Stress tensor on null boundaries

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Using the Brown-York prescription for the definition of quasilocal gravitational energy-momentum tensor on a boundary and also complete canonical structure on a null boundary which has been found recently [Classical Quantum Gravity **36**, 015012 (2019)], we propose a similar stress tensor on the null boundary. Then we exploit this stress tensor to compute the quasilocal energy and angular momentum for some well-known gravitational solutions. We have found that in addition to reference spacetime method for regularizing total energy, in the case of null boundary we can add a possible counterterm thereby avoiding embedding difficulties.

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I. INTRODUCTION

In general relativity (GR), defining a local energy for gravitational field is problematic (for a review on the subject see [1]). The problem is rooted in the general covariance principle and the fact that the first derivative of metric can always vanish in a properly chosen local coordinate system. But in order to have a notion of local energy density with correct physical dimension, such quantity could be defined in terms of the metric and its first derivative. Also, a related notion to energy density in field theories is the action functional. In fact, the standard covariant action in GR, i.e., the Hilbert-Einstein (HE) action, contains second order derivatives of the metric, which is another consequence of the general covariance principle. As it is well known, this action does not have a well-posed variational principle and needs to be complemented with additional terms defined on the boundaries of the spacetime. Brown and York [2] have pointed out that having an action with a well-posed variational principle, by using Hamilton-Jacobi analysis, one can define the quasilocal energy of the system.

A careful variation of the HE action in addition to suggesting the proper boundary term (boundary action), to ensure a well-posed variational principle, also provides dynamical degrees of the theory as well as its canonical structure (see, e.g., [3]). An important point, in finding the complete canonical structure by such a procedure, is the necessity of the condition that one should not suppress any degree of freedom beforehand by imposing restrictions. The proper boundary action complementing the HE action, when the boundary is timelike or spacelike, is the well-known Gibbons-Hawking-York (GHY) term [4,5]. Applying variational principle in GR for null boundaries has been a subject of investigation in recent years [6–10]. In these papers, the authors are either interested in finding the proper boundary terms on null boundaries, and therefore ignored those terms that are fixed by boundary conditions, or imposed some restrictions on variations in such a way that the resulting canonical structure is not complete. One of such restrictions is that the variations keep the character of the boundary unchanged. But one of the metric degrees of freedom is responsible for variations which alter the boundary character. This can be easily seen as follows: If the boundary is specified by $\phi = \text{const.}$ for some scalar field ϕ, then the normal to the boundary is proportional to $\partial_a\phi$. For a null boundary, we have $\partial_a\phi\partial^a\phi = 0$, while for a general metric variation we find:

$$\partial_a\phi\partial^a\phi \to \partial_a\phi\partial^a\phi - \delta g^{ab}\partial_a\phi\partial_b\phi. \quad (1)$$

Therefore, unless we set variations of the metric in the direction of $\partial_a\phi$ to zero, the boundary does not remain null. Recently we have shown that such variations appear in canonical structure [11]. In addition, it has been shown that, in order to preserve such variations, a general double-foliation framework is needed. The conjugate momentum to these variations is a scalar Ξ (for definition see Sec. III or Ref. [11]). In this article we will show that this scalar provides the quasilocal energy density on a null boundary For the definition of quasilocal stress tensor on null boundary we follow the same method presented by Brown and York in the case of quasilocal stress tensor on timelike boundary which means that the derivative of the action with respect to the metric variations which are tangential to the boundary [2,12].

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The content of the paper is organized as follows. In the next section we first briefly review the original Brown-York construction of quasilocal gravitational energy-momentum tensor. In Sec. III, we follow the same analysis for the case where instead of a timelike boundary we have a null one. Finally, in Sec. IV we calculate the energy and angular momentum in various space times using the proposed stress tensor.

Throughout this work we have set $G = c = 1$. We do not use various indices for specification of different parts of spacetime and use Latin indices $\{a, b, c, \dots\}$ everywhere; instead, different names are given to objects when defined on different structures. For example, we use K_{ab} for extrinsic curvature of a spacelike hypersurface while χ_{ab} is preserved for extrinsic curvature of a timelike hypersurface.

II. THE BROWN-YORK TENSOR

In this section we review the Brown-York [2] definition of gravitational stress-tensor on the boundary. To illustrate the main idea beyond Brown-York definition of quasilocal energy momentum, it is useful to start with an example in classical mechanics.

A. The Hamilton-Jacobi method

Let $L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$ be the classical Lagrangian for a particle. The action functional is $I[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)dt$, for initial and final configuration, i.e., $(\boldsymbol{q}_1, t_1)$ and $(\boldsymbol{q}_2, t_2)$, respectively. A general variation of this action for a given history is

$$\delta I[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)(\delta \boldsymbol{q} - \dot{\boldsymbol{q}}\delta t)dt + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta \boldsymbol{q}\Big|_{t_1}^{t_2} - \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} - L \right)\delta t \Big|_{t_1}^{t_2}. \tag{2}$$

Extremizing the action and imposing the boundary condition by fixing $\boldsymbol{q}$ and t at the end points, provides the equations of motion that are given by the first term in the integrand which are known as the Euler-Lagrange equations. Moreover, the Hamilton-Jacobi principal function $S(q_1, t_1; q_2, t_2)$ is defined as the value of the action for a solution $\boldsymbol{q}(t)$ of the equation of motion from $(\boldsymbol{q}_1, t_1)$ to $(\boldsymbol{q}_2, t_2)$.

Thus, according to (2), the derivative of the principal function S with respect to $\boldsymbol{q}$, i.e., $\frac{\delta S}{\delta \boldsymbol{q}}$, gives the canonical momenta $\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}$ while the derivative with respect to t, i.e., $\frac{\delta S}{\delta t}$, yields the minus of energy: $H = (\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} - L)$.

An important remark is that in order to ensure that the above procedure works well, the variational principle must be well-posed beforehand. Indeed, a Lagrangian with no well-posed variational principle does not lead to the correct relations for momenta and energy of the system. For example, the action $S_1 = \int \frac{1}{2} m\dot{x}^2 dt$ leads to the equations of motion and correct momentum and energy for a free particle. However, by considering another action as $S_2 = S_1 - \int \frac{1}{2} m \frac{d}{dt}(x\dot{x})dt = -\int \frac{1}{2} m x\ddot{x} dt$, one could see that, although the difference with the first action is a total derivative and the equations of motion are unchanged, by varying this action one finds:

$$\delta S_2 = -\int (m\ddot{x})\delta x + \left[\frac{1}{2}(\dot{x}\delta x - x\delta\dot{x}) \right]_{t_1}^{t_2}, \tag{3}$$

which means that in order to get the equation of motion we need to determine both x and $\dot{x}$ at both endpoints. This leads to inconsistency with second order equation of motion, because according to that we need to just fix the position at the ends. Therefore, for this action the variational principle is not well posed.[1] As a consequence, the derivative of the principal function does not lead to the correct definition for momentum and energy. However, we must note that all total derivatives do not spoil the variational principle, e.g., changing the action by $S \to S + \int_{t_1}^{t_2} \frac{dh}{dt} dt$ for arbitrary function $h(\boldsymbol{q}(t), t)$ is allowed. Hence, the action and the principal function are not unique; as a result, momenta and energy are also not exclusive. However, this arbitrariness can be fixed by choosing the zero point of energy, for example set the arbitrary function h so that it yields to zero energy for free particle at rest in the above example.

B. Stress tensor on timelike boundary

Having learned enough from the above simple mechanical example, lets begin with HE action in d dimension:

$$\mathcal{S}_{\rm EH} = \frac{1}{16\pi} \int \mathrm{d}^d x \sqrt{-g} R, \tag{4}$$

in which R is the Ricci scalar. By varying the action, if we consider the boundary segments to be either timelike or spacelike, one gets:

$$\begin{aligned}\delta\mathcal{S}_{\rm EH} = &\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g} G^{ab}\delta g_{ab} \\ &+ \frac{1}{16\pi}\sum_i \Bigg[2\delta\left(\int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\sqrt{|h|}K + \int_{\mathcal{C}_i} \mathrm{d}^{d-2}x\sqrt{|q|}\vartheta \right) \\ &+ \int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\sqrt{|h|}(K^{ab} - Kh^{ab})\delta h_{ab} \\ &+ \int_{\mathcal{C}_i} \mathrm{d}^{d-2}x\sqrt{|q|}\vartheta q^{ab}\delta q_{ab} \Bigg],\end{aligned} \tag{5}$$

where the sum is over each boundary segment $\mathcal{B}_i$ and every corner $\mathcal{C}_i$ at intersection of two neighboring segments of

[1] Of course this argument is valid if we are interested in Dirichlet or Neumann boundary conditions. By choosing a Robin boundary conditions one could revive the variational principle.

boundary. Moreover, K is the extrinsic curvature of each segment and ϑ is the angle or the boost parameter between segments, depending on their character. In addition, δh_{ab} is the variation of metric on each boundary while δq_{ab} is the variation on each codimension two joint. For the details of the calculations and specially the method by which one can take into account the contribution of joints, see Refs. [7,9,12] or [11]. We see that the variation of HE action includes both the metric and its normal derivative (the extrinsic curvature) on the boundary, thus, the variational principle is not well-posed for this action if we demand the Dirichlet boundary conditions.[2] This variation also suggests the correct action with well-posed variational principle as:

$$\mathcal{S} = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}R - \frac{1}{8\pi}\sum_i \left[\int_{\mathcal{B}_i} \mathrm{d}^{d-1}x\sqrt{|h|}K + \int_{\mathcal{C}_i} \mathrm{d}^{d-2}x\sqrt{|q|}\vartheta\right]. \tag{6}$$

Now consider a region in space-time as in Fig. 1 which is generated by evolution of a spacelike surface Σ from Σ_1 to Σ_2 and a timelike boundary $\mathcal{T}$. The intersection of Σ leafs with $\mathcal{T}$ are codimension two surfaces S from S_1 to S_2. Now in this region of space-time we calculate the variation of the action (6), and impose the equations of motions, i.e., $G_{ab} = 0$. Then we may find:

$$\delta\mathcal{S} = \frac{1}{16\pi}\left[\int_{\Sigma_1}^{\Sigma_2} \mathrm{d}^{d-1}x P^{ab}\delta h_{ab} + \int_{\mathcal{T}} \mathrm{d}^{d-1}x\Pi^{ab}\delta\gamma_{ab} + \int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{|q|}\vartheta q^{ab}\delta q_{ab}\right]. \tag{7}$$

Here, the symbol $\int_{\Sigma_1}^{\Sigma_2}$ is a shorthand for $\int_{\Sigma_2} - \int_{\Sigma_1}$, h_{ab} is the metric on each spacelike boundary and γ_{ab} is the induced metric on the timelike boundary $\mathcal{T}$. P^{ab} and Π^{ab} are respectively the gravitational momenta of Σ and $\mathcal{T}$:

$$P^{ab} = \sqrt{h}(K_{ab} - Kh_{ab})|_{\Sigma},$$
$$\Pi^{ab} = \sqrt{-\gamma}(\chi_{ab} - \chi\gamma_{ab})|_{\mathcal{T}}, \tag{8}$$

where K_{ab} and χ_{ab} are their corresponding extrinsic curvatures. In the original work of Brown and York [2], they assumed the boundaries to be orthogonal, thus, the contribution of the joints was missing. However, after the work of Hayward [14], who emphasized the importance of this term for the variational principle to be well posed, this term appeared in later works [12,15,16].

[2]For Neumann boundary condition in four dimensions, there is no need to any boundary term in the action to make a well-defined variational principle, see for example [13]. But usually in gravity one requires a Dirichlet boundary condition for which the metric on the boundary is fixed.

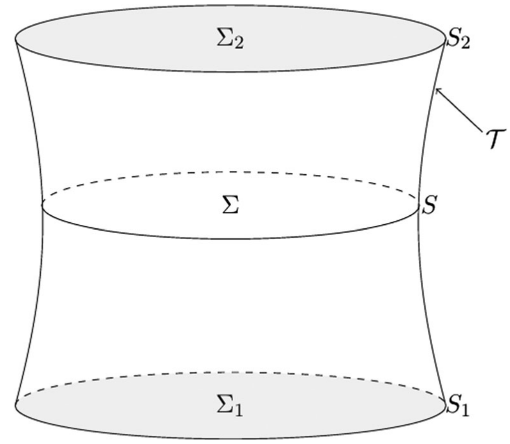

FIG. 1. Space time region with timelike and spacelike boundaries.

If we ignore these joints and consider the boundaries to be orthogonal, then in an analogous way to the mechanical example one may define the gravitational canonical momentum as the derivative of the principal functions with respect to the induced metric on spacelike segments of the boundary and gravitational energy-momentum-stress tensor as the derivative with respect to the induced metric on timelike segment. In fact, the Π^{ab} has the same expression as for ADM canonical momentum. Thus, the energy-momentum-stress tensor (or just the stress tensor for abbreviation) will be defined as:

$$T^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}} = \frac{1}{8\pi}(\chi_{ab} - \chi\gamma_{ab}). \tag{9}$$

To define conserved quantities, now we choose a spacelike codimension two surface S in $\mathcal{T}$ with unit timelike normal u^a. Then, the metric γ_{ab} is further decomposed as $\gamma^{ab} = q^{ab} - u^a u^b$. Here, u^a defines local the flow of time in $\mathcal{T}$. Moreover, according to Brown-York [2] for an isometry in the boundary generated by the Killing vector ξ^a, the conserved charge associated to this symmetry is defined by:

$$Q_\xi = \int_S d^{d-2}x\sqrt{q}T_{ab}u^a\xi^b. \tag{10}$$

In fact, there is some points regarding to the expression (9) and (10). The first one is that, as we pointed out before, these expressions are not unique. One can append a subtraction term S_0 to the boundary action without affecting variational problem when S_0 depends on fixed boundary data, $S_0 = S_0(h_{ab}, \gamma_{ab})$, which leads to ambiguities in the definitions of energy and momenta. According to the Brown and York interpretation these ambiguities are a consequence of freedom to choose the zero point of energy and redefine system momenta with canonical transformation [2,12]. On the other hand, Eq. (10) leads to infinities when calculated for large spheres in general systems in spacetime. The Brown-York proposal is to choose subtraction term S_0 such that the modified action $S - S_0$ leads to zero energy for flat spacetime. Therefore, the zero point is chosen to be the flat spacetime while the scheme for other geometries is to embed their boundary in flat spacetime. Thus, the modified expression for the stress tensor will be

$$T^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_0}{\delta\gamma_{ab}} = \frac{1}{8\pi}(\chi_{ab} - \chi\gamma_{ab}) - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_0}{\delta\gamma_{ab}}. \tag{11}$$

As another remark, if we want to consider the effect of joints or nonorthogonal boundaries, we must care about the components of $\delta\gamma_{ab}$ contained in δq_{ab} that appeared in the last term of (7). Also, we must be careful about the question that with respect to which observer we desire to calculate the quasilocal quantities. Note that when the boundaries are nonorthogonal, the Eulerian observers orthogonal to Σ constant are different from the observers orthogonal to S constant in the boundary $\mathcal{T}$. In these cases, a further decomposition of the induced metric γ_{ab} with the assistance of the vector $u_a = N\nabla_a t$ is required, where t is a foliation of $\mathcal{T}$. By this decomposition we get [12]:

$$\delta\gamma_{ab} = \delta q_{ab} - \frac{2}{N}u_{(a}\delta V_{b)} - u_a u_b \frac{\delta N}{N},$$

where N and V^a are lapse and shifts of decomposition. The components of $\frac{\delta S}{\delta\gamma_{ab}}$ can be calculated as

$$\epsilon \equiv u^a u^b T_{ab} = -\frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}}{\delta N}, \tag{12}$$

$$j^a \equiv q^{ac}u^b T_{cb} = -\frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}}{\delta V_a}, \tag{13}$$

$$s^{ab} \equiv q^{ac}q^{bd}T_{cd} = \frac{2}{N\sqrt{q}}\frac{\delta \mathcal{S}}{\delta q_{ab}}, \tag{14}$$

which are known respectively as the quasilocal energy density, tangential momentum density and spatial stress. The details of calculations for different observers can be found in [12,16]. In these cases it has been shown that a double foliation of space-time is a natural setup for calculations as described in appendices of [12]. There, the relation between quantities, as measured by different observers, has been obtained. The relations between two sets of normals to S as depicted in Fig. 2 is

$$\bar{n}_a = \gamma(n_a - v u_a) \tag{15}$$

$$\bar{u}_a = \gamma(u_a - v n_a). \tag{16}$$

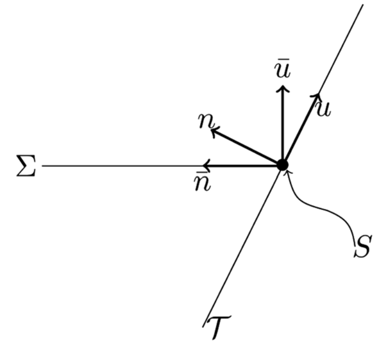

FIG. 2. Nonorthogonal boundaries.

The quasilocal energy density associated with the two surfaces S as seen by the observers orthogonal to Σ constant is [12]

$$\epsilon = \frac{1}{8\pi}(\gamma\mathbf{k} + \gamma v\mathbf{l}) - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_0}{\delta\gamma_{ab}}u^a u^b, \tag{17}$$

where $\mathbf{k}_{ab} = -q_a^c q_b^d \nabla_c \bar{n}_d$ and $\mathbf{l}_{ab} = -q_a^c q_b^d \nabla_c \bar{u}_d$ are defined so that $\mathbf{k}$ and $\mathbf{l}$ are their trace, respectively. In the case for which two observers are at rest with respect to each other, i.e., $v = 0$ or, in other words, the boundaries are orthogonal, the quasilocal energy density becomes:

$$\epsilon = \frac{1}{8\pi}(\mathbf{k} - \mathbf{k}_0), \tag{18}$$

where $\mathbf{k}_0$ is the corresponding extrinsic curvature as embedded in flat space. So, the total quasilocal energy on S becomes [2,12]:

$$E = \frac{1}{8\pi}\int_S \mathrm{d}^{d-2}x\sqrt{q}(\mathbf{k} - \mathbf{k}_0). \tag{19}$$

Evaluating the above integral provides an expression for E as a function of r, $E = E(r)$. However, we must note that because of the Eq. (17), in general the form of this function varies for different observers. For example, in the case of Schwarzschild black hole the function for static, radially infalling and boosted observers has been obtained in [16].

As the last point, let us note that for the above relations, and for different observers, it is always assumed that these observers do not move in null geodesics. In fact, null observers cannot be contained in the above setup because the normal vectors (or four-velocities) are always normalized to unity. As the name indicates, due to the fact that the norm vanishes for null observers, normalization does not make sense in this case. In fact, here, the normal vectors or four-velocities are both normal and tangent to the hypersurface, the induced metric becomes degenerate, and the usual extrinsic curvature does not make sense.

In the next section we will use standard treatment for null hypersurfaces. The resulting stress tensor can provide the quasilocal quantities as measured by null observers.

C. Asymptotic AdS spacetime and counterterm method

Finding a proper subtraction term by embedding in reference spacetime is a difficult task. In fact, it is not possible to embed a boundary with an arbitrary intrinsic metric in the reference spacetime. In the case of asymptotically AdS spacetimes, there is an attractive proposal without necessity of embedding in reference spacetime as proposed in [17]. This approach is inspired from AdS/CFT duality [18,19] by interpreting divergences which appear in stress tensor when the boundary is moved to infinity as dual to standard ultraviolet divergences of quantum field theory; then, they can be removed by adding local counterterms to the boundary action. For instance, for AdS_4 the counterterm Lagrangian in the boundary proposed to be

$$L_{ct} = -\frac{2}{\ell}\sqrt{-\gamma}\left(1 - \frac{\ell^2}{4}\mathcal{R}\right), \tag{20}$$

where $\mathcal{R}$ is the scalar curvature of induced metric on the boundary, and ℓ is the AdS radius. Adding this term to the usual GHY term in the timelike boundary, then variation of action yields to a regularized stress tensor such that energy becomes finite at the $r \to \infty$ limit.[3] The main advantage of this method, beside the fact that counterterms are covariant and do not spoil the variational principle, is the lack of embedding difficulties in the BY method. However, these counterterms are known just for asymptotically AdS spaces. For asymptotic flat space, since there is no length scale ℓ, finding such counterterms is problematic. Also, the limit $\ell \to \infty$ in Eq. (20) does not lead to a unique covariant expression [21].

III. HAMILTON-JACOBI ANALYSIS ON NULL BOUNDARY

In this section we are going to repeat the calculation of the previous section when instead of a timelike boundary we have a null one. The corresponding spacetime region is illustrated in Fig. 3. In order to calculate the variation of the action in this region we need to express surface divergences in the variation of HE action in terms of geometric objects of spacelike surfaces, i.e., Σ_1, Σ_2 and the null boundary $\mathcal{N}$. Variation on Σ_1 and Σ_2 is similar to the previous section; however, for a null surface (because of degeneracy of induced metric and divergence of extrinsic curvature) the calculation is completely different. In this section, we first introduce the basic tools for general investigation of null boundaries (without any gauge fixing) by introducing a general double foliation. Then, we find canonical momenta in this hypersurface so that by varying the action with respect to the metric components on this hypersurface we find stress tensor on the boundary.

[3] In [20] a topological method is proposed for finding standard counterterm series of AdS gravity.

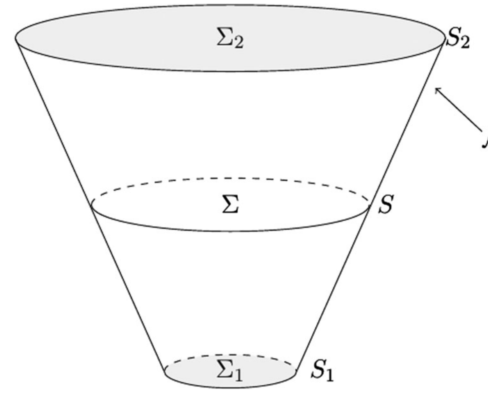

FIG. 3. Space-time region with a null and space-like boundaries.

A. The setup

A segment $\mathcal{N}$ of the spacetime boundary, characterized by $\phi_0 = 0$, is called a null hypersurface if $\nabla_a\phi_0\nabla^a\phi_0 = 0$. This feature of null boundary indicates that the normal vector to the null surface is also tangent to it. This property is the origin of some difficulties when dealing with such hypersurfaces; because, as a consequence, the induced metric becomes degenerate and therefore constructing a projector to the null surface just from its normal is not possible. One standard remedy to this problem is to introduce an auxiliary null vector k^a which lays out of the hypersurface and therefore $\ell_a k^a \neq 0$, when ℓ_a is the null normal to the boundary, e.g., $\ell_a \propto \nabla_a\phi_0$ on the boundary. For more details about the geometry of null hypersurfaces we refer the interested reader to [22,23].

By defining ℓ_a as the normal to the null boundary, we introduce the auxiliary null form k_a and take the normalization of these null forms to be everywhere as:

$$\ell_a\ell^a = 0, \qquad k_a k^a = 0 \quad \text{and} \quad \ell_a k^a = -1. \tag{21}$$

With the aid of ℓ_a and k_a, we can define the projector as:

$$q^a{}_b = \delta^a{}_b + \ell^a k_b + k^a \ell_b \tag{22}$$

This projector is not in fact a projector on null surface; instead, it essentially projects spacetime vectors onto the codimension two surface S, to which ℓ_a and k_a are orthogonal.

A systematic approach to define these codimension two surfaces is to use a double-foliation by two scalar fields (ϕ_0, ϕ_1). The intersection of level surfaces of ϕ_0 and ϕ_1 are the codimension two surfaces S. In this foliation ℓ_a and k_a can be expanded generally as:

$$\ell_a = A\nabla_a\phi_0 + B\nabla_a\phi_1 \tag{23}$$

$$k_a = C\nabla_a\phi_0 + D\nabla_a\phi_1 \tag{24}$$

Three out of four coefficients in the above expansions are determined by normalization conditions (21) and one remains free, due to the rescaling or boost gauge freedom ($\ell_a \to \alpha\ell_a, k_a \to \frac{1}{\alpha}\kappa_a$). On the boundary, we set the coefficient $B = 0$, so that the boundary is a level surface of ϕ_0 and we have $\ell_a \overset{\mathcal{B}}{=} A\nabla_a\phi_0$, and it is null because $\nabla_a\phi_0\nabla^a\phi_0 \propto \ell^a\ell_a = 0$. Note that ℓ_a and k_a are form fields defined in whole space-time and everywhere we have $\ell^2 = k^2 = 0$, whereas $\nabla_a\phi_0\nabla^a\phi_0 = 0$ is satisfied just on the boundary. In fact, in general we have $\nabla_a\phi_0\nabla^a\phi_0 = \frac{2BD}{AD-BC}$ from which one could specify the location of null boundary as $B = 0$. The other point regarding to this foliation comparing to single foliation is that the vectors ℓ_a and k_a are not in general hypersurface orthogonal. In fact using Eqs. (23) and (24), one can easily evaluate $\ell_{[a}\nabla_b\ell_{c]}$ or $k_{[a}\nabla_b k_{c]}$, which for general value of functions $\{A, B, C, D\}$ do not vanish. So according to the Frobenius theorem, vectors ℓ_a and k_a are not in general hypersurface orthogonal.

In this double foliation framework, the spacetime metric becomes:

$$g_{ab}dx^a dx^b = H_{ij}d\phi^i d\phi^j + q_{AB}(d\sigma^A + \beta_i^A d\phi^i) \times (d\sigma^B + \beta_j^B d\phi^j), \tag{25}$$

in which $\{i, j\} \in \{0, 1\}$ whereas $\{A, B\} \in \{2, \ldots, d-1\}$. Here σ^A are coordinates on codimension two surface S and β_i^A are shift vectors. The normal metric H_{ij} consists of lapse functions as

$$H_{ij} = -\begin{pmatrix} 2AC & BC + AD \\ BC + AD & 2BD \end{pmatrix}. \tag{26}$$

By covariant differentiation of vectors ℓ_a and k_a and projecting them in different directions, using q_b^a, ℓ^a, and k^a, we can define the following geometric objects from $\nabla_a\ell_b$ and $\nabla_a\ell_b$:

$$\nabla_a\ell_b = -\Theta_{ab} - \omega_a\ell_b - \ell_a\eta_b - k_a a_b + \kappa k_a\ell_b - \bar{\kappa}\ell_a\ell_b, \tag{27}$$

$$\nabla_b k_b = -\Xi_{ab} + \omega_a k_b - k_a\bar{\eta}_b - \ell_a\bar{a}_b - \kappa k_a k_b + \bar{\kappa}\ell_a k_b. \tag{28}$$

These relations are generalizations of the relation $\nabla_a n_b = -K_{ab} + n_a a_b$ to current case where decomposition has been done with two null vectors. The definitions are as follows:

$$\begin{aligned}
\Theta_{ab} &= -q^c{}_a q^d{}_b\nabla_a\ell_b, \qquad \Xi_{ab} = -q^c{}_a q^d{}_b\nabla_a k_b, \\
\eta_a &= q^c{}_a k^b\nabla_b\ell_c, \qquad \bar{\eta}_b = q^c{}_a\ell^b\nabla_b k_c, \\
\omega_a &= q^c{}_a k^b\nabla_c\ell_b = -q^c{}_a\ell^b\nabla_c k_b, \\
a_a &= q^c{}_a\ell^b\nabla_b\ell_c, \qquad \bar{a}_a = q^c{}_a k^b\nabla_b k_c, \\
\kappa &= \ell^a k^b\nabla_a\ell_b = -\ell^a\ell^b\nabla_a k_b, \\
\bar{\kappa} &= k^a\ell^b\nabla_a k_b = -k^a k^b\nabla_a\ell_b,
\end{aligned} \tag{29}$$

where Θ_{ab} and Ξ_{ab} are extrinsic curvatures of S while ω_a, η_a, and $\bar{\eta}_a$ are twists. In addition, a_a and $\bar{a}_a$ are tangent accelerations of ℓ^a and k^a to S, respectively. Moreover, κ and $\bar{\kappa}$ are in-affinity parameters.[4]

As a side remark, let us point out that this general double foliation described above is different from double null foliation as considered by various authors [24,25]. Although, the double null foliation is useful for initial value problem as shown by Sachs [26], one could show that for variational principle it suffers from having a partial gauge fixing condition which is a disadvantage [27]. This is because requiring that the level sets of a coordinate ϕ being null, fixes one of the metric components:

$$g^{\phi\phi} = g^{-1}(d\phi, d\phi) = 0.$$

The double null foliation is a special case of the above set up if we set $B = C = 0$ in whole spacetime. If fact, we will see in the next section that the variation of such components is important for finding complete canonical structure, and more importantly the canonical momenta of such variations is the quasilocal energy density of system.

B. Variation of Hilbert-Einstein action

The main point in calculating the variations on a null surface is that the operators δ for variations and the covariant derivative ∇ are defined in spacetime, whereas the condition for boundary, in the above setup $B = 0$, is valid only on the null boundary. Thus, we must first apply these operators and then impose the condition $B = 0$. For example consider variation of the vector ℓ_a given by Eq. (23). By variation of this vector on the boundary, one finds $\delta\ell_a = \delta A\nabla_a\phi_0 + \delta B\nabla_a\phi_1$. On the other hand, if we set B to zero first and then vary the equation, we find that $\delta\ell_a = \delta A\nabla_a\phi_0$. This variation is not valid, because variation of one degrees of freedom in metric has been killed in this relation, i.e., $\delta B \propto \ell^a\ell^b\delta g_{ab} = 0$. This variation is responsible for taking out the boundary from being null. To avoid losing any degree of freedom, in the following we will not set $B = 0$ until the end of calculations.

[4] The quantity κ is called surface gravity in the case of a black hole horizon null surface.

We consider a hypersurface to be a leaf of one of the foliations, let it be $\phi_0 = \text{const}$. Variation of HE action on such hypersurface in the double foliation determined by Eqs. (23) to (25) leads to [11]:

$$\begin{aligned}\delta\mathcal{S}_{\mathrm{HE}} &= \frac{1}{8\pi}\delta\left(\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}[D(\Theta+\kappa) - B(\Xi+\bar{\kappa})] + \int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}\ln D\sqrt{H}\right) \\ &+ \frac{1}{16\pi}\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}[D(\Theta^{ab} - q^{ab}(\Theta+\kappa)) - B(\Xi^{ab} - q^{ab}(\Xi+\bar{\kappa}))]\delta q_{ab} \\ &+ 2\omega^a\delta\beta_{1a} - 2\Xi\delta B + 2\Theta\delta D + \frac{1}{16\pi}\int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}(\ln D\sqrt{H}q^{ab})\delta q_{ab},\end{aligned} \tag{30}$$

where H is the determinant of H_{ij} defined above. The first line as a total variation suggests appropriate boundary term that must be subtracted in order to have a well-posed variational principle. The total variation term in the boundary integral, i.e.,

$$\sqrt{q}[D(\Theta+\kappa) - B(\Xi+\bar{\kappa})], \tag{31}$$

has a nice geometric interpretation. It can be rewritten as $\sqrt{-g}v_a\mathcal{K}^a$, where $v_a = \nabla_a\phi_0 = \frac{1}{\sqrt{H}}(D\ell_a - Bk_a)$ is the normal to the hypersurface, and $\mathcal{K}^a{}_{bc} = -(k^a\nabla_b\ell_c + l^a\nabla_b k_c)$ is defined such that $\mathcal{K}^a$ is its trace on two last indices.[5]

On a null boundary now we set $B = 0$; thus, we find the action with a well-posed variational principle to be

$$\begin{aligned}\mathcal{S} &= \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}R - \frac{1}{8\pi}\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}[D(\Theta+\kappa)] \\ &+ \frac{1}{8\pi}\int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}\ln D\sqrt{H}.\end{aligned} \tag{32}$$

Let us recall that, as mentioned in previous subsection, there is a scaling boost gauge symmetry in null hypersurface description by two vectors ℓ_a and k_a. We can use this gauge freedom to set $D = 1$ in this gauge and by putting $B = 0$ we have $\sqrt{H} = A$. Therefore, the action becomes:

$$\begin{aligned}\mathcal{S} &= \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}R - \frac{1}{8\pi}\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}(\Theta+\kappa) \\ &+ \frac{1}{8\pi}\int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}\ln A.\end{aligned} \tag{33}$$

This is what was found in [7,8]. In [6] the authors also have set the lapse $A = 1$, so the last term vanishes. Using the above action, the variation of the principal function on the null boundary becomes:

$$\begin{aligned}\delta\mathcal{S} &= \frac{1}{16\pi}\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}([(\Theta^{ab} - q^{ab}(\Theta+\kappa))]\delta q_{ab} \\ &+ 2\omega_a\delta\beta_1^a - 2\Xi\delta B) + \frac{1}{16\pi}\int_{S_1}^{S_2} \mathrm{d}^{d-2}x\sqrt{q}(\ln A q^{ab})\delta q_{ab}.\end{aligned} \tag{34}$$

A similar expression for canonical structure on the null boundary has been found in [8]. There, the authors considered only the variations that keeps the boundary to remain null, thus, the last term of the first line was missing in their analysis. Note that in this expression δq_{ab}, $\delta\beta_{1a}$, and $(-2\delta B)$ are all variations of the metric components tangential to the boundary as easily can be seen from the decomposition (25). This is an important point because, as in the non-null case, the variational principle tells us that for the Dirichlet boundary condition we only need to fix the tangential metric components. Here also the number of degrees of freedom matches with the non-null case; for example, in four dimensions δh_{ab} has six components while $(\delta q_{ab}, \delta\beta_{1a}, -2B)$ altogether have $3 + 2 + 1 = 6$ components.

C. The stress tensor

Having found the variation of action on the null boundary, we can use the BY prescription to find the stress tensor on the boundary. In doing so we must differentiate the principal function in space-time region, illustrated in Fig. 3, with respect to the metric component tangential to the null boundary segments $\mathcal{N}$. Note that differentiating with respect to metric components in Σ_1 and Σ_2 gives the canonical momenta P^{ab} similar to the previous section. According to the Eq. (25), the metric components tangential to $\mathcal{N}$ are $(\delta q_{ab}, \delta\beta_{1a}, -2B)$. Note that, in contrast to the previous timelike boundary presented in the last section, there is no induced metric on null surface; thus, we must differentiate with respect to each component of metric separately according to:

$$\begin{aligned}\frac{2}{\sqrt{q}}g_{ac}g_{bd}\frac{\delta\mathcal{S}|_{\mathcal{N}}}{\delta g_{cd}} &= \frac{2}{\sqrt{q}}\left(q_{ac}q_{bd}\frac{\delta\mathcal{S}}{\delta q_{cd}}\right. \\ &\left. - q_{c(a}k_{b)}\frac{\delta\mathcal{S}}{\delta\beta_{1c}} + k_a k_b\frac{\delta\mathcal{S}}{\delta(-2B)}\right).\end{aligned} \tag{35}$$

[5] We note that the common Gibbons-Hawking-York term can be rewritten as: $\sqrt{|h|}K = \sqrt{-g}\nabla_a\phi(n^a K)$, with $K = -\nabla_a n^a$, in this sense, the above term is a generalization of GHY term for null surfaces.

Where the notation $|_{\mathcal{N}}$ means differentiating with respect to the metric component tangential to the null boundary. On the other hand, the variational principle does not fix the boundary action completely; therefore, we can subtract any functional $\mathcal{S}_0$ of fixed boundary data from the action (33). Using the expression (34), hence, the stress tensor components read:

$$\epsilon \equiv \ell^a \ell^b T_{ab} = -\frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}}{\delta B} = \frac{1}{8\pi}[\Xi - \Xi_0], \tag{36}$$

$$j^a \equiv q^{ac}\ell^b T_{cb} = \frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}}{\delta \beta_{1a}} = \frac{1}{8\pi}[\omega^a - \omega_0^a], \tag{37}$$

$$\begin{aligned} s^{ab} &\equiv q^{ac}q^{bd}T_{cd} \\ &= \frac{2}{\sqrt{q}}\frac{\delta \mathcal{S}}{\delta q_{ab}} = \frac{1}{8\pi}\left[\Theta^{ab} - q^{ab}(\Theta + \kappa) - \frac{2}{\sqrt{q}}\frac{\delta \mathcal{S}_0}{\delta q_{ab}}\right], \end{aligned} \tag{38}$$

where ω_0^a and Ξ_0 are defined as:

$$\omega_0^a = \frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}_0}{\delta \beta_{1a}}, \qquad \Xi_0 = -\frac{1}{\sqrt{q}}\frac{\delta \mathcal{S}_0}{\delta B}.$$

According to the BY prescription, Eqs. (36) to (38) may be used to define quasilocal energy density ϵ, tangential momentum density j^a, and spatial stress s^{ab}, respectively.

The subtraction term $\mathcal{S}_0$ is related to the zero point of energy and must be well chosen so that the stress tensor leads to finite energy for different systems and conventionally zero energy for Minkowski space time. The expression (36) is similar to the Eq. (18) for timelike boundary. Ξ is defined by $\Xi = -q^{ab}\nabla_a k_b$ while $\boldsymbol{k}$ reads as $\boldsymbol{k} = -q^{ab}\nabla_a n_b$. The similarity is in the sense that for null boundary the only vector pointing out of the boundary is k_a. For non-null boundaries, studied in the previous section, the normal to boundary was also pointing out of it; however, note that for null hypersurfaces the normal is within the boundary.

IV. QUASILOCAL QUANTITIES

In this section we examine the proposed stress tensor by evaluating conserved charges of various space-times and then we compare it with the known results. Here, as to the timelike boundary, we propose the quasilocal quantity to be

$$Q_\xi = \int_S d^{d-2}x\sqrt{q}T_{ab}\ell^a\xi^b, \tag{39}$$

where ℓ^a has the role of time flow on the boundary while ξ^a is a Killing vector which generates an isometry of the boundary. If ℓ^a is the generator of time translation symmetry, then the total energy becomes:

$$E = \int_S d^{d-2}x\sqrt{q}\epsilon, \tag{40}$$

where the quasilocal energy density ϵ is defined by (36). When there is a rotational Killing vector ζ^a, then its corresponding angular momentum is

$$J = \int_S d^{d-2}x\sqrt{q}j_a\zeta^a. \tag{41}$$

In the following we will calculate the above quantities for some well-known gravitational solutions.

A. Minkowski space

The simplest example for investigation is Minkowski spacetime. In retarded-spherical coordinates the metric is as follows:

$$ds^2 = -du^2 - 2dudr + r^2 d\Omega^2 \tag{42}$$

where $d\Omega^2 = \gamma_{AB}d\sigma^A d\sigma^B$ is the metric on a unit sphere. It is evident that $u =$ const. is a null surface. Comparing the above line element with Eq. (25) and using the foliation relations (23) and (24) yields $\ell_a = \nabla_a u$, $k_a = \nabla_a r + \frac{1}{2}\nabla_a u$ and $q_{AB} = r^2\gamma_{AB}$. From the definitions (29) one easily finds:

$$\Theta_{AB} = -2\Xi_{AB} = r\gamma_{AB}, \qquad \omega_A = 0. \tag{43}$$

As a result $\Xi = -\frac{1}{r}$, and the integral $\int d\theta d\phi\sqrt{q}\Xi = -\int d\theta d\phi r\sin^2\theta$ will be infinite as $r \to \infty$. Subtracting Ξ_0 leads to zero energy for flat space as expected. In fact, by using its definition we calculate Ξ_0 as embedding of boundary in flat space. Then, one could show that $\Xi = \Xi_0$ and as a result the energy density vanishes. In other words, in this case, the physical and reference spacetimes are the same.

B. Schwarzschild black hole

Our next simple example is the Schwarzschild black hole. In retarded Eddington-Finkelstein coordinates the metric is given by:

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\Omega^2. \tag{44}$$

In this case we have $\ell_a = \nabla_a u$ and $k_a = \frac{1}{2f(r)}\nabla_a u + \nabla_a r$. Here, by calculating the quasilocal energy density, we get:

$$\epsilon = \frac{1}{8\pi}[\Xi - \Xi_0] = \frac{1}{8\pi}\left[-\frac{f(r)}{r} + \frac{1}{r}\right]. \tag{45}$$

Here, we have used the fact $\Xi_0 = -\frac{1}{r}$, as a result of embedding in flat space, and $\Xi = -\frac{f(r)}{r}$ which can easily be found. By replacing $f(r) = 1 - \frac{2M}{r}$, we obtain $\epsilon = \frac{M}{4\pi r^2}$. Thus the total energy becomes:

$$E = \int_0^{\pi} \int_0^{2\pi} d\theta d\phi r^2 \sin\theta \epsilon = 4\pi r^2 \left(\frac{M}{4\pi r^2}\right) = M. \qquad (46)$$

It is interesting that the quasilocal energy, as calculated for the null observers in $\mathcal{N}$, is independent of the distance r for which the energy is calculated. At first sight, it may seems strange because the usual BY expression for energy is just equal to the ADM mass at $r \to \infty$. However, notice that as mentioned previously, in general the function $E(r)$ is observer dependent. The in-dependency of energy to the distance also has been observed for boosted foliation of Schwarzschild black hole in Ref. [12].

C. AdS-Schwarzschild black hole

In the above example, we considered the usual asymptotic flat Schwarzschild black hole. The validity of the above procedure for asymptotic AdS/dS black holes could be justified as follows. In fact, our method is similar to the usual Brown-York strategy for quasilocal quantities. Let us point out that in timelike case, for every geometry that can be contained in a spacetime region depicted in Fig. 1, one must be able to calculate quasilocal quantities. The only subtlety here is the problem of choosing the reference spacetime. For asymptotic flat space, the natural choice is Minkowski space. On the other hand, the preferred reference for asymptotic AdS spacetime is the AdS space (see, e.g., Ref. [28]). For spacetime region in this study, namely Fig. 3, the same story is true. Moreover, for AdS-Schwarzschild black hole we have to embed the null hypersurface in AdS space in order to obtain an expression for the reference term. Consider the metric

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\Omega^2 \qquad (47)$$

with $f(r) = 1 + \frac{r^2}{l^2} - \frac{2M}{r}$, for AdS-Schwarzschild black hole. One can easily calculate Ξ_0 and Ξ as

$$\Xi_0 = -\frac{1}{r} - \frac{r}{l^2}, \qquad \Xi = -\frac{f(r)}{r} \qquad (48)$$

from which the quasilocal energy density will be obtained as $\epsilon = \frac{M}{4\pi r^2}$. Thus, simple integration leads to the ADM mass M for these black holes. Furthermore, the same results could be derived in the case of asymptotic dS black hole if we choose the reference spacetime to be dS.

D. Slow-rotating black hole

Here we examine conserved quantities for slow rotating Kerr black holes. Again we write the metric in the retarded Eddington-Finkelstein coordinates:

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\Omega^2 + \frac{2J}{r}\sin^2\theta du d\phi + \frac{2J}{rf(r)}\sin^2\theta dr d\phi. \qquad (49)$$

Comparing with metric (25) we arrive at:

$$A = 1, \qquad B = 0, \qquad C = \frac{1}{2f}, \qquad D = 1,$$
$$\beta_{1\phi} = \frac{2J}{r}\sin^2\theta, \qquad \beta_{2\phi} = \frac{2J}{rf}\sin^2\theta. \qquad (50)$$

Up to the first power in J, one finds the same expression for energy as the Schwarzschild case. The angular momentum quantity is related to ω_a, using its definition we find:

$$\xi^a \omega_a = \frac{3J\sin^2\theta}{r^2}, \qquad (51)$$

where $\xi = \partial_\phi$ is rotational Killing symmetry. Therefore, the total angular momentum becomes:

$$Q_\xi = \frac{1}{8\pi}\int_0^{\pi}\int_0^{2\pi} d\theta d\phi r^2 \sin\theta \frac{3J\sin^2\theta}{r^2} = J. \qquad (52)$$

E. Asymptotic flat spacetime and the Bondi mass

Here, we want to study the quasilocal quantities for gravitational theories in which the metric has an asymptotic flat space behavior. In order to do that, we suppose the metric in the Bondi coordinates. In this gauge, the most general four-dimensional metric takes the form:

$$ds^2 = -UVdu^2 - 2Vdudr + q_{AB}(d\sigma^A + U^A du)(d\sigma^B + U^B du) \qquad (53)$$

where $\partial_r \det(\frac{q_{AB}}{r^2}) = 0$. By comparison the above equation with Eq. (25), we find:

$$A = V, \qquad B = 0, \qquad C = \frac{U}{2}, \qquad D = 1,$$
$$\beta_0^A = U^A, \qquad \beta_1^A = 0. \qquad (54)$$

The expressions for Θ_{AB} and Ξ_{AB} in the metric (25) have been calculated explicitly in [11] and are:

$$\Theta_{AB} = -\frac{1}{2\sqrt{H}}(B\partial_0 q_{AB} - A\partial_1 q_{AB} - 2B\mathcal{D}_{(A}\beta_{0B)} + 2A\mathcal{D}_{(A}\beta_{1B)}), \qquad (55)$$

$$\Xi_{AB} = -\frac{1}{2\sqrt{H}}(-D\partial_0 q_{AB} + C\partial_1 q_{AB} + 2D\mathcal{D}_{(A}\beta_{0B)} - 2C\mathcal{D}_{(A}\beta_{1B)}), \qquad (56)$$

where $\mathcal{D}$ is covariant derivative on two sphere $\mathcal{S}$, compatible with the metric q_{AB}.[6] Using standard asymptotic expansions [29]:

$$U = 1 - \frac{2m_B}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad V = 1 + \mathcal{O}\left(\frac{1}{r^2}\right)$$
$$\beta_0^A = \frac{W^A}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad q_{AB} = r^2\gamma_{AB} + \mathcal{O}(r)$$

we can easily find the following leading terms for Ξ:

$$\Xi = -\frac{1}{r} + \frac{2m_B(u,\sigma^A)}{r^2} + \frac{\mathcal{D}_A W^A}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right).$$

By embedding in flat space we have the reference term $\Xi_0 = -\frac{1}{r}$. The term $\mathcal{D}_A W^A$ is a total derivative on compact two sphere $\mathcal{S}$ which vanishes by integration. Thus, finally the total energy becomes:

$$E = \frac{1}{8\pi}\int_S d^2x\sqrt{q}\epsilon = \frac{1}{4\pi}\int_S d\Omega m_B(u,\sigma^A) \tag{57}$$

which is the expression known as Bondi mass.

F. A possible counterterm for asymptotic flat spacetime

We have seen that for asymptotic AdS space there is a counterterm method for regularizing quasilocal quantities that has some advantages. The first benefit is that the dependency on the reference spacetime and mathematical difficulty of embedding is avoided. Furthermore, the counterterms have a direct interpretation in the dual field theory and have an important role for building the dictionary of AdS/CFT. Then the natural question is that: why is there no similar counterterms for flat space? For flat space there is not a length scale counterpart to the AdS length scale; thus, we cannot have terms like: $\ell^{2n-1}\mathcal{R}^n$. In addition, the counterterms should not spoil the variational principle so that the extrinsic curvature terms are forbidden. Therefore, on the timelike boundary we do not have a viable candidate as counterterm in asymptotic flat spacetime. However, as we have seen in this paper, null boundaries are special. Consider the following term on the null boundary:

$$\alpha\int_{\mathcal{N}} \mathrm{d}^{d-1}x\sqrt{q}B\Theta \tag{58}$$

with arbitrary numerical coefficient α. The first point is that adding such term is compatible with Dirichlet boundary conditions. Also, we must note that although the above term vanishes on the null boundary, its variation is not zero. Therefore, it has a contribution to the stress tensor and energy though it is zero for on-shell action. Fortunately, we can set the coefficient $\alpha = -\frac{1}{2}$ so that its contribution to energy make the total finite. By adding this term to the boundary action, the quasilocal energy density becomes:

$$\epsilon = \frac{1}{8\pi}\left[\Xi + \frac{1}{2}\Theta\right]. \tag{59}$$

For asymptotic flat spacetime studied previously, using (55), Θ is easily computed and its leading terms is as:

$$\Theta = \frac{2}{r} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{60}$$

Therefore the expression (59) leads to correct total energy without need to embedding and reference spacetime.

V. CONCLUSION AND OUTLOOK

In this work we have extended the Brown-York prescription to the case of null boundaries. For the achievement of this goal we needed a general double foliation framework. The mathematics of this general double foliation is described in [11] and reviewed and clarified here. The main reason for considering such framework is that a single or double null foliation is basically gauge fixed, and are not appropriate for a variational problem, because some degrees of freedom are already fixed by considering a null foliation:

$$g^{\phi\phi} = g^{-1}(d\phi, d\phi) = 0.$$

Variation of Hilbert-Einstein is calculated on such general double foliation in [11]. The main unanswered question in that article was to determine the physical meaning of metric variation on such null boundary. This question is answered here. We especially have seen that the variation of one metric component is responsible for taking away the boundary from being null. It was shown that an exact derivation with this variation gives the expression for quasilocal energy density. In addition, the expressions for total energy and angular momentum were examined for some known spacetimes. Moreover, a special property of the calculated energy for null observers was found to be the in-dependency to radial distance. Furthermore, it was shown that in the case of null boundaries, there is a possible counterterm which is consistent with the variational principle and can be added to boundary action so that the total energy becomes finite, without the necessity of embedding in reference spacetime.

Regarding the similarity with the standard AdS/CFT dictionary, one may interpret the introduced stress tensor as expectation value of stress tensor in dual field theory to flat space. Because, according to Penrose diagrams for

[6] These relations are counterpart to the well-known relation for extrinsic curvature in $3+1$ decomposition: $K_{ij} = -\frac{1}{2N}(\partial_t h_{ij} - 2D_{(i}N_{j)})$.

asymptotic flat spacetimes, the null hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$ are regarded as the boundaries of spacetime; thus, one application of the stress tensor found in this paper may be flat holography. In this context, the relation between asymptotic symmetries and some special limits—for example, Carrolian symmetry—is worthwhile to investigate [30].

Another application of the formalism presented in this article and [11] is to revisit the gravity in light-front. The usual investigation of gravity in light-front uses the double-null foliation of spacetime [31,32]. However, as noted above, the double-null foliation leads to partial gauge fixing of the metric. It is in contrast to field theory formulation in the light-front coordinate which no gauge fixing is required. As we have seen a general double foliation can preserve all degrees of freedom. Thus, it is motivating to revisit gravity in light-front.

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[1] L. B. Szabados, Quasi-local energy-momentum and angular momentum in general relativity, Living Rev. Relativity **12**, 4 (2009).

[2] J. D. Brown and J. W. York, Jr., Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D **47**, 1407 (1993).

[3] T. Padmanabhan, A short note on the boundary term for the Hilbert action, Mod. Phys. Lett. A **29**, 1450037 (2014).

[4] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).

[5] J. W. York, Jr., Role of Conformal Three Geometry in the Dynamics of Gravitation, Phys. Rev. Lett. **28**, 1082 (1972).

[6] K. Parattu, S. Chakraborty, B. R. Majhi, and T. Padmanabhan, A boundary term for the gravitational action with null boundaries, Gen. Relativ. Gravit. **48**, 94 (2016).

[7] L. Lehner, R. C. Myers, E. Poisson, and R. D. Sorkin, Gravitational action with null boundaries, Phys. Rev. D **94**, 084046 (2016).

[8] F. Hopfmüller and L. Freidel, Gravity degrees of freedom on a null surface, Phys. Rev. D **95**, 104006 (2017).

[9] I. Jubb, J. Samuel, R. Sorkin, and S. Surya, Boundary and corner terms in the action for general relativity, Classical Quantum Gravity **34**, 065006 (2017).

[10] F. Hopfmüller and L. Freidel, Null conservation laws for gravity, Phys. Rev. D **97**, 124029 (2018).

[11] S. Aghapour, G. Jafari, and M. Golshani, On variational principle and canonical structure of gravitational theory in double-foliation formalism, Classical Quantum Gravity **36**, 015012 (2019).

[12] J. D. Brown, S. R. Lau, and J. W. York, Jr., Action and energy of the gravitational field, arXiv:gr-qc/0010024.

[13] C. Krishnan and A. Raju, A Neumann boundary term for gravity, Mod. Phys. Lett. A **32**, 1750077 (2017).

[14] G. Hayward, Gravitational action for space-times with nonsmooth boundaries, Phys. Rev. D **47**, 3275 (1993).

[15] S. W. Hawking and C. J. Hunter, The gravitational Hamiltonian in the presence of nonorthogonal boundaries, Classical Quantum Gravity **13**, 2735 (1996).

[16] I. S. Booth and R. B. Mann, Moving observers, nonorthogonal boundaries, and quasilocal energy, Phys. Rev. D **59**, 064021 (1999).

[17] V. Balasubramanian and P. Kraus, A stress tensor for Anti-de Sitter gravity, Commun. Math. Phys. **208**, 413 (1999).

[18] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).

[19] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[20] O. Miskovic, R. Olea, and M. Tsoukalas, Renormalized AdS action and critical gravity, J. High Energy Phys. 08 (2014) 108.

[21] R. N. Caldeira Costa, Aspects of the zero Λ limit in the AdS/CFT correspondence, Phys. Rev. D **90**, 104018 (2014).

[22] E. Gourgoulhon and J. L. Jaramillo, A $3+1$ perspective on null hypersurfaces and isolated horizons, Phys. Rep. **423**, 159 (2006).

[23] C. G. Torre, Null surface geometrodynamics, Classical Quantum Gravity **3**, 773 (1986).

[24] P. R. Brady, S. Droz, W. Israel, and S. M. Morsink, Covariant double null dynamics: $(2+2)$ splitting of the Einstein equations, Classical Quantum Gravity **13**, 2211 (1996).

[25] R. A. d'Inverno, P. Lambert, and J. A. Vickers, Hamiltonian analysis of the double null $2+2$ decomposition of general relativity expressed in terms of self-dual bivectors, Classical Quantum Gravity **23**, 4511 (2006).

[26] R. K. Sachs, On the characteristic initial value problem in gravitational theory, J. Math. Phys. (N.Y.) **3**, 908 (1962).

[27] J. N. Goldberg, D. C. Robinson, and C. Soteriou, Null hypersurfaces and new variables, Classical Quantum Gravity **9**, 1309 (1992).

[28] J. D. Brown, J. Creighton, and R. B. Mann, Temperature, energy and heat capacity of asymptotically anti-de Sitter black holes, Phys. Rev. D **50**, 6394 (1994).

[29] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems, Proc. R. Soc. A **269**, 21 (1962).

[30] L. Ciambelli and C. Marteau, Carrollian conservation laws and Ricci-flat gravity, Classical Quantum Gravity **36**, 085004 (2019).

[31] S. Alexandrov and S. Speziale, First order gravity on the light front, Phys. Rev. D **91**, 064043 (2015).

[32] V. Shyam, Scale invariance in gravity on the light-front, arXiv:1507.08493.