cosmological background is neglected (with the sensible justification that the cosmological evolution is slow compared to the typical timescales of, say, the Solar System). In this case, therefore, the role of the cosmological background reduces purely to setting the boundary conditions for the metric and the scalar curvature, when the distance from the star tends to infinity. Allowing for a nonzero cosmological value R_0 for the scalar curvature is indeed crucial for a generic $f(R)$ theory, since $R = 0$ is not necessarily an allowed configuration of the theory [for example, it is not allowed if $f(R)$ contains negative powers of R]. However, for the Starobinsky model the cosmological background given by $R_0 = 0$ *is* an allowed configuration. As a matter of fact, when the Starobinsky model is studied in [48], they set to zero both the cosmological energy-momentum tensor and the background value of the scalar curvature, that is they take $R_0 = 0$. This shows that considering asymptotically flat solutions not only is compatible with the approach of [48], being equivalent to considering a homogeneous and static cosmological background with $R_0 = 0$, but actually fits nicely in that approach.

A natural further development of this work would be to generalize its results for a generic function $f(R)$, at least qualitatively [since a numerical analysis without specifying a functional form for $f(R)$ is impossible]. Moreover, in a forthcoming paper [59] we are analyzing a realistic model of neutron star, focusing on the mass-radius relation, in order to understand how the use of the two definitions of mass introduced in the present paper influence the analysis of the properties of a neutron star and their estimation from observational data.

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