

Metastability in quadratic gravity

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Quadratic gravity is a UV completion of general relativity, which also solves the hierarchy problem. The presence of four derivatives implies via the Ostrogradsky theorem that the *classical* Hamiltonian is unbounded from below. Here we solve this issue by showing that the relevant solutions are not unstable but metastable. When the energies are much below a threshold (that is high enough to describe the whole cosmology) runaways are avoided. Remarkably, the chaotic inflation theory of initial conditions ensures that such a bound is satisfied and we work out testable implications for the early Universe. The possible instability occurring when the bound is violated not only is compatible with cosmology but would also explain why we live in a homogeneous and isotropic universe.

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I. INTRODUCTION

Let us start with some basic definitions. The action of quadratic gravity (QG) in the Jordan frame is (modulo total derivatives)

$$S = \int d^4x \sqrt{-g} \left(\frac{R^2}{6f_0^2} - \frac{W^2}{2f_2^2} - \frac{\bar{M}_{\text{Pl}}^2 R}{2} + \mathcal{L}_m \right), \quad (1)$$

where $W^2 \equiv W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$, $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and $\mathcal{L}_m$ is the matter piece (a cosmological constant can be included, but we neglect it here given its tiny value). $\mathcal{L}_m$ includes nonminimal couplings between the scalar fields ϕ^a and the Ricci scalar of the form $-\xi_{ab}\phi^a\phi^b R/2$. The parameters f_0^2 and f_2^2 are positive to avoid tachyons¹ [1,2].

The R^2 and W^2 terms render gravity renormalizable [1]. However, renormalizability requires the space of states to be endowed with an indefinite inner product. It was recently realized that this does not preclude a physical interpretation as the probabilities involving observable states can be computed with positive norms [2–4] (see also Refs. [5–17] for related approaches).

Another remarkable feature of QG is the possibility to solve the hierarchy problem (why is the Higgs mass much smaller than $\bar{M}_{\text{Pl}}$?): this requires² $f_2 \lesssim 10^{-8}$ [18–21]. Note that, as will be reviewed in the article, the Starobinsky inflationary model [22] (which is built in here thanks to the presence of the R^2 term) requires a very small f_0 to match the observed curvature power spectrum; this is another independent reason to think that a very small value of f_2 is natural.

¹In this work the flat metric $\eta_{\mu\nu}$ has the mostly minus signature: $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

²One could go up to $f_2 \sim 10^{-7}$ with specific matter contents, but we quote here the most general bound.

Still the classical theory may harbor some issues: given that QG features four time derivatives in the Lagrangian, the Ostrogradsky theorem [23] establishes that the classical Hamiltonian is not bounded from below. This manifests itself through the presence of a ghost with spin 2 and mass [1,24]

$$M_2 = \frac{f_2 \bar{M}_{\text{Pl}}}{\sqrt{2}}, \quad (2)$$

which is due to the W^2 term. In the present paper this problem will be addressed.

The basic idea exploits two key elements: (i) the ghost is not tachyonic, and (ii) its coupling f_2 [21] is very small. Indeed, as is well known, a nontachyonic decoupled ghost does not suffer from any instability [3,25]. By introducing an order-one coupling to normal particles one expects, from effective field theory arguments, that the theory remains stable up to energies of order M_2 (below which the ghost is not excited). But, given that the ghost coupling is tiny to solve the hierarchy problem, this energy threshold is lifted to a much higher value. In the rest of the paper we confirm this expectation.

At the end, we also work out testable predictions for inflation.

II. GHOST METASTABILITY

A. A four-derivative scalar field example

In order to illustrate our argument in a clear way we start by presenting it in a four-derivative theory of a real scalar field ϕ . Indeed the potential issues due to the Ostrogradsky theorem are present in this simple case too. In Sec. II B we will then turn to QG.

The Lagrangian is given by

$$\mathcal{L}_\phi = -\frac{1}{2}\phi\Box\phi - \frac{c_4}{2}\phi\Box^2\phi - V(\phi), \quad (3)$$

where c_4 is a real parameter, $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and the function V represents some interaction. At low energy the second term in Eq. (3) is negligible and one obtains the standard Lagrangian of a scalar field.

The four-derivative terms in Eq. (3) can be eliminated by introducing an auxiliary field A : one adds

$$\frac{c_4}{2}\left(\Box\phi - \frac{A - \phi/2}{c_4}\right)^2 \quad (4)$$

(which is zero by using the field equation of A) to $\mathcal{L}_\phi$, which then becomes

$$\begin{aligned} \mathcal{L}_\phi &= -\frac{1}{2}\phi\Box\phi - \frac{c_4}{2}\phi\Box^2\phi + \frac{c_4}{2}\left(\Box\phi - \frac{A - \phi/2}{c_4}\right)^2 - V(\phi) \\ &= -A\Box\phi + \frac{1}{2c_4}\left(A - \frac{\phi}{2}\right)^2 - V(\phi). \end{aligned} \quad (5)$$

One can diagonalize the kinetic terms by defining $\phi_\pm \equiv \phi/2 \pm A$, that is

$$A \equiv \frac{1}{2}(\phi_+ - \phi_-) \quad \phi \equiv \phi_+ + \phi_-, \quad (6)$$

to obtain

$$\mathcal{L}_\phi = -\frac{1}{2}\phi_+\Box\phi_+ + \frac{1}{2}\phi_-\Box\phi_- + \frac{m^2}{2}\phi_-^2 - V(\phi_+ + \phi_-), \quad (7)$$

where $m^2 \equiv 1/c_4$. The corresponding field equations are

$$\begin{aligned} \Box\phi_+ &= -V'(\phi_+ + \phi_-), \\ \Box\phi_- &= -m^2\phi_-^2 + V'(\phi_+ + \phi_-). \end{aligned} \quad (8)$$

We observe that the theory includes two two-derivative scalars, which are decoupled in the noninteracting case $V = 0$: a standard one ϕ_+ , which is massless and a ghost ϕ_- with mass m (note that $c_4 > 0$ in order for ϕ_- not to be a tachyon, a condition which we assume here). ϕ_+ and ϕ_- are analogous to the massless graviton and the ghost in QG, respectively.

Let us now assume some nontrivial interaction: $V(\phi) = \lambda\phi^4/4$, where λ is a positive coupling constant. Then we see that V tends to stabilize the motion of ϕ_+ and destabilize the one of ϕ_- . If $m^2 \leq 0$ the solution $\phi_+ = \phi_- = 0$ would be unstable. However, for $m^2 > 0$ the situation is much better as long as the values and the

derivatives of the fields are taken below certain thresholds that we now determine.

Let us denote with φ the typical order of magnitude of field values. Then ϕ_- feels a potential of the form $v(\varphi) \equiv m^2\varphi^2/2 - V(\varphi) = m^2\varphi^2/2 - \lambda\varphi^4/4$. This is a potential with a local minimum at $\varphi = 0$ and two maxima at $\varphi = \pm m/\sqrt{\lambda}$ with potential barriers $v(\pm m/\sqrt{\lambda}) = m^4/(4\lambda) \equiv E_d^4$, where we have introduced

$$E_d \equiv \frac{m}{(4\lambda)^{1/4}}. \quad (9)$$

Since E_d gives the height of the barrier, when the typical energy scale E associated with the field derivatives is much below E_d the runaways are avoided. Thus the condition on the field derivatives to ensure that the motion is bounded is

$$\boxed{E \ll E_d} \quad (\text{condition on field derivatives}). \quad (10)$$

There is a different condition on the field values that can be computed by equating the two terms (the stable and the unstable one) in v , that is $m^2\varphi^2/2 = \lambda\varphi^4/4$, which gives φ equal to

$$E_f \equiv \frac{m}{\sqrt{\lambda/2}}, \quad (11)$$

which is larger than E_d for small λ . Thus the condition on the field values to have a bounded motion is

$$\boxed{\varphi \ll E_f} \quad (\text{condition on field values}). \quad (12)$$

Both Eqs. (10) and (12) have to be satisfied by the boundary conditions in order for the motion to be bounded (that is to avoid the Ostrogradsky instabilities). An important point is that when λ is small both E_d and E_f become larger than the ghost mass, m . Because of the presence of a potential barrier the solution $\phi_+ = \phi_- = 0$ is not unstable, but *metastable*. The geometrical meaning of the thresholds E_d and E_f is illustrated in Fig. 1, which shows a typical potential with a metastable minimum. This resembles the Higgs potential in the Standard Model for the current central value of the top mass.

In Fig. 2 we show a spatially homogeneous but time-dependent solution for the simple quartic interaction $V(\phi) = \lambda\phi^4/4$. Whenever the initial conditions are chosen to satisfy the bounds in Eqs. (10) and (12) we observe indeed a bounded motion (the runaways are avoided) like in the plot.³ Note that the small values of $\phi_\pm$ on the vertical axes just reflect the fact that Condition (12) is enforced.

³The existence of an “island of stability” was noted in the simple one-dimensional toy model obtained from Eq. (3) by neglecting the spatial derivatives and choosing $V(\phi) = \lambda\phi^4/4$ [26]. However, Ref. [26] did not identify the thresholds (9)–(11).

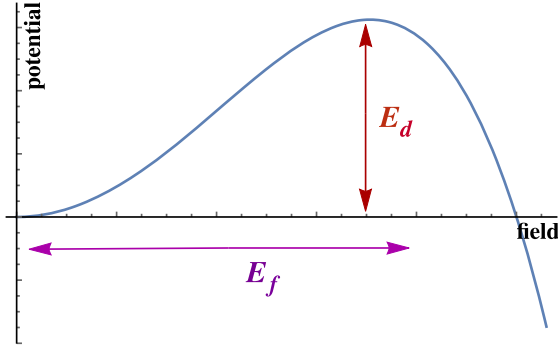


FIG. 1. Geometrical meaning of the thresholds E_f and E_d [defined in Eqs. (11) and (9)] for field values and derivatives, respectively.

B. The case of quadratic gravity

Having illustrated our argument in a simple theory, let us now turn to QG. In analogy with what was done in the previous subsection we start by rewriting the R^2 and W^2 terms as two extra two-derivative fields.

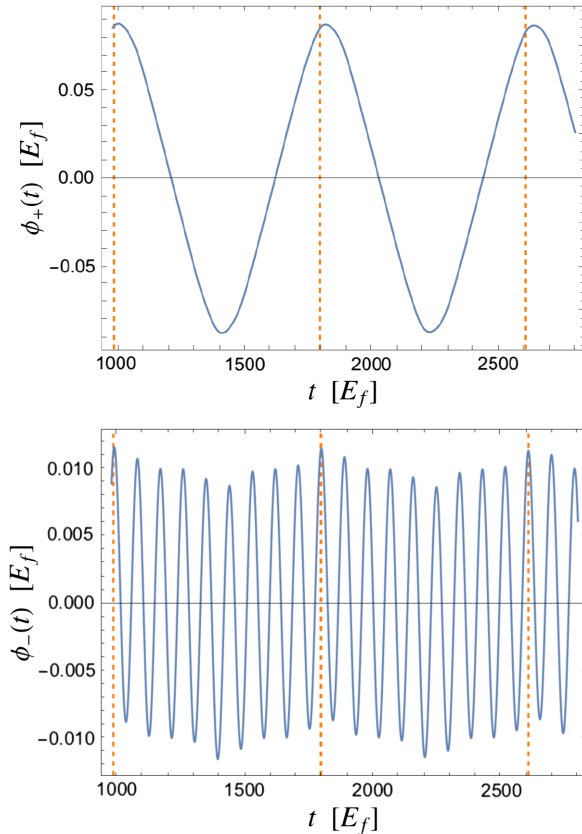


FIG. 2. Homogeneous time-dependent solution for $V(\phi) = \lambda\phi^4/4$ with $\lambda = 10^{-2}$. The initial conditions are chosen as follows: $\phi_+(0) = 10^{-2}E_f$, $\phi_-(0) = 10^{-2}E_f$, $\dot{\phi}_+(0) = (1.5 \times 10^{-1}E_d)^2$ and $\dot{\phi}_-(0) = -(10^{-2}E_d)^2$, where a dot denotes the derivative with respect to time t . The vertical dashed lines indicate the period.

Let us first perform the field redefinition

$$g_{\mu\nu} \rightarrow \frac{\bar{M}_{\text{Pl}}^2}{f} g_{\mu\nu}, \quad f \equiv \bar{M}_{\text{Pl}}^2 + \xi_{ab} \phi^a \phi^b - \frac{2R}{3f_0^2} > 0, \quad (13)$$

where the Ricci scalar above is computed in the Jordan-frame metric (the one before the redefinition). The transformation (13) gives the Einstein-frame action [2,20,27,28]

$$S = \int d^4x \sqrt{-g} \left(-\frac{W^2}{2f_0^2} - \frac{\bar{M}_{\text{Pl}}^2}{2} R + \mathcal{L}_m^E \right). \quad (14)$$

The Einstein-frame matter Lagrangian, $\mathcal{L}_m^E$, also contains an effective scalar ω , which corresponds to the R^2 term in Eq. (1) and is defined in terms of f by

$$\omega = \sqrt{6} \bar{M}_{\text{Pl}} \ln(\sqrt{f}/\bar{M}_{\text{Pl}}). \quad (15)$$

The part of the Lagrangian that depends only on ω is given by

$$\mathcal{L}_m^\omega = \frac{(\partial\omega)^2}{2} - U, \quad U = \frac{3f_0^2 \bar{M}_{\text{Pl}}^4}{8} (1 - e^{-2\omega/\sqrt{6}\bar{M}_{\text{Pl}}})^2. \quad (16)$$

The complete form of the Einstein-frame matter Lagrangian, which includes the most general matter sector, can be found in Ref. [2] (see also Ref. [20], where the reheating in this class of theories has been studied).

It is also possible to make the ghost explicit by considering an auxiliary field⁴ $\gamma_{\mu\nu}$:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_2^2 \bar{M}_{\text{Pl}}^2}{8} (\gamma_{\mu\nu} \gamma^{\mu\nu} - \gamma^2) - \frac{\bar{M}_{\text{Pl}}^2}{2} G_{\mu\nu} \gamma^{\mu\nu} - \frac{\bar{M}_{\text{Pl}}^2}{2} R + \mathcal{L}_m^E \right], \quad (17)$$

where $G_{\mu\nu}$ is the Einstein tensor and $\gamma \equiv \gamma_{\mu\nu} g^{\mu\nu}$. Equation (17) can be proved simply by noting that if we insert the solution of the $\gamma_{\mu\nu}$ equations,

$$G_{\mu\nu} = \frac{M_2^2}{2} (\gamma_{\mu\nu} - \gamma g_{\mu\nu}), \quad (18)$$

that is

$$\gamma_{\mu\nu} = \frac{2}{M_2^2} \left(R_{\mu\nu} - \frac{g_{\mu\nu} R}{6} \right),$$

into Eq. (17) we recover Eq. (14) (modulo total derivatives). Expanding around the flat metric $\eta_{\mu\nu}$ gives a mixing between $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ and $\gamma_{\mu\nu}$ that can be removed by

⁴We extend the analysis of Ref. [29] to a generic matter sector.

expressing $h_{\mu\nu} = \bar{h}_{\mu\nu} - \gamma_{\mu\nu}$. The tensors $\bar{h}_{\mu\nu}$ and $\gamma_{\mu\nu}$ represent the graviton and the ghost, respectively.

Equation (17) is useful because it allows us to understand the mass and interaction terms of the ghost. For example, it tells us that the ghost interactions vanish as $f_2 \rightarrow 0$. This can be seen by inserting Eq. (18) into Eq. (17) and by noting that the $f_2 \rightarrow 0$ limit of the result gives back general relativity (GR).

Let us consider first the term $\frac{M_2^2 \bar{M}_{\text{Pl}}^2}{8} (\gamma_{\mu\nu} \gamma^{\mu\nu} - \gamma^2)$ in Eq. (17). If one rewrites it in terms of $\bar{h}_{\mu\nu}$ and $\gamma_{\mu\nu}$ it leads to mass and interaction terms of the schematic form

$$\frac{M_2^2}{2} \left(\phi_2^2 + \frac{\phi_2^3}{\bar{M}_{\text{Pl}}} + \frac{\phi_2^4}{\bar{M}_{\text{Pl}}^2} + \dots \right), \quad (19)$$

where we have understood Lorentz indices and order-one factors and denoted the spin-2 fields with ϕ_2 (which we also canonically normalized: $\phi_2 \rightarrow \phi_2 / \bar{M}_{\text{Pl}}$). The mass term has the same order of magnitude as the interactions for $\phi_2 \sim \bar{M}_{\text{Pl}}$, which, therefore, represents the maximal spin-2 field value to avoid the runaways. This maximal value gives $M_2^2 \phi_2^2 / 2 = M_2^4 / f_2^2 \equiv E_2^4$, where

$$E_2 \equiv \frac{M_2}{\sqrt{f_2}} = \sqrt{\frac{f_2}{2}} \bar{M}_{\text{Pl}}. \quad (20)$$

For energies

$$\boxed{E \ll E_2} \quad (\text{in the spin-2 sector}) \quad (21)$$

the Ostrogradsky instabilities are avoided. Indeed, if Eq. (21) is satisfied the mass of the ghost dominates its interactions and, as is well known, a decoupled ghost does not suffer from runaways. Condition (21) applies to the derivatives of the spin-2 fields because, as we have seen above, the threshold for the spin-2 field values is much larger (of order $\bar{M}_{\text{Pl}}$). If Condition (21) is satisfied by the boundary conditions the runaways are avoided. The quantity E_2 is analogous to E_d in the simple scalar field theory of Sec. II A.

The same result holds if one considers the second and third terms in Eq. (17). The ghost interactions from them are at most of order $f_2 E^2 \phi_2^n / \bar{M}_{\text{Pl}}^{n-2}$ ($n = 1, 3, 4, \dots$), where we have introduced at least a factor of f_2 (the ghost decouples for $f_2 \rightarrow 0$) and the energy squared E^2 appears because of the two derivatives present there. These interactions for $\phi_2 \lesssim \bar{M}_{\text{Pl}}$ are smaller than $\sim f_2 E^2 \bar{M}_{\text{Pl}}^2$ which gives $\sim M_2^2 \bar{M}_{\text{Pl}}^2 \sim (M_2 / \sqrt{f_2})^4 = E_2^4$ when evaluated at $E = M_2 / \sqrt{f_2} \equiv E_2$.

Finally, let us consider $\mathcal{L}_m^E$. The ghost-matter interactions have size of order

$$f_2 E^4 \phi_2^n / \bar{M}_{\text{Pl}}^n \quad (n = 1, 2, 3, \dots), \quad (22)$$

where now one should interpret E as due to either derivative or mass terms of matter fields or matter field values times coupling constants; by requiring these interactions to be less than E_2^4 and $\phi_2 \lesssim \bar{M}_{\text{Pl}}$ one obtains that runaways are avoided for

$$\boxed{E \ll E_m}, \quad E_m \equiv \sqrt[4]{f_2} \bar{M}_{\text{Pl}}, \quad (\text{in the matter sector}), \quad (23)$$

which is larger than E_2 for small f_2 . Indeed, Eq. (21) regards only the energy E in the spin-2 sector (where the ghost is). Condition (23), like Eq. (21), should be satisfied by the boundary conditions. Notice that multiplying E_m by $\sqrt[4]{f_2}$ [to obtain the size of the ghost-matter interactions for the maximal field values; see Eq. (22)] one obtains (as one should) a scale of order E_2 . The quantity E_m vaguely corresponds to E_f in the simple scalar field theory of Sec. II A. There is a difference, however: gravity is sourced by other fields (the matter sector) while the scalar ϕ of Sec. II A was assumed to be sourced by itself.⁵

Therefore, one finds that the ghost of quadratic gravity is not associated with Ostrogradsky instabilities, but rather with metastability: there exists an energy barrier [given in Eqs. (20) and (23)] that prevents the fields from running away.

Note that the thresholds E_2 and E_m are both larger than M_2 for a weakly coupled ghost. This leads us to a very interesting situation: there exists an energy range in which the predictions of QG deviate from those of GR, but without activating runaway solutions. We will see some of these predictions in Sec. IV below. Nevertheless, it is important to observe that Conditions (21) and (23) apply to any positive choice of f_2 . If one takes $f_2 \sim 1$ both E_2 and E_m are at the Planck scale and again the runaways are avoided for any physical situation that has a chance to be observable. This effect is due to the fact that M_2 increases when f_2 grows, as is clear from Eq. (2). The disadvantage of the $f_2 \sim 1$ case is that, unlike the $f_2 \ll 1$ case, there is no hope to have a large energy window in which there are observational consequences of the ghost without Ostrogradsky instabilities.

For a natural Higgs mass ($f_2 \lesssim 10^{-8}$, $M_2 \lesssim 10^{10}$ GeV) E_2 and E_m can still be as high as $10^{-4} \bar{M}_{\text{Pl}}$ and $10^{-2} \bar{M}_{\text{Pl}}$, respectively. It is clear that inflation (and the preceding epoch) is the only stage of the Universe that can provide us information about such high scales. For energies much below M_2 (which can be many orders of magnitude above the TeV scale for a natural Higgs mass and is at the Planck scale for $f_2 \sim 1$) the theory reduces to Einstein gravity and, therefore, all the observations related to low-energy

⁵One could make the two cases more similar by adding to the theory of Sec. II A other fields, which mimic the matter sector in QG. We do not do it here for the sake of simplicity.

astrophysical systems (involving typical energies much smaller than M_2) are reproduced just as in Einstein gravity. Let us then focus on inflation and the preinflationary epoch.

Note that $W_{\mu\nu\rho\sigma} = 0$ on a Friedmann-Robertson-Walker (FRW) metric because such a metric is conformally flat. Therefore, only perturbations that violate homogeneity and/or isotropy could destabilize the Universe. In the chaotic theory [30] (a key element to understand the naturalness of inflation) one assumes that the fields took random values including inhomogeneous and anisotropic ones before inflation. But we live in one of those patches where the energy scales of inhomogeneities ($1/L$) and anisotropies (A) were small enough [31]:

$$L \gg |\Phi/U'(\Phi)|^{1/2}, \quad A \ll H, \quad (24)$$

where H is the inflationary Hubble rate, Φ is a generic canonically normalized inflaton field and U is its potential. The conditions above justify the use of homogeneous and isotropic solutions to describe the classical part of inflation (which is regularly done in the literature on inflation). On the one hand, the experimental bound [32]

$$H < 2.7 \times 10^{-5} \bar{M}_{\text{Pl}} \quad (95\% \text{C.L.}) \quad (25)$$

implies that the second condition in Eq. (24) ensures both Eqs. (21) and (23) (at least for the maximal ghost mass compatible with Higgs naturalness). On the other hand, by identifying U with the one in Eq. (16) (Starobinsky inflation⁶ [22]) the first condition in Eq. (24) becomes $1/L \ll 10^{-6} \bar{M}_{\text{Pl}}$, which again agrees with both Eqs. (21) and (23).

In other words, remarkably, the chaotic theory automatically ensures that the conditions to avoid runaway solutions are satisfied.

III. EXPLICIT NONLINEAR CALCULATIONS

In this section we solve the nonlinear gravity equations,⁷ namely

$$G_{\mu\nu} + \frac{2}{M_2^2} B_{\mu\nu} = \frac{T_{\mu\nu}^E}{M_{\text{Pl}}^2}, \quad (26)$$

where $B_{\mu\nu} \equiv (\nabla^\rho \nabla^\sigma + \frac{R^{\rho\sigma}}{2}) W_{\mu\rho\nu\sigma}$ is the Bach tensor and $T_{\mu\nu}^E$ is the energy-momentum tensor in the Einstein frame. We do so with an ansatz that violates the symmetries of the FRW metric to see the nonlinear effect of the ghost. In order to understand how things work in practice, we consider the following anisotropic ansatz:⁸

⁶We find similar results with other successful models such as Higgs inflation [33] or hilltop inflation [34].

⁷From now on we perform the calculations in the Einstein frame (14) unless otherwise stated.

⁸For the study of other anisotropic metrics in QG and conformal gravity see Refs. [35,36] and [37], respectively.

$$ds^2 = dt^2 - a(t)^2 \sum_{i=1}^3 e^{2\alpha_i(t)} dx^i dx^i. \quad (27)$$

The scale factor a and the α_i are generic functions of the cosmic time, t . As usual H is defined in terms of a by $H \equiv \dot{a}/a$, where a dot denotes the derivative with respect to t . This allows us to probe the nonlinear dynamics by solving ordinary differential equations. Note, however, that the argument of the previous section also applies to inhomogeneities. To the best of our knowledge this is the first article where anisotropic metrics in QG are studied in the Einstein frame, which allows us to compare theoretical predictions with inflationary observations (as we will do in Sec. IV). In this section we focus on Starobinsky inflation, a natural option in QG.

Given that $\sum_i \alpha_i$ can be included in a redefinition of a we can take

$$\alpha_1 \equiv \beta_+ + \sqrt{3}\beta_-, \quad \alpha_2 \equiv \beta_+ - \sqrt{3}\beta_-, \quad \alpha_3 \equiv -2\beta_+. \quad (28)$$

Therefore, the amount of anisotropy is encoded only in the functions $\beta_\pm$ and we can measure it through

$$A \equiv \sqrt{\dot{\beta}_+^2 + \dot{\beta}_-^2}. \quad (29)$$

By inserting this ansatz into the trace of the gravity equations one obtains

$$R = \frac{3p - \rho}{\bar{M}_{\text{Pl}}^2}, \quad \left(\rho = \frac{\dot{\omega}^2}{2} + U, p = \frac{\dot{\omega}^2}{2} - U \right). \quad (30)$$

The Weyl-squared term does not contribute to this equation. If the anisotropy is zero, the other equations do not receive contributions either. However, for $A \neq 0$ the Weyl-squared term contributes to some equations and leads to terms with four derivatives. One can reduce the gravity system to first-order equations through the definitions

$$\gamma_\pm \equiv \dot{\beta}_\pm, \quad \delta_\pm \equiv \dot{\gamma}_\pm, \quad \epsilon_\pm \equiv \dot{\delta}_\pm. \quad (31)$$

The tt component of the gravity equations then is

$$H^2 = \frac{\rho}{3\bar{M}_{\text{Pl}}^2} + A^2 - \frac{1}{M_2^2} \left[\frac{RA^2}{3} + H^2 A^2 + 14A^4 - 4H(\gamma_+ \delta_+ + \gamma_- \delta_-) - 2(\gamma_+ \epsilon_+ + \gamma_- \epsilon_-) + \delta_+^2 + \delta_-^2 \right]. \quad (32)$$

Note that, by using Eq. (30), this equation becomes a second-order algebraic equation for H . As usual we choose the solution that supports the Universe's expansion. The ii components of the gravity equations lead instead to the $\epsilon_\pm$ equations:

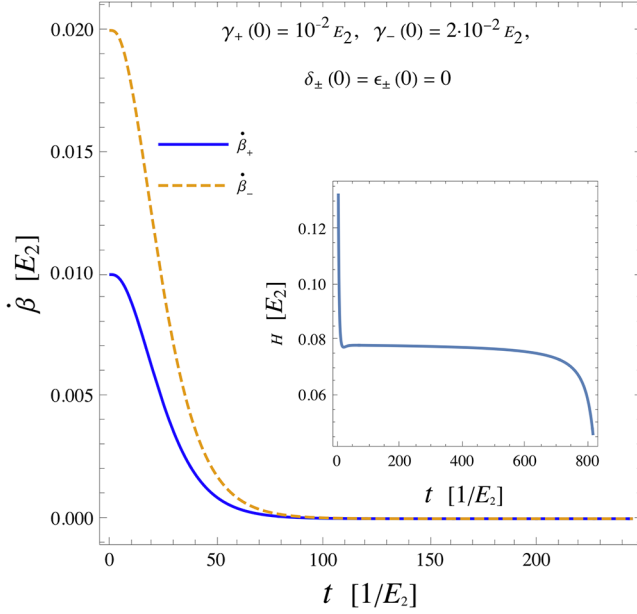


FIG. 3. The anisotropy vs the cosmic time. We set $f_2 = 10^{-8}$, $f_0 \approx 1.6 \times 10^{-5}$, $\phi(0) \approx 5.5 \bar{M}_{\text{Pl}}$ and $\sqrt{\pi_\phi(0)} \approx 7.1 \times 10^{-6} \bar{M}_{\text{Pl}}$. In the inset we show the corresponding Hubble rate.

$$\begin{aligned} \dot{\epsilon}_\pm = & -M_2^2(3H\gamma_\pm + \delta_\pm) \\ & + \left[\frac{\dot{R}}{6} + \frac{RH}{2} + 27HA^2 + 18(\gamma_+\delta_+ + \gamma_-\delta_-) \right] \gamma_\pm \\ & + \left(\frac{2R}{3} - 3H^2 + 12A^2 \right) \delta_\pm - 6H\epsilon_\pm, \end{aligned} \quad (33)$$

where we have used the tt component and the trace of the gravity equations. Finally, we also reduce the inflaton equation to two first-order differential equations:

$$\dot{\omega} = \pi_\omega, \quad \dot{\pi}_\omega + 3H\pi_\omega = -\frac{dU}{d\omega}. \quad (34)$$

By using the first of these equations and Eq. (30) one can express R in terms of ω and π_ω . Equations (31)–(34) then form a set of 11 equations in 11 unknowns ($\beta_\pm, \gamma_\pm, \delta_\pm, \epsilon_\pm, H, \omega, \pi_\omega$). We find, as we should, that there are no other independent equations.

In Fig. 3 we show how small initial values for the anisotropy in the sense of Eqs. (21)–(23) ($|\gamma_\pm(0)| \ll E_2$, $\sqrt{|\delta_\pm(0)|} \ll E_2$, $\sqrt[3]{|\epsilon_\pm(0)|} \ll E_2$ and⁹ $H \ll E_2$) do not create problems: the anisotropy quickly goes to zero and one recovers the GR behavior; no runaway solutions are observed in agreement with the general argument of Sec. II.

⁹By using the definitions in Eqs. (20) and (23) one can equivalently write the last condition $H \ll E_2$ as $\sqrt{\bar{M}_{\text{Pl}}H} \ll E_m$. This is because we can also regard $\bar{M}_{\text{Pl}}H$ as generated (through the Einstein equations) by the energy density stored in the matter sector.

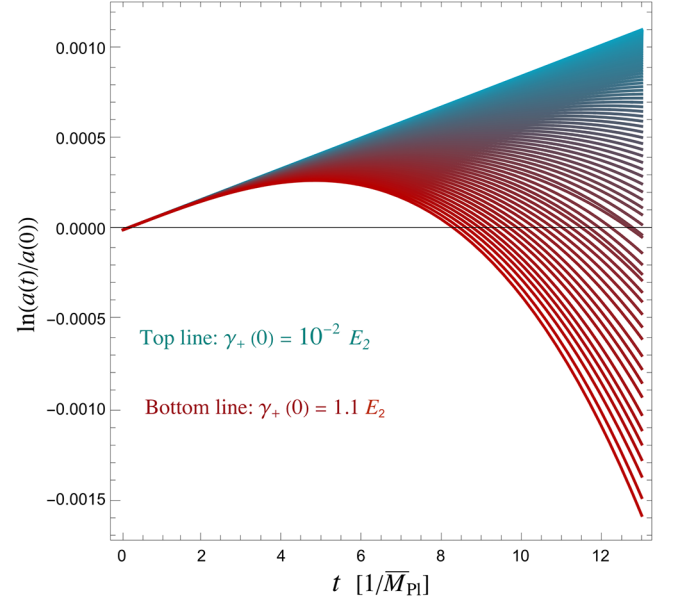


FIG. 4. The scale factor in the Jordan frame by varying the initial condition for γ_+ . We set $\gamma_-(0) = 10^{-1} E_2$ and all the other initial conditions for the anisotropy functions to zero. Furthermore, $f_2 = 10^{-8}$, $f_0 \approx 1.6 \times 10^{-5}$, $R(0) \approx 1.3 \times 10^2 f_0^2 \bar{M}_{\text{Pl}}^2$ and $H(0) = 1.2 E_2$.

Note that the smallness of $\dot{\beta}_\pm$ simply reflects the fact that Condition (21) is enforced because the plot is presented in units of E_2 ; for the chosen value of f_2 , the quantities $\dot{\beta}_\pm$ in units of M_2 are actually much larger than one at the beginning. Regarding H , after a short (preinflationary) time, it quickly reaches a plateau (inflation) and then decreases again when inflation ends. The inflaton initial conditions in Fig. 3 were chosen to obtain $N \approx 62$. We scanned the possible values of initial conditions, not just the one used in Fig. 3, and always found qualitatively the same result.

For the considered anisotropies we find that when Eqs. (21)–(23) are *not* satisfied the Universe collapses as shown in¹⁰ Fig. 4. In that plot we have set all the initial conditions of the anisotropy functions [other than $\gamma_\pm(0)$] to zero. By turning on the other initial conditions one finds similar behaviors: when the energy scales associated with them are much smaller than E_2 the anisotropy goes to zero as time passes by and the Universe inflates; when they are comparable or larger than E_2 the collapse occurs.

The interpretation is that the regions of space with large initial anisotropies have eventually zero size as compared to those satisfying Eqs. (21)–(23), which instead lead to inflation. What we have shown provides an explicit mechanism to implement the original chaotic inflation idea

¹⁰The scale factor in Fig. 4 is the one in the Jordan frame because we find that the field redefinition in Eq. (13) is ill defined when collapse occurs [the inequality in Eq. (13) is not satisfied for some time].

by Linde [30]. Indeed, it is not clear if in Einstein gravity the patches that were largely inhomogeneous and anisotropic [where Conditions (24) were violated] are incompatible with life: the fact that inflation does not occur in that case is not sufficient to reach this conclusion. On the other hand, the classical runaways that are triggered when those conditions are violated in the presence of the W^2 term for a natural Higgs mass ($f_2 \sim 10^{-8}$) certainly render the Universe inhospitable. Indeed, it is very interesting to note that the maximal Hubble rate allowed by Eq. (25) is, remarkably, just slightly smaller than the value of E_2 for the natural choice, $f_2 \sim 10^{-8}$. Therefore, the Weyl-squared term combined with Higgs naturalness provides an explicit mechanism to implement the original chaotic inflation idea.

IV. LINEAR ANALYSIS AND OBSERVATIONAL PREDICTIONS

A general check of ghost metastability can be performed by studying the complete set of linear perturbations around the de Sitter (dS) spacetime (which is the relevant one according to the results of Sec. II): one should find no runaway solutions there for arbitrary energies. The linear dS modes were found in Refs. [38–48]. Here we show that they are all bounded (and thus they do not suffer from runaways) for any wave number q .

First, one should recall that for $M_2 > H$ one recovers the Einstein modes, which are bounded. This expected decoupling was rigorously shown in Ref. [47]. Therefore, we focus here on the case $M_2 < H$, which, taking into account Eq. (25), implies that f_2 has to be very small. In the following we choose to work in the conformal Newtonian gauge.

Let us start with the scalar perturbations. They are similar to those in GR with one exception: there is one more isocurvature mode B (the helicity-0 component of the ghost) [46,47]. Its modes are g_B and g_B^* , where¹¹ [47]

$$g_B(\eta, q) \equiv \frac{H}{\sqrt{2q}} \left(\frac{3}{q^2} + \frac{3i\eta}{q} - \eta^2 \right) e^{-iq\eta} + \mathcal{R} \text{ terms} \quad (35)$$

and η is the conformal time ($\eta < 0$ and $\eta \rightarrow 0^-$ corresponds to large t). The terms due to the curvature perturbation $\mathcal{R}$ are not shown since they are the same as in GR and thus are bounded. Also the first term in Eq. (35) is bounded: for given initial conditions at a negative value of η the super-horizon limit $\eta \rightarrow 0$ is finite.

The vector modes are instead given by¹²

$$V_I \equiv \sqrt{-q\eta} I_{\frac{\sqrt{\rho-4}}{2\sqrt{\rho}}}(-q\eta), \quad (36)$$

¹¹In Ref. [46] it was proved that no physical singularity can be present in any other gauge. This proof was later extended to the most general matter sector in Ref. [47].

¹²For the derivation of the vector modes see Ref. [47], where a previous calculation in Ref. [42] was corrected.

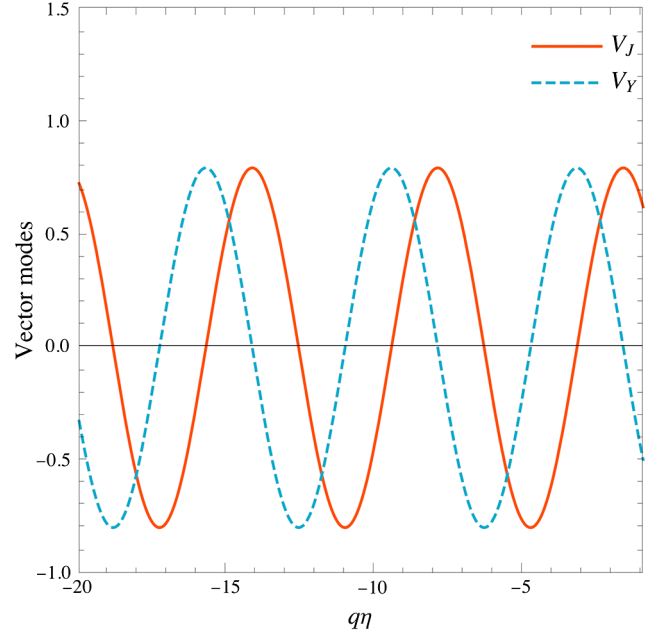


FIG. 5. Vector modes for $\rho \equiv H^2/M_2^2 = 10^4$.

where $I = \{J, Y\}$, J_α and Y_α are the Bessel functions of the first and second kind, respectively, and $\rho \equiv H^2/M_2^2$. We plot them in Fig. 5 to show that they are bounded.

In the tensor sector we have four modes (the two helicity components of the graviton and the ghost). They are [38,40,47]

$$g_a \equiv \cos(q\eta) + q\eta \sin(q\eta), \quad g_b \equiv q\eta \cos(q\eta) - \sin(q\eta), \quad (37)$$

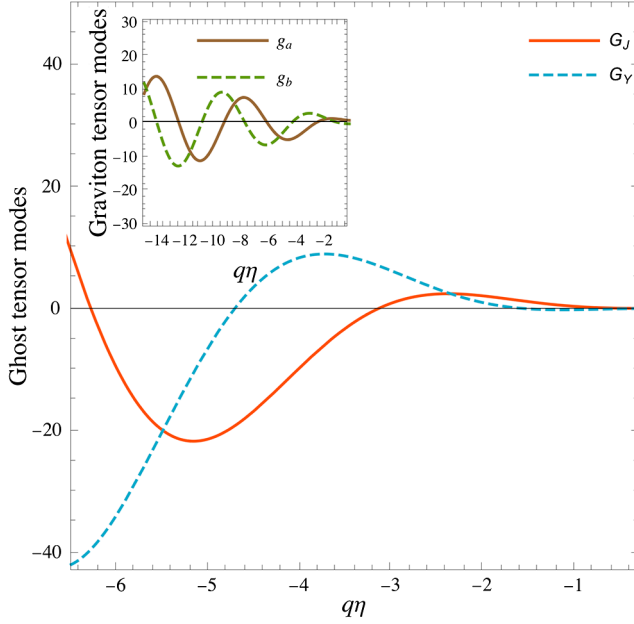
$$G_I \equiv (-q\eta)^{3/2} I_{\frac{\sqrt{\rho-4}}{2\sqrt{\rho}}}(-q\eta), \quad (\text{where } I = \{J, Y\}) \quad (38)$$

for the graviton (g) and the ghost (G) respectively. The graviton tensor modes g_a and g_b coincide with the modes one finds in GR. In Fig. 6 we show that the ghost tensor modes G_Y and G_J are also bounded (we plot the corresponding graviton tensor modes too for comparison).

In Figs. 5 and 6 we chose $\rho = 10^4$, which is a typical value in QG with a natural Higgs mass. The results do not change qualitatively as long as $H \gg M_2$. For ghost masses above H one instead recovers the modes of GR.

Note that the modes presented in this section would reduce to a linear analysis of the ansatz in Eq. (27) if one considers the space-independent limit of the perturbations and matches the two different definitions of time used here and in Sec. III.

The linear analysis also provides observational predictions of the theory. There are two differences compared to GR as shown in Ref. [47]. The first one is a suppression of the tensor power spectrum such that the tensor-to-scalar

FIG. 6. Tensor modes for $\rho \equiv H^2/M_2^2 = 10^4$.

ratio r is¹³ $r = r_E/(1 + 2H^2/M_2^2)$, where r_E is the tensor-to-scalar ratio in GR. In QG a natural Higgs mass corresponds typically [49] to $H \gg M_2$ so r is highly suppressed. For example, for the parameter values and inflationary model chosen in Fig. 3 $r_E \approx 0.003$, $r \sim 10^{-9}$, the curvature power spectrum $P_{\mathcal{R}} \approx 2.1 \times 10^{-9}$ and $n_s \approx 0.968$. All predictions are in agreement with the most recent Planck data [32].

The second difference is the presence of B , whose power spectrum is

$$P_B = \frac{3}{2\bar{M}_{\text{Pl}}^2} \left(\frac{H}{2\pi} \right)^2. \quad (39)$$

Given that P_B is *not* suppressed (and in fact it is $3/16$ times the tensor power spectrum in GR) we compute here its dependence on q , which is required to compare it with the Planck constraints on isocurvature modes [32]. By defining the spectral index as $n_B \equiv 1 + \frac{d \ln P_B}{d \ln q}$ one finds $n_B = 1 - 2\epsilon$, where ϵ is the first slow-roll parameter (in single-field inflation $\epsilon = r_E/16$). Then the q dependence is

$$P_B(q) = P_B(q_0) \left(\frac{q_0}{q} \right)^{2\epsilon}. \quad (40)$$

In Fig. 7 we compare P_B with Planck data. As shown in the plot, models with $r_E \approx 0.2$ are compatible with the data, unlike in GR. Moreover, given that the spectral index n_B is close to 1, B fulfills the bounds on the spectral index of isocurvature modes given in Ref. [32]. The CMB-S4

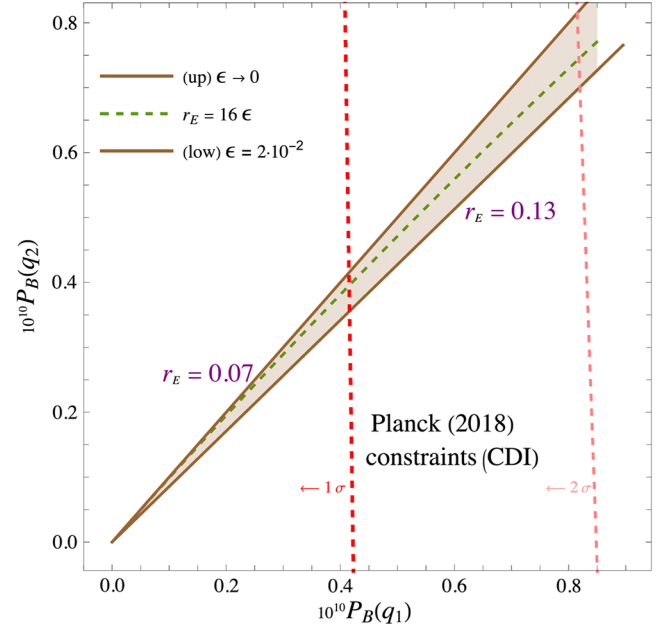


FIG. 7. The ghost-isocurvature power spectrum P_B computed at two different scales ($q_1 = 0.002$ and $q_2 = 0.1 \text{ Mpc}^{-1}$). The most precise determination of $P_{\mathcal{R}}(q_0)$ (the curvature power spectrum at the pivot scale $q_0 = 0.05 \text{ Mpc}^{-1}$) by Planck (2018) is used. The strongest constraints from Planck at the $1 - 2\sigma$ level are also shown.

Collaboration will be able to improve the sensitivity to isocurvature modes [50] and, therefore, this scenario can be further tested in the future.

V. CONCLUSIONS

We have shown that the possible classical runaways of quadratic gravity do not occur if the energies satisfy Conditions (21) and (23), which regard the boundary conditions for the spin-2 sector and the matter sector, respectively. Those conditions are weak enough to accommodate the entire history of the Universe. For a natural Higgs mass with $f_2 \sim 10^{-8}$, E_2 and E_m are still so high that they can be tested only via inflation (and preinflation dynamics). In that context, those energies represent the deviations from a homogeneous and isotropic metric (given that the ghost is inactive in a conformally flat metric).

To illustrate how this argument works we have solved numerically the nonlinear equations for anisotropic metrics. We found that the regions where Conditions (21) and (23) are satisfied quickly become isotropic and inflate; the others undergo collapse. The possible lethal instabilities occurring whenever the energy bounds are violated not only are avoided in our Universe, but would also explain (for a natural Higgs mass) why we live in a homogeneous and isotropic universe: life can only emerge from those patches that are isotropic and homogeneous enough in the sense of Eqs. (21) and (23).

¹³The power spectra are at horizon exit $q = aH$.

As a check of the general argument, we have also shown that the linear perturbations around dS are bounded for any q . Those linear modes also encode important and testable predictions of the theory, most notably a gravity-isocurvature mode that satisfies all current bounds and can be tested with cosmic microwave background observations in the near future.

It is also appropriate to mention here some advantages of our approach with respect to an alternative proposed in Refs. [15,16] where the ghost is projected out from the spectrum and the classical limit is taken: first our argument holds for generic metrics while the method of Refs. [15,16] was developed (so far) only for flat and purely FRW metrics (and the general perturbations around FRW are crucial to make contact with observations); second the classical

runaways, which appear when Conditions (21) and (23) are violated, give us an explanation of the quasihomogeneity and isotropy of the preinflationary initial conditions.

Finally, our results render legitimate and motivate the study of classical solutions in QG, such as black holes and horizonless spherical solutions (partially explored in Refs. [51–55] and [56,57]), wormholes (see Ref. [58] for the conformal gravity case) and gravitational waves.

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- [1] K. S. Stelle, Renormalization of higher derivative quantum gravity, *Phys. Rev. D* **16**, 953 (1977).
 - [2] A. Salvio, Quadratic gravity, *Front. Phys.* **6**, 77 (2018).
 - [3] A. Salvio and A. Strumia, Quantum mechanics of 4-derivative theories, *Eur. Phys. J. C* **76**, 227 (2016).
 - [4] A. Strumia, Interpretation of quantum mechanics with indefinite norm, *MDPI Phys.* **1**, 17 (2018).
 - [5] S. W. Hawking and T. Hertog, Living with ghosts, *Phys. Rev. D* **65**, 103515 (2002).
 - [6] C. M. Bender and P. D. Mannheim, No-Ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model, *Phys. Rev. Lett.* **100**, 110402 (2008).
 - [7] C. M. Bender and P. D. Mannheim, Exactly solvable PT-symmetric Hamiltonian having no Hermitian counterpart, *Phys. Rev. D* **78**, 025022 (2008).
 - [8] B. Holdom and J. Ren, QCD analogy for quantum gravity, *Phys. Rev. D* **93**, 124030 (2016).
 - [9] A. Salvio, Solving the Standard Model problems in softened gravity, *Phys. Rev. D* **94**, 096007 (2016).
 - [10] M. Raidal and H. Veerme, On the quantisation of complex higher derivative theories and avoiding the Ostrogradsky ghost, *Nucl. Phys. B* **916**, 607 (2017).
 - [11] D. Anselmi and M. Piva, A new formulation of Lee-Wick quantum field theory, *J. High Energy Phys.* **06** (2017) 066.
 - [12] J. F. Donoghue, Quartic propagators, negative norms and the physical spectrum, *Phys. Rev. D* **96**, 044007 (2017).
 - [13] D. Anselmi, On the quantum field theory of the gravitational interactions, *J. High Energy Phys.* **06** (2017) 086.
 - [14] A. Salvio, A. Strumia, and H. Veerme, New infra-red enhancements in 4-derivative gravity, *Eur. Phys. J. C* **78**, 842 (2018).
 - [15] D. Anselmi, Fakeons, Microcausality and the classical limit of quantum gravity, *Class. Quantum Grav.* **36**, 065010 (2019).
 - [16] D. Anselmi, Fakeons and the classicization of quantum gravity: The FLRW metric, *J. High Energy Phys.* **04** (2019) 61.
 - [17] J. F. Donoghue and G. Menezes, Gauge assisted quadratic gravity: A framework for UV complete quantum gravity, *Phys. Rev. D* **97**, 126005 (2018).
 - [18] A. Salvio and A. Strumia, Agravity, *J. High Energy Phys.* **06** (2014) 080.
 - [19] G. F. Giudice, G. Isidori, A. Salvio, and A. Strumia, Softened gravity and the extension of the Standard Model up to infinite energy, *J. High Energy Phys.* **02** (2015) 137.
 - [20] K. Kannike, G. Htsi, L. Pizza, A. Racioppi, M. Raidal, A. Salvio, and A. Strumia, Dynamically induced planck scale and inflation, *J. High Energy Phys.* **05** (2015) 065.
 - [21] A. Salvio and A. Strumia, Agravity up to infinite energy, *Eur. Phys. J. C* **78**, 124 (2018).
 - [22] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, *Phys. Lett. B* **91**, 99 (1980).
 - [23] M. Ostrogradsky, Memoires sur les équations différentielles relatives au problème des isopérimètres, *Mem. Ac. St. Petersburg VI* 385 (1850). Pdf available online. For a review see Ref. [2].
 - [24] K. S. Stelle, Classical gravity with higher derivatives, *Gen. Relativ. Gravit.* **9**, 353 (1978).
 - [25] A. Pais and G. E. Uhlenbeck, On field theories with non-localized action, *Phys. Rev.* **79**, 145 (1950).
 - [26] M. Pavšič, Stable self-interacting Pais-Uhlenbeck oscillator, *Mod. Phys. Lett. A* **28**, 1350165 (2013); Pais-Uhlenbeck oscillator and negative energies, *Int. J. Geom. Methods Mod. Phys.* **13**, 1630015 (2016).
 - [27] A. Salvio and A. Mazumdar, Classical and quantum initial conditions for Higgs inflation, *Phys. Lett. B* **750**, 194 (2015).
 - [28] A. Salvio and A. Mazumdar, Higgs stability and the 750 GeV Diphoton excess, *Phys. Lett. B* **755**, 469 (2016).
 - [29] A. Hindawi, B. A. Ovrut, and D. Waldram, Consistent spin two coupling and quadratic gravitation, *Phys. Rev. D* **53**, 5583 (1996).
 - [30] A. D. Linde, Chaotic inflation, *Phys. Lett.* **129B**, 177 (1983).

- [31] See e.g., S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008), p. 593.
- [32] P. A. R. Ade *et al.* (Planck Collaboration), Planck 2015 results. XX. Constraints on inflation, *Astron. Astrophys.* **594**, A20 (2016); Y. Akrami *et al.* (Planck Collaboration), Planck 2018 results. X. Constraints on inflation, [arXiv:1807.06211](#).
- [33] F. Bezrukov and M. Shaposhnikov, Higgs inflation at the critical point, *Phys. Lett. B* **734** (2014) 249; A. Salvio, Initial conditions for critical Higgs inflation, *Phys. Lett. B* **780**, 111 (2018).
- [34] L. Boubekeur and D. H. Lyth, Hilltop inflation, *J. Cosmol. Astropart. Phys.* **07** (2005) 010.
- [35] S. Cotsakis, J. Demaret, Y. D. Rop, and L. Querella, Mix-master universe in fourth-order gravity theories, *Phys. Rev. D* **48**, 4595 (1993).
- [36] J. D. Barrow and S. Hervik, Anisotropically inflating universes, *Phys. Rev. D* **73** (2006) 023007.
- [37] J. Demaret, L. Querella, and C. Scheen, Hamiltonian formulation and exact solutions of the Bianchi type I space-time in conformal gravity, *Classical Quantum Gravity* **16**, 749 (1999).
- [38] T. Clunan and M. Sasaki, Tensor ghosts in the inflationary cosmology, *Classical Quantum Gravity* **27**, 165014 (2010).
- [39] N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, Inflation with a Weyl term, or ghosts at work, *J. Cosmol. Astropart. Phys.* **03** (2011) 040.
- [40] N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, Lorentz-violating vs ghost gravitons: the example of Weyl gravity, *J. High Energy Phys.* **09** (2012) 009.
- [41] F. d. O. Salles and I. L. Shapiro, Do we have unitary and (super)renormalizable quantum gravity below the Planck scale?, *Phys. Rev. D* **89**, 084054 (2014); Erratum, *Phys. Rev. D* **90**, 129903(E) (2014).
- [42] Y. S. Myung and T. Moon, Primordial massive gravitational waves from Einstein-Chern-Simons-Weyl gravity, *J. Cosmol. Astropart. Phys.* **08** (2014) 061.
- [43] Y. S. Myung and T. Moon, Primordial gravitational waves from conformal gravity, [arXiv:1407.0441](#).
- [44] I. L. Shapiro, A. M. Pelinson, and F. de O. Salles, Gravitational waves and perspectives for quantum gravity, *Mod. Phys. Lett. A* **29**, 1430034 (2014).
- [45] Y. S. Myung and T. Moon, Scale-invariant tensor spectrum from conformal gravity, *Mod. Phys. Lett. A* **30**, 1550172 (2015).
- [46] M. M. Ivanov and A. A. Tokareva, Cosmology with a light ghost, *J. Cosmol. Astropart. Phys.* **12** (2016) 018.
- [47] A. Salvio, Inflationary perturbations in no-scale theories, *Eur. Phys. J. C* **77**, 267 (2017).
- [48] P. Peter, F. D. O. Salles, and I. L. Shapiro, On the ghost-induced instability on de Sitter background, *Phys. Rev. D* **97**, 064044 (2018).
- [49] Exceptions are low-scale models with $H \ll 10^{10}$ GeV, see e.g., G. German, G. G. Ross, and S. Sarkar, Low scale inflation, *Nucl. Phys. B* **608**, 423 (2001); P. Q. Hung, E. Masso, and G. Zsembinszki, Low-scale inflation in a model of dark energy and dark matter, *J. Cosmol. Astropart. Phys.* **12** (2006) 004; F. Takahashi, W. Yin, and A. H. Guth, QCD axion window and low-scale inflation, *Phys. Rev. D* **98**, 015042 (2018).
- [50] K. N. Abazajian *et al.* (CMB-S4 Collaboration), *CMB-S4 Science Book*, 1st ed. See also the website Forecastfiso planck, https://cmb-s4.org/wiki/index.php/Forecastfiso_planck.
- [51] H. Lu, A. Perkins, C. N. Pope, and K. S. Stelle, Black Holes in Higher-Derivative Gravity, *Phys. Rev. Lett.* **114**, 171601 (2015).
- [52] H. Lü, A. Perkins, C. N. Pope, and K. S. Stelle, Spherically symmetric solutions in higher-derivative gravity, *Phys. Rev. D* **92**, 124019 (2015).
- [53] K. Goldstein and J. J. Mashiyane, Ineffective higher derivative Black Hole hair, *Phys. Rev. D* **97**, 024015 (2018).
- [54] H. Lü, A. Perkins, C. N. Pope, and K. S. Stelle, Lichnerowicz modes and Black Hole families in Ricci quadratic gravity, *Phys. Rev. D* **96**, 046006 (2017).
- [55] R. Svarc, J. Podolsky, V. Pravda, and A. Pravdova, Exact Black Holes in Quadratic Gravity with Any Cosmological Constant, *Phys. Rev. Lett.* **121**, 231104 (2018).
- [56] B. Holdom, On the fate of singularities and horizons in higher derivative gravity, *Phys. Rev. D* **66**, 084010 (2002).
- [57] B. Holdom and J. Ren, Not quite a black hole, *Phys. Rev. D* **95**, 084034 (2017).
- [58] M. Hohmann, C. Pfeifer, M. Raidal, and H. Veerme, Wormholes in conformal gravity, *J. Cosmol. Astropart. Phys.* **10** (2018) 003.