Mass-to-radius profiles of Newtonian polytropic spheres: Low-mass main-sequence stars in Palatini gravity versus dark-matter-admixed low-mass stars in general relativity

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We investigate properties of Newtonian (nonrelativistic) polytropic stars in two different scenarios: on the one hand in the Starobinsky model in Palatini formalism, and on the other hand in general relativity assuming that the star contains both ordinary and dark matter. We obtain numerical solutions to the structure equations, and we show the mass-to-radius profiles of both scenarios in the same figure for comparison. Our findings show that (a) contrary to the Palatini gravity, where the mass may be an increasing or decreasing function of the radius depending on the polytropic index, in admixed dark matter stars the mass is always a decreasing function of the radius, and (b) if the α parameter of the Starobinsky model is positive the two scenarios give distinct predictions, while if the α parameter is negative, the two scenarios exhibit similar behavior in the $n = 2$ case.

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I. INTRODUCTION

Many modern well-established observational data coming from astrophysics and cosmology indicate that the Universe is expanding at an accelerating rate dominated by dark energy and dark matter [1]. The concordance cosmological model, which is based on cold dark matter and a cosmological constant (ΛCDM), is the most economical model that successfully describes the structure formation of the Universe on large (cosmological) scales. The determination of the particles that play the role of dark matter in the Universe is one of the biggest challenges of particle physics and modern cosmology, since the origin and nature of dark matter still remain unknown.

The ΛCDM model, although very successful in describing a vast amount of observational data, suffers from the cosmological constant problem [2]. Therefore, many other possibilities have been explored over the years, such as dynamical dark energy models, with a time varying equation-of-state parameter, or geometrical dark energy models, where it is assumed that an alternative theory of gravity modifies Einstein's general relativity (GR) at cosmological scales.

Perhaps the simplest choice to describe current cosmic acceleration without introducing either extra dimensions or new dynamical fields is to generalize GR in a straightforward manner, and study $f(R)$ theories of gravity [3,4], where the Ricci scalar R in the Einstein-Hilbert term of GR is replaced by a generic function. Although nowadays the main motivation to study $f(R)$ theories of gravity is to explain the undergoing acceleration of the Universe (for a review on modified theories of gravity and cosmology see [5]), the astrophysical implications of this class of theories, too, should be investigated. A simple, well-motivated, and well-studied $f(R)$ theory model in the literature is the Starobinsky model [6], or R^2 model, which can describe the inflationary Universe [7]. Higher order in the curvature terms are natural in the Lovelock gravity [8], and also they appear in the effective equations of string theory [9].

Any $f(R)$ theory of gravity may be studied either in the usual metric approach or in the Palatini formalism, where the connection and the metric tensor are treated as independent quantities [3]. In this work we work in the Palatini gravity in the Starobinsky model [6]. For studies in relativistic stars in R^2 gravity see, e.g., [10–16] and references therein, while for works on polytropic stars in the Palatini gravity see, e.g., [17–20] and references therein.

Dark matter may be accumulated inside stars and modify their properties, such as mass-to-radius profiles or the frequencies of radial and nonradial oscillation modes. Even if dark matter does not have any direct couplings to ordinary matter, it may still have a significant impact on the properties of stars [21–31]. To the best of our knowledge we study for the first time here dark-matter-admixed low-mass stars (treated as nonrelativistic, Newtonian polytropic stars). Since both dark matter and alternative theories of gravity are expected to modify properties of stars, we take the initiative in the present work to compare the

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mass-to-radius profiles of low-mass stars obtained in two distinct scenarios, namely (i) stars containing only ordinary matter in Palatini gravity, and (ii) dark-matter-admixed low-mass in GR. This comparative study will help us to have a better understanding of how these two distinct scenarios (modified gravity without dark matter versus dark matter particles in GR) can be differentiated between them, and in which cases one solution is more adequate than the other.

Our work is organized as follows: In the next section we summarize the theoretical framework, while in Sec. III we show and discuss our results. Finally we conclude our work in the fourth section.

II. POLYTROPIC STARS IN GR

To set the notation, although it is a textbook knowledge, let us first briefly summarize how Newtonian (nonrelativistic) polytropic stars are treated in GR assuming an equation of state (EoS) $p(\rho)$, with p being the pressure and ρ being the energy density, of the form

$$p = K\rho^{\gamma} = K\rho^{(1+1/n)}, \tag{1}$$

where K, γ are constants, and polytropic index $n = 1/(\gamma - 1)$. Under the approximations, $\rho \gg p$, $m \gg pr^3$, $1 \gg m/r$, with $m(r)$ being the mass function, the nonrelativistic version of the Tolman-Oppenheimer-Volkoff equations [32,33] may be combined to derive the Lane-Emden equation [34]

$$\frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right] = -\xi^2\theta^n \tag{2}$$

for any index n, supplemented with the following two initial conditions,

$$\theta(0) = 1, \tag{3}$$

$$\theta'(0) = 0, \tag{4}$$

where the primes denote differentiation with respect to ξ.

The new variables (ξ, θ) are related to the original ones (r, ρ) via $\xi = r/a$ and $\theta^n = \rho/\rho_c$, with ρ_c being the central energy density, while the constant a is defined to be (setting Newton's constant to unity, $G_N = 1$)

$$a^2 = \frac{(n+1)p_c}{4\pi\rho_c^2}. \tag{5}$$

The radius R of the star is determined by the condition $\theta(\xi_n) = 0$. Once the root ξ_n (for a given polytropic index n) is known, the radius and the mass of the star are computed by $R = a\xi_n$ with

$$M = 4\pi a^3 \rho_c J, \tag{6}$$

where J is given by

$$J \equiv \int_0^{\xi_n} z^2[\theta(z)]^n dz. \tag{7}$$

Combining the two equations and eliminating the central energy density we obtain the mass-to-radius profile of the form $M(R) \sim R^{(n-3)/(n-1)}$. Clearly, the case $n = 3$ corresponds to a trivial profile, $M(R) =$ constant, and therefore it is not considered here.

It is known that one can obtain exact analytical solutions in three cases, for a polytropic index $n = 0$, 1, 5. In the latter case there is no finite radius for the star, and therefore it is not interesting for our study. The first case, $n = 0$, corresponds to a uniform density star, $\rho(r) = \bar{\rho} =$ constant, which is not realistic, since both pressure and energy density should decrease from the center of the star to its surface. The solution to the Lane-Emden equation is computed to be $\theta(\xi) = 1 - \xi^2/6$ with $\xi_0 = \sqrt{6} \simeq 2.449$. The second case, $n = 1$, corresponds to an EoS of the form $p = K\rho^2$, which describes for example bosonic condensed dark matter [35,36], and the solution is given by $\theta(\xi) = \sin(\xi)/\xi$ with $\xi_1 = \pi \simeq 3.142$. For completeness we report the solution for $n = 5$, which is the following $\theta(\xi) = (1 + \xi^2/3)^{-1/2}$, and since it is always positive there is no root satisfying the equation $\theta(\xi) = 0$; therefore $\xi_5 = \infty$. All three exact solutions are shown in Fig. 1.

Sir A. Eddington pointed out in [37] that ordinary gaseous stars may be better described using a varying polytropic index that decreases from a value n_2 at the surface to a value n_1 at the center. Although he discussed the $n_1 = 1.5$ and $n_2 = 3.5$, more recent data and analyses show that the best fit model for our Sun corresponds to the values $n_1 = 1.58$ and $n_2 = 3.94$ [38]. Therefore in the following we consider the fiducial cases $n = 2$ and $n = 4$ [39]. These solutions are also shown in Fig. 1.

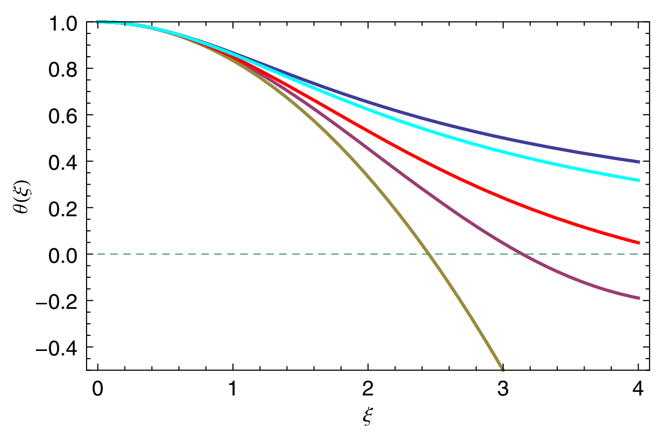

FIG. 1. Five solutions (both analytical and numerical) to the Lane-Emden equation in GR for polytropic indices $n = 0$ (green curve), $n = 1$ (magenta curve), $n = 5$ (dark blue curve), $n = 2$ (red curve), and $n = 4$ (cyan curve).

For the discussion to follow we find it convenient to introduce an energy density that corresponds to a uniform density star with mass $M_\odot$ and radius $R_\odot$, $\bar{\rho} = 3\ M_\odot/(4\pi R_\odot^3)$, and we express pressure and energy density in terms of that quantity. It is easy to verify that one can obtain a solarlike star, with a mass of the order of the solar mass, $M \sim M_\odot$, and a radius of the order of the solar radius, $R \sim R_\odot$, if we choose the constant K to be $K = k_{bm}\bar{\rho}^{-1/2}$ for $n = 2$ and $K = k_{bm}\bar{\rho}^{-1/4}$ for $n = 4$, with $k_{bm} = 6 \times 10^{-7}$. Then the central density of the star is comparable to $\bar{\rho}$.

III. NEWTONIAN POLYTROPIC STARS IN NONSTANDARD SCENARIOS

Here we proceed to study Newtonian polytropic stars in two nonstandard scenarios, namely (a) in GR assuming that the star contains both ordinary and dark matter, and (b) in modified gravity. These two classes of models are defined as follows:

(a) Dark-matter-admixed star: We assume that ordinary matter [with the baryonic density $\rho_{bm}(r)$] is described by the EoS mentioned in the end of the previous section, while dark matter [with the dark matter density $\rho_m(r)$] inside the star is modeled as a condensate characterized by an EoS of the form $p_{dm} = K_{dm}\rho_{dm}^2$ [35,36], for which $n = 1$. The constant K_{dm} is computed in terms of two free parameters m, l [35,36],

$$K_{dm} = \frac{2\pi l}{m^3}, \tag{8}$$

with m being the mass of the dark matter particle, while l is the scattering length, which determines the two-body self-interaction cross section, $\sigma = 4\pi l^2$, in a cold, dilute boson gas [35,36]. Self-interacting dark matter is constrained to take values in the range [40–42]

$$1.75 \times 10^{-4}\ \frac{\mathrm{cm}^2}{g} < \frac{\sigma}{m} < (1-2)\ \frac{\mathrm{cm}^2}{g}. \tag{9}$$

The two fluids interact through gravity only without any direct interaction between them. In this case we need to integrate the structure equations in the two-fluid formalism [43,44],

$$m'(r) = 4\pi r^2(\rho_{bm}(r) + \rho_{dm}(r)), \tag{10}$$

$$p'_{bm}(r) = -\frac{m(r)\rho_{bm}(r)}{r^2}, \tag{11}$$

$$p'_{dm}(r) = -\frac{m(r)\rho_{dm}(r)}{r^2}. \tag{12}$$

The total mass of the star M has two contributions $M = M_{bm} + M_{dm}$, where the mass of ordinary matter $M_{bm}(\equiv M_o)$ and the mass of dark matter M_{dm} are given by

$$M_{bm} = 4\pi \int_0^R dr\, \rho_{bm}(r) r^2 \tag{13}$$

and

$$M_{dm} = 4\pi \int_0^R dr\, \rho_{dm}(r) r^2. \tag{14}$$

Finally, we define the parameter $f \equiv p_c^{dm}/p_c^T$, where p_c^T $(\equiv p_c^{bm} + p_c^{dm})$ is the total central pressure.

To integrate the system of equations numerically we need to specify the numerical values of m, l, f, and the initial conditions. These are (with $z = p_c^T/\bar{\rho}$): $\rho_c^{bm} = ((1-f)z/k_{bm})^{2/3}\bar{\rho}$ for ordinary matter when $n = 2$, $\rho_c^{bm} = ((1-f)z/k_{bm})^{4/5}\bar{\rho}$ for ordinary matter when $n = 4$, and $\rho_c^{dm} = (fz/k_{dm})^{1/2}\bar{\rho}$ for dark matter. For the f factor we may use a numerical value in the range used in [23] by us, where we studied the impact of condensed dark matter on relativistic compact objects, and therefore in the following we consider $f = 0.08$. Finally, we write the constant K_{dm} in terms of $\bar{\rho}$ as follows, $K_{dm} = k_{dm}\bar{\rho}^{-1}$, and for k_{dm} we consider a numerical value of the order of k_{bm}, $k_{dm} = 8 \times 10^{-7}$. This can be achieved by assuming $m = 2 \times 10^{-4}$ GeV and $l = 0.06$ fm, for which the self-interaction cross section $\sigma/m = 1\ \mathrm{cm}^2/g$, which lies within the observationally allowed range mentioned before.

In Fig. 2 we show the mass-to-radius profiles for dark-matter-admixed stars for $f = 0.08$ corresponding to $\sim$10% dark matter mass fraction for $n = 2$ (blue) and $n = 4$ (red). Although these two polytropic models have quite distinct mass-to-radius profiles, in both cases the mass of these stars decreases as their radius increases. Moreover, the overall relation mass-to-radius profile is quite distinct of other compact objects; see, e.g., [16,23].

(b) Palatini gravity: Here we consider the Starobinsky model [6], $R + bR^2$, which can describe inflation without a

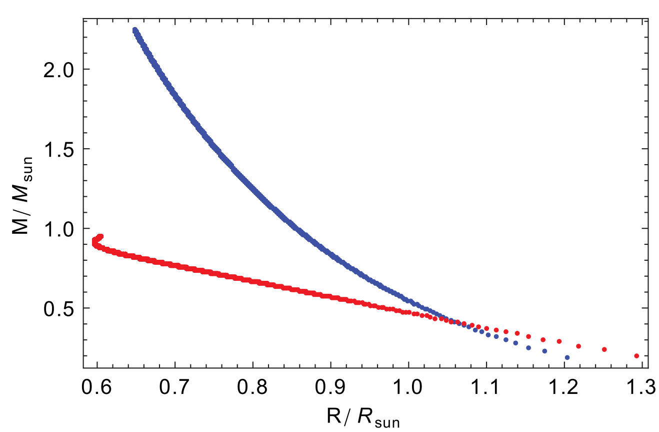

FIG. 2. Mass-to-radius profiles (in solar units) for dark-matter-admixed Newtonian polytropic star with a dark matter fraction $f = 0.08$. The blue curve corresponds to $n = 2$, while the red curve corresponds to $n = 4$.

TABLE I. Roots $\xi_n(\alpha)$ and integrals J for different values of the Starobinsky α parameter and $n = 2$ or $n = 4$. The GR value $\alpha = 0$ is shown as well for comparison reasons.

Lane-Emden equation in the Palatini gravity			
n	α	$\xi_n(\alpha)$	J
2	0.3	5.40	5.44
⋯	0.2	4.96	4.10
⋯	0.0	4.35	2.41
⋯	−0.2	4.11	1.49
⋯	−0.3	4.16	1.22
4	0.3	11.57	3.42
⋯	0.2	12.06	2.69
⋯	0	14.97	1.80
⋯	−0.2	24.15	1.41
⋯	−0.3	32.90	1.36

scalar field [7]. From a theoretical point of view R^2 gravity is well motivated, since higher order in R terms are natural in Lovelock theory [8], and also higher order curvature corrections appear in the low-energy effective equations of superstring theory [9]. In studying $f(R)$ theories of gravity there are two formalisms, namely, the usual metric approach and the Palatini formalism [3] where the metric tensor and the connection are treated as independent quantities. In the metric formalism there is an additional degree of freedom, which corresponds to a scalar field in the Einstein frame. The mass of the scalar field is constrained from solar system tests, and it is not allowed to be very light [45–48]. In the Palatini formalism, however, it is easier for $f(R)$ theories of gravity to pass the solar system tests [49,50]. Therefore, in the present work we work in the Palatini formalism, and the parameter b of the Starobinsky model is a free, unbounded, parameter.

In this case one can obtain a modified Lane-Emden equation [20] with the same initial conditions as before,

$$\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}(1+2\alpha\theta^n)+\alpha\xi^3\theta^{2n}\right]=\xi^2(-\theta^n+3\alpha\theta^{2n}), \qquad (15)$$

where $\alpha = -8\pi b\rho_c$, and it may be either positive or negative provided that $\alpha > -1/2$ [20]. Table I and Fig. 3 gives the properties of polytropic stellar models in Palatini gravity.

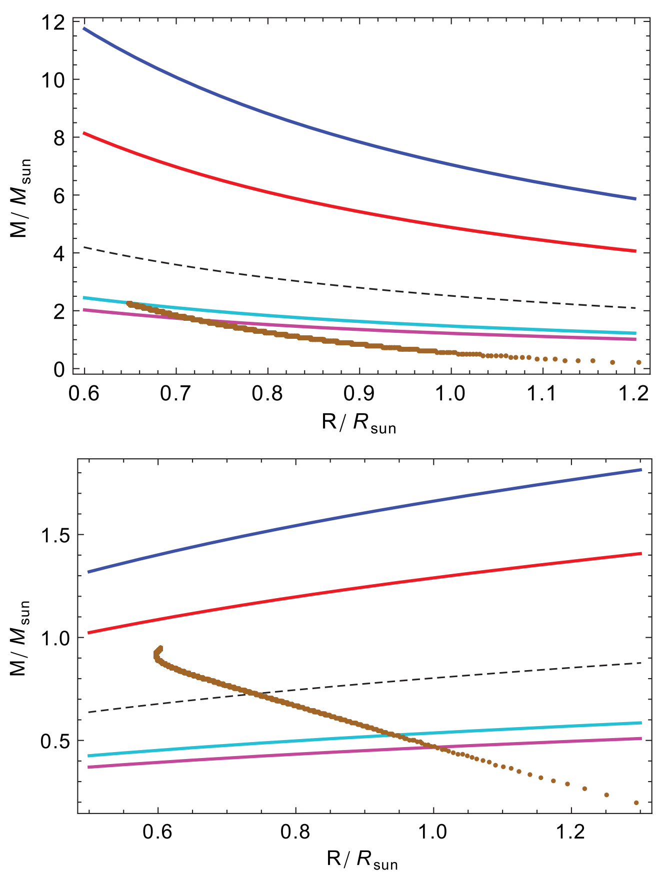

FIG. 3. Mass-to-radius profiles (in solar units) of Newtonian polytropic stars in the two scenarios discussed in the text for $n = 2$ (top panel) and $n = 4$ (bottom panel). The points correspond to the dark matter admixed star with a dark matter fraction of $f = 0.08$, while the curves correspond to the Palatini gravity. From the bottom to the top $\alpha = -0.3, -0.2, 0, 0.2, 0.3$.

IV. CONCLUSIONS

We have investigated properties of low-mass stars in the main sequence, treated as Newtonian, nonrelativistic, polytropic stars, in two different scenarios: On the one hand in the Starobinsky model in Palatini formalism, and on the other hand in GR assuming that the star contains both ordinary and dark matter. We have assumed for ordinary matter a polytropic index $n = 2$ and $n = 4$, while for dark matter we have assumed $n = 1$. We have integrated the equations numerically, and we have shown some properties of the numerical solutions of the modified Lane-Emden equation in Table I. In the Palatini gravity the mass of the star may be a decreasing (cf. Fig. 3) or an increasing function of its radius depending on the value of the polytropic index, while in the case of the dark matter admixed stars our results show that the mass decreases as the radius increases for both values of the polytropic index considered here. The mass-to-radius profiles we have obtained are shown in Fig. 2 for $n = 2$ (blue) and $n = 4$ (red). Our findings indicate that if the α parameter of the Starobinsky model is positive the two scenarios give distinct predictions, while if the α parameter is negative, the two scenarios exhibit similar behavior in the $n = 2$ case.

The significance of the main result obtained in this work may be summarized as follows: On the one hand, the two scenarios considered here—modified gravity without dark matter and dark matter particles in GR—predict quite distinct mass-radius relations for the same cluster of stars in the same stage of evolution. On the other hand, it is possible, at least in principle, to obtain a mass-to-radius profile using observational data (i.e., mass-radius pairs) of stars formed in a certain location in the Universe. Then this mass-radius relation can be used to distinguish between the two classes of solutions (as shown in Fig. 3). Accordingly, if the formation of stars occurs in a location of the Universe that is dominated by dark matter, modified theory of gravity, or both, this leads to a unique mass-to-radius profile for a given population of stars. Therefore, if one or both of such mass-radius relations (with quite distinct features) are found for groups of stars formed in different locations of the Universe, this will be a strong clue that processes could be operating in such locations of the Universe.

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