Exact Foldy-Wouthuysen transformation for a Dirac theory revisited

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The exact Foldy-Wouthuysen transformation method is generalized here. In principle, it is not possible to construct the exact Foldy-Wouthuysen transformation for any Hamiltonian. The transformation conditions are the same, but the involution operator has a new form. We took a particular example and constructed explicitly the new involution operator that allows one to perform the transformation. We treat the case of the Hamiltonian with 160 possible CPT-Lorentz breaking terms, using this new technique. The transformation was performed, and physics analysis of the equations of motion is shown.

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I. INTRODUCTION

The study of the possible candidates to break CPT-Lorentz symmetry is very important nowadays [1]. There are a large number of studies over the last ten years that show the possible experiments that could give the more prominent physical effect to measure one of these fields [2]. Until now, none of them has been directly observed. The most prominent theoretical approaches that consider these cases are based on indirect physical effects, as is shown in Refs. [3,4]. In other words, the search for these manifestations starts with an action that considers at least two independent fields, as one can see in recent papers [5]. For the nonrelativistic scenario, the results are well established in Refs. [6–9], for the torsion field, for example. It is very interesting to see [10] that the torsion field could be generated from the symmetry breaking. Some recent theoretical studies have been developed with the same phenomenological background [11–14]. In Ref. [15], a relativistic description of a Dirac particle in the torsion field has been fulfilled.

Another possible phenomenological approach to this problem can be constructed step by step by searching for new terms in the Hamiltonian that describes this situation. Thinking this way, it is possible to find, in the transformed Hamiltonian, a term with explicit mix between a known external field and one of the CPT-Lorentz terms. The most interesting scenario is found when the external field has an amplitude which is big enough to compensate the weakness of the CPT-Lorentz term.

The idea is the same as that shown in Ref. [16], in which the strong magnetic field could, in principle, change the trajectory of the Dirac particle that interacts with gravitational waves. It is important to take into account the corrections, made with canonical Foldy-Wouthuysen transformation (FWT), to these results that are shown in Ref. [17]. The massive linearized gravity is studied in Ref. [18], and the general relativistic description of a Dirac particle in a gravitational wave and a magnetic field has been carried out (with canonical FWT) in Ref. [19], wherein some possible experiments that could measure indirect effects of gravitational waves on Dirac fermions are indicated. However, solving the Dirac equation for the general case is not a simple procedure [20]. It is well known in the literature that working with the exact Foldy-Wouthuysen transformation (EFWT) is a more prominent approach to interpreting a Dirac Hamiltonian than the canonical transformation [21,22]. But this is true not only for the fact that it can give us new terms but also that it is a faster and more economic (in terms of algebraic calculation) procedure [16,23–25]. One can see this transformation as an improvement of the usual FWT.

Let us perform a comparison of the two procedures. It is possible to see that in the usual FWT the multiplication on each step (on each order on $1/m$) by the term that makes the Hamiltonian even generates a maximum of $1+2n$ even terms, where n represents the number of terms of the previous Hamiltonian (see, for example, pp. 48–51 in

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Ref. [26]). The maximum number of terms in the nth Hamiltonian is straightforwardly obtained by the fact that this is an expansion in power series of an operator. The factor 2 on $1 + 2n$ expression is obtained in case in which it does not commute with all original terms.

On the other hand, the EFWTs impose the multiplication of all terms of the Hamiltonian by themselves. Analogous arguments give us the maximum of $1 + 2n^2$ on the expanded Hamiltonian. If the parameter of expansion here is also taken to be $1/m$, one can see that the possibility of having new terms in comparison with the usual method is greater. In many particular known cases [24,27,28], the anticommutators on both cases are such that the results are the same. But it is not the general case. This was explicitly shown in Ref. [21]. In this paper, we show another case in which it happens.

In Ref. [29], the author performs in a very didactic way the formal comparison between the two methods. He also describes which is the most efficient method for each possible application. The explicit calculations are performed in the series of three works in which the generality for the exact procedure becomes evident [30–32].

References [33–35] are a series of papers in which the EFWT conditions are not satisfied. In these articles, the study of the CPT-Lorentz violating terms is used as a background for this transformation. It is possible to see in Ref. [36] the diagonalized Hamiltonian for all possible terms that allows this procedure.

Using the result of Ref. [35], we develop an algorithm to construct a generalized involution operator for the EFWT. We show a method to construct the explicit form of the operator that allows the Hamiltonian to be diagonalized. In some sense, the logic here is inverse: we do not test if it is possible to perform the EFWT, but we search for the operator that gives us this possibility.

By showing the explicit analytic form of this operator, the EFWT usual algorithm can be applied to the initial Hamiltonian. We construct the general operator, and the complete case of CPT-Lorentz interacting with the Dirac field [20] is studied here using the EFWT technique. We also compare the result with the usual transformation, and two new terms show up.

II. COMPLETE HAMILTONIAN FOR A DIRAC THEORY WITH *CPT*-LORENTZ INVARIANCE VIOLATION

In Ref. [25], the authors present a table that specifies the 80 cases of CPT and Lorentz violating terms in the modified Dirac equation. A complete study of the EFWT, taking into account these 80 cases, is presented in Ref. [36].

However, it should be noted that a sort of terms was not considered in Refs. [25,36]. To perform the EFWT study of the complete set of cases, it is necessary for the Hamiltonian to admit the involution operator [21,24,27, 28,36]. In this work, we present a new table corresponding to all the CPT-Lorentz breaking terms. The main point is the search for an involution operator J, which satisfies the anticommutation relation,

$$JH + HJ = 0, \tag{1}$$

for the complete set of terms, presented in Table I.

The quantities a_μ, b_μ, m_5, $c^{\mu\nu}$, $d^{\mu\nu}$, e^μ, f^μ, $g^{\mu\nu\lambda}$, and $H_{\mu\nu}$ represent the CPT-Lorentz violating parameters [37–39]. We adopt notations as described in Ref. [26] for Dirac matrices and the useful notations for P_i, used in Ref. [25].

The terms highlighted in bold font in Table I, correspond to the empty spaces, in the table presented in Ref. [25]. These terms do not obey the anticommutation relation (1), if one takes into account the following form of the involution operator:

$$J = i\gamma^5\gamma^0. \tag{2}$$

The set of terms that obey relation (1), considering the involution operator (2), is presented in Ref. [25]. From now

TABLE I. Interaction coefficients.

	m $P^*_\nu e^\nu$	a_l $P^*_\nu c^{l\nu}$ $\bar{P}_l$	b_0 $P^*_\nu d^{0\nu}$	H^{lj} $P^*_\nu g^{lj\nu}$	m_5 $P^*_\nu f^\nu$	b_l $P^*_\nu d^{l\nu}$	a_0 $P^*_\nu c^{0\nu}$ $\bar{P}_0$	$H^{0\mu}$ $P^*_\nu g^{0\mu\nu}$
γ^0	1	γ^l	$-\gamma^0\gamma^5$	$\frac{1}{2}\sigma^{lj}$	$\boldsymbol{i\gamma_5}$	$\boldsymbol{\gamma_5\gamma_l}$	$\boldsymbol{\gamma_0}$	$\boldsymbol{\frac{1}{2}\sigma^{0\mu}}$
c^{00}	$-\gamma^0$	$-\alpha^l$	γ^5	$-\frac{1}{2}\gamma^0\sigma^{lj}$	$\boldsymbol{-i\gamma_0\gamma_5}$	$\boldsymbol{\gamma_5\alpha_l}$	$\boldsymbol{-1}$	$\boldsymbol{-\frac{1}{2}\gamma^0\sigma^{0\mu}}$
f^0	$i\gamma^5$	$i\gamma^5\gamma^l$	$i\gamma^0$	$\frac{i}{2}\gamma^5\sigma^{lj}$	$\boldsymbol{-1}$	$\boldsymbol{i\gamma^l}$	$\boldsymbol{i\gamma^5\gamma^0}$	$\boldsymbol{\frac{i}{2}\gamma^5\sigma^{0\mu}}$
d^{i0}	$-i\gamma^i\gamma^5$	$i\gamma^i\gamma^5\gamma^l$	$-\alpha^i$	$-\frac{1}{2}\gamma^i\gamma^5\sigma^{lj}$	$\boldsymbol{i\gamma^i}$	$\boldsymbol{\gamma^i\gamma^l}$	$\boldsymbol{\gamma^5}\alpha^i$	$\boldsymbol{\frac{1}{2}\gamma^i\gamma^5\sigma^{0\mu}}$
g^{i00}	$-\frac{i}{2}\alpha^i$	$-\frac{i}{2}\alpha^i\gamma^l$	$-\frac{i}{2}\gamma^i\gamma^5$	$-\frac{1}{4}\alpha^i\sigma^{lj}$	$\boldsymbol{\frac{1}{2}}\alpha^i\boldsymbol{\gamma_5}$	$\boldsymbol{-\frac{i}{2}\alpha^i\gamma_5\gamma_l}$	$\boldsymbol{-\frac{i}{2}\alpha^i\gamma_0}$	$\boldsymbol{-\frac{i}{4}\alpha^i\sigma^{0\mu}}$
d^{00}	$\boldsymbol{-\gamma^0\gamma^5}$	$\boldsymbol{\gamma^5\alpha^l}$	$\boldsymbol{-1}$	$\boldsymbol{\frac{1}{2}\gamma_5\gamma_0\sigma^{lj}}$	$-i\gamma^0$	$-\alpha^l$	$-\gamma^5$	$-\frac{1}{2}\sigma^{0\mu}\gamma^0\gamma^5$
e^0	$\boldsymbol{-1}$	$\boldsymbol{-\gamma^l}$	$\boldsymbol{-\gamma^5\gamma^0}$	$\boldsymbol{-\frac{1}{2}\sigma^{lj}}$	$-i\gamma^5$	$-\gamma^5\gamma^l$	$-\gamma^0$	$-\frac{1}{2}\sigma^{0\mu}$
c^{i0}	$\boldsymbol{\gamma^i}$	$\boldsymbol{\gamma^i\gamma^l}$	$\boldsymbol{\gamma^5}\alpha^i$	$\boldsymbol{\frac{1}{2}\gamma^i\sigma^{lj}}$	$i\gamma^i\gamma^5$	$-i\gamma^i\gamma^5\gamma^l$	$-\alpha^i$	$\frac{1}{2}\gamma^i\sigma^{0\mu}$
g^{ik0}	$\boldsymbol{-\frac{1}{2}\sigma^{ik}}$	$\boldsymbol{-\frac{1}{2}\sigma^{ik}\gamma^l}$	$\boldsymbol{\frac{1}{2}\sigma^{ik}\gamma^0\gamma^5}$	$\boldsymbol{-\frac{1}{4}\sigma^{ik}\sigma^{lj}}$	$-\frac{i}{2}\sigma^{ik}\gamma^5$	$-\frac{1}{2}\sigma^{ik}\gamma^5\gamma^l$	$-\frac{1}{2}\sigma^{ik}\gamma^0$	$-\frac{1}{4}\sigma^{ij}\sigma^{0\mu}$

on, we shall call the quantities in bold font new terms and the quantities that are not in bold font old terms.

To understand how the Hamiltonian can be obtained, directly from Table I, let us present a simple example. The rule is based on the product of the line terms by the terms in the rows. We shall consider, for instance, the first line times the first row: $\gamma^0 \times 1 \times m = \gamma^0 m$. We get, in this case, the free Dirac equation term, which is the most trivial one.

Let us consider another example. The product of the sixth line by the first row. The terms inside Table I must also be taken into account. Such multiplication gives two terms:

$$d^{00} \times (-\gamma^0\gamma^5) \times m = -md^{00}\gamma^0\gamma^5 \quad \text{and}$$
$$d^{00} \times (-\gamma^0\gamma^5) \times P^*_\nu e^\nu = -md^{00}\gamma^0\gamma^5 P^*_\nu e^\nu. \tag{3}$$

Observe that both of them break C, P, PT, and CT [2,39]. It is remarkable to say that the study of this kind of terms, with EFWT considerations, depends on the correct choice of the involution operator, such that relation (1) is contemplated.

The general form of the involution operator [24,40] has the structure

$$\hat{J} = M \times \hat{F}, \tag{4}$$

where M and $\hat{F}$ are operators. They act on the matrices and functions space, respectively. In particular, the choices $M = i\gamma^5\gamma^0$ and $\hat{F} = \hat{1}$ corresponds to the usual operator used in previous works [25,36]. However, as already mentioned above, the new terms in Table I do not satisfy the anticommutation relation (1) for such a choice. The main point here is the following: the choice of an appropriate involution operator, for a specific term of Table I, involves the knowledge of exactly what symmetry is being broken (for each term of Table I).

An interesting case is the vectorial part of the torsion field, b_l. As one can check [2,39], this term breaks T, CT, PT, and CPT. In the Hamiltonian, the torsion field is founded by the product of line 0 by row 6. It gives $b_l\gamma^0\gamma^5\gamma^l$. It has been shown that $M = i\gamma^5\gamma^0$ and $\hat{F} = T$ represents a specific choice for the involution operator, such that the anticommutation relation is obeyed [36]. However, it is not the only possible choice. In particular, $\hat{F} = CT$, $\hat{F} = PT$, and $\hat{F} = CPT$ would work equally well.

To perform the EFWT for all the terms in Table I, we present in the next section a proposal of a new involution operator that anticommutes with all the terms of Table I.

III. INVOLUTION OPERATOR, AN APPROPRIATE CHOICE

We begin with an appropriate representation of the Hamiltonian with CPT-Lorentz breaking terms:

$$\mathcal{H} = \phi_1^A H_{AB} \phi_2^B. \tag{5}$$

Throughout this paper, the quantities with latin indices A and B are only associated with possible positions in Table I. We would like to emphasize that these indices are not space-time indices. The possible values for A are running horizontally in Table I, from 0 to 8. In the case of B, the possible values are running vertically, from 0 to 9. In addition, the quantities ϕ_1^A and ϕ_2^B are the fields that appear in the top row and in the left column of Table I, respectively, and the quantity H_{AB} represents the terms contained in the cells of Table I.

As shall be better understood in the next section, the EFWT works if, and only if, one can write the Hamiltonian in the form of Eq. (5). It may seem cumbersome, at first sight, but it is not. Let us consider an example. The choice $A = 0$ and $B = 6$ (first column and the seventh row, respectively) leads us to $\phi_1^0 = m + P^*_\nu e^\nu$, $\phi_2^6 = d_{00}$, and $H_{0,6} = -\gamma^0\gamma^5$. It gives exactly the two terms described in Eq. (3).

Taking into account these considerations, we present, as a next step, an involution operator that anticommutes with the complete set of terms of the Hamiltonian (5),

$$\hat{J} = (i\gamma^5\gamma^0) \times (C^{\mathcal{O}'_{AB}} P^{\mathcal{O}''_{AB}} T^{\mathcal{O}'''_{AB}})^{\theta_{IK}}, \tag{6}$$

where C, P, and T are the known charge, parity, and time operators, respectively [2,39]. Observe that Eq. (6) obeys the structure of Eq. (4), with the M and $\hat{F}$ choice

$$M = i\gamma^5\gamma^0 \quad \text{and} \quad \hat{F} = (C^{\mathcal{O}'_{AB}} P^{\mathcal{O}''_{AB}} T^{\mathcal{O}'''_{AB}})^{\theta_{IK}}, \tag{7}$$

where we define

$$I = A - 5 \quad \text{and} \quad K = B - 6. \tag{8}$$

The quantity θ_{IK} is defined in order to assume the value 0 or 1. If $I \times K > 0$, $\theta_{IK} = 0$. On the other hand, if $I \times K < 0$, $\theta_{IK} = 1$. Actually, the product between I and K tells us if we are dealing with the new or old terms of Table I. The quantities $\mathcal{O}_{AB}$ also assume the 0 or 1 value. They are determined by the previous knowledge of which symmetry is being broken.

Let us consider an example, by setting $A = 6$ and $B = 0$. Then, $\phi_1^6 = b_l + P^*_\nu d^{ln}$, $\phi_2^0 = \gamma^0$, and $H_{6,0} = \gamma^5\gamma^l$. According to Eq. (5), the Hamiltonian for this case is given by

$$\mathcal{H} = \gamma^0\gamma^5\gamma^l (b_l + P^*_\nu d^{ln}). \tag{9}$$

The next step is the choice of the $\mathcal{O}_{AB}$ quantities. In Refs. [2,39], there is a table with the properties of operators for Lorentz violation in QED. According to this table, one can consider that $\mathcal{O}''_{AB} = 0$ and $\mathcal{O}'_{AB} = \mathcal{O}'''_{AB} = 1$. Observe that from Eq. (8) $I = 1$, $K = -6$, and $I \times K = -6$;

for this reason, we have $\theta_{6,0} = 1$. With these considerations, the corresponding involution operator is

$$\hat{J} = i\gamma^5\gamma^0 PT. \tag{10}$$

As one can check, the anticommutation relation is obeyed when the quantities $\mathcal{H}$ and $\hat{J}$ are described by the relations (9) and (10), respectively.

One can see that for the old terms of Table I the product between I and K is always positive and the quantity θ in Eq. (6) is equal to zero. Consequently, in what concerns the old part of Table I, we shall have, as expected, $M = i\gamma^5\gamma^0$ and $\hat{F} = \hat{1}$.

IV. EXACT TRANSFORMATION WITH *CPT*

We present in this section the EFWT of the Hamiltonian for a free spin-1/2 Dirac fermion Ψ of mass m in the standard model extension [3,20]. Let us begin with the following Hamiltonian:

$$\begin{aligned}\mathcal{H} = m\left(\gamma^0 - \gamma^0 c_{00} - e_0 - d_{j0}\gamma^5\gamma^j + \frac{1}{2}g_{ik0}\sigma^{ik}\right)\\
+ P^k\Big(-\alpha_k + 2d_{0k}\gamma^5 - c_{jk}\alpha^j + c_{00}\alpha_k + if_k\gamma^5\gamma^0\\
- 2g_{0jk}\gamma^0\sigma^{0j} - 2c_{0k} + d_{jk}\gamma^5\alpha^j - d_{00}\gamma^5\alpha_k\\
- e_k\gamma^0 + \frac{1}{2}\gamma^0\sigma^{ij}g_{ijk} - ig_{i00}\gamma_k\alpha^i\Big) + a_j\alpha^j\\
- b_0\gamma^5 + iH_{0j}\gamma^j - b_j\gamma^5\alpha^j - \frac{1}{2}\gamma^0\sigma^{ij}H_{ij}.\end{aligned} \tag{11}$$

This Hamiltonian can be constructed directly from Table I presented in the last section. However, it is not the most complete Hamiltonian that one can extract from Table I. The main point of this work is the development of the operator described in (7). As it is being used for the first time, it is worthwhile to deal with a Hamiltonian which we could know at least the qualitative diagonalized result. On the other hand, it would be very interesting from the physical point of view if the new EFWT generates unexpected terms in comparison with the usual transformation for the same action. We decide to pick just the terms represented in Eq. (11) because in Ref. [20] the authors perform the usual FWT, taking into account this Hamiltonian. Performing the transformation for it, we could validate our algorithm and also search for physical quantities mixed in a new form. The transformed Hamiltonian (with usual FWT) is the [41]

$$\tilde{H}^{\mathrm{tr}} = \beta m + \frac{1}{2m}\{(1+\tilde{A})[(\delta_{ij}+\tilde{B}_{ij})\bar{P}^i + \tilde{C}_j]^2 + \tilde{D}\}, \tag{12}$$

where

$$\begin{aligned}\tilde{A} &= -2c_{00}\gamma^0,\\
\tilde{B}_{ij} &= \frac{1}{2}[4(d_{0i}+d_{i0})\gamma^5\gamma^j - 4c_{ij}\gamma^0\\
&\quad + 4\epsilon^l{}_{mj}(g_{l0i}+g_{li0})\gamma^5\gamma^0\gamma^m],\\
\tilde{C}_j &= \frac{1}{2}[-4m(c_{0j}+c_{j0}) + 4md_{ij}\gamma^5\gamma^0\gamma^i - 4md_{00}\gamma^5\gamma^0\gamma^j\\
&\quad - 4me_j\gamma^0 + 2m\epsilon^{kl}{}_m g_{klj}\gamma^5\gamma^m - 4m\epsilon^{ij}{}_l g_{i00}\gamma^5\gamma^l\\
&\quad + 4a_j\gamma^0 - 4b_0\gamma^5\gamma^j + 4\epsilon^{jk}{}_l H_{0k}\gamma^5\gamma^0\gamma^l],\\
\tilde{D} &= -2m^2c_{00}\gamma^0 - 2m^2e_0 - 2m^2d_{j0}\gamma^5\gamma^j\\
&\quad - m^2\epsilon^{ik}{}_l g_{ik0}\gamma^5\gamma^0\gamma^l + 2ma_0\\
&\quad - 2mb_j\gamma^5\gamma^0\gamma^j + m\epsilon^{ij}{}_l H_{ij}\gamma^5\gamma^l.\end{aligned} \tag{13}$$

Besides that EWFT is more economic in algebra, it presents more detailed information with respect to the nonrelativistic approximation [42–45].

For a first step in performing the EFWT, we calculate the squared Hamiltonian $\mathcal{H}^2$. To simplify the algebra, we shall write this quantity as

$$\mathcal{H}^2 = m^2\left(1 + \frac{\bar{\mathcal{H}}^2}{m^2}\right), \tag{14}$$

where $\bar{\mathcal{H}}^2$ is given by

$$\bar{\mathcal{H}}^2 = (1+\bar{A})[(\delta_{ij}+\bar{B}_{ij})\bar{P}^i + \bar{C}_j]^2 + \bar{D}. \tag{15}$$

The quantities $\bar{A}$, $\bar{B}_{ij}$, $\bar{C}_j$, and $\bar{D}$ are written in the form

$$\begin{aligned}\bar{A} &= -2c_{00} - d_{00}\gamma^5 + 2ig_{i00}\gamma^0\alpha^i,\\
\bar{B}_{ij} &= \frac{1}{2}[-8d_{0i}\gamma^5\alpha_j - 4c_{ij} + 8g_{0li}\gamma^0\epsilon^{jlm}\Sigma_m\\
&\quad + 8c_{0i}\alpha_j + 4d_{ij}\gamma^5 + 4g_{lmi}\epsilon^{lmj}\gamma^0\gamma^5\\
&\quad + 4ig_{ilj}\gamma^0\gamma^5\Sigma^l + 4ig_{i00}\gamma^0\alpha_j],\\
\bar{C}_j &= \frac{1}{2}[-8m\gamma^0c_{0j} + 4md_{ij}\gamma^0\gamma^5\alpha^i - 4md_{00}\gamma^0\gamma^5\alpha_j\\
&\quad - 4me_j + 2mg_{klj}\sigma^{kl} - 4img_{j00} - 4mg_{i00}\epsilon^{ijl}\Sigma_l\\
&\quad + 4me_0\alpha_j + 4imd_{k0}\gamma^0\gamma^5\epsilon^{jkl}\Sigma_l - 2mg_{il0}\epsilon^{ilj}\gamma^5\\
&\quad + 4a_j + 4b_0\gamma^5\alpha_j - 4H_{0k}\gamma^0\epsilon^{jkl}\Sigma_l - 4a_0\alpha_j\\
&\quad - 4b_j\gamma^5 - 4H_{kl}\epsilon^{klj}\gamma^0\gamma^5 + 4iH_{lj}\gamma^0\gamma^5\Sigma^l],\\
\bar{D} &= -2m^2c_{00} - 2m^2\gamma^0e_0 + 2m^2d_{j0}\gamma^5\alpha^j\\
&\quad + m^2\gamma^0\sigma^{ik}g_{ik0} + 2m\gamma^0a_0 - 2m\gamma^0\gamma^5\alpha^jb_j - m\sigma^{ij}H_{ij}\\
&\quad + (1 - 2c_{00} + 2d_{00}\gamma^5 - 2ig_{i00}\gamma^0\alpha^i)\frac{i\hbar e}{mc}\Sigma_kB^k.\end{aligned} \tag{16}$$

There are, in the last equation, even and odd terms. In the FW context, even and odd operators are written as

$$M_{(\mathrm{EVEN})} = \frac{1}{2}(M + \gamma^0 M \gamma^0)$$
$$M_{(\mathrm{ODD})} = \frac{1}{2}(M - \gamma^0 M \gamma^0). \tag{17}$$

In the situation in which there are many odd terms, one must take into account the relation [36]

$$\mathcal{H}^{\mathrm{tr}} = \hat{J}\frac{1}{2}(\sqrt{\mathcal{H}^2} - \gamma^0\sqrt{\mathcal{H}^2}\gamma^0) + \gamma^0\frac{1}{2}(\sqrt{\mathcal{H}^2} + \gamma^0\sqrt{\mathcal{H}^2}\gamma^0), \tag{18}$$

where $\hat{J}$ is given by Eq. (6). The transformed Hamiltonian is denoted by $\mathcal{H}^{\mathrm{tr}}$, which presents only even terms. For this reason, $\mathcal{H}^{\mathrm{tr}}$ does not mix spinor components. Naturally, the calculation of $\sqrt{\mathcal{H}^2}$ should be performed, and the result must be inserted in Eq. (18). Let us consider that $m^2 \gg \bar{\mathcal{H}}^2$ in Eq. (14), such that

$$\sqrt{\mathcal{H}} = m\left(1 + \frac{\bar{\mathcal{H}}^2}{2m^2}\right). \tag{19}$$

After some algebra, the transformed Hamiltonian is given by

$$\mathcal{H}^{\mathrm{tr}} = \gamma^0 m + \frac{1}{2m}\{(1 + A^{\mathrm{tr}})[(\delta_{ij} + B^{\mathrm{tr}}_{ij})\bar{P}^i + C^{\mathrm{tr}}_j] + D^{\mathrm{tr}}\}, \tag{20}$$

where

$$\begin{aligned}
A^{\mathrm{tr}} &= -2\gamma^0 c_{00} - 2i\gamma^0 d_{00} + 2g_{i00}\Sigma^i,\\
B^{\mathrm{tr}}_{ij} &= \frac{1}{2}[8d_{0i}\gamma^0\Sigma_j - 4\gamma_0 c_{ij} - 8g_{0li}\epsilon^{jlm}\Sigma_m - 8ic_{0i}\gamma^0\Sigma_j\\
&\quad + 4i\gamma^0 d_{ij} + 4ig_{lmi}\epsilon^{lmj} - 4g_{ilj}\Sigma^l + 4g_{i00}\Sigma^i],\\
C^{\mathrm{tr}}_j &= \frac{1}{2}[8mc_{0j} + 4md_{ij}\Sigma^i - 4md_{00}\Sigma_j - 4m\gamma^0 e_j\\
&\quad + 2mg_{klj}\gamma^0\epsilon^{klm}\Sigma_m - 4img_{j00}\gamma^0 - 4mg_{i00}\gamma^0\epsilon^{ijl}\Sigma_l\\
&\quad - 4ime_0\gamma^0\Sigma_j - 4md_{k0}\epsilon^{jkl}\Sigma_l - 2img_{il0}\epsilon^{ilj}\gamma^0\\
&\quad + 4\gamma^0 a_j - 4b_0\gamma^0\Sigma_j + 4H_{0k}\epsilon^{jkl}\Sigma_l + 4ia_0\gamma^0\Sigma_j\\
&\quad - 4ib_j\gamma^0 - 4i\epsilon^{klj}H_{kl} - 4H_{lj}\Sigma^l],\\
D^{\mathrm{tr}} &= -2m^2\gamma^0 c_{00} + 2m^2 e_0 - 2m^2 d_{j0}\gamma^0\Sigma^j\\
&\quad - m^2 g_{ik0}\epsilon^{ikl}\Sigma_l - 2ma_0 - 2mb_j\Sigma^j - m\gamma^0\epsilon^{ijl}H_{ij}\Sigma_l\\
&\quad + \gamma^0(1 + 2c_{00} + 2id_{00} - 2g_{i00}\gamma^0\Sigma^i)\frac{i\hbar e}{mc}\Sigma_k B^k.
\end{aligned} \tag{21}$$

We have considered Eq. (11) as the starting point, in order to obtain the transformed Hamiltonian (21). It is possible to see that there are nine new terms in (21) when compared to (13). The new terms are one in the quantity $\bar{A}$ related to the coefficient d_{00}; two in $\bar{B}_{ij}$ related to the coefficients c_{0i} and d_{ij}; four in $\bar{C}_j$ related to the coefficients a_0, b_j, e_0, and g_{il0}; and two terms in $\bar{D}$ related to the coefficients c_{00} and d_{00} with the magnetic field. Nevertheless, the exact process has some advantages when compared to the usual one, as commented above. For instance, the new terms that appear in D are relevant when the bound state of the theory is considered.

V. BOUND STATE OF THE THEORY

The determination of which kind of experimental tests, like the Penning trap, clock-comparison, torsion pendulum, and others (see Refs. [46–50], and references cited therein), has a significant relevance in the scope of the standard model extension (SME) [3]. To determine the kind of experimental test that should be performed, considering the CPT-Lorentz violation terms, presented in the Dirac equation, one should derive the bound state of the theory. In Ref. [2], the authors present a table with a set of many possible bound states. It is expected that the bound associated with transformed Hamiltonian (21) could be found in such a table. Hence, with the knowledge of the bound, in combination with its magnitude and the original Hamiltonian, one can determine the kind of appropriate experimental test should be performed (see Refs. [3,51–53] for a theoretical framework about CPT-Lorentz breaking tests).

In this section, we derive the bound state of the Hamiltonian (21). Let us begin by taking into account the two-component spinor

$$\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}\exp^{-imt}. \tag{22}$$

From this point, one can write, after some algebra, the Dirac equation in the Schrödinger form $i\partial_t\psi = \mathcal{H}\psi$. With these considerations, the Hamiltonian to ϕ is written as

$$\mathcal{H} = \frac{1}{2m}\{(1 + A)[(\delta_{ij} + B_{ij})\bar{P}^i + C_j]^2 + D\}, \tag{23}$$

where

$$
\begin{aligned}
A &= -2c_{00} - 2id_{00} + 2g_{i00}\sigma^i, \\
B_{ij} &= 4d_{0i}\sigma_j - 2c_{ij} - 4g_{0li}\epsilon^{jlm}\sigma_m - 4ic_{0i}\sigma_j \\
&\quad + 2id_{ij} + 2ig_{lmi}\epsilon^{lmj} - 2g_{ilj}\sigma^l + 2g_{i00}\sigma^i, \\
C_j &= 4mc_{0j} + 2md_{ij}\sigma^i - 2md_{00}\sigma_j - 2me_j \\
&\quad + mg_{klj}\epsilon^{klm}\sigma_m - 2img_{j00} - 2mg_{i00}\epsilon^{ijl}\sigma_l \\
&\quad - 2ime_0\sigma_j - 2md_{k0}\epsilon^{jkl}\sigma_l - img_{il0}\epsilon^{ilj} \\
&\quad + 2a_j - 2b_0\sigma_j + 2H_{0k}\epsilon^{jkl}\sigma_l + 2ia_0\sigma_j \\
&\quad - 2ib_j - 2i\epsilon^{klj}H_{kl} - 2H_{lj}\sigma^l, \\
D &= -2m^2c_{00} + 2m^2e_0 - 2m^2d_{j0}\sigma^j \\
&\quad - m^2g_{ik0}\epsilon^{ikl}\sigma_l - 2ma_0 - 2mb_j\sigma^j - m\epsilon^{ijl}H_{ij}\sigma_l \\
&\quad + [1 + 2c_{00} + 2id_{00} - 2g_{i00}\sigma^i]\frac{i\hbar e}{mc}\sigma_k B^k. \qquad (24)
\end{aligned}
$$

The bound state of the Hamiltonian (21) can be calculated by taking into account the Lorentz violating potential V, which corresponds to the term D in the last equation. Actually, this potential obeys the relation [47]

$$
V = -\tilde{b}_j\sigma^j, \qquad (25)
$$

where σ represents the spin matrices. From this point, one can calculate the bound state of the theory:

$$
\begin{aligned}
\tilde{b}_j &= b_j + \frac{1}{2}\epsilon^{lmj}H_{lm} + md_{j0} + \frac{1}{2}m\epsilon^{lmj}g_{lm0} \\
&\quad - [1 + 2c_{00} + 2id_{00} - 2g_{i00}\sigma^i]\frac{i\hbar e}{2m^2c}B_j. \qquad (26)
\end{aligned}
$$

As was expected, this bound state is a specific combination of two parts related to the SME coefficients. The first part includes the coefficients b_j, H_{lm}, d_{j0}, and g_{lm0}, and the bound is based on atomic clock and other nonrelativistic experiments [54] that can involve a maser/magnetometer (see, for example, Table VII in Ref. [2]). In the second part, there is the presence of a magnetic field, which can be a remarkable and very important result from the experimental point of view. As is known, the external fields in the Eq. (26) are very weak. However, the modulus of $\mathbf{B}$ may be sufficiently high, in order to compensate the weakness of the interactions c_{00}, d_{00}, and g_{i00}. In another words, with a strong enough magnetic field, one can have indications, in principle, of the kind of motion generated by the external field commented above. It is an indirect way of performing measurements of such weak external fields.

VI. CONCLUSIONS AND DISCUSSIONS

The exact Foldy-Wouthuysen transformation was performed in the context of a Dirac field interacting with many possible external fields associated with *CPT*-Lorentz violation.

The first result of the work is written in the form of Table I, representing the Hamiltonian with the complete set *CPT*-Lorentz violating terms in the Dirac equation. In the table, the terms highlighted in bold font do not anticommute with the usual involution operator (2).

Another result of the work is the appropriate involution operator, given by Eq. (6), such that the anticommutation relation with the Hamiltonian of the problem is achieved. Actually, Eq. (6) introduces the new possibility of performing EFWT. From now on, a large class of Hamiltonians may admit the exact transformation, since the new involution operator in Eq. (6) is used.

In Sec. IV, the usual EFWT algorithm was applied to the initial Hamiltonian, and the exact transformation was performed. As was expected, the EWFT approach presented a transformed Hamiltonian (13) with additional terms, when compared to the Hamiltonian (21), in which the usual FWT was used.

In the last section, we derived the bound state of the theory, given by Eq. (26). It is worth mentioning that the possibility of the weakness of *CPT*-Lorentz terms could be compensated by the presence of a strong magnetic field. Thus, one can understand the particle behavior due to the interactions with external field; it gives the possibility to measure the external fields in an indirect way.

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