Charged spherically symmetric black holes in $f(R)$ gravity and their stability analysis

G. G. L. Nashed[1,2,*] and S. Capozziello[3,4,5,6,†]

[1]*Centre for Theoretical Physics, The British University in Egypt, P.O. Box 43, El Sherouk City, Cairo 11837, Egypt*
[2]*Department of Mathematics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt*
[3]*Dipartimento di Fisica "E. Pancini", Universitá di Napoli "Federico II", Complesso Universitario di Monte Sant' Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy*
[4]*Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli, Complesso Universitario di Monte Sant'Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy*
[5]*Gran Sasso Science Institute, Viale F. Crispi, 7, I-67100, L'Aquila, Italy*
[6]*Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia*

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A new class of analytic charged spherically symmetric black hole solutions, which behave asymptotically as flat or (anti-)de Sitter spacetimes, is derived for specific classes of $f(R)$ gravity, i.e., $f(R) = R - 2\alpha\sqrt{R}$ and $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$, where Λ is the cosmological constant. These black holes are characterized by the dimensional parameter α that makes solutions deviate from the standard solutions of general relativity. The Kretschmann scalar and squared Ricci tensor are shown to depend on the parameter α, which is not allowed to be zero. Thermodynamical quantities, like entropy, Hawking temperature, quasilocal energy, and the Gibbs free energy are calculated. From these calculations, it is possible to put a constraint on the dimensional parameter α to have $0 < \alpha < 0.5$ so that all thermodynamical quantities have a physical meaning. The interesting result of these calculations is the possibility of a negative black hole entropy. Furthermore, present calculations show that for negative energy particles inside a black hole,behave as if they have a negative entropy. This fact gives rise to instability for $f_{RR} < 0$. Finally, we study the linear metric perturbations of the derived black hole solution. We show that for the odd-type modes our black hole is always stable and has a radial speed with fixed value equal to 1. We also use the geodesic deviation to derive further stability conditions.

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I. INTRODUCTION

Challenging problems ranging from quantum gravity to dark energy (DE) and dark matter (DM) give support to searching for other gravitational theories beyond the standard Einstein general relativity (GR). Actually, GR has many unsolved issues like singularities, the nature of DE and DM, etc. All these issues encourage scientists to modify GR or extend it in view of addressing shortcomings at UV and IR scales [1]. In other words, viable modified/extended theories should be compatible with the current experimental constraints and should give motivations on issues in quantum gravity and cosmology. Thus, it is straightforward to directly extend GR, considering it as a limit of a more general theory of gravitation.[1] Among the possible extensions of GR, the so-called $f(R)$ gravity generalizes the Einstein-Hilbert action by substituting the Ricci scalar R by an analytic differentiable function. The fundamental reasons for this approach come out of the formulation of any quantum field theory on curved spacetime [3]. $f(R)$ gravity has some important applications, like the Starobinsky model, $f(R) = R + \alpha R^2$, $\alpha > 0$, which is successful in explaining inflationary behavior of early Universe [4–6]. Furthermore, $f(R)$ gravity is capable of explaining the observed cosmic acceleration without

*nashed@bue.edu.eg
†capozzie@na.infn.it

[1]In *modified gravity*, it is not necessary to recover GR, however, an equivalent form like the teleparallel equivalent of general relativity can be recovered. Extended gravity means that in a given limit, or for a given choice, GR is recovered. For a discussion, see Ref. [2].

assuming the cosmological constant. Possible toy models have the form $f(R) = R - \frac{\beta}{R^n}$, where β and n have positive values [7–9]. Nevertheless, this model suffers from instability problems because of the second derivative of the function f that has a negative value, i.e., $f_{RR} < 0$ [10–18]. Lately, this problem has been tackled [19] and cosmological stable models have been achieved using some limitations on the parameter space. There are many viable cosmological models constructed using $f(R)$ [20,21]. Finally, $f(R)$ models give interesting results for structure formation, as the modification of the spectra of galaxy clustering, cosmic microwave background, weak lensing, etc., [22–30]. There are many applications of $f(R)$ from the astrophysical point of view [31–38]; for general reviews of $f(R)$ gravity, see Refs. [1,39–41].

From the viewpoint of mathematically, modified/extended gravity poses the issue to establish or modified well-known facts of GR like the stability of solutions, initial value problem, and problem of deriving new black hole solutions [42–46]. As is well known, in addition to the cosmological solutions, there exist axially symmetric as well as spherical ones that could have a main role in several astrophysical problems spanning from black hole solutions to galactic nuclei. Modified gravitational theories must include black hole solutions like Schwarzschild in order to be compatible with GR results and, in principle, must give new black hole solutions that might have physical interest. Accordingly to this fact, the way to find out exact or approximate black hole solutions is highly important to investigate if observations can be matched to modified/extended gravity [47,48].

In the framework of $f(R)$ gravity, there is specific interest for spherically symmetric black hole solutions. They have been derived using the constant Ricci scalar [49]. Moreover, spherically symmetric black hole solutions, including perfect fluid matter, have been analyzed [50]. Additionally, by using the method of Noether symmetry, many spherically symmetric black holes have been derived [50]. Hollenstein and Lobo [51] derived exact solutions of static spherically symmetric spacetimes in $f(R)$ coupled to nonlinear electrodynamics. For the readers interested in the static black holes, we refer to Refs. [52–87], and the references therein. Using the Lagrangian multiplier, new analytic solutions with the dynamical Ricci scalar have been derived [88]. It is the purpose of the present study, by using the field equation of $f(R)$, to generalize these black hole solutions [88] and derive new charged black hole solutions with a dynamical Ricci scalar asymptotically converging towards flat or (anti-)de Sitter [(A)dS] spacetimes.

Gravitational stability of a black hole solution is considered a main problem for checking the adequateness of any black hole solutions [89,90]. However, the stability analysis appears to be not directly applicable to $f(R)$ black hole solutions because it involves fourth-order derivative terms in the linearized equations [91,92]. In that case, it is necessary that the black holes are free from tachyon and ghost instabilities that would come into the game as soon as one is considering $f(R)$ gravity [93]. Therefore, one may transform $f(R)$ gravity into the corresponding scalar-tensor theory to remove the fourth-order derivative terms [94]. It was suggested that the stability of black hole solutions does not rely on the frame due to the fact that it is a classical solution that is considered as the ground state [95]. It is well known that a nonminimally coupled scalar makes the linearized GR field equations around the black hole very intricate when compared to a minimally coupled scalar in the context of GR [96]. Because of this intricacy, some people have used conformal transformations to find the corresponding theory in the Einstein frame where a minimally coupled scalar appears. Taking into account these difficulties, several perturbation studies on the black holes in different modified gravitational theories have been developed. See, e.g., Refs. [93,97–100].

The paper is organized as follows. In Sec. II, a summary of Maxwell-$f(R)$ gravity is provided. In Sec. III, a spherically symmetric ansatz is applied to the field equation of Maxwell-$f(R)$ theory, and an exact solution is derived. In Sec. IV, the same spherically symmetric ansatz is applied to the field equation of Maxwell-$f(R)$ theory that includes a cosmological constant. Solving the resulting differential equations, we derive a new black hole solution that behaves as (A)dS. In Sec. V, the characteristic properties of these black holes are analyzed. In Sec. VII C, thermodynamical quantities like entropy, quasilocal energy, Hawking temperature, and Gibbs energy are calculated. We show that the entropy of the derived black hole solutions is not proportional to the horizon area and show some regions of the parameter space where the entropy becomes negative. The main reason for this result is due to the parameter α related to the higher-order correction. In Sec. VII, we study the linear stability using the odd perturbations to the black holes derived in Secs. III and IV. Furthermore, in Sec. VII, we derive the stability conditions considering the geodesic motion. In Sec. VIII, we discuss the main results of the present study and draw conclusions.

II. MAXWELL–$f(R)$ GRAVITY

The theory of gravity that will be considered in this work is the $f(R)$ gravity, which was first taken into account in Ref. [101]. See also Refs. [1,7–9,102]:

$$\mathcal{S} := \mathcal{S}_g + \mathcal{S}_{\rm E.M.}, \tag{1}$$

where $\mathcal{S}_g$ is the gravitational action given by

$$\mathcal{S}_g := \frac{1}{2\kappa}\int d^4x\sqrt{-g}(f(R) - \Lambda), \tag{2}$$

where Λ represents the cosmological constant, R is the Ricci scalar, κ is the gravitational constant, g is the determinant of the metric, and $f(R)$ is an analytic differentiable function. In this study, $\mathcal{S}_{\rm E.M.}$ is the action of the nonlinear electrodynamics field, which takes the form

$$\mathcal{S}_{\text{E.M.}} \coloneqq -\frac{1}{2}F^{2s}, \tag{3}$$

where $s \geq 1$ is an arbitrary parameter that is equal to 1 for the standard Maxwell theory and $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = 2A_{[\mu,\nu]}$ with A_μ being the gauge potential 1-form and the comma denotes the ordinary differentiation[2] [82].

The field equations of $f(R)$ gravitational theory can be obtained by carrying out the variations of the action given by Eq. (1) with respect to the metric tensor $g_{\mu\nu}$ and the strength tensor F that yield the form of the field equations [103,104]

$$I_{\mu\nu} = R_{\mu\nu}f_R - \frac{1}{2}g_{\mu\nu}f(R) - 2g_{\mu\nu}\Lambda + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R - 8\pi T_{\mu\nu} \equiv 0, \tag{4}$$

$$\partial_\nu(\sqrt{-g}\mathrm{F}^{\mu\nu}F^{s-1}) = 0, \tag{5}$$

with $R_{\mu\nu}$ being the Ricci tensor defined by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = 2\Gamma^\rho{}_{\mu[\nu,\rho]} + 2\Gamma^\rho{}_{\beta[\rho}\Gamma^\beta{}_{\nu]\mu},$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel symbols of second kind. The d'Alembert operator $\Box$ is defined as $\Box = \nabla_\alpha\nabla^\alpha$, where $\nabla_\alpha V^\beta$ is the covariant derivatives of the vector V^β and $f_R = \frac{df(R)}{dR}$. In this study $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} \coloneqq \frac{1}{4\pi}\left(s\mathrm{g}_{\rho\sigma}\mathrm{F}_\nu{}^\rho\mathrm{F}_\mu{}^\sigma F^{s-1} - \frac{1}{4}\mathrm{g}_{\mu\nu}F^{2s}\right), \tag{6}$$

which is the energy-momentum tensor of the nonlinear electrodynamic field. When $s = 1$, we get the standard energy-momentum tensor of Maxwell field.

The trace of Eq. (4) is

$$Rf_R - 2f(R) - 8\Lambda + 3\Box f_R = T, \quad \text{where } T = F^s(sF - F^s). \tag{7}$$

It is worth noticing that, for $s = 1$, it is $T = 0$. This property means that the Maxwell field is conformally invariant. In the following, we are going to assume some form of the field equations (4) without and with a cosmological constant to derive exact solutions that asymptotically behave as flat or (A)dS spacetimes.

III. EXACT CHARGED BLACK HOLE SOLUTION

Let us derive a charged black hole solution adopting the model $f(R) = R - 2\alpha\sqrt{R}$. To this aim, we are going to use the spherically symmetric ansatz[3]

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2, \tag{8}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta$ is the line element on the unit sphere. The Ricci scalar of the metric (8) has the form

$$R = \frac{2 - r^2B'' - 4rB' - 2B}{r^2}. \tag{9}$$

Applying the ansatz (8) to Eqs. (4), (5), and (7), after using (9) and putting the parameter $s = 1$, we get the non-vanishing field equations[4]

$$\begin{aligned}
I_t{}^t = \frac{1}{2r^{10}\sqrt{R^9}}\{&r^6\sqrt{R^7}[r^4BB'''' + B'''(1/2r^4B' + 6r^3B) + 2r^2B''(B + rB') - r^2B'^2 + 2B'(r - 3rB) + 4B(B-1)]\\
&+ r^4\sqrt{R^5}[4r^2B'^2 + r^6RBB'''' - r^6BB'''^2 + 1/2r^3B'''\{r^2B''(rB' - 4B) + 2B'(29rB - r) + 40B(B-1)\}\\
&+ 2r^4B''^2(r^2q'' - 6B + 2rB' - 1) + r^2B'' \times [23r^2B'^2 + 2rB'(23B + 8r^2q'^2 - 13) + 8(B-1)(6B + r^2q'^2 - 1)]\\
&+ 28r^3B'^3 + B'^2(34r^2B + 32r^4q'^2 - 54r^2) + 4rB'(B-1) \times (7B + r^2q'^2 - 9) + 8(B-1)^2[r^2q'^2 - 1]]\\
&+ r^4R^2[B\sqrt{R}[4r^2B'' + r^3B''' - 2rB' + 4(1-B)]^2 + \alpha\{r^6RBB'''' - 3/2r^6BB'''^2 + 1/2r^3B'''[r^2B''(rB' - 12B)\\
&+ 4r^2B'^2 + 2B'(31rB - r) + 48B(B-1)] - r^6B''^3 - 100r^3B'^3 + 2r^2B'^2[96B - 85] + B''^2(8r^4 - 12r^5B' - 30r^4B)\\
&+ r^2B''[57r^2B'^2 + 14rB'[3B-5] + 20 + 4B(4-3B)] - 4rB'(B-1)[27B - 23] - 16(B-1)^2(2B-1)\}]\} = 0,
\end{aligned}$$

[2]The square brackets represent the antisymmetrization, i.e., $A_{[\mu,\nu]} = \frac{1}{2}(A_{\mu,\nu} - A_{\nu,\mu})$, and the symmetric one is represented by $A_{(\mu,\nu)} = \frac{1}{2}(A_{\mu,\nu} + A_{\nu,\mu})$.

[3]The reason to take the ansatz (8) is to be able to find an exact solution. Other forms make the field equations very complicated and not easy to solve.

[4]Here and through all this study, $B \equiv B(r)$, $B' = \frac{dB(r)}{dr}$, $B'' = \frac{d^2B(r)}{dr^2}$, etc. Also, in this application, we put $\Lambda = 0$.

$$
\begin{aligned}
I_r{}^r = &-\frac{1}{4r^8\sqrt{R^7}}\{r^4\sqrt{R^5}[(rB'+4B)[4(1-B)-2rB'+r^2(4B''+rB''')]\\
&-(r^3B'''[rB'+4B]+4r^2B''[5B+r^2q'^2+2rB'-1]+14r^2B'^2+B'[16r^3q'^2+12rB-20r]\\
&+8(B-1)[r^2q'^2-B-1])]-\alpha r^4R^2(r^3B'''[4B+rB']-2r^4B''^2+B''[4r^2(B+3)-16r^3B']-50r^2B'^2\\
&+4rB'(15-17B)-16[2B-1][B-1])\}=0,\\
I_\theta{}^\theta = I_\phi{}^\phi = &\frac{-1}{2r^{10}\sqrt{R^9}}\{r^6\sqrt{R^7}[r^4BB''''+r^3B'''(rB'+5B)+2r^2B''(2rB'-B)+2rB'(2-3B)-2r^2B'^2+8B(B-1)]\\
&-r^5\sqrt{R^5}[r^5BRB''''-r^5BB'''^2+r^2B'''\{r^2B''(rB'-3B)+4r^2B'^2+2rB'(13B-2)+18B(B-1)\}\\
&+r^5B''^3-2r^3B''^2(r^2q'^2+2-7rB'+7B)+2rB''(23r^2B'^2-rB'[14+8r^2q'^2-17B]\\
&+2(B-1)[1-10B+2r^2q'^2])+24r^2B'^3-4rB'^2(3+8r^2q'^2)+4B'(B-1)(3B-r^2q'^2)-8rq'^2(B-1)^2]\\
&+r^4R^2[\sqrt{R}(B[(B-1)-r^3B'''-4r^2B''+2rB']^2)+\alpha(r^6BB''''R+3/2r^6BB'''^2-r^3B'''[r^2B''\{rB'-7B\}\\
&+4r^2B'^2+2rB'[14B-1]+22B(B-1)]+2r^6B''^3+2r^4B''^2[18B+9rB'-5]\\
&+2r^2B''[33r^2B'^2+rB'(27B-34)-2(9B^2-5B-4)]+104r^3B'^3+2r^2B'^2(74-81B)+4rB'(B-1)(15B-16)\\
&-8\{1-2B^3-4B+5B^2\})]\}=0,\\
I = &\frac{-3}{2r^{10}\sqrt{R^9}}\{r^6\sqrt{R^7}[r^4BB''''+r^3B'''(rB'+6B)+2r^2B''(B+2rB')-2r^2B'^2+4rB'(1-2B)+4B(B-1)]\\
&+r^4\sqrt{R^5}[r^6RBB''''+r^6BB'''^2-r^3B'''\{r^2B''[rB'-2B]+4r^2B'^2+2rB'(15B-1)+20B(B-1)\}+2/3r^6B''^3\\
&+2r^4B''^2(6rB'-5B-2)+2r^2B''\{23r^2B'^2+2rB'(14B-9)+4(6B^2-7B+1)\}\\
&+104/3r^3B'^3+4r^2B'^2(6B-11)+8rB'(B-1)(2B-3)-8/3(B^3-2)+8B]\\
&-r^4R^2[B\sqrt{R}(4+r^3B'''-4B+4r^2B''-2rB')^2-\alpha(r^6RBB''''+3/2r^6BB'''^2\\
&-r^3B'''\{r^2B''(rB'-6B)+4r^2B'^2+2rB'(16B-1)+24B(B-1)\}+2r^6B''^3+2r^4B''^2\{10rB'+17B-6\}\\
&+2r^2B''\{41r^2B'^2+2rB'(16B-23)+4(3-4B^2+B)\}+136r^3B'^3-2r^2B'^2(106-117B)\\
&+8rB'(B-1)(15B-13)+16(2B-1)(B-1)^2)]\}=0,
\end{aligned}
\tag{10}
$$

where q is the gauge potential, which is defined as

$$A \coloneqq q(r)dt. \tag{11}$$

If we subtract $I_t{}^t$ from $I_r{}^r$ and solve the system $I_t{}^t - I_r{}^r$ and $I_\theta{}^\theta$, which is a closed system for the two unknown functions $B(r)$ and $q(r)$, we get the following exact solution:

$$B(r)=\frac{1}{2}-\frac{1}{3\alpha r}+\frac{1}{3\alpha r^2}, \qquad A=\frac{1}{\sqrt{3\alpha r}}. \tag{12}$$

The analytic solution (12) satisfies the system of differential equations (10) including the trace equation I. Using Eq. (9), we get the Ricci scalar in the form

$$R=\frac{1}{r^2}, \tag{13}$$

which is also a consistency check for the whole procedure. The metric of the above solution takes the form

$$
\begin{aligned}
ds^2 = &\left(\frac{1}{2}-\frac{1}{3\alpha r}+\frac{1}{3\alpha r^2}\right)dt^2-\left(\frac{1}{2}-\frac{1}{3\alpha r}+\frac{1}{3\alpha r^2}\right)^{-1}dr^2\\
&-r^2d\Omega^2,
\end{aligned}
\tag{14}
$$

which asymptotically behaves as a flat spacetime. We have to stress the fact that solution (12) is different from that obtained in Ref. [88] due to the fact that the authors of that reference derived their solution using the form $f(R)=R+\alpha\sqrt{R}$. Therefore, our solution is identical with theirs when we neglect the term $\frac{1}{3\alpha r^2}$, which is responsible for the electric charge, and reverse the negative sign to be positive to satisfy the field equation of $f(R)=R+\alpha\sqrt{R}$. We must stress the fact that the dimensional parameter α must take a positive value so that solution (12) satisfies the field equations (4), (5), and (7).

IV. EXACT (A)dS CHARGED BLACK HOLE SOLUTION

Let us now derive a charged (A)dS black hole solution for the model $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$.[5] Applying the anzatz (8) to the field Eqs. (4), (5), and (7), after using (9) and putting $s = 1$, we get the following nonvanishing field equations:

$$
\begin{aligned}
I_t{}^t ={}& \frac{1}{2r^{10}\sqrt{\mathcal{R}^9}}\{r^6\sqrt{\mathcal{R}^7}[r^4BB'''' + B'''(1/2r^4B' + 6r^3B) + 2r^2B''(B + rB') - r^2B'^2 + 2rB'(1 - 3B) + 4B(B - 1)]\\
&+ r^4\sqrt{\mathcal{R}^5}[4r^2B'^2 + r^6\mathcal{R}BB'''' - r^6BB^{2'''} + 1/2r^3B'''\{r^2B''(rB' - 4B) + 2B'(29rB - r + 4r^2\Lambda) + 8B(5B + 12r^2\Lambda - 5)\}\\
&+ 4r^4B''^2(r^2q'' + 2r^2\Lambda - 6B + 2rB' - 1) + 2r^2B''[23r^2B'^2 + 2rB'(23B + 8r^2q'^2 - 13 + 40r^2\Lambda) + 8r^2q'^2(B - 1 + 4r^2\Lambda)\\
&+ 48B^2 + 8B(8r^2\Lambda - 7) + 16r^2\Lambda(4r^2\Lambda - 3) + 8] + 28r^3B'^3 + r^2B'^2(34B + 32r^2q'^2 + 184r^2\Lambda - 54)\\
&+ 4rB'(B - 1 + 4r^2\Lambda)(7B + 8r^2q'^2 - 9 + 24r^2\Lambda) + 8r^2q'^2(B - 1 + 4r^2\Lambda)^2 + 8B^2[14r^2\Lambda - 1]\\
&+ 16B(16r^4\Lambda^2 - 12r^2\Lambda + 1) + 8(2r^2\Lambda - 1)(4r^2\Lambda - 1)^2] + r^4\mathcal{R}^2[B\sqrt{\mathcal{R}}[4r^2B'' + r^3B''' - 2rB' + 4(1 - B)]^2\\
&+ \alpha\{r^6\mathcal{R}BB'''' - 3/2r^6BB'''^2 + 1/2r^3B'''[r^2B''(rB' - 12B) + 4r^2B'^2 + 2rB'(31B - 1 + 4r\Lambda) + 48B(B - 1 + 2r^2\Lambda)]\\
&- r^6B''^3 - 100r^3B'^3 + 2r^2B'^2[96B - 85 + 324r^2\Lambda] + 2r^4B''^2(4 - 6rB' - 15rB + 16r^2\Lambda)\\
&+ r^2B''[57r^2B'^2 + 2rB'[21B - 35 + 68r^2\Lambda] + 4B(4 - 3B - 4r^2\Lambda) + (4r^2\Lambda - 1)^2] - 4rB'[27B^2 + 4B(47r^2\Lambda - 50) - 23\\
&- 4r^2\Lambda(88r^2\Lambda - 45)] - 16[(B - 1)^2(2B - 1) + 2B^2r^2\Lambda + 5Br^2\Lambda(3r^2\Lambda - 11) + (4r^2\Lambda - 1)^3]\}]\} = 0,\\
I_r{}^r ={}& -\frac{1}{4r^8\sqrt{\mathcal{R}^7}}\{r^4\sqrt{\mathcal{R}^5}[(rB' + 4B)[4(1 - B) - 2rB' + r^2(4B'' + rB''')] - (r^3B'''[rB' + 4B] + 4r^2B''[5B + r^2q'^2 + 2rB' - 1\\
&+ 2r^2\Lambda] + 14r^2B'^2 + rB'[16r^2q'^2 + 12B - 20 + 64r^2\Lambda] + 8(B - 1 + 8r^2\Lambda)r^2q'^2 + 8 + 64r^4\Lambda^2 - 8B^2 - 48r^2\Lambda(1 + B))]\\
&- \alpha r^4\mathcal{R}^2(r^3B'''[4B + rB'] - 2r^4B''^2 + r^2B''[4(B + 3) + 48r^2\Lambda - 16rB'] - 50r^2B'^2 + 4rB'(15 - 17B - r^2\Lambda)\\
&- 16[2B - 1][B - 1] + 32r^2\Lambda(4B + 8r^2\Lambda + 1))\} = 0,\\
I_\theta{}^\theta = I_\phi{}^\phi ={}& \frac{-1}{2r^{10}\sqrt{\mathcal{R}^9}}\{r^6\sqrt{\mathcal{R}^7}[r^4BB'''' + r^3B'''(rB' + 5B) + 2r^2B''(2rB' - B) + 2rB'(2 - 3B) - 2r^2B'^2 + 8B(B - 1)]\\
&- r^5\sqrt{\mathcal{R}^5}[r^5BRB'''' - r^5BB'''^2 + r^2B'''\{r^2B''(rB' - 3B) + 4r^2B'^2 + 2rB'(13B - 1 + 2r^2\Lambda) + 2B(9B - 9 + 20r^2\Lambda)\}\\
&+ r^5B''^3 - 2r^3B''^2(r^2q'^2 + 2 - 7rB' + 7B - 10r^2\Lambda)] + 2rB''(23r^2B'^2 - rB'[14 + 8r^2q'^2 - 17B - 80r^2\Lambda]\\
&+ 4(B - 1 - 4r^2\Lambda)r^2q'^2 - B[10B + 8r^2\Lambda - 11] + 4r^2\Lambda[3 + 8r^2\Lambda] + 1) + 24r^2B'^3 - 4rB'^2(3 + 8r^2q'^2 - 44r^2\Lambda)\\
&+ 4B'[(B - 1 + 4r^2\Lambda)r^2q'^2 + B[3B - 3 + 20r^2\Lambda] + 24r^2\Lambda[4r^2\Lambda - 1] - r[q'^2(B - 1 + 4r^2\Lambda) - 2B(5B + 4r^2\Lambda - 3)\\
&+ 4(r^2\Lambda - 1)^2]] + r^4\mathcal{R}^2[\sqrt{\mathcal{R}}B[B - 1 - r^3B''' - 4r^2B'' + 2rB']^2 + \alpha(r^6\mathcal{R}BB'''' + 3/2r^6BB'''^2 - r^3B'''[r^2B''\{rB' - 7B\}\\
&+ 4r^2B'^2 + 2rB'[14B - 1 + 4r^2\Lambda] + 2B(11B - 11 + 20r^2\Lambda)] + 2r^6B''^3 + 2r^4B''^2[18B + 9rB' - 5 + 24r^2\Lambda]\\
&+ 2r^2B''[33r^2B'^2 + rB'(27B - 34 + 160r^2\Lambda) - 2(9B^2 - 5B - 4) - r^2\Lambda\{88B + 192r^2\Lambda + 80\}] + 104r^3B'^3\\
&+ 2r^2B'^2(74 - 81B + 328r^2\Lambda) + rB'[15B^2 - 31B + 156Br^2\Lambda + 352r^4\Lambda^2 + 16 - 152r^2\Lambda]\\
&- 8\{1 - 2B^3 - 8r^2\Lambda - 4B + 5B^2 - 90r^4\Lambda^2 + 24r^2\Lambda\} + (8r^2\Lambda - 1)(4r^2\Lambda - 1)^2)]\} = 0,
\end{aligned}
$$

[5] We define $\mathcal{R} = R - 8\Lambda$.

$$
\begin{aligned}
I = {} & \frac{-3}{2r^{10}\sqrt{\mathcal{R}^9}}\{r^6\sqrt{\mathcal{R}^7}[r^4BB'''' + r^3B'''(rB'+6B) + 2r^2B''(B+2rB') - 2r^2B'^2 + 4rB'(1-2B) + 4B(B-1)] \\
& + r^4\sqrt{\mathcal{R}^5}[r^6\mathcal{R}BB'''' + r^6BB'''^2 - r^3B'''\{r^2B''[rB'-2B] + 4r^2B'^2 + 2rB'(15B-1+4r^2\Lambda) + 4B(5B-5+12r^2\Lambda)\} \\
& + 2/3r^6B''^3 + 2r^4B''^2(6rB'-5B-2+8r^2\Lambda) + 2r^2B''\{23r^2B'^2 + 2rB'(14B-9+40r^2\Lambda) + 4B(6B-7+10r^2\Lambda) \\
& + (4r^2\Lambda-1)^2\} + 104/3r^3B'^3 + 4r^2B'^2(6B-11+60r^2\Lambda) + 8rB'(B-1+4r^2\Lambda)(2B-3+16r^2\Lambda) - 8/3B^3 + 8B \\
& + 96B^2r^2\Lambda + 32r^2B\Lambda[8r^2\Lambda-5] + 64/3(4r^2\Lambda-1)^3] - r^4\mathcal{R}^2[B\sqrt{\mathcal{R}}(4+r^3B''' - 4B + 4r^2B'' - 2rB')^2 \\
& - \alpha(r^6\mathcal{R}BB'''' + 3/2r^6BB'''^2 - r^3B'''\{r^2B''(rB'-6B) + 4r^2B'^2 + 2rB'(16B-1+4r^2\Lambda) + 24B(B-1+2r^2\Lambda)\} \\
& + 2r^6B''^3 + 2r^4B''^2\{10rB' + 17B - 6 + 80/3r^2\Lambda\} + 2r^2B''\{41r^2B'^2 + 2rB'(16B-23+296/3r^2\Lambda) \\
& + 4(3-4B^2+B+176/3r^2\Lambda) + 16/3r^2\Lambda(r^2\Lambda-20)\} + 136r^3B'^3 - 2r^2B'^2(106-117B-1304/3r^2\Lambda) \\
& + 8rB'(15B^2 + B[344/3r^2\Lambda - 84] + r^2\Lambda/3[704r^2\Lambda-332] + 13) + 32B^3 + 16B^2[34/3r^2\Lambda - 5] \\
& + 32B[2 - 37/3r^2\Lambda + 88/3r^4\Lambda^2] + 16(4r^2\Lambda-1)^2 \times (16r^2\Lambda-3))]\} = 0.
\end{aligned} \tag{15}
$$

If we subtract the component $I_t{}^t$ from the component $I_r{}^r$ and solve the system $I_t{}^t - I_r{}^r$ and $I_\theta{}^\theta$, which is a closed system for the two unknown functions $B(r)$ and $q(r)$, we get the exact solution

$$B(r) = \frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3\alpha r} + \frac{1}{3\alpha r^2}, \qquad A = \frac{1}{\sqrt{3\alpha}r}. \tag{16}$$

Using Eq. (16) in (9), we get the Ricci scalar in the form

$$R = \frac{8r^2\Lambda + 1}{r^2}. \tag{17}$$

The metric of the above solution takes the form

$$
\begin{aligned}
ds^2 = {} & \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3\alpha r} + \frac{1}{3\alpha r^2}\right)dt^2 \\
& - \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{1}{3\alpha r} + \frac{1}{3\alpha r^2}\right)^{-1}dr^2 - r^2 d\Omega^2,
\end{aligned} \tag{18}
$$

which behaves asymptotically as (A)dS spacetime. Solution (16) is different from that derived in Ref. [88] for the reason discussed for solution (12). The same constraint put on the parameter α in the noncharged case is also true here.

V. PHYSICAL PROPERTIES OF THE BLACK HOLES

The metric of solution (12) can be rewritten in the form

$$
\begin{aligned}
& ds^2 = \left(\frac{1}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(\frac{1}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 - r^2 d\Omega^2, \\
& \text{where } M = \frac{1}{6\alpha}, \quad q = \frac{1}{\sqrt{3\alpha}},
\end{aligned} \tag{19}
$$

which shows clearly that the dimensional parameter α cannot be equal to zero, and in that case, the line element coincides with the Reissner-Nordström spacetime. Also, the metric of solution (16) may be rewritten as

$$
\begin{aligned}
ds^2 = {} & \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 \\
& - \left(\frac{1}{2} - \frac{2r^2\Lambda}{3} - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 - r^2 d\Omega^2, \\
& \text{where, again } M = \frac{1}{6\alpha} \quad \text{and} \quad q = \frac{1}{\sqrt{3\alpha}},
\end{aligned} \tag{20}
$$

which shows that the line element coincides with the (A)dS Reissner-Nordström spacetime. Equations (19) and (20) show in a clear way that the dimensional parameter α must not equal zero.

Let us now study the regularity of solutions (12) and (16) when $B(r) = 0$. For solution (12), we evaluate the scalar invariants and get

$$
\begin{aligned}
& R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{56 + 9r^4\alpha^2 + 12\alpha r^3 + 12r^2[\alpha+1] + 48r}{9\alpha^2 r^8}, \\
& R^{\mu\nu}R_{\mu\nu} = \frac{9r^4\alpha^2 - 12\alpha r^2 + 8}{18\alpha^2 r^8}, \qquad R = \frac{1}{r^2},
\end{aligned} \tag{21}
$$

where $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$, $R^{\mu\nu}R_{\mu\nu}$, and R are the Kretschmann scalars, the Ricci tensor square, the Ricci scalar, respectively. Equations (21) show that the solutions, at $r = 0$, have true singularities and the dimensional parameter $\alpha \neq 0$. Also, Eq. (12) as well as Eq. (14) show clearly that the dimensional parameter α cannot be equal to zero, which ensures that solution (12) cannot reduce to GR. This means that this solution is a new exact charged one in the frame of $f(R)$ gravitational theory.

Using Eq. (16) we get the scalar invariants in the form

$$R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{96r^8\Lambda^2\alpha^2 + 24r^6\Lambda\alpha^2 + 9r^4\alpha^2 + 12\alpha r^3 + 12r^2[\alpha+1] + 48r + 56}{9\alpha^2 r^8},$$
$$R^{\mu\nu}R_{\mu\nu} = \frac{288r^8\Lambda^2\alpha^2 + 72r^6\Lambda\alpha^2 + 9\alpha^2 r^4 - 12\alpha r^2 + 8}{18\alpha^2 r^8}, \qquad R = \frac{8r^2\Lambda + 1}{r^2}. \tag{22}$$

The same considerations carried out for solution (12) can also be applied for solution (16) which insure also that solution (16) is a novel charged one in the framework of $f(R)$ gravity that cannot reduce to GR.

VI. BLACK HOLE THERMODYNAMICS

Now, we are going to explore the thermodynamics of the new black hole solutions derived in the previous sections. The Hawking temperature is defined as [105–108]

$$T_+ = \frac{B'(r_+)}{4\pi}, \tag{23}$$

where the event horizon is located at $r = r_+$, which is the largest positive root of $B(r_+) = 0$ that fulfills $B'(r_+) \neq 0$. The Bekenstein-Hawking entropy in the framework of $f(R)$ gravity is given as [105–110]

$$S(r_+) = \frac{1}{4}Af_R(r_+), \tag{24}$$

where A is the area of the event horizon. The form of the quasilocal energy in the framework of $f(R)$ gravity is defined as [105–110]

$$E(r_+) = \frac{1}{4}\int [2f_R(r_+) + {r_+}^2\{f(R(r_+)) - R(r_+)f_R(r_+)\}]dr_+. \tag{25}$$

At the horizon, one has the constraint $B(r_+) = 0$, which gives

$$r_{+_{\mathrm{Eq.(12)}}} = \frac{1}{3\alpha}\left[1 + \sqrt{1+6\alpha}\right],$$
$$r_{-_{\mathrm{Eq.(12)}}} = \frac{1}{3\alpha}\left[1 - \sqrt{1+6\alpha}\right]$$
$$r_{+_{\mathrm{Eq.(16)}}} = \mathrm{Root}(4x^4\alpha\Lambda - 3\alpha x^2 + 2x + 2), \tag{26}$$

where $\mathrm{Root}(4x^4\alpha\Lambda - 3\alpha x^2 + 2x + 2)$ is the roots of the equation $(4x^4\alpha\Lambda - 3\alpha x^2 + 2x + 2 = 0)$. It is clear from the first equation of Eqs. (26) that α should not be equal to zero to ensure that the black hole (12) has no analogy with GR. Moreover, Eqs. (26) tell us that the dimensional parameter α should be positive so that the horizons have a positive real value. Therefore, we must put on the restriction $\alpha > 0$; otherwise, we get a nonreal value for the horizon. This constraint is consistent with the relation given by Eqs. (19) and (20), which allows the mass parameter to have the correct sign in the metric, and the charge parameter has a real value. Moreover, if the parameter α takes a negative value, the solutions (12) and (16) do not satisfy the field equations (4), (5), and (7).

The relation between the radial coordinate r and the dimensional parameter α of the black hole (12) is represented in Fig. 1. From this figure, we can see the root of $B(r)$ defining the black hole outer event horizon r_+ [111]. We can continue the study of thermodynamics assuming $\alpha > 0$ according to the pervious analysis and taking into account the outer event horizon r_+ only, which is consistent with $\alpha > 0$.

Using Eq. (24), the entropies of the black holes (12) and (16) are computed as

$$S_{+_{\mathrm{Eq.(12)}}} = \frac{\pi}{27\alpha^2}\left[1 + \sqrt{1+6\alpha}\right]^2\left[2 - \sqrt{1+6\alpha}\right],$$
$$S_{+_{\mathrm{Eq.(16)}}} = \frac{\pi {r_+}^2}{4}[1 - \alpha r_+]. \tag{27}$$

The first equation of Eq. (27) shows that, in order to have a positive entropy, the dimensional parameter α must take the value $0 < \alpha < 0.5$. The second equation of (27) tells us that

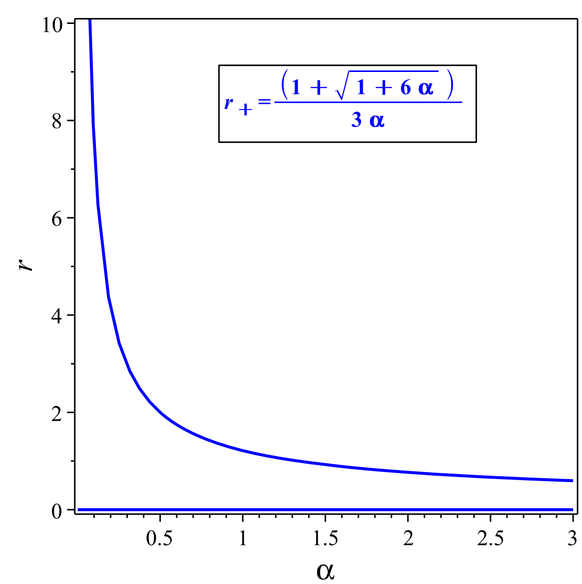
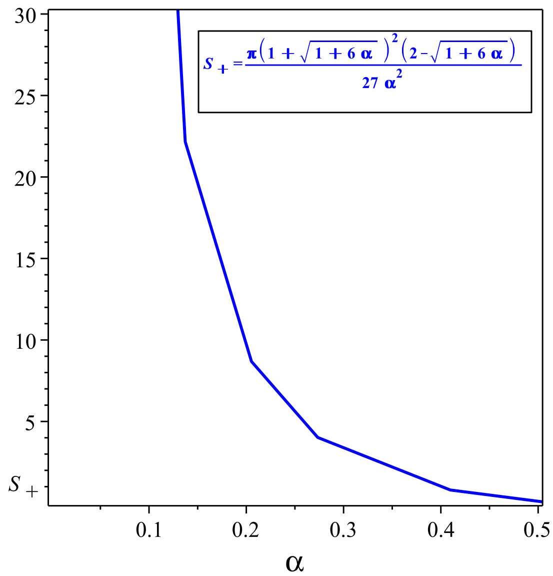
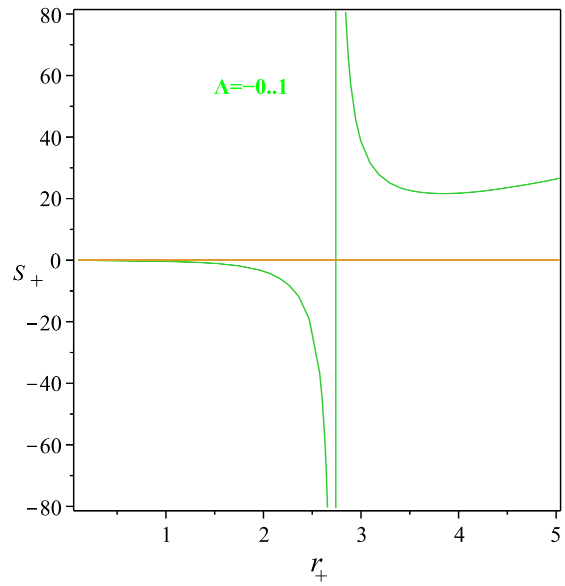


FIG. 1. Schematic plot of the radial coordinate r vs the dimensional parameter α that characterizes the spherically symmetric black hole [12].



(a) The entropy of the black hole solution (12)



(b) The entropy of the black hole solution (16)

FIG. 2. Schematic plot of the entropy of the two black holes (12) and (16) vs the dimensional parameter α and r_+ respectively.

we must have $\alpha < \frac{1}{r_+}$ for positive entropy. Equations (27) are drawn in Fig. 2. As one can see, from Fig. 2(a), for $0.5 > \alpha > 0$, the black hole (12) has $+ve$ entropy. For the black hole[6] (16), as Fig. 2(b) shows, we have a phase transition at 2.738612788, and then the entropy has a $-ve$ value for $0 < \alpha < 2.738612788$, and it evolves to $+ve$ at $r = 2.187$. The following remarks must be taken into account. It is remarkable that the entropy S is not proportional to the area of the horizon due to Eq. (24). We should also note that the entropy S is proportional to the area if there is no Ricci scalar squared term, i.e., $f_R = 1$. The Hawking temperatures associated with the black hole solutions (12) and (16) are

$$T_{+_{\mathrm{Eq.(12)}}} = \frac{3\alpha(1+\sqrt{1+6\alpha}+6\alpha)}{4\pi(1+\sqrt{1+6\alpha})^3},$$
$$T_{+_{\mathrm{Eq.(16)}}} = \frac{r_+ - 4\alpha\Lambda {r_+}^4 + 2}{12\pi\alpha {r_+}^3}, \tag{28}$$

where T_+ is the Hawking temperature at the event horizon. We represent the Hawking temperature in Fig. 3. Figure 3(a), which is related to the black hole (12), shows that we have a positive temperature when the parameter α has the value $0 < \alpha < 0.5$. Figure 3(b) is related to the black hole (16). Here, the temperature always has a $+ve$ value. From Eq. (25), the quasilocal energies of the two black holes (12) and (16) are calculated as

$$E_{+_{\mathrm{Eq.(12)}}} = \frac{1+\sqrt{1+6\alpha}-3\alpha}{12\alpha},$$
$$E_{+_{\mathrm{Eq.(16)}}} = \frac{r_+}{8}(4 - 3\alpha r_+ + 4\Lambda\alpha {r_+}^3). \tag{29}$$

The first equation of (29) shows that the dimensional parameter has to be $\alpha \neq 0$. We plot the energy in Fig. 4, which shows that for Fig. 4(a) the quasilocal energy has a $+ve$ value when $0 < \alpha < 0.5$. In the other case, we have a negative value for the quasilocal energy until $r_+ = 1$, and then the energy becomes positive as Fig. 4(b) shows.

The free energy in the grand canonical ensemble, also called Gibbs free energy, can be defined as [110,112]

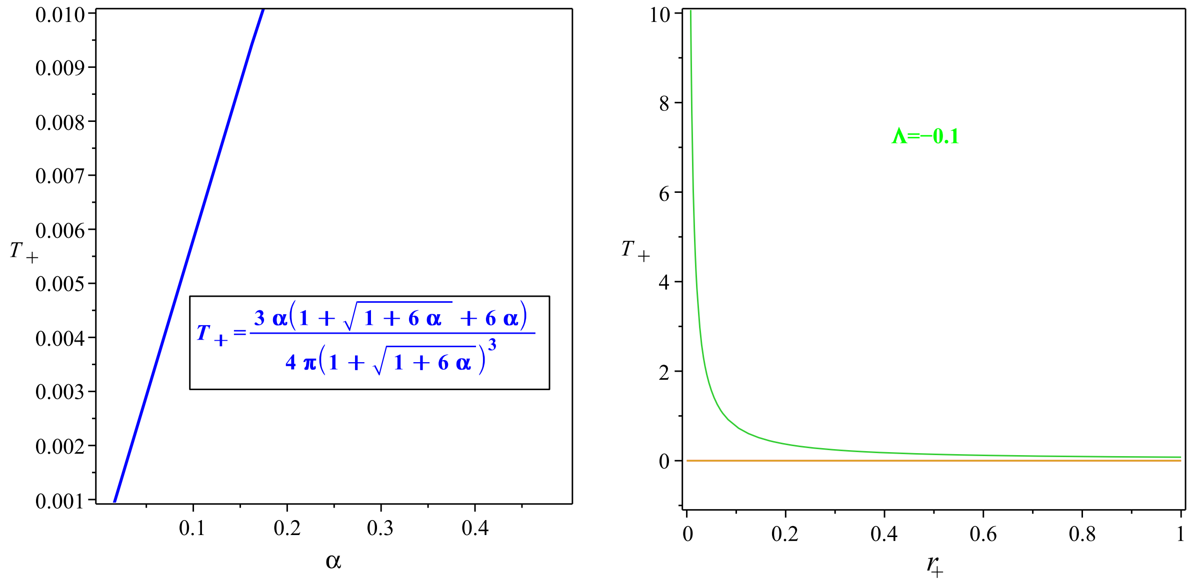
$$G(r_+) = E(r_+) - T(r_+)S(r_+), \tag{30}$$

where $E(r_+)$, $T(r_+)$, and $S(r_+)$ are the quasilocal energy, the temperature, and entropy at the event horizons, respectively. Using Eqs. (24), (26), (27), and (29) in (30), we get

$$G_{+_{\mathrm{Eq.(12)}}} = \frac{1+\sqrt{1+6\alpha}-3\alpha}{12\alpha} - \frac{\alpha {r_+}^2(1+\sqrt{1+6\alpha}+6\alpha)(2-\sqrt{1+6\alpha})}{4(1+\sqrt{1+6\alpha})^3},$$
$$G_{+_{\mathrm{Eq.(16)}}} = \frac{r_+(4-3\alpha r + 4\Lambda\alpha {r_+}^3)}{8} - \frac{(r - 4\Lambda\alpha r^3 + 2)(1-\alpha r)}{48\alpha r_+}. \tag{31}$$

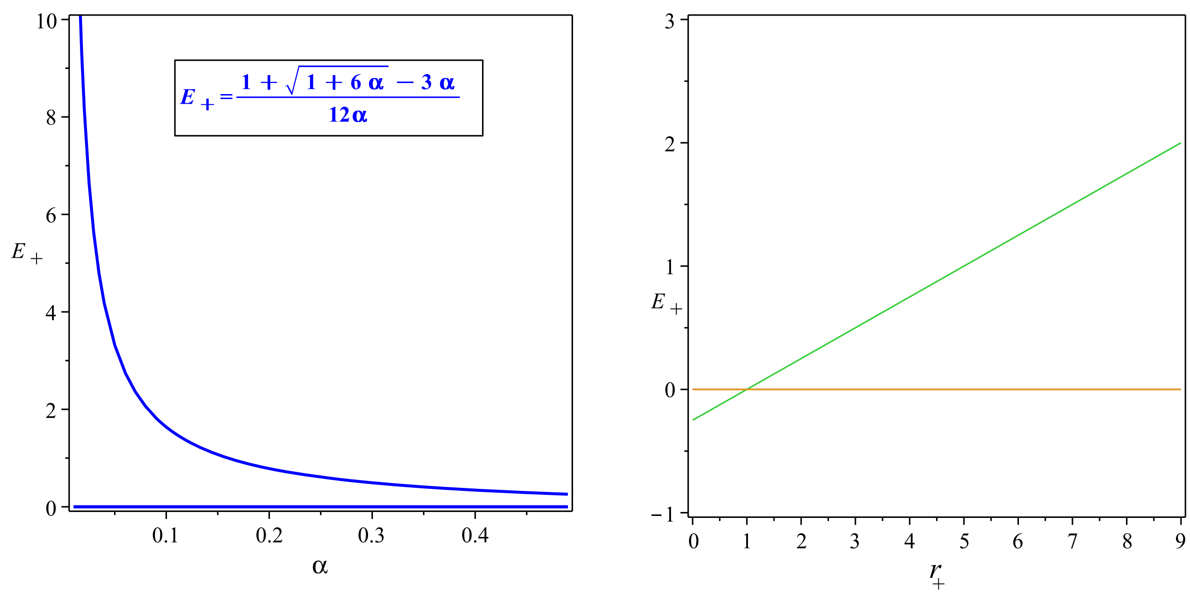
The behaviors of the Gibbs energy of our black holes are presented in Figs. 5(a) and 5(b) for particular

[6] We substitute the value of α in terms of Λ using Eq. (16) throughout this section.



(a) The Hawking temperature of the black hole solution (12) (b) The Hawking temperature of the black hole solution (16)

FIG. 3. Schematic plot of the Hawking temperature of the two black holes (12) and (16) vs the dimensional parameter α and r_+, respectively.



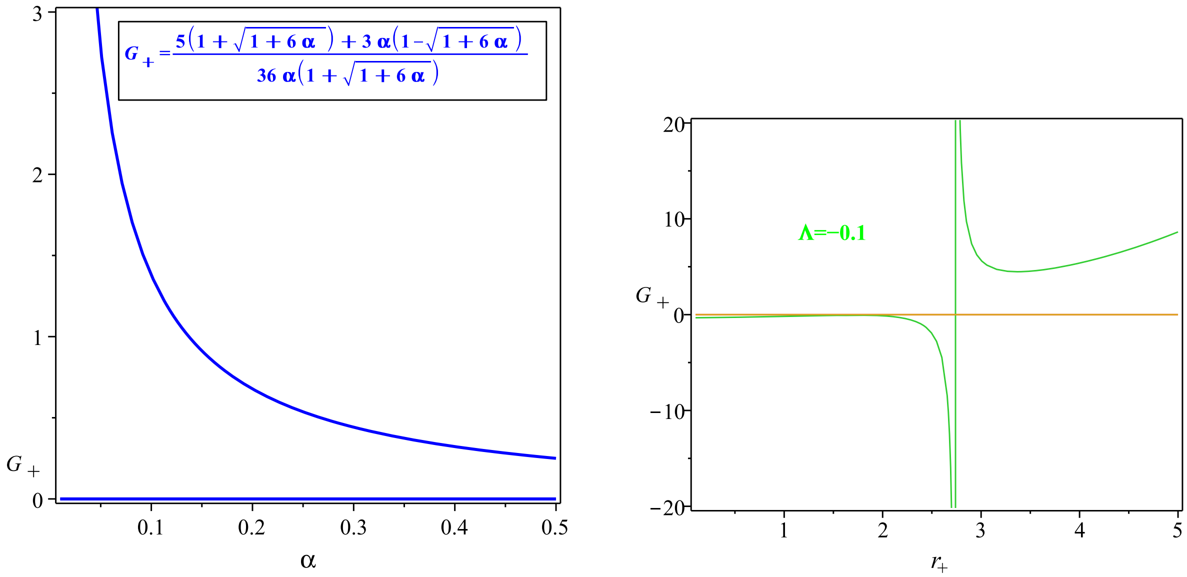
(a) The quasilocal energy of the black hole solution (12) (b) The quasilocal energy of the black hole solution (16)

FIG. 4. Schematic plot of the quasilocal energy of the black holes (12) and (16) vs the dimensional parameter α and r_+, respectively.

values of the model parameters. As Fig. 5(a) shows, for the black hole solution (12), the Gibbs energy is positive when $0 < \alpha < 0.5$, which means that it is more globally stable. For the black hole solution (16), the Gibbs energy has a phase transition. It has a negative value when $r < 2.73$ and a positive one when $2.73 < r$.

VII. STABILITY OF CHARGED BLACK HOLE SOLUTIONS IN $f(R)$ GRAVITY

To study the stability of the above black hole solutions, it is better recast $f(R)$ gravity in terms of the corresponding scalar- tensor theory. Discarding the cosmological term, the Lagrangian (2) can be rewritten as



(a) The free energy of the black hole solution (12)

(b) The free energy of the black hole solution (16)

FIG. 5. Schematic plot of the free energy of the black holes (12) and (16) vs the dimensional parameter α and r_+, respectively.

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi R - V(\phi)], \tag{32}$$

where ϕ is a scalar field coupled to the Ricci scalar R and $V(\phi)$ is the potential (see Ref. [1] for details). Here, we will discuss the behavior of the perturbations about a static spherically symmetric vacuum background, the metric of which is written as above, that is,

$$ds^2 = g^0_{\mu\nu}dx^\mu dx^\nu$$
$$= B(r)dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{33}$$

where $g^0_{\mu\nu}$ is the background metric. Considering the above black hole solutions, we want to investigate whether these backgrounds are stable or not against linear perturbations and what we can learn in terms of speed of propagation for the scalar gravitational modes. For such a theory, the background equations of motion read

$$V = -\frac{4B\phi'}{r} - \frac{2\phi B'}{r} - \phi' B' + \frac{2\phi}{r^2} - \frac{2B\phi}{r^2},$$
$$\phi'' = 0, \qquad R = \frac{dV}{d\phi}, \tag{34}$$

where $'$ stands for differentiation with respect to r.

A. Outline of the Regge-Wheeler-Zerilli formalism

Before studying the metric perturbation of static spherically symmetric spacetime of $f(R)$ gravity, let us give a brief summary of the formalism developed by Regge and Wheeler [113] and Zerilli [114] to decompose the metric perturbations according to their transformation properties under two-dimensional rotations. Although Regge, Wheeler, and Zerilli considered the perturbations of the Schwarzschild spacetime in GR, the formalism depends on the properties of spherical symmetry and then can be applied to $f(R)$ gravity as well.

Let us denote the metric slightly perturbed from a static spherically symmetric spacetime by $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ represents infinitesimal quantities. In the lowest, linear approximation, the perturbations are supposed to be very small with respect to the background, that is, $g^0_{\mu\nu} \gg h_{\mu\nu}$. Then, under two-dimensional rotations on a sphere, h_{tt}, h_{tr}, and h_{rr} transform as scalars; h_{ta} and h_{ra} transform as vectors; and h_{ab} transforms as a tensor (a, b are either θ or ϕ). Any scalar quantity Φ can be expressed in terms of the spherical harmonics $Y_{\ell m}(\theta,\phi)$:

$$\Phi(t,r,\theta,\phi) = \sum_{\ell,m}\Phi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi). \tag{35}$$

In the spherically symmetric spacetimes, the solution will be independent of the index m; therefore, this subscript can be omitted, and we take into account only the index ℓ, which represents the multipole number, which arises from the separation of angular variables by the expansion into spherical harmonics,

$$\Delta_{\theta,\phi}Y_\ell(\theta,\phi) = -\ell(\ell+1)Y_\ell(\theta,\phi), \tag{36}$$

exactly in the same way as it happens for the hydrogen atom problem in quantum mechanics when dealing with the Schrödinger equation. Any vector V_a can be decomposed into a divergence part and a divergence-free part as

$$V_a(t,r,\theta,\phi) = \nabla_a\Phi_1 + E_a^b\nabla_b\Phi_2, \tag{37}$$

where Φ_1 and Φ_2 are two scalars and $E_{ab} \equiv \sqrt{\det\gamma}\epsilon_{ab}$, with γ_{ab} being the two-dimensional metric on the sphere and ϵ_{ab} being the totally antisymmetric symbol with $\epsilon_{\theta\varphi} = 1$. Here, ∇_a represents the covariant derivative with respect to the metric γ_{ab}. Since V_a is a two-component vector, it is completely specified by the quantities Φ_1 and Φ_2. Then, we can apply the scalar decomposition (35) to Φ_1 and Φ_2 to decompose the vector quantity V_a into spherical harmonics.

Finally, any symmetric tensor T_{ab} can be decomposed as

$$T_{ab}(t,r,\theta,\phi) = \nabla_a\nabla_b\Psi_1 + \gamma_{ab}\Psi_2 + \frac{1}{2}(E_a{}^c\nabla_c\nabla_b\Psi_3 + E_b{}^c\nabla_c\nabla_a\Psi_3), \tag{38}$$

where Ψ_1, Ψ_2, and Ψ_3 are scalars. Since T_{ab} has three independent components, Ψ_1, Ψ_2, and Ψ_3 completely specify T_{ab}. Then, we can again apply the scalar decomposition (35) to Ψ_1, Ψ_2, and Ψ_3 to decompose the tensor quantity T_{ab} into spherical harmonics. We refer to the variables accompanied by E_{ab} by odd-type variables and the others by even-type variables. What makes these decompositions useful is that, in the linearized equations of motion (or equivalently, in the second-order action) for $h_{\mu\nu}$, odd-type and even-type perturbations are completely decoupled. This fact reflects the invariance of the background spacetime under parity transformations. Therefore, one can study odd-type perturbations and even-type ones separately, as we will do in the following.

B. Perturbations in $f(R)$ gravity

1. Odd modes

It is well known that there are two classes of vector spherical harmonics (polar and axial), which are built out of combinations of the Levi-Civita volume form and the gradient operator acting on the scalar spherical harmonics. The difference between the two families is their parity. Under the parity transformation, the operator π a spherical harmonic with index ℓ transforms as $(-1)^\ell$, the polar class of perturbations transform under parity in the same way, as $(-1)^\ell$ and the axial perturbations as $(-1)^{\ell+1}$.

Using the Regge-Wheeler formalism, the odd-type metric perturbations can be written as

$$h_{tt} = 0, \qquad h_{tr} = 0, \qquad h_{rr} = 0, \tag{39}$$

$$h_{ta} = \sum_{\ell,m} h_{0,\ell m}(t,r)E_{ab}\partial^b Y_{\ell m}(\theta,\varphi), \tag{40}$$

$$h_{ra} = \sum_{\ell,m} h_{1,\ell m}(t,r)E_{ab}\partial^b Y_{\ell m}(\theta,\varphi), \tag{41}$$

$$h_{ab} = \frac{1}{2}\sum_{\ell,m} h_{2,\ell m}(t,r)[E_a{}^c\nabla_c\nabla_b Y_{\ell m}(\theta,\varphi) + E_b{}^c\nabla_c\nabla_a Y_{\ell m}(\theta,\varphi)]. \tag{42}$$

Using the gauge transformation $x^\mu \to x^\mu + \xi^\mu$, where ξ^μ are infinitesimal, we can show that not all the metric perturbations are physical and some of them can be set to vanish. For the odd-type perturbation, we can consider the gauge transformation

$$\xi_t = \xi_r = 0, \qquad \xi_a = \sum_{\ell m} \Lambda_{\ell m}(t,r)E_a{}^b\nabla_b Y_{\ell m}, \tag{43}$$

where $\Lambda_{\ell m}$ can always set $h_{2,\ell m}$ to vanish (Regge-Wheeler gauge). By this procedure, $\Lambda_{\ell m}$ is completely fixed, and there are no remaining gauge degrees of freedom. Then, after substituting the metric into the action (32) and performing integrations by parts, we find that the action for the odd modes becomes

$$S_{\rm odd} = \frac{1}{2\kappa}\sum_{\ell,m}\int dt\,dr\mathcal{L}_{\rm odd} = \frac{1}{4\kappa}\sum_{\ell,m}\int dt\,dr j^2\left[\phi(\dot h_1 - h_0')^2 + \frac{4h_0\dot h_1\phi}{r} + \frac{h_0^2}{r^2}\left[2r\phi' + 2\phi + \frac{(j^2-2)\phi}{B}\right] - \frac{(j^2-2)B\phi h_1^2}{r^2}\right], \tag{44}$$

where we neglect the suffix ℓ for the fields and $j^2 = \ell(\ell+1)$. Variation of (44) with respect to h_0 yields

$$[\phi(h_0' - \dot h_1)]' = \frac{1}{r^2}\left[r\phi' + j^2\phi + \frac{(j^2-2)\phi}{2B}\right]h_0 + \frac{2\phi\dot h_1}{r}, \tag{45}$$

which cannot be solved for h_0. Let us now rewrite the above action as

$$L_{\rm odd} = \frac{j^2\phi}{2}\left(\dot h_1 - h_0' + \frac{2h_0}{r}\right)^2 - \frac{j^2(\phi + r\phi')h_0{}^2}{r^2} + \frac{2j^2h_0^2}{r^2}\left[r\phi' + \phi + \frac{(j^2-2)\phi}{2B}\right] - \frac{j^2(j^2-2)B\phi h_1^2}{2r^2} \tag{46}$$

so that all the terms containing $\dot h_1$ are inside the first squared term. Using a Lagrange multiplier Q, we can rewrite Eq. (46) as follows:

$$L_{\rm odd} = \frac{j^2\phi}{2}\left[2Q\left(\dot{h}_1 - h_0' + \frac{2h_0}{r}\right) - Q^2\right] - \frac{j^2(\phi + r\phi')h_0{}^2}{r^2} + \frac{2j^2h_0^2}{r^2}\left[r\phi' + \phi + \frac{(j^2-2)\phi}{2B}\right] - \frac{j^2(j^2-2)B\phi h_1^2}{2r^2}. \tag{47}$$

Equation (47) shows that both fields h_0 and h_1 can be integrated out by using their own equations of motion, which can be written as

$$h_1 = -\frac{r^2\dot{Q}}{(j^2-2)B}, \tag{48}$$

$$h_0 = \frac{r}{\phi(j^2-2)}[(\phi + r\phi')Q + r\phi Q']. \tag{49}$$

These relations link the physical modes h_0 and h_1 to the auxiliary field Q. Once Q is known, then also h_0 and h_1 are. After substituting these expressions into the Lagrangian and performing an integration by parts for the term proportional to $Q'Q$, one finds the Lagrangian in the canonical form

$$L_{\rm odd} = \frac{j^2r^2\phi}{2(j^2-2)B}\dot{Q}^2 - \frac{j^2B\phi r^2}{2(j^2-2)}Q'^2 - \mu_1{}^2Q^2, \tag{50}$$

where

$$\mu_1{}^2 = \frac{j^2[j^2\phi^2 - Br^2\phi\phi'' + 2B\phi^2 - r^2\phi\phi'B' + r^2B\phi'^2 - 2\phi^2 - 2r\phi^2B']}{2\phi(j^2-2)}. \tag{51}$$

From Eq. (50), we can derive the no-ghost conditions

$$j^2 \geq 2, \quad \text{and} \quad B \geq 0.$$

For solutions proportional to $e^{i(\omega t - kr)}$ with large k and ω, we have the radial dispersion relation

$$\omega^2 = B^2k^2,$$

where we made use of the background equations of motion. Finally, the expression for the radial speed reads

$$c_{\rm odd}^2 = \left(\frac{dr_*}{d\tau}\right)^2 = 1,$$

where we used the radial tortoise coordinate ($dr_*^2 = dr^2/B$) and the proper time ($d\tau^2 = Bdt^2$).

C. Black hole stability: Geodesic

The trajectories of a test particle in a gravitational field are described by the geodesic equations

$$\frac{d^2x^\sigma}{d\lambda^2} + \left\{ {\sigma \atop \mu\nu} \right\}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{52}$$

where λ is an affine parameter along the geodesic. The geodesic deviation takes the form [115]

$$\frac{d^2\xi^\sigma}{d\lambda^2} + 2\left\{ {\sigma \atop \mu\nu} \right\}\frac{dx^\mu}{d\lambda}\frac{d\xi^\nu}{ds} + \left\{ {\sigma \atop \mu\nu} \right\}_{,\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\xi^\rho = 0, \tag{53}$$

with ξ^ρ being the deviation 4-vector. Applying (52) and (53) into (8), we get for the geodesic equations

$$\frac{d^2t}{d\lambda^2} = 0, \qquad \frac{1}{2}B'(r)\left(\frac{dt}{d\lambda}\right)^2 - r\left(\frac{d\phi}{d\lambda}\right)^2 = 0,$$
$$\frac{d^2\theta}{d\lambda^2} = 0, \qquad \frac{d^2\phi}{d\lambda^2} = 0, \tag{54}$$

and for the geodesic deviation,

$$\frac{d^2\xi^1}{d\lambda^2} + B(r)B'(r)\frac{dt}{d\lambda}\frac{d\xi^0}{d\lambda} - 2rB(r)\frac{d\phi}{d\lambda}\frac{d\xi^3}{d\lambda} + \left[\frac{1}{2}(B'^2(r) + B(r)B''(r))\left(\frac{dt}{d\lambda}\right)^2 - (B(r) + rB'(r))\left(\frac{d\phi}{d\lambda}\right)^2\right]\xi^1 = 0,$$
$$\frac{d^2\xi^0}{d\lambda^2} + \frac{B'(r)}{B(r)}\frac{dt}{d\lambda}\frac{d\zeta^1}{d\lambda} = 0, \qquad \frac{d^2\xi^2}{d\lambda^2} + \left(\frac{d\phi}{d\lambda}\right)^2\xi^2 = 0,$$
$$\frac{d^2\xi^3}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{d\xi^1}{d\lambda} = 0, \tag{55}$$

where $B(r)$ is defined by the metric (19) or (20), $B'(r) = \frac{dB(r)}{dr}$. Using the circular orbit

$$\theta = \frac{\pi}{2}, \qquad \frac{d\theta}{d\lambda} = 0, \qquad \frac{dr}{d\lambda} = 0, \tag{56}$$

we get

$$\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{B'(r)}{r(2B(r) - rB'(r))},$$
$$\left(\frac{dt}{d\lambda}\right)^2 = \frac{2}{2B(r) - rB'(r)}. \tag{57}$$

Equations (55) can be rewritten as

$$\frac{d^2\xi^1}{d\phi^2}+B(r)B'(r)\frac{dt}{d\phi}\frac{d\xi^0}{d\phi}-2rB(r)\frac{d\xi^3}{d\phi}$$
$$+\left[\frac{1}{2}(\eta'^2(r)+\eta(r)\eta''(r))\left(\frac{dt}{d\phi}\right)^2-(\eta(r)+r\eta'(r))\right]\zeta^1=0,$$
$$\frac{d^2\xi^2}{d\phi^2}+\xi^2=0,\quad \frac{d^2\xi^0}{d\phi^2}+\frac{B'(r)}{B(r)}\frac{dt}{d\phi}\frac{d\xi^1}{d\phi}=0,$$
$$\frac{d^2\xi^3}{d\phi^2}+\frac{2}{r}\frac{d\xi^1}{d\phi}=0. \tag{58}$$

The second equation of (58) shows that it is a simple harmonic motion, which means that the motion in the plan $\theta=\pi/2$ is stable. Now, the solutions of the remaining equations of (58) are given by

$$\xi^0=\zeta_1 e^{i\sigma\phi},\qquad \xi^1=\zeta_2 e^{i\sigma\phi},\quad \text{and}\quad \xi^3=\zeta_3 e^{i\sigma\phi}, \tag{59}$$

where ζ_1, ζ_2, and ζ_3 are constants, and the variable ϕ has to be determined. Substituting (59) in (58), we get

$$\frac{3BB'-\omega^2B'-2rB'^2+rBB''}{B'}>0, \tag{60}$$

which is the stability condition for any charged static spherically symmetric spacetime. The condition (60) for the black holes (19) and (20) can be rewritten as

$$r+\frac{2q^2}{5M}>0,\qquad r+12M>0,\quad \text{and}\quad 1+\frac{4r^3\Lambda}{31M}>0, \tag{61}$$

which are the stability conditions according to the values of the parameters Λ, M, and q.

VIII. DISCUSSION AND CONCLUSIONS

Spherically symmetric spacetimes constitute an essential part of black hole physics because all the fundamental properties of the black holes can be explained and can further be used to recognize and hence generalize in any eligible more general scenario [116]. In this paper, we have discussed two main issues. In the first part, we focused on a spherically symmetric spacetime in the framework of $f(R)$ gravitational theories. We derived new black hole charged solutions for the specific forms $f(R)=R-2\alpha\sqrt{R}$ and $f(R)=R-2\alpha\sqrt{R-8\Lambda}$. The main merits of these black holes are the facts that they depend on the dimensional parameter α and have dynamical Ricci scalar; i.e., $R=\frac{1}{r^2}$ for the first model of $f(R)$ and $R=\frac{8r^2\Lambda+1}{r^2}$ for the second one. These solutions are new and cannot reduce to the standard solutions of GR due to the fact that the parameter α is not allowed to have a zero value. We calculated the scalar invariance of those black holes and found that the Kretschmann and Ricci tensor square invariants depend on the dimensional parameter α. All of the invariants show true singularity at $r=0$. In the second part, we studied the thermodynamical properties of these black holes to extract more physical information from them. The first important thing in $f(R)$ gravity is the fact that entropy is not always proportional to the area of the horizon [117–119]. We showed that, for some constraint on the parameter $0<\alpha<0.5$, we have a positive value of the entropy. However, for the black hole solution (16), there is a region in which the entropy has a negative value [117–120]. This is not the first time that a black hole with negative entropy has been found. Several black holes with negative entropy have been found as well in charged Gauss-Bonnet (A)dS gravity [117–119]. As our calculations show, negative entropy may be interpreted as a region where the parameter α has transitions into forbidden regions related to some phase transition. The complete understanding of gravitational entropy of a nontrivial solution in the framework of $f(R)$ gravitational theories remains the subject of future research.

We also calculated the thermodynamical quasilocal energy and showed that it has a positive value when $0<\alpha<0.5$. Moreover, we calculated the Hawking temperature and have shown that it also depends on the parameter α. Also, we showed that the Hawking temperature always has a positive value when $0.5>\alpha>0$ for the black holes (12) and (16). In fact, this is the case presented in Fig. 3(a) for the $\alpha>0$ regime. As for the black hole (16), the Hawking temprature always shows a positive value as Fig. 3(b). Finally, we calculated the Gibb's free energy and showed that our black hole (12) is globally stable when $0<\alpha<0.5$. However, the black hole (16) is not globally stable when $r<2.73$ and becomes stable when $r>2.73$. The main reason that makes this black is unstable comes from the contribution of entropy, which has a negative value in the region $r<2.73$, as Fig. 2(b) shows. The results obtained here, together with other results in the literature, seem to indicate that the thermodynamical origin of $f(R)$ gravitational theories, when horizons are present, has a broad of validation. To confirm this statement, we need to know more about the novel black holes derived in this paper. This will be done in future studies.

Finally, we have studied the linear perturbations around the static spherically symmetric charged spacetime derived in $f(R)$ gravity. Because $f(R)$ gravity is a fourth-order theory, we have rewritten its Lagrangian as a Ricci scalar coupled with a scalar field to make the study of perturbation more easy to deal with. We have derived the gradient instability condition for our black holes using the odd-type modes. Furthermore, we have calculated the radial propagation speed and showed that it is equal 1. To make the picture more complete, we have derived the stability conditions using also the geodesic deviation for the black holes. These conditions are different with respect to the charged black hole of GR, the Reissner-Nordström spacetime. This difference is due to the fact that the charged black hole derived in this study is a solution in the context of $f(R)$ only and cannot be reduced to GR.

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