Nonlinearly charged scalar-tensor black holes in (2 + 1) dimensions

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The action of three-dimensional charged Einstein-dilaton gravity theory has been obtained from that of scalar-tensor modified gravity theory by utilizing the suitable conformal transformations. The field equations of the Einstein-dilaton gravity coupled to the power Maxwell nonlinear electrodynamics have been solved and two new classes of static and spherically symmetric charged dilatonic black holes, as the exact solutions to the coupled scalar, electromagnetic and gravitational field equations, have been obtained. Also, the dilaton potential has been written as the linear combination of two Liouville-type potentials. The black hole conserved charges and thermodynamic quantities have been calculated by utilizing the geometrical and thermodynamical methods, separately. The compatibility of the results obtained from these two alternative approaches confirms the validity of the first law of black hole thermodynamics for both of the new black hole solutions in the Einstein frame. A black hole stability or phase transition analysis has been performed in the context of the canonical ensemble. By calculating the black hole heat capacity, with the black hole charge as a constant, the type one and type two phase transition points have been determined. Also, the ranges of the black hole horizon radii at which the Einstein black holes are thermally stable have been identified for both of the new black hole solutions. Then making use of the inverse conformal transformations, two new classes of the scalar-tensor black holes have been obtained from their Einstein frame counterparts. The thermodynamic properties and thermal stability of the new scalar-tensor black holes have been investigated. It has been found that the new charged black holes have the same thermodynamic behaviors in both of the Einstein and Jordan frames.

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I. INTRODUCTION

The fact that, unlike the result of the standard Friedmann model, our Universe is in the accelerated expanding phase has created increasing interest in the alternative theories of modified gravity [1–5]. There are a variety of modified gravity theories which are proposed with the aim of explaining the failures of Einstein's theory of gravity [6–14]. The scalar-tensor gravity theory is one of the alternative theories of modified gravity which is conformally related to Einstein's theory of relativity minimally coupled to a scalar field, the well-known Einstein-dilaton gravity theory [15–18].

The action of scalar-tensor gravity theory, in which a scalar field interferes with the Ricci scalar, can be described in two popular frameworks. One is called the Jordan frame in which the scalar field appears in the action as a function multiplied by the Ricci scalar. Also in the field equations, derived from the action by means of a variational principle, it produces a strong coupling between the scalar and gravity field equations. In the second frame, named the Einstein frame, the theory appears to be simpler and similar to Einstein's general relativity with the coupled scalar fields which appear as additional terms in the action [19–24]. It is well known that the actions of these two frames are related to each other via a conformal transformation. Indeed the nonminimal coupling of the gravity and scalar fields, in the action and consequently in the field equations, is broken by use of a conformal transformation in the form of $g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$ [25,26]. It has been shown that these two formalisms are equivalent at the quantum level [27].

The Lagrangian of Maxwell's electrodynamics remains invariant under the transformations $g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$ and $A_\mu \to A_\mu$ in the four-dimensional space-times. This conformal invariance is violated in the space-times with dimensions other than 4 [28]. The mentioned failure and the other challenges of the classical theory of electrodynamics, such as appearance of the infinite field and self-energy for the pointlike charges, were the original motivations for introducing the nonlinear theories of electrodynamics such as Born-Infeld [29–31], logarithmic [32–35], exponential [36–39] and power-law [40,41] nonlinear electrodynamics. The models of nonlinear electrodynamics are written as the functions of the Maxwell invariant $F^{\alpha\beta}F_{\alpha\beta}$. Many believe that the powers of the Maxwell invariant, which have been neglected in Maxwell's (or linear) electrodynamics, can be interpreted as photon-photon interactions. They are important when electromagnetic fields are very strong [42,43].

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Interestingly, the Lagrangian of power-law nonlinear electrodynamics preserves its conformal invariance in three-, four-, and higher-dimensional space-times. Indeed, this model of nonlinear electrodynamics is generally conformal invariant provided that the power is chosen equal to one-fourth of the space-time dimensions [44–46]. Motivated by this significant property we prefer to investigate the charged scalar-tensor black hole solutions with the power-law nonlinear electrodynamics in a three-dimensional space-time.

On the other hand, the findings of Hawking *et al.*, which state that the black holes are thermodynamic objects with the well-defined temperature and entropy, are the more outstanding achievements in the context of the black hole physics [47–52]. The black hole thermodynamics and thermodynamic stability are the interesting topics to be investigated. The black hole thermal stability analysis has several popular methods for its description. In the method of geometrical thermodynamics the stability of the black holes can be studied regarding the thermodynamical Ricci scalar [35,53]. The information related to the black hole thermal stability can be extracted from the black hole heat capacity if the method of canonical ensemble is used [38,54,55]. In the grand canonical ensemble approach one is able to explore the black hole thermodynamic stability by calculating the Hessian determinant [46,56,57].

The exact black hole solutions of the four-dimensional scalar-tensor gravity theory, in the presence of the nonlinear electrodynamics, with the related thermodynamic properties have been studied in Ref. [25]. The thermodynamic phase transition or stability of charged scalar-tensor black holes with Maxwell's theory of electrodynamics has been considered in my previous work [26]. Here, in the same direction and in order to extend this idea to the case of nonlinear electrodynamics, we turn to the investigation on the scalar-tensor gravity theory in the presence of power-law electrodynamics with the aim of finding the exact three-dimensional black hole solutions. Then we consider the thermodynamic properties and especially prove the validity of the thermodynamical first law. Eventually, we perform a stability analysis by treating the black holes as the canonical ensembles.

This paper is outlined as follows. In Sec. II, it is shown that the action of three-dimensional scalar-tensor modified gravity theory, coupled to the power-law nonlinear electrodynamics, is related to that of Einstein-power Maxwell-dilaton gravity theory via the suitable conformal transformations. In Sec. III the explicit forms of the Einstein-dilaton gravity field equations are obtained in a spherically symmetric geometry. Section IV is devoted to solving the coupled field equations of the theory. The dilatonic potential, as the solution to the scalar field equation, is written as the combination of two Liouville-type potentials. Also, two new classes of three-dimensional dilatonic black hole solutions are obtained. In Sec. V the conserved and thermodynamic quantities related to the new dilatonic black holes are calculated from the geometrical and thermodynamical approaches. It is proved that these quantities satisfy the first law of thermodynamics. Section VI is dedicated to analyzing the local stability or phase transition of the new dilatonic black hole solutions, obtained here. Regarding the black hole heat capacity, with the black hole charge as a constant, the type one and type two phase transition points as well as the range of horizon radii for the black holes to be locally stable are determined. In Sec. VII, by applying the inverse transformation relations, two new classes of the scalar-tensor black holes are obtained from their Einstein counterparts. The thermodynamical properties as well as the thermal stability or phase transition of the new scalar-tensor black holes are analyzed in the Jordan frame. The results are summarized and discussed in Sec. VIII.

II. THE ACTION IN THE JORDAN AND EINSTEIN FRAMES

The action of three-dimensional scalar-tensor theories can be written in the Jordan frame or in the Einstein frame. In the Jordan frame there is a nonminimal coupling between the gravity and scalar fields. In the Einstein frame, which is related to the Jordan frame through a conformal transformation, the coupling between scalar and gravitational fields is removed and finding of the solutions is easier. Let us start with the action of the scalar-tensor theory in the Jordan frame [19], that is

$$I^{(ST)} = -\frac{1}{16\pi}\int \sqrt{-\tilde{g}}[X(\psi)\tilde{\mathcal{R}} + Y(\psi)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\nu\psi - 2Z(\psi) + L(\tilde{\mathcal{F}})]d^3x. \tag{2.1}$$

Here, $\tilde{\mathcal{R}} = \tilde{g}^{\mu\nu}\tilde{\mathcal{R}}_{\mu\nu}$ is the Ricci scalar in the space-time identified by the metric components $\tilde{g}_{\mu\nu}$. It is multiplied by a function of scalar field ψ labeled by $X(\psi)$. Other functions of ψ [i.e., $Y(\psi)$ and $Z(\psi)$] are arbitrary functions to be determined. In this frame, the covariant derivative with respect to $\tilde{g}_{\mu\nu}$ is denoted by $\tilde{\nabla}$. The last term is the matter field which is considered as the function of the Maxwell invariant $\tilde{\mathcal{F}} = \tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$. The covariant and contravariant forms of the electromagnetic tensors are shown via $\tilde{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $\tilde{F}^{\rho\lambda} = \tilde{g}^{\rho\alpha}\tilde{g}^{\lambda\beta}\tilde{F}_{\alpha\beta}$, respectively. We consider the explicit form of the matter field in the functional form [45]

$$L(\tilde{\mathcal{F}}) = (-\tilde{\mathcal{F}})^p, \tag{2.2}$$

which is known as the power-law model of nonlinear electrodynamics with p as the power.

Although the field equations of Jordan formalism can be obtained by varying the action (2.1) with respect to different fields, because of the strong coupling between the scalar and gravitational fields, they are too difficult to be solved directly. Thus, by use of a suitable conformal transformation, we translate it to the Einstein-dilaton action in the Einstein frame [25]. For this purpose, by use of the method of Stefanov *et al.* [20–22], we proceed with the following conformal transformations,

$$\tilde{g}_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\Omega(\psi))^2 g_{\mu\nu}, \tag{2.3}$$

which relate the components of the Jordan metric to those of the Einstein metric $g_{\mu\nu}$. Also, we introduce a new scalar field ϕ which is related to ψ via the following differential equation [19]:

$$\left(\frac{d\ln\Omega(\psi)}{d\psi}\right)^2 + \frac{Y(\psi)}{2\Omega(\psi)} = \left(\frac{d\phi}{d\psi}\right)^2. \tag{2.4}$$

It is worth mentioning that in order for the energy carried by the scalar field to be positive valued, $\Omega(\psi)$ must be positive, and noting Eq. (2.4) one obtains [19]

$$2\left(\frac{d\Omega(\psi)}{d\psi}\right)^2 + \Omega(\psi)Y(\psi) \geq 0. \tag{2.5}$$

Now, by use of the conformal transformations (2.3) and using the following definitions,

$$\Omega(\psi) = X^{-1}(\psi), \quad \text{and} \quad Y(\psi) = -2X(\psi), \quad \text{and}$$
$$V(\phi) = 2Z(\psi)X^{-3}(\psi), \tag{2.6}$$

$$L(\mathcal{F},\phi) = X^{-3}(\phi)L(X^4(\phi)\mathcal{F}), \quad \text{with} \quad \psi = \psi(\phi), \tag{2.7}$$

one finds that the action (2.1) transforms to its new form in the Einstein frame. That is

$$I = -\frac{1}{16\pi}\int \sqrt{-g}[\mathcal{R} - V(\phi) - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + L(\mathcal{F},\phi)]d^3x. \tag{2.8}$$

Here, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is the Ricci scalar and ∇ is the covariant derivative in the Einstein frame identified by the metric $g_{\mu\nu}$. The scalar field ϕ is assumed to coupled to itself via the functional form $V(\phi)$ and $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ being the Maxwell invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the electromagnetic potential. Indeed Eq. (2.8) is just the action of three-dimensional Einstein-Maxwell-dilaton gravity theory provided that $L(\mathcal{F},\phi)$ is chosen as [58,59]

$$L(\mathcal{F},\phi) = (-\mathcal{F}e^{-2\alpha\phi})^p. \tag{2.9}$$

The parameter α is known as the scalar-electromagnetic coupling constant. The power p is known as the nonlinearity parameter and by setting $p = 1$ the Lagrangian density $L(\mathcal{F},\phi)$ reduces to the scalar coupled Maxwell field. Equations (2.7) and (2.9) show that

$$X(\phi) = e^{-\frac{2p\alpha}{4p-3}\phi}, \text{ and } \Omega(\phi) = e^{\frac{2p\alpha}{4p-3}\phi}, \quad p \neq \frac{3}{4}. \tag{2.10}$$

Note that $p = \frac{3}{4}$ corresponds to the case of conformally invariant power-law electrodynamics which will be considered later. In the next section we solve the field equations obtained by varying the action of Eq. (2.8) with respect to the different fields.

III. FIELD EQUATIONS IN THE EINSTEIN FRAME

The gravitational, electromagnetic and scalar field equations can be obtained by varying the action (2.8) with respect to the corresponding fields. They are

$$\mathcal{R}_{\mu\nu} = V(\phi)g_{\mu\nu} + 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}L(\mathcal{F},\phi) + 2L_\mathcal{F}(\mathcal{F},\phi)(\mathcal{F}g_{\mu\nu} - F_{\mu\sigma}F^\sigma_\nu), \tag{3.1}$$

$$\nabla_\mu[L_\mathcal{F}(\mathcal{F},\phi)F^{\mu\nu}] = 0, \qquad L_\mathcal{F}(\mathcal{F},\phi) \equiv \frac{\partial}{\partial\mathcal{F}}L(\mathcal{F},\phi), \tag{3.2}$$

$$4\Box\phi = \frac{dV(\phi)}{d\phi} + 2\alpha pL(\mathcal{F},\phi), \qquad \phi = \phi(r). \tag{3.3}$$

We are interested in obtaining the solution of these field equations in a static and spherically symmetric geometry. Therefore, we start with the following line element as an ansatz:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -W(r)dt^2 + \frac{1}{W(r)}dr^2 + r^2R^2(r)d\theta^2, \tag{3.4}$$

where $W(r)$ and $R(r)$ are two unknown functions to be determined. $W(r)$ is named as the metric function and $R(r)$ is a dimensionless function of r which indicates the impacts of dilaton field on the space-time geometry. It is evident that in the absence of a dilaton field it must reduce to unity. A notable point is that the line element presented in Eq. (3.4) is not the most general form of the spherically symmetric and static metric. It can be considered as the first simplification that one can make in order to find black hole solutions.

Making use of Eqs. (3.1) and (3.4), we arrive at the following differential equations:

$$e_{tt} \equiv rR(r)W''(r) + [R(r) + rR'(r)]W'(r) + 2rR(r)[V(\phi) + (p-1)L(\mathcal{F},\phi)] = 0, \tag{3.5}$$

$$e_{rr} \equiv e_{tt} + 2[rR''(r) + 2R'(r) + 2rR(r)\phi'^2(r)]W(r) = 0, \tag{3.6}$$

$$e_{\theta\theta} \equiv [R(r) + rR'(r)]W'(r) + [rR''(r) + 2R'(r)]W(r) + rR(r)[V(\phi) + (2p-1)L(\mathcal{F},\phi)] = 0, \tag{3.7}$$

for tt, rr and $\theta\theta$ components, respectively.

Assuming as a function of r, the only nonvanishing component of the electromagnetic field is $F_{tr} = -A_t'(r)$, and we have $\mathcal{F} = -2F_{tr}^2 = -2(-A_t'(r))^2$. Throughout the paper, prime means derivative with respect to the argument. By use of Eqs. (3.2) and (3.4), one can show that, with respect to a constant q_1, F_{tr} takes the following form:

$$F_{tr} = \frac{-q_1}{[rR(r)]^{\frac{1}{2p-1}}} e^{\frac{2p\alpha\phi}{2p-1}}, \qquad p \neq -\frac{1}{2}. \tag{3.8}$$

Noting Eqs. (3.5) and (3.6) we obtain

$$rR''(r) + 2R'(r) + 2rR(r)\phi'^2(r) = 0. \tag{3.9}$$

Now, it is the time to explore the consistency of the equations as well as to determine the number on independent equations. To do these, one can show that Eqs. (3.5) and (3.7) satisfy the following equation (see the Appendix):

$$\frac{de_{\theta\theta}}{dr} = \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)(e_{tt} - e_{\theta\theta}). \tag{3.10}$$

Therefore, the solution of Eq. (3.7) satisfies Eq. (3.5) and they are mutually consistent. Also, based on Eq. (3.10), Eqs. (3.5) and (3.7) are not independent. Indeed, we have five unknown functions $R(r)$, F_{tr}, $\phi(r)$, $V(\phi)$ and $W(r)$ which obey Eqs. (3.2), (3.3), (3.5), (3.6) and (3.7). This means that the number of unknown functions is one more than the independent equations. In order to overcome this problem, following the works of Chan and Mann [60,61], we start with an ansatz of the form

$$R(r) = \left(\frac{r}{r_0}\right)^N, \tag{3.11}$$

where r_0 is a dimensional constant. By substituting Eq. (3.11) into Eq. (3.9), one can show that

$$\phi(r) = \beta \ln\left(\frac{b}{r}\right), \quad \text{with} \quad \beta = \sqrt{-N(N+1)/2}. \tag{3.12}$$

It is valid for positive b and the N-values in the range $-1 < N \leq 0$. A similar power-law solution has been used previously for finding the three-, four-, and higher-dimensional black hole solutions [25,60–62].

In the following sections we proceed to obtain the solution of the field equations, making use of the power-law solution and the scalar field ϕ given in Eqs. (3.11) and (3.12), respectively.

IV. THE EINSTEIN-DILATON BLACK HOLES

In this section we explore the full solution of the field equations. For this purpose we reconsider Eq. (3.8) as the solution to the electromagnetic field equation. Making use of Eqs. (3.8), (3.11), and (3.12), and by use of a properly fixed integration constant q_1, one obtains

$$F_{tr} = -qr^{-\frac{B+1}{2p-1}}, \quad \text{with} \quad B = N + 2p\alpha\beta, \tag{4.1}$$

and noting the relation $F_{tr} = -\partial_r A_t(r)$ we have

$$A_t(r) = \frac{(2p-1)q}{2(p-1) - B} r^{1-\frac{B+1}{2p-1}}. \tag{4.2}$$

Here q is another constant related to the charge of a black hole. It must be noted that, in order for the potential $A_t(r)$ to be physically reasonable (i.e., zero at infinity) the following condition must be fulfilled:

$$\frac{B+1}{2p-1} > 1. \tag{4.3}$$

With the help of the above solutions in Eqs. (3.3) and (3.7), after some manipulations, we obtain the following differential equations:

$$W'(r) + \frac{N}{r}W(r) + \frac{r}{N+1}[V(\phi) + (2p-1)L(\mathcal{F},\phi)] = 0, \tag{4.4}$$

$$\frac{dV(\phi)}{d\phi} - \frac{4\beta}{N+1}V(\phi) + 2\left[\alpha p - \frac{2\beta(2p-1)}{N+1}\right]L(\mathcal{F},\phi) = 0, \tag{4.5}$$

for the metric function $W(r)$ and the dilatonic potential $V(\phi)$, respectively. The solution to the first order differential equation (4.5) can be easily obtained as

$$V(\phi) = C_2 e^{\frac{4\beta}{N+1}\phi} + \frac{2^p q^{2p}}{b^{\frac{2p(B+1)}{2p-1}}} \frac{\beta(2p-1)[2\beta(2p-1) - \alpha p(N+1)]}{(N+1)(p - N + 3Np + \alpha p\beta)} \times e^{\frac{2p(1+N+\alpha\beta)}{\beta(2p-1)}\phi}. \tag{4.6}$$

Since, in the absence of the dilaton field (i.e., $\phi = 0$ or equivalently $N = 0 = \beta$), the action (2.8) reduces to the action of Einstein-Maxwell gravity with a cosmological constant, one can obtain the constant C_2 by imposing the condition $V(\phi = 0) = 2\Lambda = -2\ell^{-2}$. By imposing this condition one obtains $C_2 = 2\Lambda$. Thus the dilaton potential takes the following form:

$$V(\phi) = 2\Lambda e^{2a\phi} + 2\Lambda_1 e^{2a_1\phi}, \tag{4.7}$$

where

$$a=\frac{2\beta}{N+1},\quad a_1=\frac{p(1+N+\alpha\beta)}{\beta(2p-1)},$$
$$\Lambda_1=-\frac{2^{p-1}(2p-1)q^{2p}}{b^{\frac{2p(B+1)}{2p-1}}}\Upsilon_1,\quad \Upsilon_1=\frac{p(2N+\alpha\beta)-N}{p(1+3N+\alpha\beta)-N}.$$

Regarding Eq. (4.7) one can argue that the dilaton potential can be written as the linear combination of two Liouville-type dilatonic potentials. Now, by substituting Eq. (4.7) into Eq. (4.4) and solving the related differential equation, the metric function $W(r)$ can be calculated as

$$W(r)=\begin{cases}-mr^{2/3}-3\left[2\Lambda b^2(\frac{r}{b})^{\frac{2}{3}}\ln(\frac{r}{\ell})-\frac{3p(2)^{p-1}q^{2p}}{b^{\eta}}\left(\frac{2p-1}{3p-p\alpha-2}\right)^2 r^2(\frac{b}{r})^{\frac{2p(\alpha+1)}{3(2p-1)}}\right], & N=-\frac{2}{3},\\ -mr^{-N}-\frac{1}{N+1}\left[\frac{2\Lambda b^2}{2+3N}(\frac{r}{b})^{2(1+N)}+\frac{(2p-1)^2(2q^2)^p(1-\Upsilon_1)}{(2p-2-B)b^{\lambda}}r^2(\frac{b}{r})^{2a_1\beta}\right], & N\neq-\frac{2}{3},\end{cases}\tag{4.8}$$

where m is the constant of integration and sometimes is named as the mass parameter and

$$\eta=\frac{2p(2p\alpha+1)}{3(2p-1)},\qquad \lambda=\frac{2p(B+1)}{2p-1}.\tag{4.9}$$

It must be noted that based on the condition presented in Eq. (4.4), the denominator of the second terms in the square brackets are nonzero. With the aim of finding the number of horizon radii, the plots of $W(r)$ vs r are shown in Figs. 1 and 2. The plots, which indicate the impacts of the important parameters, show that the black holes with two horizons, those with one horizon and naked singularity black holes can occur provided that the parameters are fixed properly.

An important point to be noted is that for the exact solutions presented in Eq. (4.8) to be considered as black holes it is necessary that the following two requirements be fulfilled, simultaneously. (1) The existence of at least one horizon radius, which has been clarified via plots of Figs. 1 and 2. (2) The appearance of curvature singularities which can be studied through the curvature scalars such as Ricci and Kretschmann scalars. It is a matter of calculation to show that they can be written in the following forms:

$$\mathcal{R}=-W''-(N+1)\frac{W'}{r}-2N(N+1)\frac{W}{r^2},\tag{4.10}$$

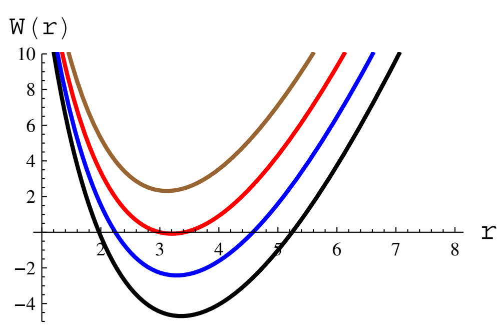
$$\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda}=(W'')^2+2N(N+2)\left(\frac{W'}{r}\right)^2+4N(N+1)^2\frac{WW'}{r^3}+4N^2(N+1)^2\left(\frac{W}{r^2}\right)^2.\tag{4.11}$$

By use of the metric functions (4.8) into Eqs. (4.10) and (4.11), as a matter of calculation, one can show that the Ricci and Kretschmann scalars are finite for finite values of r, and they fulfill the following relations:

$$\lim_{r\to\infty}\mathcal{R}=0,\qquad \lim_{r\to0}\mathcal{R}=\infty,\tag{4.12}$$

$$\lim_{r\to\infty}\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda}=0,\qquad \lim_{r\to0}\mathcal{R}^{\mu\nu\rho\lambda}\mathcal{R}_{\mu\nu\rho\lambda}=\infty.\tag{4.13}$$

Thus, there is an essential (not coordinate) singularity located at $r=0$, which is covered by the existence of the horizon radii. Based on the analysis performed here, we conclude that the exact solutions obtained in this work are



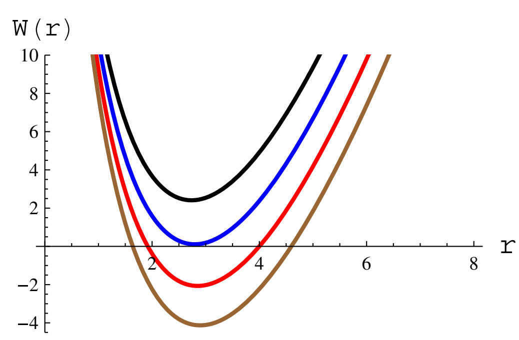


FIG. 1. $W(r)$ vs r for $N=-\frac{2}{3}$, $\Lambda=-1$, $b=2.5$, $Q=1$, $M=2$, $r_0=1$ [Eq. (4.8)]: (a) $\alpha=2.8$ and $p=1.16$, 1.18, 1.2, 1.22 for the black, blue, red and brown curves, respectively; (b) $p=1.5$ and $\alpha=3.14$, 3.17, 3.2, 3.23 for the black, blue, red and brown curves, respectively.

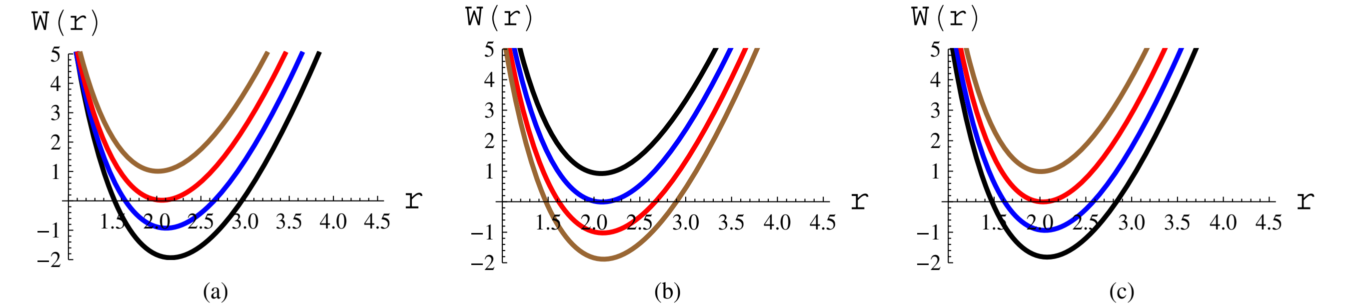


FIG. 2. $W(r)$ vs r for $\Lambda=-1$, $b=2.5$, $Q=1$, $M=2.5$, $r_0=1$ Eq. (4.8): (a) $\alpha=2.8$, $N=-0.48$ and $p=1.14, 1.19, 1.235, 1.28$ for black, blue, red and brown curves, respectively; (b) $p=1.2$, $N=-0.48$ and $\alpha=2.7, 2.755, 2.82, 2.88$ for the black, blue, red and brown curves, respectively; (c) $p=1.2$, $\alpha=3$ and $N=-0.488, -0.492, -0.496, -0.5$ for the black, blue, red and brown curves, respectively.

really black holes. In the following section we study their thermodynamic properties.

V. THERMODYNAMIC PROPERTIES OF THE EINSTEIN-DILATON BLACK HOLES

The main goal of the present section is to investigate the validity of the first law of black hole thermodynamics for the new black holes identified in the previous section. For this purpose we need to calculate the thermodynamic quantities relevant to these black holes. One of the important thermodynamic quantities is the temperature associated with the black hole horizon. It can be calculated with the help of the surface gravity. It is well known that the Hawking temperature is related to the surface gravity via $T=\frac{\kappa}{2\pi}$ and, in terms of the vector $\chi^\mu=(-1,0,0)$, one can show that

$$\kappa=\sqrt{-\frac{1}{2}(\nabla_\mu\chi_\nu)(\nabla^\mu\chi^\nu)}=\frac{1}{2}\left(\frac{dW(r)}{dr}\right)_{r=r_h}. \tag{5.1}$$

Now, making use of Eq. (4.8) one obtains

$$T=\begin{cases}-\frac{3}{4\pi}\left[2\Lambda b(\frac{b}{r_h})^{\frac{1}{3}}-\frac{p(2q^2)^p}{b^\eta}(\frac{2p-1}{3p-p\alpha-2})r_h(\frac{b}{r_h})^{\frac{2p(\alpha+1)}{3(2p-1)}}\right], & N=-\frac{2}{3},\\ -\frac{r_h}{4\pi(N+1)}\left[2\Lambda(\frac{b}{r_h})^{-2N}-\frac{(2p-1)(2q^2)^p}{b^\lambda}(\Upsilon_1-1)(\frac{b}{r_h})^{2a_1\beta}\right], & N\neq-\frac{2}{3}.\end{cases} \tag{5.2}$$

It must be noted that the condition $W(r_h)=0$ has been used for eliminating the mass parameter m from Eq. (5.2). It is worth mentioning that black holes with zero temperature known as the extreme black holes can occur if the charge (i.e., $q=q_{\rm ext}$) and the horizon radius (i.e., $r_h=r_{\rm ext}$) of the black holes are fixed such that the relation $T(r_{\rm ext},q_{\rm ext})=0$ is satisfied. Therefore, we arrived at

$$r_{\rm ext}=b\left[\frac{(2p-1)(2q^2)^p}{2\Lambda b^\lambda}(\Upsilon_1-1)\right]^{\frac{1}{2(N+a_1\beta)}},$$
$$\text{for } N=-\frac{2}{3},\qquad N\neq-\frac{2}{3}. \tag{5.3}$$

The plots of T vs r_h are shown in Figs. 3 and 4 (black and blue curves). They show that, for the properly fixed parameters, the physically reasonable black holes, with positive temperature, occur for $r_h>r_{\rm ext}$ and the unphysical black holes, having negative temperature, are in the range $r_h<r_{\rm ext}$.

The black hole entropy is the other important thermodynamic quantity to be calculated. Based on the outstanding Hawking discoveries the black hole entropy, as a pure geometrical object, obeys the entropy-area law. According to this famous law the black hole entropy is equal to one-fourth of the black hole horizon area. Thus, one can write

$$S=\frac{2\pi r_h R(r_h)}{4}=\frac{\pi r_h}{2}\left(\frac{r_h}{r_0}\right)^N. \tag{5.4}$$

Note that Eq. (5.4) reduces to its standard form in the three-dimensional space-times when the dilaton scalar field is absent (i.e., $N=0$).

The black hole electric potential is also an important thermodynamic quantity which is needed to be calculated. With the temporal component of the electromagnetic potential (4.2) in hand and making use of the following standard relation [63],

$$\Phi = A_\mu \chi^\mu|_{\text{reference}} - A_\mu \chi^\mu|_{r=r_h}, \tag{5.5}$$

it is possible to obtain the electric potential relative to a reference point located at infinity relative to the black hole horizon. In terms of an arbitrary constant C, which will be determined later, we have [28,64,65]

$$\Phi = \frac{C(2p-1)q}{2(p-1)-B} r_h^{1-\frac{B+1}{2p-1}}. \tag{5.6}$$

The black hole electric charge as a conserved quantity can be found by calculating the flux of the electromagnetic field at infinity (i.e., $r \to \infty$) [66–68]. By use of Gauss's electric law in the form of [69]

$$Q = \frac{1}{4\pi} \int \sqrt{-g} \mathcal{L}_{\mathcal{F}}(\phi, \mathcal{F}) F_{tr} d\Omega \tag{5.7}$$

and making use of Eqs. (3.4) and (4.1) after some simple calculations we arrive at

$$Q = \frac{p(2)^{p-2}}{r_0^N b^{2p\alpha\beta}} q^{2p-1}, \tag{5.8}$$

which is compatible with the charge of Banados-Teitelboim-Zanelli (BTZ) black holes when the dilaton field vanishes.

The black hole mass is the other conserved quantity to be calculated. For the purpose of identifying the black hole mass, following the works of Refs. [31,60,61] we must write the line element in the following standard form:

$$ds^2 = -f^2(\rho)dt^2 + \frac{d\rho^2}{g^2(\rho)} + \rho^2 d\theta^2. \tag{5.9}$$

Then, provided that the matter field does not contain derivatives of the metric, the quasilocal black hole mass $\mathcal{M}$ can be obtained by use of the following relation:

$$\mathcal{M} = 2f(\rho)[g_0(\rho) - g(\rho)], \tag{5.10}$$

in which $g_0(\rho)$ is an arbitrary function which determines the zero of the mass. In our case, making use of the transformation relation $\rho = rR(r)$, we have

$$dr^2 = \frac{d\rho^2}{(1+N)^2 R^2}. \tag{5.11}$$

Therefore, one obtains

$$f^2(\rho) = W(r(\rho)), \text{ and } g^2(\rho) = (1+N)^2 (R(\rho))^2 W(r(\rho)). \tag{5.12}$$

By use of these quantities in Eq. (5.10) and taking the limit $\rho \to \infty$ the black hole mass, in terms of the mass parameter m, can be calculated as

$$M = \frac{N+1}{8} r_0^{-N} m, \tag{5.13}$$

provided that the N values are restricted to the range $-2/3 \leq N \leq 0$. Note that the black hole mass presented in Eq. (5.13) recovers the BTZ black hole mass in the absence of a dilaton field (i.e., $N = 0$). Also, it is compatible with the black hole mass identified in Ref. [25].

In order to calculate the thermodynamic and conserved quantities, related to our new black holes from the thermodynamical methods, we need to have a Smarr-type mass formula. It can be derived from Eq. (4.8) by imposing the condition $W(r_+) = 0$. The Smarr mass formula which gives the black hole mass is obtained as

$$M(r_h, q) = \begin{cases} -\frac{1}{8}\left(\frac{r_0}{b}\right)^{\frac{2}{3}}\left[2\Lambda b^2 \ln\left(\frac{r_h}{\ell}\right) - \frac{3p(2)^{p-1}q^{2p}}{b^\eta}\left(\frac{2p-1}{3p-p\alpha-2}\right)^2 (br_h^2)^{\frac{2}{3}}\left(\frac{b}{r_h}\right)^{\frac{2p(\alpha+1)}{3(2p-1)}}\right], & N = -\frac{2}{3}, \\ -\frac{r_h^2}{8}\left(\frac{r_h}{r_0}\right)^N \left[\frac{\Lambda}{2+3N}\left(\frac{r_h}{b}\right)^{2N} + \frac{(2p-1)^2(2q^2)^p(1-\Upsilon_1)}{(2p-2-B)b^\lambda}\left(\frac{b}{r}\right)^{2a_1\beta}\right], & N \neq -\frac{2}{3}. \end{cases} \tag{5.14}$$

Noting Eqs. (5.4) and (5.8) the black hole mass given in Eq. (5.14) can be considered as the function of thermodynamical extensive parameters S and Q. Now, we can calculate the intensive parameters T and Φ, conjugate to the black hole entropy and charge, respectively. As a matter of calculation one can show that

$$\left(\frac{\partial M}{\partial S}\right)_Q = T, \qquad \left(\frac{\partial M}{\partial Q}\right)_S = \Phi,$$
$$\text{for both} \quad N = -\frac{2}{3} \quad \text{and} \quad N \neq -\frac{2}{3}, \tag{5.15}$$

provided that the constant C in Eq. (5.6) is chosen equal to $1 - \Upsilon_1$ [28,64,65]. Note that $C = 1$ in the absence of a dilaton field and it is compatible with the results of Refs. [54,55].

Therefore, we proved that the first law of black hole thermodynamics is valid, for both classes of the charged dilatonic three-dimensional black holes, in the following standard form:

$$dM(S, Q) = TdS + \Phi dQ. \tag{5.16}$$

VI. STABILITY ANALYSIS IN THE CANONICAL ENSEMBLE METHOD

It is well known that, in the canonical ensemble method, the stability information of a thermodynamic system can be extracted from its heat capacity. The heat capacity of a black hole, as a thermodynamic system, can be obtained through the following relation:

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q. \tag{6.1}$$

Here, the notations means that the derivative of entropy relative to temperature is calculated by treating the black hole charge as a constant and the subscript Q is used to remember this fact. Now, by use of Eq. (5.15) and identifying $M_{SS} \equiv (\partial^2 M/\partial S^2)_Q$, Eq. (6.1) takes the following form:

$$C_Q = \frac{T}{M_{SS}}. \tag{6.2}$$

According to the canonical ensemble method a physical black hole (it means $T > 0$) is thermally stable if its heat capacity (or equivalently M_{SS}) is positive. An unstable black hole undergoes a thermodynamic phase transition to be stabilized. A type one phase transition occurs at the vanishing points of the black hole heat capacity. Thus, the real roots of $T = 0$ are known as the type one phase transition points. In addition, a type two thermodynamic phase transition occurs at the points where the black hole heat capacity diverges. This means that the real roots of the denominator of the black hole heat capacity are the locations of the type two phase transition [66,70–72]. Keeping these facts in mind, we proceed to perform a detailed analysis on the thermal stability or phase transition of the new Einstein-dilaton black hole solutions identified here.

A. The black holes with $N = -\frac{2}{3}$

The numerator of the black hole heat capacity is given by Eq. (5.2). Now, by use of Eqs. (5.4) and (5.14), as a matter of calculation, one can show that the denominator of the black hole heat capacity can be written as

$$M_{SS} = \frac{3}{2\pi^2}\left(\frac{r_h}{r_0}\right)^{\frac{2}{3}}\left[2\Lambda\left(\frac{b}{r_h}\right)^{\frac{4}{3}} + \frac{p(2q^2)^p}{b^\eta}\left(\frac{4p-2p\alpha-3}{3p-p\alpha-2}\right)\left(\frac{b}{r_h}\right)^{\frac{2p(1+\alpha)}{3(2p-1)}}\right]. \tag{6.3}$$

The real roots of Eq. (6.3) indicate the type two phase transition points. The locations of these points can be obtained by solving the equation $M_{SS} = 0$, which are identified as

$$r_h \equiv r_1 = b^{\frac{(2p-1)(p\alpha+2)}{3p-p\alpha-2}}\left[-\frac{p(2q^2)^p}{2\Lambda}\left(\frac{4p-2p\alpha-3}{3p-p\alpha-2}\right)\right]^{\frac{3(2p-1)}{2(2-3p+p\alpha)}}. \tag{6.4}$$

Therefore, the black holes with the horizon radius equal to r_1 undergo a type two phase transition. Also, the black holes with the horizon radius $r_h \equiv r_{\text{ext}}$, given by Eq. (5.3), experience a type one phase transition. Finally, as it is shown by Fig. 3 this class of new black holes with the horizon radii in the interval $r_{\text{ext}} < r_h < r_1$ is locally stable.

B. The black holes with $N \neq -\frac{2}{3}$

The denominator of the heat capacity of this class of black holes can be obtained by use of Eqs. (5.4) and (5.14). It can be written in the following explicit form:

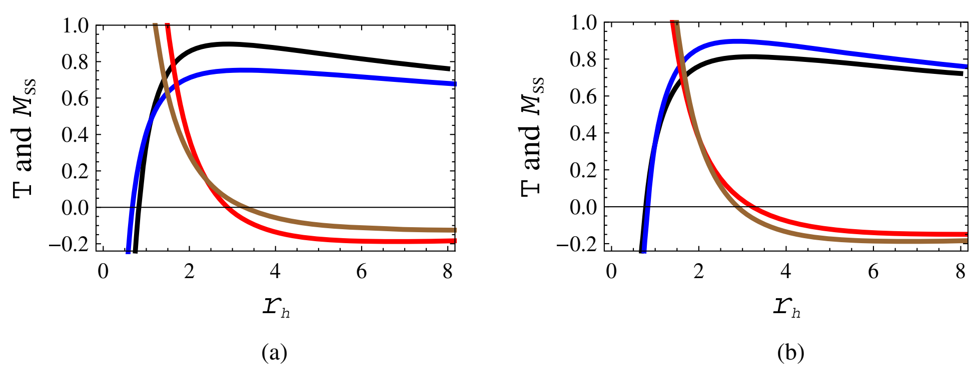


FIG. 3. T and M_{SS} vs r_h, for $N = -\frac{2}{3}$, $\Lambda = -1$, $Q = 0.5$, $b = 2.5$, $r_0 = 1$ [Eqs. (5.2) and (6.3)]: (a) $\alpha = 3.5$, [$T : p = 1.2$(black), 1.8(blue)], [$M_{SS} : p = 1.2$(red), 1.8(brown)]; (b) $p = 1.2$, [$T : \alpha = 3$(black), 3.5(blue)], [$M_{SS} : \alpha = 3$(red), 3.5(brown)].

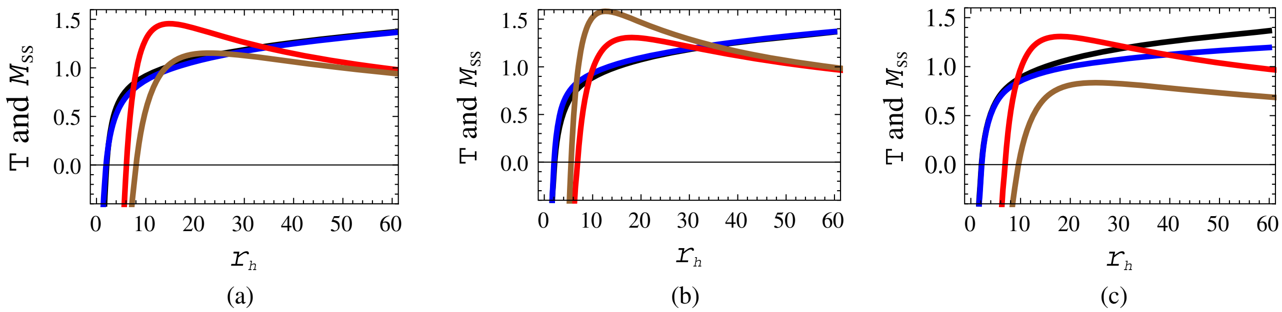


FIG. 4. T and M_{SS} vs r_h for $\Lambda = -1$, $b = 2.8$, $Q = 1$, $r_0 = 1$ [Eqs. (5.2) and (6.5)]: (a) $\alpha = 3$, $N = -0.4[T: p = 1.2(\text{black}), 1.8(\text{blue})]$ and $[40M_{ss}: p = 1.2(\text{red}), 1.8(\text{brown})]$; (b) $p = 1.2$, $N = -0.4[T: \alpha = 2.5(\text{black}), 3.5(\text{blue})]$ and $[40M_{ss}: \alpha = 2.5(\text{red}), 3.5(\text{brown})]$; (c) $p = 1.2$, $\alpha = 2.5[T: N = -0.4(\text{black}), -0.43(\text{blue})]$ and $[40M_{ss}: N = -0.4(\text{red}), -0.43(\text{brown})]$.

$$M_{SS} = -\frac{1+2N}{2\pi^2(N+1)^2}\left(\frac{r_h}{r_0}\right)^{-N}\left[2\Lambda\left(\frac{r_h}{b}\right)^{2N} + \frac{(2q^2)^p}{b^\lambda}\frac{(\Upsilon_1-1)(2pN+2p\alpha\beta+1)}{1+2N}\left(\frac{b}{r_h}\right)^{2a_1\beta}\right]. \tag{6.5}$$

From the viewpoint of the canonical ensemble method, the real root(s) of equation $M_{SS} = 0$ indicate the locations of the type two phase transition points. Therefore, the type two phase transition point is located at

$$r_h \equiv r_2 = b\left[\frac{(2q^2)^p(1-\Upsilon_1)}{2\Lambda b^\lambda}\left(\frac{2pN+2p\alpha\beta+1}{2N+1}\right)\right]^{\frac{1}{2(N+a_1\beta)}}. \tag{6.6}$$

Also, the points of the type one phase transition are identified as the vanishing points of the black hole temperature presented in Eq. (5.2). In order to obtain the type one and type two transition points exactly, we have plotted T and M_{SS} vs r_h in Fig. 4. The plots show that $r_h = r_{\text{ext}}$ and $r_h = r_2$ are the points of type one and type two phase transition, respectively. The black holes corresponding to $N \neq -\frac{2}{3}$ are stable for the horizon radii in the interval $r_h > r_2$.

VII. THERMODYNAMICS OF THE SCALAR-TENSOR BLACK HOLES

The calculations presented in the previous sections show the black hole solutions and the related thermodynamic properties in the Einstein-dilaton gravity theory, and they are valid in the Einstein frame only. Now, we are in a situation to explore the scalar-tensor black hole solutions and their properties in the Jordan frame. This is possible with the help of inverse transformations introduced in Sec. II. At first, using Eqs. (3.12) and (4.7) into Eq. (2.6) and noting (2.10), it is easily shown that the explicit forms of the functions in the action (2.1) are as follows:

$$X(\phi) = e^{-\frac{2p\alpha}{4p-3}\phi}, \qquad Y(\phi) = -2X(\phi),$$
$$Z(\phi) = \frac{1}{2}V(\phi)e^{-\frac{6p\alpha}{4p-3}\phi}, \tag{7.1}$$

and by solving the differential equation (2.4) for the functional form of $\psi(\phi)$ we arrive at

$$\psi(\phi) = e^{\frac{2p\alpha}{4p-3}\phi}\sqrt{1-\left(\frac{4p-3}{2p\alpha}\right)^2}, \quad -1 < \frac{4p-3}{2p\alpha} < 1. \tag{7.2}$$

Now, in order to obtain the scalar-tensor black hole solutions, we start with the following line element,

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -F(r)dt^2 + \frac{1}{G(r)}dr^2 + r^2H^2(r)d\theta^2, \tag{7.3}$$

as the solution to the gravitational field equations in the Jordan frame. The functions $F(r)$, $G(r)$, and $H(r)$ are unknown functions of r, which can be determined from their Einstein counterparts by applying the inverse transformations of Sec. II. Thus we have $\tilde{g}_{\mu\nu} = g_{\mu\nu}(\frac{b}{r})^{\frac{4p\alpha\beta}{4p-3}}$, which results in

$$F(r) = \left(\frac{b}{r}\right)^{\frac{4p\alpha\beta}{4p-3}}W(r), \qquad G(r) = \left(\frac{b}{r}\right)^{-\frac{4p\alpha\beta}{4p-3}}W(r),$$
$$H(r) = \left(\frac{b}{r}\right)^{\frac{2p\alpha\beta}{4p-3}}R(r), \tag{7.4}$$

where the functions $R(r)$ and $W(r)$ are given by Eqs. (3.11) and (4.8), respectively.

The plots of $G(r)$ vs r are shown in Figs. 5 and 6 for the $N = -\frac{2}{3}$ and $N \neq -\frac{2}{3}$ cases, respectively. The plots show that, for the suitably fixed parameters, the scalar-tensor black holes with two horizons, as well as extreme and naked singularity black holes, can occur.

The Hawking temperature associated with the horizon of the scalar-tensor black holes can be calculated by use of the black hole's surface gravity. It is obtained as

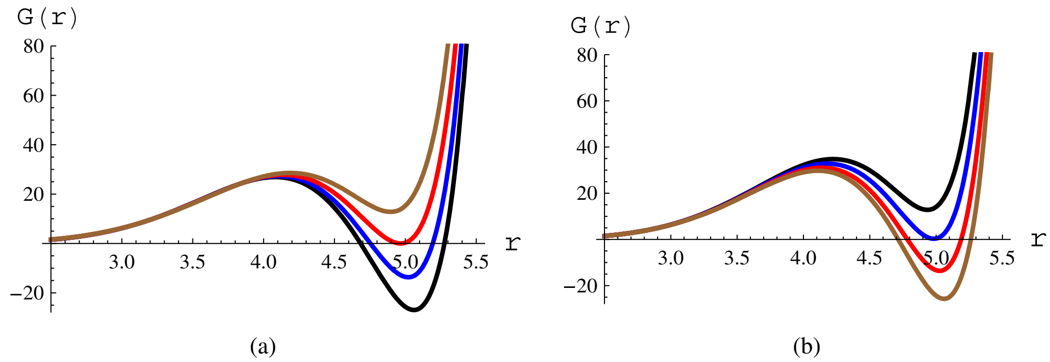


FIG. 5. $0.5G(r)$ vs r for $N = -\frac{2}{3}$, $\Lambda = -1$, $b = 3$, $Q = 1$, $M = 2.5$, $r_0 = 1$ [Eq. (7.4)]: (a) $\alpha = 2.1$ and $p = 0.81$, 0.8014, 0.8019, 0.8025 for the black, blue, red and brown curves, respectively; (b) $p = 0.8$ and $\alpha = 2.0946$, 2.0956, 2.0966, 2.0974 for the black, blue, red and brown curves, respectively.

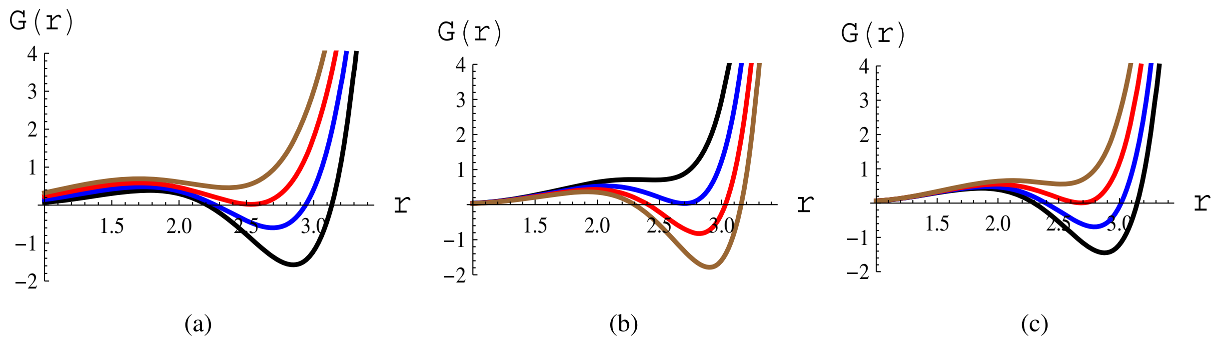


FIG. 6. $G(r)$ vs r for $\Lambda = -1$, $Q = 1$, $M = 2.5$, $b = 2.5$, $r_0 = 1$ [Eq. (7.4)]: (a) $N = -0.48$, $\alpha = 2.3$ and $p = 0.86$, 0.88, 0.9, 0.92 for the black, blue, red and brown curves, respectively; (b) $N = -0.48$, $p = 0.84$ and $\alpha = 2.18$, 2.212, 2.24, 2.265 for the black, blue, red and brown curves, respectively; (c) $p = 0.85$, $\alpha = 2.2$ and $N = -0.475$, -0.4765, -0.4782, -0.48 for black, blue, red and brown curves, respectively.

$$\tilde{T} = \frac{1}{4\pi}\left(\sqrt{\frac{G(r)}{F(r)}}\frac{dF(r)}{dr}\right)_{r=r_h} = \frac{1}{4\pi}\left(\frac{b}{r_h}\right)^{-\frac{4p\alpha\beta}{4p-3}}\frac{d}{dr}\left[\left(\frac{b}{r}\right)^{\frac{4p\alpha\beta}{4p-3}}W(r)\right]_{r=r_h} = \frac{1}{4\pi}\frac{dW(r)}{dr}\bigg|_{r=r_h} = T. \quad (7.5)$$

It means that the Hawking temperature is just the same for the scalar-tensor and the Einstein-dilaton black holes. Also, one can show that the black hole charge, mass, entropy and electric potential are identical in the Einstein and Jordan frames. Making use of the condition $G(r_h) = 0$ and obtaining the mass parameter m of the scalar-tensor black holes one is able to show that it is equal to that of the Einstein black holes. It leads to the same Smarr-type mass formula for the scalar-tensor and Einstein black holes. As a result we obtain

$$\tilde{\Phi} = \frac{\partial\tilde{M}(\tilde{S},\tilde{Q})}{\partial\tilde{Q}}, \quad \text{and} \quad \tilde{T} = \frac{\partial\tilde{M}(\tilde{S},\tilde{Q})}{\partial\tilde{S}}, \quad (7.6)$$

from which one can argue that the first law of black thermodynamics is valid for the new scalar-tensor black holes in the following form:

$$d\tilde{M}(\tilde{S},\tilde{Q}) = \frac{\partial\tilde{M}(\tilde{S},\tilde{Q})}{\partial\tilde{S}}d\tilde{S} + \frac{\partial\tilde{M}(\tilde{S},\tilde{Q})}{\partial\tilde{Q}}d\tilde{Q}. \quad (7.7)$$

It must be noted that, as the heat capacities of the scalar-tensor and Einstein-dilaton black holes are identical, they have the same points of type one and type two phase transitions. Also, the ranges at which the black holes remain stable are identical in the Einstein and Jordan frames.

VIII. CONCLUSION

With the aim of finding the charged scalar-tensor black hole solutions and studying their thermodynamic properties, we started from the general form of the three-dimensional scalar-tensor action coupled to the power-law nonlinear electrodynamics. In this action the Ricci scalar is multiplied by a scalar function which produces a strong coupling between the field equations and makes them too difficult to be solved directly. To overcome this problem we used a suitable conformal transformation to translate the scalar-tensor action from the Jordan frame to the Einstein frame. It, in general, leads to the action of Einstein-dilaton gravity in the presence of the scalar coupled electromagnetic Lagrangian. We found that there is an especial case, by a suitable choice of the nonlinearity parameter p (i.e., $p = 3/4$), which preserves the Lagrangian of the electrodynamics invariant. It is the so-called conformally invariant electrodynamics, which will be considered in a forthcoming paper.

By varying the action of three-dimensional Einstein-dilaton gravity theory in the presence of the scalar coupled power-law electrodynamics we obtained the coupled scalar, electromagnetic and gravitational field equations. We calculated the exact solutions in a static and spherically symmetric geometry. The results show that the dilaton potential can be written as the linear combination of two Liouville-type potentials. Also, two new classes of nonlinearly charged dilatonic black hole solutions have been obtained which are asymptotically nonflat and non-AdS. They can produce black holes with two horizons, as well as extreme and naked singularity black holes, provided that the parameters in the theory are chosen properly (Figs. 1 and 2).

In order to investigate the thermodynamic behavior of these new dilatonic black hole solutions, we calculated the black hole temperature, entropy, charge, mass and electric potential by utilizing the geometrical and thermodynamical methods. Through a Smarr-type mass formula, we identified the black hole total mass as a function of the black hole charge and entropy as the extensive thermodynamic parameters. Then, by use of the Smarr mass formula, we calculated the black hole temperature and electric potential as the thermodynamic intensive parameters conjugate to the entropy and electric charge, respectively. We showed that the conserved and thermodynamic quantities obtained from the geometrical and thermodynamical approaches satisfy the thermodynamical first law in its standard form. Thus, the first law of black hole thermodynamics remains valid even in the presence of dilaton fields.

Then, we analyzed the thermodynamic phase transition or thermal stability of the new dilatonic black holes that we just obtained. Making use of the canonical ensemble method and by calculating the black hole heat capacity, we identified the location of the type one and type two phase transition points, exactly. Also, regarding the signature of the black hole temperature and black hole heat capacity, we determined the ranges of the horizon radii in which the black holes are locally stable (Figs. 3 and 4).

Next, we proceeded to study the exact charged three-dimensional scalar-tensor black hole solutions in the Jordan frame. To do so, we obtained the Jordan frame black hole solutions from their Einstein frame counterparts by applying the inverse conformal transformations. Interestingly we found that, just like the dilatonic black holes, the exact solutions of the scalar-tensor gravity theory can produce the two horizon, extreme and naked singularity black holes (Figs. 5 and 6). Also, we showed that the conserved and thermodynamic quantities related to the scalar-tensor black holes are just identical to those obtained for the Einstein-dilaton black holes. As a result the first law of black hole thermodynamics remains valid for both of the charged scalar-tensor black holes introduced here. Also, they have the same points of type one and type two thermodynamic phase transitions as the dilaton black holes. The ranges at which the dilatonic and scalar-tensor black holes remain stable are identical.

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APPENDIX: DETAILS OF DERIVATION OF EQ. (3.10)

With the purpose of finding the relation between Eqs. (3.5) and (3.7), we start with the following definitions:

$$E_{tt} \equiv \frac{e_{tt}}{rR(r)} = W''(r) + \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)W'(r) + 2[V(\phi) + (p-1)\mathcal{L}(\phi, \mathcal{F})], \tag{A1}$$

$$E_{\theta\theta} \equiv \frac{e_{\theta\theta}}{rR(r)} = \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)W'(r) + \left(\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)}\right)W(r) + V(\phi) + (2p-1)\mathcal{L}(\phi, \mathcal{F}). \tag{A2}$$

Taking the derivative of Eq. (A.2) with respect to r and making use of Eqs. (A.1) and (3.9), after some algebraic simplifications, we arrive at

$$\begin{aligned}\frac{dE_{\theta\theta}}{dr} = {}& \left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)(E_{tt} - 2E_{\theta\theta}) - 4\left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)(\phi')^2 W(r) - 4(\phi')^2 W'(r) - 4r\phi'\phi'' W(r) \\ & + 2p\left(\frac{1}{r} + \frac{R'(r)}{R(r)}\right)\mathcal{L}(\phi, \mathcal{F}) + \frac{d}{dr}[V(\phi) + (2p-1)\mathcal{L}(\phi, \mathcal{F})],\end{aligned} \tag{A3}$$

in which Eq. (A.2) has been used once again. Now, we can write

$$\frac{d}{dr}V(\phi) = \phi' \frac{d}{d\phi}V(\phi), \tag{A4}$$

and noting Eq. (3.3) one obtains

$$\frac{d}{dr}V(\phi) = 4(\phi')^2\left(\frac{1}{r}+\frac{R'(r)}{R(r)}\right)W(r) + 4(\phi')^2W'(r) + 4\phi'\phi''W(r) - 2p\alpha\phi'\mathcal{L}(\phi,\mathcal{F}). \tag{A5}$$

Also, one can show that

$$\frac{d}{dr}\mathcal{L}(\phi,\mathcal{F}) = \left(\frac{2p}{F_{tr}}\frac{dF_{tr}}{dr} - p\alpha\phi'\right)\mathcal{L}(\phi,\mathcal{F}). \tag{A6}$$

Substituting Eqs. (A.5) and (A.6) into Eq. (A.3), after some algebraic simplifications, one obtains

$$\frac{dE_{\theta\theta}}{dr} - \left(\frac{1}{r}+\frac{R'(r)}{R(r)}\right)(E_{tt}-2E_{\theta\theta}) = 2p\left[\frac{2p-1}{F_{tr}}\frac{dF_{tr}}{dr} + \frac{1}{r} + \frac{R'(r)}{R(r)} - 2p\alpha\phi'\right]\mathcal{L}(\phi,\mathcal{F}). \tag{A7}$$

Regarding Eq. (3.8), we have

$$\frac{dF_{tr}}{dr} = \frac{F_{tr}}{2p-1}\left[2p\alpha\phi' - \left(\frac{1}{r}+\frac{R'(r)}{R(r)}\right)\right]. \tag{A8}$$

By combining Eqs. (A.7) and (A.8) we arrive at Eq. (3.10).

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