Class of Lorentz-Invariant compact structures

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We endow a class of Lorentz Invariant Lagrangians with a simple shape factor and demonstrate that it induces Q balls and radial Q vortices that, as is to be expected from particlelike modes, have a *genuinely compact support* with the latter turning above a critical azimuthal number into planar rings with *sharp inner and outer radii*.

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I. THE PROBLEM

Following the introduction of compactons [1]—solitary waves with a compact support, in scalar dispersive systems, see [1] or a primer on the subject [2]—we have introduced in [3,4], a Lorenz Invariant class of complex Lagrangians that induce *Q balls with a strict compact support* and thus dispense with the perennial nuisance of infinite tails of Q balls [5–7] and strictly compact planar vortices which above a critical azimuthal number become *genuine multinodal radial rings of a finite span* [2–4] (see, also, [8–9]). Such solutions can in earnest be viewed as particlelike modes. In this paper, we present yet another complex class of compactness inducing Lorenz-Invariant Lagrangians, but from a different angle. Whereas in [3,4] to induce compactness the conventional potential was appended with a subquadratic part, here we equip the Lorenz-Invariant Lagrangian with a shape factor. Shape factors were already employed in different contexts. In the complex Sine-Gordon equation [10–11], the shape factor enabled integrability, whereas in [12], the coordinate-depending scale factors enable to construct self-dual modification of the Skyrme model. In contradistinction, in the present work, the chosen shape factor degenerates at the ground state. The resulting local loss of solution's uniqueness is then used to enforce compactification of the resulting Q balls and Q vortices. To demonstrate the idea, the simplest possible shape factor is used. It will then become patently clear how to extend it to more realistic setups.

We start with the (x, t) domain. Let $\mathbf{Z}$ be complex in (x, t) and $\mathcal{L}$ the Lagrangian,

$$\mathcal{L} = F(|\mathbf{Z}|)|\mathbf{Z}_\alpha|^2 - P(|\mathbf{Z}|) \quad \text{where } \alpha = 1, 2, \tag{1}$$

and $\mathbf{Z}_\alpha$ denotes partial differentiation of $\mathbf{Z}$. The shape factor F will be presented shortly. From (1), we have

$$F(|\mathbf{Z}|)\partial_\alpha^2\mathbf{Z} + \frac{\bar{\mathbf{Z}}}{2|\mathbf{Z}|}F'(|\mathbf{Z}|)\mathbf{Z}_\alpha^2 + P'(|\mathbf{Z}|)\frac{\mathbf{Z}}{2|\mathbf{Z}|} = 0. \tag{2}$$

Since a Lorenz boost is available, it suffices to seek radial stationary compact Q balls and assume that $\mathbf{Z} = \exp(\imath\omega t)R(x)$. Equation (2) then takes the simple form,

$$\sqrt{F(|R|)}(\sqrt{F(|R|)}R_x)_x + \{P'_\omega(|R|) - P'(|R|)\}\frac{R}{2|R|} = 0, \tag{3}$$

and $P'_\omega(|R|) = \omega^2 F(|R|)|R|^2$. Upon one integration and dropping the constant of integration,

$$F(|R|)\{R_x^2 + \omega^2 R^2\} - P(|R|) = 0, \tag{4}$$

or

$$R_x^2 + P_{\mathrm{eff}}(|R|) = 0, \quad \text{where } P_{\mathrm{eff}} \doteq \omega^2 R^2 - \frac{P(|R|)}{F(|R|)}. \tag{5}$$

We now proceed to the crucial issue of picking a suitable shape factor F and the potential P which enable formation of compact solitary structures. Perhaps the simplest choice which exemplifies our assertion is

$$F(|R|) = |R|^{1-\delta}, \qquad 0 \le \delta < 1 \quad \text{and} \quad P = R^2(1 + \mu|R|), \tag{6}$$

which yields the effective potential P_{eff},

$$P_{\mathrm{eff}} = \omega^2 R^2 - |R|^{1+\delta}(1 + \mu|R|), \tag{7}$$

with μ being a non-negative free parameter.

We now pause for the following observations; whereas in (4), the compactification is due to the degeneracy of the shape factor F at the ground state, the reduction from (4) to (7), possible only in 1D, shifts the action to the effective

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potential $P_{\rm eff}$ which is *subquadratic* and thus supports compact structures. It may seem that one could have assumed such potential *ab initio* and thus reduce the problem to the one already addressed in [3], but the apparent equivalence holds only on the ode level of Eq. (5). The true dynamics governed by the underlying PDEs may be quite different. Moreover, as we shall see shortly, the apparent mathematical equivalence holds only in 1D. However, in spite the different origin of the effective potential insofar as the ensuing patterns are concerned, the final effect is the same: *a local loss of a solution's uniqueness* at the vicinity of the ground state enables us to "glue" the solution across the singular manifold with a vacuum solution to form a finite span entity: the compacton. Clearly, such a solution has a finite degree of smoothness, but since it is bounded, its energy is finite [1–4].

Returning to the problem at hand, we turn to solve

$$R_x^2 + \omega^2 R^2 - R^{1+\delta}(1+\mu R) = 0, \qquad (8)$$

or, using normalized coordinates, $s = \omega x, R = R_0 v$, $R_0 = 1/\omega^{\frac{2}{1-\delta}}$, and $\sigma = \mu/\omega^{\frac{2}{1-\delta}}$;

$$v_s^2 + P_{\rm eff}(v) = 0, \qquad P_{\rm eff}(v) \doteq v^2 - v^{1+\delta}(1+\sigma v). \qquad (9)$$

$P_{\rm eff}(v)$ is shown in Fig. (1) for $0 \le \sigma$, with σ_* the critical value [0.385 in Fig. (1)] and the corresponding v_* given as, $0 < \delta < 1$,

$$v_* = (1/\delta)^{1/(1-\delta)} \quad \text{and} \quad \sigma_* = v_*^{-\delta} - v_*^{-1}.$$

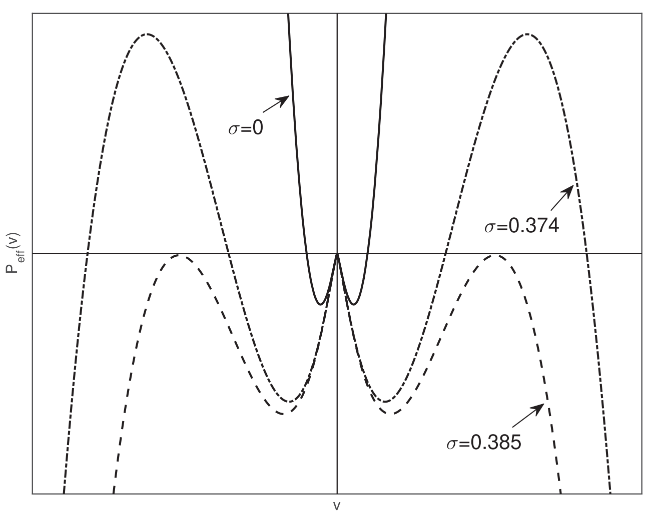

FIG. 1. A normalized effective potential $P_{\rm eff} = v^2 - |v|^{1+\delta}(1+\sigma|v|)$, $0 \le \mu$, $\delta = 1/3$. Similarly, the conventional case, in higher dimensions, its doubly humped shape for $\sigma < \sigma_*$ induces a countable sequence of multinodal compact Q balls which condense near the top; however, unlike the conventional case, the potential's subquadratic nature near the origin terminates particles motion in a "finite time," which amounts to a finite span of the resulting mode.

For compact structures to emerge, we need $\sigma < \sigma_*$ which elevates ω above a critical threshold,

$$\omega > \omega_* = \left(\frac{\mu}{\sigma_*}\right)^\gamma \quad \text{where } \gamma = \frac{1-\delta}{2}. \qquad (10)$$

When $\delta = 0$, we assume $\omega > \sqrt{\mu}$ and define $\kappa^2 = \omega^2 - \mu$. The resulting compacton solution,

$$R = \frac{1}{\kappa^2}\cos^2\left(\frac{\kappa x}{2}\right)H(\pi - \kappa|x|), \qquad (11)$$

and vanishes elsewhere. Similarly, for $\sigma = 0$,

$$R = \frac{1}{\omega^\gamma}\cos^{\frac{1}{\gamma}}(\omega\gamma x)H(\pi - 2\omega\gamma|x|). \qquad (12)$$

For $\delta \ne 0$ and $\sigma > 0$, one has to solve

$$s = \frac{1}{\gamma}\int_0^{v^\gamma}\frac{du}{\sqrt{1+\sigma u^{1/\gamma} - u^2}}, \qquad (13)$$

which has a finite span and in a number of cases may be expressed explicitly in terms of elliptic functions. In any case, what matters is not the explicit form but the fact that near the origin $v \sim s^{\frac{2}{1-\delta}}$, integral (13) converges and yields a finite support. As noted, a solution's compactness implies its nonanalyticity (otherwise, it would have to extend indefinitely). However, since both the solution and its gradient are bounded, its finite span implies finite energy. The same will be shortly seen to hold in higher dimensions.

Finally, since the presented solutions are in earnest breathers, we note that for $\mu < 0$ strictly static solutions are available. Proceeding as before, we readily obtain

$$\gamma x = \int_0^{R^\gamma}\frac{dw}{\sqrt{1 - |\mu| w^{1/\gamma}}}, \qquad (14)$$

which for $\delta = 0$ yields

$$R(x) = \frac{1}{\sqrt{|\mu|}}\cos^2(2\sqrt{|\mu|}x)H(\pi - 4\sqrt{|\mu|}x). \qquad (15)$$

II. COMPACT Q BALLS AND Q VORTICES

As aforementioned, in N dimensions, $N = 2, 3$, the shortcut toward solution used in 1D is no longer available, and we have to address directly the underlying PDE. Let $\tilde{x}\epsilon \mathsf{R}^N$ and rewrite Eq. (2),

$$F(|\mathbf{Z}|)(\mathbf{Z}_{tt} - \nabla^2\mathbf{Z}) + \frac{\bar{\mathbf{Z}}}{2|\mathbf{Z}|}(\mathbf{Z}_t{}^2 - (\nabla\mathbf{Z}^2))F'(|\mathbf{Z}|) - \frac{\mathbf{Z}}{2|\mathbf{Z}|}P'(|\mathbf{Z}|) = 0. \quad (16)$$

Let $|\mathbf{Z}| = \exp(\imath\omega t)R(r)$, where $r = |\tilde{x}|$, then

$$\sqrt{F(|R|)}\nabla_r \cdot (\sqrt{F(|R|)}\nabla_r R) + \{P'_\omega(|R|) - P'(|R|)\}\frac{R}{2|R|} = 0, \quad (17)$$

which amounts to $x \to r$ in Eq. (3) + the N-dimensional Laplacian trace $\frac{(N-1)}{r}R_r F(|R|)$. A formal integration then yields

$$F(|R|)\{R_r^2 + \omega^2R^2 - P_{\mathrm{eff}}(|R|)\} = -2(N-1)\int F(|R|)R_r^2\frac{dr}{r} + 2E_0, \quad (18)$$

where $E_0 = \mathrm{const}$.

As in conventional cases [5–7], given the choice of the potential P and the shape factor F in (6), the doubly humped effective potential, see Fig. (1) for details, assures a countable sequence of flat top multinodal solutions condensing near the top, provided that the vibration frequency ω is held above the critical threshold ω_*. This mimics the standard procedure with a crucial distinction: the degeneracy of F and the resulting subquadratic potential assure a finite span of the pattern. The compactons finite span becomes crucial when their interactions are considered; for unlike the conventional Q balls, they are oblivious of each other up to the moment of their contact. For the cases studied in [3], interactions in the relativistic range were ballistic and quite clean, whereas interactions at low speeds exhibited fission-fusion features [2–3]. The nature of interactions in the present case has yet to be studied.

Q-Vortices. To construct compact planar vortices, we let $N = 2$ and $|\mathbf{Z}| = \exp[\imath(\omega t + m\theta)]R(r)$. Since we limit ourself to the simple choice (6) and an even $\gamma = p/q$, we may further simplify the description via $u = R^{\gamma+1}$, to obtain

$$\frac{d^2u}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{du}{d\tilde{r}} + \left(\varpi^2 - \frac{\tilde{m}^2}{\tilde{r}^2}\right)u - u^{\gamma_*}\left(1 + \frac{3\mu}{2}|u|^{\frac{1}{1+\gamma}}\right) = 0, \quad (19)$$

where $\tilde{r} = r\sqrt{1+\gamma}$, $\varpi = \omega\sqrt{1+\gamma}$, $\tilde{m} = m\sqrt{1+\gamma}$ and

$$\gamma_* = \frac{1-\gamma}{1+\gamma} = \frac{1+\delta}{3-\delta} < 1. \quad (20)$$

Note that whereas m is integer, $\tilde{m}$ is not. Since $\gamma_* < 1$, ode (19) is sublinear. The simplicity of the assumed shape plays further into our hands, for the reduced Eq. (19) was recently addressed by us in a different context; see [2] or [4], with a particular attention paid to a detailed characterization of the plethora of compact patterns which Eq. (19) supports. Thus, in principle, we could delegate the reader there, but to round off the presentation, we shall briefly outline the landscape of compact vortex rings detailed in [4]. To this end, consider the power expansion around a vortex's center at the origin,

$$u \simeq u_0\tilde{r}^{\tilde{m}} + u_1\tilde{r}^{2+\tilde{m}\gamma_*},$$

viable only if $\tilde{m} < 2/(1-\gamma_*)$ or, in terms of original parameters, if

$$m < m_\infty \doteq \frac{\sqrt{1+\gamma}}{\gamma} = \frac{\sqrt{2(3-\delta)}}{1-\delta} \geq \sqrt{6}. \quad (21)$$

Notably, the only modes that satisfy (20) for all δs are $m = 1,\ 2$. This delineates three distinctive regimes of azimuthal modes. For $m = 1$ and $m = 2$, all vortices, whether uninodal or multinodal have a sharp finite span and extend to the origin. On the other end, if for a given δ, $m_\infty(\delta) < m$, then all corresponding uni- and multinodal solutions have finite inner and outer radii and are thus *genuine radial rings*.

Finally, in the intermediate regime wherein $2 < m < m_\infty(\delta)$, the plot thickens. Here, for a given δ, we find a sequence $2 < m_1 < m_2 < \ldots < m_\infty$ and a corresponding sequence of solutions such that

(1) As in the first case, for $m < m_1$, there exists a countable set of finite span multinodal vortices which start at the origin.
(2) For $m_1 \leq m < m_2$, the uninodal vortex solution is detached from the origin and thus turns into a true radial ring, but higher nodal compact vortices extend to the origin.
(3) For $m_k \leq m < m_{k+1}$, one finds k-rings Q vortices (a uninodal, two nodal, up to a k-nodal vortex), whereas all $k + j,\ j = 1, 2\ldots$ nodal vortices extend to the origin.
(4) The k-nodal ring vortex is nested within the $k + 1$ vortex. As a by-product of their embedding, the high k's ring vortices become very wide with the inner radius coming very close to the origin, whereas ring's outer radius becomes very large.

Note that since the physically admissible m's are integers, depending on the chosen δ, some of the described mathematical features may be not realized. A far more detailed exposition of these features accompanied by their approximate analytical description and a graphical display is presented in [2,4].

III. SUMMARY

Using a simple shape factor to append a Lorentz-Invariant Lagrangian, we have demonstrated how one complex scalar field may induce compact Q balls and compact ring vortices. The assumed form of the Lagrangian is complimentary to our earlier work wherein the compactification was due to an *ab initio* assumed subquadratic site potential.

Similarly to the conventional theory [5–7], the efforts to unfold multidimensional patterns is richly rewarded with the single 1D mode, whether soliton or compacton, turning into *a whole spectrum of multinodal modes and vortices*. Moreover, both our earlier and the present work reveal that whenever compact structures are admissible additional structures, unavailable in the conventional cases, of both radial and nonradial nature may join the gallery of patterns [1–4]. Notable among those are multinodal ring vortices of finite radii. Both the assumed shape factor and the employed potential were perhaps the simplest forms that support formation of compact Q balls and Q vortices. Far more general forms could be employed to achieve more realism. With the only essential part being the nature of shape-factors degeneracy at the ground state.

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