GUT inspired $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification

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An $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification model inspired by $SO(11)$ gauge-Higgs grand unification is constructed in the Randall-Sundrum warped space. The 4D Higgs boson is identified with the Aharonov-Bohm phase in the fifth dimension. Fermion multiplets are introduced in the bulk in the spinor, vector and singlet representations of $SO(5)$ such that they are implemented in the spinor and vector representations of $SO(11)$. The mass spectrum of quarks and leptons in three generations is reproduced except for the down-quark mass. The small neutrino masses are explained by the gauge-Higgs seesaw mechanism which takes the same form as in the inverse seesaw mechanism in grand unified theories in four dimensions.

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I. INTRODUCTION

The existence of the Higgs boson of a mass 125 GeV has been firmly established at the LHC [1]. It supports the unification scenario of electromagnetic and weak forces. So far almost all of the experimental results and observations have been consistent with the standard model (SM) based on the gauge group $\mathcal{G}_{\rm SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$. Yet it is not clear whether or not the observed Higgs boson is precisely what the SM assumes. All of the Higgs couplings to other fields and to itself need to be determined with better accuracy. Furthermore, the SM is afflicted with the gauge hierarchy problem which becomes apparent when the model is generalized to incorporate grand unification. The fundamental problem is the lack of a principle which regulates the Higgs sector, quite in contrast to the gauge sector which is controlled by the gauge principle.

There have been several attempts to overcome these difficulties. Supersymmetric theory is one of them which has been extensively investigated. An alternative approach is gauge-Higgs unification in which the Higgs boson is identified with the zero mode of the fifth-dimensional component of the gauge potential. It appears as a fluctuation mode of the Aharonov-Bohm (AB) phase θ_H in the fifth dimension [2–7]. Already a realistic gauge-Higgs unification (GHU) model has been constructed. It is the $SO(5) \times U(1)_X$ gauge theory in the Randall-Sundrum (RS) warped space with quark and lepton multiplets in the vector representation of $SO(5)$ [8–16]. It has been shown that the $SO(5) \times U(1)_X$ GHU yields nearly the same phenomenology at low energies as the SM. Deviations of the gauge couplings of quarks and leptons from the SM values are less than 10^{-3} for $\theta_H \sim 0.1$. Higgs couplings of quarks, leptons, W, and Z are approximately the SM values times $\cos\theta_H$, the deviation being about 1%. The Kaluza-Klein (KK) mass scale is about $m_{\rm KK} \sim 8$ TeV for $\theta_H \sim 0.1$. Implications of GHU to dark matter and Majorana neutrino masses are also under intensive study [17–21].

The model predicts Z' bosons, which are the first KK modes of γ, Z, and Z_R [$SU(2)_R$ gauge boson], in the 7–9 TeV range for $\theta_H = 0.1$–0.07. They have broad widths and can be produced at 14 TeV LHC [12,13]. The current nonobservation of Z' signals puts the limit $\theta_H < 0.11$. Right-handed quarks and charged leptons have rather large couplings to Z'. It has been pointed out recently that the interference effects of Z' bosons can be clearly observed at a 250 GeV e^+e^- linear collider (ILC)[14,16]. For instance, in the process $e^+e^- \to \mu^+\mu^-$ the deviation from the SM amounts to -4% with the electron beam polarized in the right-handed mode by 80% ($P_{e^-} = 0.8$) for $\theta_H \sim 0.09$, whereas there appears negligible deviation with the electron beam polarized in the left-handed mode by 80% ($P_{e^-} = -0.8$). In the forward-backward asymmetry $A_{FB}(\mu^+\mu^-)$, the deviation from the SM becomes -2% for

$P_{e^-} = 0.8$. These deviations can be seen at 250 GeV ILC with 250 fb^{-1} data, namely in the early stage of the ILC project [22–24].

At this point one may pause to ask a question. Is there an alternative way of introducing quark-lepton multiplets in the $SO(5) \times U(1)_X \times SU(3)_C$ GHU? A different choice may lead to different predictions for the Z' couplings.

In this paper we present an alternative way of introducing fermions in the $SO(5) \times U(1)_X \times SU(3)_C$ GHU based on the compatibility with the grand unification of forces. Many gauge-Higgs grand unification models have been proposed [25–30]. Among them the $SO(11)$ GHU generalizes the gauge structure of the previous $SO(5) \times U(1)_X \times SU(3)_C$ model, yielding the 4D Higgs boson as an AB phase [31–36]. Fermions are introduced in the spinor and vector representations of $SO(11)$. The current $SO(11)$ GHU models in either 5D or 6D warped space are not completely satisfactory, however. The models yield exotic light fermions in addition to quarks and leptons at low energies.

In the framework of grand unification, the representation in an $SO(5)$ and $U(1)_X$ charge are not independent. Only certain combinations are allowed. For instance, fields with quantum numbers of up-type quarks are contained in an $SO(11)$ spinor, but not in an $SO(11)$ vector. This fact immediately implies that the fermion content in the previous $SO(5) \times U(1)_X \times SU(3)_C$ model, in which all quark multiplets are introduced in the vector representation of $SO(5)$, needs to be modified to be consistent with the $SO(11)$ unification. The purpose of the present paper is to formulate an $SO(5) \times U(1)_X \times SU(3)_C$ GHU which is compatible with the $SO(11)$ GHU scheme. Models must yield phenomenology of the SM at low energies. In particular, the mass spectrum and gauge couplings of quarks and leptons need to be reproduced within experimental errors.

In Sec. II we review the general structure of the group $SO(11)$ which is necessary to construct a model compatible with gauge-Higgs grand unification. A new model of $SO(5) \times U(1)_X \times SU(3)_C$ GHU is introduced in Sec. III. In Sec. IV the mass spectrum of gauge fields is determined. In Sec. V the mass spectra of various fermion fields are determined. Brane interactions become important for down-type quarks and neutral leptons. W couplings of quarks and leptons are also evaluated. Section VI is devoted to summary and discussions. Appendix A summarizes generators of $SO(5)$. Basis mode functions in the RS space are summarized in Appendix B. In Appendix B 3 mode functions for massive fermion fields are given. In Appendix C notation for Majorana fermions is summarized. In Appendix D the mass spectra and wave functions of additional dark fermion fields are derived.

II. STRUCTURE OF $SO(11)$

We would like to formulate $SO(5) \times U(1)_X \times SU(3)_C$ GHU inspired from $SO(11)$ GHU. For that purpose it is useful to review branching rules of $SO(11)$ to its subgroups. We check them for $SO(11)$ singlet, vector, spinor, and adjoint representations **1**, **11**, **32**, **55**. All the necessary information is found in Ref. [37]. First we note

$$
\begin{aligned}
SO(11) &\supset SO(6)_C \times SO(5)_W \simeq SU(4)_C \times USp(4)_W \\
&\supset SU(3)_C \times U(1)_X \times SU(2)_L \times SU(2)_R \\
&\supset SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_Z. \qquad (2.1)
\end{aligned}
$$

Here $U(1)_X$ represents $U(1)$ in $SO(6)_C \simeq SU(4)_C \supset SU(3)_C \times U(1)_X$, whereas $U(1)_Z$ represents $U(1)$ in $SO(10) \supset SU(5) \times U(1)_Z$.

The branching rules of $SO(11) \supset SO(6)_C \times SO(5)_W (\simeq SU(4)_C \times USp(4)_W)$ are given by

$$
\begin{aligned}
\mathbf{1} &= (\mathbf{1}, \mathbf{1}), \\
\mathbf{11} &= (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}), \\
\mathbf{32} &= (\mathbf{4}, \mathbf{4}) \oplus (\bar{\mathbf{4}}, \mathbf{4}), \\
\mathbf{55} &= (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{10}). \qquad (2.2)
\end{aligned}
$$

The branching rules of $SO(6)_C \simeq SU(4)_C \supset SU(3)_C \times U(1)_X$ are given by

$$
\begin{aligned}
\mathbf{1} &= (\mathbf{1})_0, \\
\mathbf{4} &= (\mathbf{3})_{\frac{1}{6}} \oplus (\mathbf{1})_{-\frac{1}{2}}, \\
\bar{\mathbf{4}} &= (\bar{\mathbf{3}})_{-\frac{1}{6}} \oplus (\mathbf{1})_{\frac{1}{2}}, \\
\mathbf{6} &= (\mathbf{3})_{-\frac{1}{3}} \oplus (\bar{\mathbf{3}})_{\frac{1}{3}}, \\
\mathbf{15} &= (\mathbf{8})_0 \oplus (\mathbf{3})_{\frac{2}{3}} \oplus (\bar{\mathbf{3}})_{-\frac{2}{3}} \oplus (\mathbf{1})_0. \qquad (2.3)
\end{aligned}
$$

Here the subscript represents the $U(1)_X$ charge Q_X. For later use, Q_X has been normalized such that the electric charge Q_{EM} is given by $Q_{\mathrm{EM}} = T_3^L + T_3^R + Q_X$, where T_a^L and T_a^R ($a = 1, 2, 3$) are generators of $SU(2)_L$ and $SU(2)_R$. From the branching rules (2.2) and (2.3), one obtains the branching rules of $SO(11) \supset SU(3)_C \times SO(5)_W \times U(1)_X$ as

$$
\begin{aligned}
\mathbf{1} &= (\mathbf{1}, \mathbf{1})_0, \\
\mathbf{11} &= (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{5})_0, \\
\mathbf{32} &= (\mathbf{3}, \mathbf{4})_{\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{4})_{-\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{4})_{-\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{4})_{\frac{1}{2}}, \\
\mathbf{55} &= (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{1})_{\frac{2}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{5})_{-\frac{1}{3}} \\
&\quad \oplus (\bar{\mathbf{3}}, \mathbf{5})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{10})_0. \qquad (2.4)
\end{aligned}
$$

The branching rules of $SO(5)_W \simeq USp(4) \supset SU(2)_L \times SU(2)_R$ are given by

$$\begin{aligned} \mathbf{1} &= (\mathbf{1},\mathbf{1}),\\ \mathbf{4} &= (\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}),\\ \mathbf{5} &= (\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1}),\\ \mathbf{10} &= (\mathbf{3},\mathbf{1})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{3}). \end{aligned} \tag{2.5}$$

(For more information, see Table 471 in Ref. [37].)

It has been shown [35,36] that in 6D $SO(11)$ gauge-Higgs grand unification in the hybrid warped space 4D SM chiral fermions and other vectorlike fermions can be extracted from 6D Weyl fermions without 6D and 4D gauge anomalies. With appropriate boundary conditions imposed, only $(\mathbf{3},\mathbf{4})_{\frac{1}{6}}\oplus(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$ of $SU(3)_C\times SO(5)_W\times U(1)_X$ have zero modes of 6D $SO(11)$ **32** Weyl fermions. Also, only either $(\mathbf{1},\mathbf{5})_0$ or $(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}\oplus(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}$ have zero modes of 6D $SO(11)$ **11** Weyl fermions.

The gauge symmetry breaking takes place in three steps:

$$\begin{aligned} &SU(3)_C\times SO(5)_W\times U(1)_X\\ &\underset{\text{BCs}}{\longrightarrow} SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)_X\\ &\underset{\langle\Phi_{(\mathbf{1},\mathbf{4})_{1/2}}\rangle\neq0}{\longrightarrow} SU(3)_C\times SU(2)_L\times U(1)_Y=G_{\rm SM}\\ &\underset{\theta_H\neq0}{\longrightarrow} SU(3)_C\times U(1)_{\rm EM}. \end{aligned} \tag{2.6}$$

In the first step $SO(5)_W$ is broken to $SO(4)\simeq SU(2)_L\times SU(2)_R$ by orbifold boundary conditions. In the second step $SU(2)_R\times U(1)_X$ is spontaneously broken to $U(1)_Y$ by the nonvanishing vacuum expectation value (VEV) of a brane scalar field $\Phi_{(\mathbf{1},\mathbf{4})_{1/2}}$. In the third step $SU(2)_L\times U(1)_Y$ is broken to $U(1)_{\rm EM}$ by the Hosotani mechanism $\theta_H\neq0$. At the moment we need to introduce an elementary brane scalar field $\Phi_{(\mathbf{1},\mathbf{4})_{1/2}}$ on the UV brane, which is not completely in harmony with the philosophy of gauge-Higgs unification. The $\Phi_{(\mathbf{1},\mathbf{4})_{1/2}}$ field not only reduces the gauge symmetry to $G_{\rm SM}$ in the second step in (2.6), but also plays a crucial role in realizing the mass spectrum of quarks and leptons through brane interactions. The origin of the brane scalar field remains to be clarified.

III. $SU(3)_C\times SO(5)_W\times U(1)_X$ GHU: NEW MODEL

A new model of $SU(3)_C\times SO(5)_W\times U(1)_X$ GHU is defined in the Randall-Sundrum warped space. The construction is guided by the $SO(11)$ gauge-Higgs grand unified model [31–36]. The metric g_{MN} of the RS warped space [38] is given by

$$ds^2=g_{MN}dx^Mdx^N=e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu+dy^2, \tag{3.1}$$

where $M,\ N=0,\ 1,\ 2,\ 3,\ 5$; $\mu,\ \nu=0,\ 1,\ 2,\ 3$; $y=x^5$; $\eta_{\mu\nu}={\rm diag}(-1,+1,+1,+1)$; $\sigma(y)=\sigma(y+2L)=\sigma(-y)$; and $\sigma(y)=ky$ for $0\le y\le L$. The topological structure of the RS space is $S_1/\mathbb{Z}_2$. In terms of the conformal coordinate $z=e^{ky}$ $(1\le z\le z_L=e^{kL})$ in the region $0\le y\le L$,

$$ds^2=\frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu+\frac{dz^2}{k^2}\right). \tag{3.2}$$

The bulk region $0<y<L$ $(1<z<z_L)$ is anti–de Sitter (AdS) spacetime with a cosmological constant $\Lambda=-6k^2$, which is sandwiched by the UV brane at $y=0$ $(z=1)$ and the IR brane at $y=L$ $(z=z_L)$. The KK mass scale is $m_{\rm KK}=\pi k/(z_L-1)\simeq\pi kz_L^{-1}$ for $z_L\gg1$.

Parity transformations around the two fixed points $(y_0,y_1)=(0,L)$ are defined as $(x^\mu,y_j+y)\to(x^\mu,y_j-y)$. We choose orbifold boundary conditions (BCs) such that they break $SO(5)_W$ to $SO(4)\simeq SU(2)_L\times SU(2)_R$ as described below.

A. Gauge fields and orbifold boundary conditions

The structure of the gauge field part is the same as in the previous $SU(3)_C\times SO(5)_W\times U(1)_X$ GHU model. We have $SU(3)_C\times SO(5)_W\times U(1)_X$ $(\mathbf{8},\mathbf{1})_0$, $(\mathbf{1},\mathbf{10})_0$, and $(\mathbf{1},\mathbf{1})_0$ gauge bosons denoted by $A_M^{SU(3)_C}$, $A_M^{SO(5)_W}$, and $A_M^{U(1)_X}$. The orbifold BCs are given by

$$\begin{pmatrix}A_\mu\\A_y\end{pmatrix}(x,y_j-y)=P_j\begin{pmatrix}A_\mu\\-A_y\end{pmatrix}(x,y_j+y)P_j^{-1} \tag{3.3}$$

for each gauge field. In terms of

$$\begin{aligned} P_{\mathbf{3}}^{SU(3)}&=I_3,\\ P_{\mathbf{4}}^{SO(5)}&={\rm diag}(I_2,-I_2),\\ P_{\mathbf{5}}^{SO(5)}&={\rm diag}(I_4,-I_1), \end{aligned} \tag{3.4}$$

$P_0=P_1=P_{\mathbf{3}}^{SU(3)}$ for $A_M^{SU(3)_C}$ and $P_0=P_1=1$ for $A_M^{U(1)_X}$. $P_0=P_1=P_{\mathbf{5}}^{SO(5)}$ for $A_M^{SO(5)_W}$ in the vector representation and $P_{\mathbf{4}}^{SO(5)}$ in the spinor representation, respectively. $P_{\mathbf{4}}^{SO(5)}$ and $P_{\mathbf{5}}^{SO(5)}$ break $SO(5)_W$ to $SO(4)$. The parity assignments of A_μ and A_y are summarized in Table I. Note that the 4D Higgs field is contained in the $(\mathbf{1},\mathbf{2},\mathbf{2})_0$ part of A_y.

TABLE I. Parity assignment $P_0=P_1$ of A_μ and A_y in $SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)_X$. G_{3221} ≔ $SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)_X$.

G_{3221}	A_μ	A_y
$(\mathbf{8},\mathbf{1},\mathbf{1})_0$	$(+,+)$	$(-,-)$
$(\mathbf{1},\mathbf{3},\mathbf{1})_0$	$(+,+)$	$(-,-)$
$(\mathbf{1},\mathbf{1},\mathbf{3})_0$	$(+,+)$	$(-,-)$
$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(-,-)$	$(+,+)$
$(\mathbf{1},\mathbf{1},\mathbf{1})_0$	$(+,+)$	$(-,-)$

B. Matter fields and orbifold boundary conditions

Matter fields are introduced both in the 5D bulk and on the UV brane. They are listed in Table II. Quark multiplets $(\mathbf{3},\mathbf{4})_{\frac{1}{6}}$ and $(\mathbf{3},\mathbf{1})^{\pm}_{-\frac{1}{3}}$ are introduced in the 5D bulk in three generations. They are denoted as $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}(x,y)$ and $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}(x,y)$ ($\alpha = 1, 2, 3$). All $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$ and $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$ intertwine with each other. Lepton multiplets in the bulk are introduced in $(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$, denoted as $\Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}(x,y)$. In addition, brane fermions $\chi^{\alpha}_{(\mathbf{1},\mathbf{1})}(x)$ in the singlet $(\mathbf{1},\mathbf{1})_0$ are introduced on the UV brane, which satisfy the Majorana condition $\chi(x)^c = \chi(x)$. $\chi^{\alpha}_{(\mathbf{1},\mathbf{1})}$ and $\Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}$ intertwine with each other to induce the seesaw mechanism for neutrino masses. Two types of dark fermion multiplets, $\Psi^{\alpha=4}_{(\mathbf{3},\mathbf{4})}(x,y)$ in $(\mathbf{3},\mathbf{4})_{\frac{1}{6}}$ and $\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}(x,y)$ ($\beta = 1, \ldots, n_F$) in $(\mathbf{1},\mathbf{5})^{\pm}_{0}$, are introduced in the bulk, which is necessary to have desired electroweak (EW) symmetry breaking with $0 < \theta_H < \frac{1}{2}\pi$. $\Psi^{\alpha=4}_{(\mathbf{3},\mathbf{4})}$ obeys orbifold boundary conditions such that no zero modes arise. Zero modes of $\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}$ appear, but $\Psi^{+\beta}_{(\mathbf{1},\mathbf{5})}$ and $\Psi^{-\beta}_{(\mathbf{1},\mathbf{5})}$ intertwine to have large Dirac masses. The brane scalar field $\Phi_{(\mathbf{1},\mathbf{4})}(x)$ is introduced in $(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$ on the UV brane. All of these fields can be implemented in the representations **1**, **11**, and **32** of $SO(11)$ as seen from (2.4). $SU(3)_C \times SO(5) \times U(1)_X$ gauge symmetry is preserved on the UV brane, which should be contrasted to the previous model in which only $SU(3)_C \times SO(4) \times U(1)_X$ symmetry is preserved on the UV brane. $(\bar{\mathbf{3}},\mathbf{1})^{\pm}_{+\frac{1}{3}}$ fermion fields accompany $(\mathbf{3},\mathbf{1})^{\pm}_{-\frac{1}{3}}$ fermion fields when they are implemented in the **11** representation in $SO(11)$ GHU. Zero modes of $(\bar{\mathbf{3}},\mathbf{1})^{+}_{+\frac{1}{3}}$ and $(\bar{\mathbf{3}},\mathbf{1})^{-}_{+\frac{1}{3}}$ couple to have large Dirac masses so that they may be ignored here. One can confirm that anomalies are canceled in the present model.

Orbifold boundary conditions for bulk fermions are specified in the following manner.

(i) Quark multiplets: $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$, $\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$

$$\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}(x, y_j - y) = -P_{\mathbf{4}}^{SO(5)}\gamma^5 \Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}(x, y_j + y),$$
$$\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}(x, y_j - y) = \mp\gamma^5 \Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}(x, y_j + y). \tag{3.5}$$

Here 5D Dirac matrices γ^a ($a = 0, 1, 2, 3, 5$) satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ [$\eta^{ab} = \mathrm{diag}(-I_1, I_4)$], and $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$.

(ii) Lepton multiplets: $\Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}$

$$\Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}(x, y_j - y) = -P_{\mathbf{4}}^{SO(5)}\gamma^5 \Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}(x, y_j + y). \tag{3.6}$$

(iii) Dark fermions: $\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}$

$$\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}(x, y_j - y) = \pm P_{\mathbf{5}}^{SO(5)}\gamma^5 \Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}(x, y_j + y). \tag{3.7}$$

Alternatively one may adopt the parity assignment $\pm(-1)^j P_{\mathbf{5}}^{SO(5)}$ instead of $\pm P_{\mathbf{5}}^{SO(5)}$ in (3.7).

(iv) Dark fermion: $\Psi_{(\mathbf{3},\mathbf{4})} \equiv \Psi_F$

$$\Psi_F(x, y_j - y) = (-1)^j P_{\mathbf{4}}^{SO(5)}\gamma^5 \Psi_F(x, y_j + y). \tag{3.8}$$

The parity assignment of the 4D left- and right-handed components of each fermion field is summarized in Table III. $\Psi^{\alpha}_{(\mathbf{3},\mathbf{4})}$ and $\Psi^{\alpha}_{(\mathbf{1},\mathbf{4})}$ ($\alpha = 1, 2, 3$) has zero modes, corresponding to one generation of quarks and leptons for each α.

C. Action

The action consists of the 5D bulk action and 4D brane action.

TABLE II. Matter fields. $SU(3)_C \times SO(5) \times U(1)_X$ content is shown. For comparison the matter content in the previous model is listed in the last column. In the previous model only $SU(3)_C \times SO(4) \times U(1)_X$ symmetry is preserved on the UV brane so that the $SU(2)_L \times SU(2)_R$ content is shown for brane fields.

	Present model Type B	Previous model Type A
Quark	$(\mathbf{3},\mathbf{4})_{\frac{1}{6}}(\mathbf{3},\mathbf{1})^{+}_{-\frac{1}{3}}(\mathbf{3},\mathbf{1})^{-}_{-\frac{1}{3}}$	$(\mathbf{3},\mathbf{5})_{\frac{2}{3}}(\mathbf{3},\mathbf{5})_{-\frac{1}{3}}$
Lepton	$(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$	$(\mathbf{1},\mathbf{5})_0(\mathbf{1},\mathbf{5})_{-1}$
Dark fermion	$(\mathbf{3},\mathbf{4})_{\frac{1}{6}}(\mathbf{1},\mathbf{5})^{+}_{0}(\mathbf{1},\mathbf{5})^{-}_{0}$	$(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$
Brane fermion	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{3},[\mathbf{2},\mathbf{1}])_{\frac{7}{6},\frac{1}{6},-\frac{5}{6}}(\mathbf{1},[\mathbf{2},\mathbf{1}])_{\frac{1}{2},-\frac{1}{2},-\frac{3}{2}}$
Brane scalar	$(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$	$(\mathbf{1},[\mathbf{1},\mathbf{2}])_{\frac{1}{2}}$
Symmetry of brane interactions	$SU(3)_C \times SO(5) \times U(1)_X$	$SU(3)_C \times SO(4) \times U(1)_X$

TABLE III. Parity assignment (P_0, P_1) of fermion fields in the bulk. The corresponding names adopted in Ref. [33] are listed in the last column for the first generation. Brane fermion and scalar fields are listed at the bottom for convenience.

Field	G_{3221}	Left	Right	Name
$\Psi^\alpha_{(\mathbf{3},\mathbf{4})}$	$(\mathbf{3},\mathbf{2},\mathbf{1})_{\frac{1}{6}}$	$(+,+)$	$(-,-)$	u_j d_j
	$(\mathbf{3},\mathbf{1},\mathbf{2})_{\frac{1}{6}}$	$(-,-)$	$(+,+)$	u'_j d'_j
$\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-\frac{1}{3}}$	$(\pm,\pm)$	$(\mp,\mp)$	$D^\pm_j$
$\Psi^\alpha_{(\mathbf{1},\mathbf{4})}$	$(\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2}}$	$(+,+)$	$(-,-)$	ν_e e
	$(\mathbf{1},\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	$(-,-)$	$(+,+)$	ν'_e e'
Ψ_F	$(\mathbf{3},\mathbf{2},\mathbf{1})_{\frac{1}{6}}$	$(-,+)$	$(+,-)$	F_{1j} F_{2j}
	$(\mathbf{3},\mathbf{1},\mathbf{2})_{\frac{1}{6}}$	$(+,-)$	$(-,+)$	F'_{1j} F'_{2j}
$\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}$	$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\pm,\pm)$	$(\mp,\mp)$	$N^\pm$ $\hat{E}^\pm$ $E^\pm$ $\hat{N}^\pm$
	$(\mathbf{1},\mathbf{1},\mathbf{1})_0$	$(\mp,\mp)$	$(\pm,\pm)$	$S^\pm$
χ^α	$(\mathbf{1},\mathbf{1},\mathbf{1})_0$	$\cdots$	$\cdots$	χ
$\Phi_{(\mathbf{1},\mathbf{4})}$	$(\mathbf{1},\mathbf{2},\mathbf{1})_{\frac{1}{2}}$	$\cdots$	$\cdots$	$\Phi_{[\mathbf{2},\mathbf{1}]}$
	$(\mathbf{1},\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\cdots$	$\cdots$	$\Phi_{[\mathbf{1},\mathbf{2}]}$

1. Bulk action

The bulk part of the action is given by

$$S_{\rm bulk} = S^{\rm gauge}_{\rm bulk} + S^{\rm fermion}_{\rm bulk}, \tag{3.9}$$

where $S^{\rm gauge}_{\rm bulk}$ and $S^{\rm fermion}_{\rm bulk}$ are bulk actions of gauge and fermion fields, respectively. The action of each gauge field, $A_M^{SU(3)_C}$, $A_M^{SO(5)_W}$, or $A_M^{U(1)_X}$, is given in the form

$$S^{\rm gauge}_{\rm bulk} = \int d^5x\sqrt{-\det G} \times\left[-{\rm tr}\left(\frac{1}{4}F^{MN}F_{MN} + \frac{1}{2\xi}(f_{\rm gf})^2 + \mathcal{L}_{\rm gh}\right)\right], \tag{3.10}$$

where $\sqrt{-\det G} = 1/kz^5$; $z = e^{ky}$; $M, N = 0, 1, 2, 3, 5$; and tr is a trace over all group generators for each group. Field strength F_{MN} is defined by

$$F_{MN} \coloneqq \partial_M A_N - \partial_N A_M - ig[A_M, A_N], \tag{3.11}$$

with each 5D gauge coupling constant g. For the gauge fixing and ghost terms we take

$$f_{\rm gf} = z^2\left\{\eta^{\mu\nu}\mathcal{D}^{\rm c}_\mu A^{\rm q}_\nu + \xi k^2 z\mathcal{D}^{\rm c}_z\left(\frac{1}{z}A^{\rm q}_z\right)\right\},$$
$$\mathcal{L}_{\rm gh} = \bar{c}\left\{\eta^{\mu\nu}\mathcal{D}^{\rm c}_\mu\mathcal{D}_\nu + \xi k^2 z\mathcal{D}^{\rm c}_z\frac{1}{z}\mathcal{D}_z\right\}c, \tag{3.12}$$

where $\mu, \nu = 0, 1, 2, 3$; $\eta^{\mu\nu} = \eta_{\mu\nu} = {\rm diag}(-1,1,1,1)$; and $A_M = A^c_M + A^q_M$. $\mathcal{D}^c_M B = \partial_M B - ig[A^c_M, B]$ and $\mathcal{D}^{c+q}_M B = \partial_M B - ig[A_M, B]$, where $B = A^q_\mu$, A^q_z/z and c. In the present paper only the A_z component of $A_M^{SO(5)}$ has nonvanishing classical background A^c_z.

Each fermion multiplet $\Psi(x,y)$ in the bulk has its own bulk-mass parameter c. The covariant derivative is given by

$$\mathcal{D}(c) = \gamma^A e_A{}^M\left(D_M + \frac{1}{8}\omega_{MBC}[\gamma^B,\gamma^C]\right) - c\sigma'(y),$$
$$D_M = \partial_M - ig_S A_M^{SU(3)} - ig_A A_M^{SO(5)} - ig_B Q_X A_M^{U(1)}. \tag{3.13}$$

Here $\sigma'(y) \coloneqq d\sigma(y)/dy$ and $\sigma'(y) = k$ for $0 < y < L$. g_S, g_A, g_B are $SU(3)_C$, $SO(5)_W$, $U(1)_X$ gauge coupling constants. Let Ψ^J collectively denote all fermion fields in the bulk. Then the action in the bulk becomes

$$S^{\rm fermion}_{\rm bulk} = \int d^5x\sqrt{-\det G}\left\{\sum_J \overline{\Psi^J}\,\mathcal{D}(c_J)\Psi^J - \sum_\alpha\left(m^\alpha_D\bar{\Psi}^{+\alpha}_{(\mathbf{3},\mathbf{1})}\Psi^{-\alpha}_{(\mathbf{3},\mathbf{1})} + {\rm H.c.}\right) - \sum_\beta\left(m^\beta_V\bar{\Psi}^{+\beta}_{(\mathbf{1},\mathbf{5})}\Psi^{-\beta}_{(\mathbf{1},\mathbf{5})} + {\rm H.c.}\right)\right\}, \tag{3.14}$$

where $\bar{\Psi} = i\Psi^\dagger\gamma^0$. m^α_D and m^β_V are "pseudo-Dirac" bulk mass terms.

In terms of $\check{\Psi}$ defined by

$$\check{\Psi} \coloneqq \frac{1}{z^2}\Psi, \qquad \left(\partial_z - \frac{2}{z}\right)\Psi = z^2\partial_z\check{\Psi}, \tag{3.15}$$

the bulk part of the fermion action becomes

$$S^{\rm fermion}_{\rm bulk} = \int d^4x\int_1^{z_L}\frac{dz}{k}\left\{\sum_J\bar{\check{\Psi}}^J\left[\gamma^\mu D_\mu + k\left(\gamma^5 D_z - \frac{c_J}{z}\right)\right]\check{\Psi}^J - \sum_\alpha\left(\frac{m^\alpha_D}{z}\bar{\check{\Psi}}^{+\alpha}_{(\mathbf{3},\mathbf{1})}\check{\Psi}^{-\alpha}_{(\mathbf{3},\mathbf{1})} + {\rm H.c.}\right) - \sum_\beta\left(\frac{m^\beta_V}{z}\bar{\check{\Psi}}^{+\beta}_{(\mathbf{1},\mathbf{5})}\check{\Psi}^{-\beta}_{(\mathbf{1},\mathbf{5})} + {\rm H.c.}\right)\right\}. \tag{3.16}$$

2. Action for the brane scalar $\Phi_{(\mathbf{1},\mathbf{4})}$

The action for the brane scalar field $\Phi_{(\mathbf{1},\mathbf{4})}(x)$ in $(\mathbf{1},\mathbf{4})_{\frac{1}{2}}$ is given by

$$S^\Phi_{\rm brane} = \int d^5x\sqrt{-\det G}\delta(y) \times\{-(D_\mu\Phi_{(\mathbf{1},\mathbf{4})})^\dagger D^\mu\Phi_{(\mathbf{1},\mathbf{4})} - \lambda_{\Phi_{(1,4)}}(\Phi^\dagger_{(\mathbf{1},\mathbf{4})}\Phi_{(\mathbf{1},\mathbf{4})} - |w|^2)^2\}, \tag{3.17}$$

where

$$D_\mu \Phi_{(\mathbf{1},\mathbf{4})} = \left\{ \partial_\mu - i g_A \sum_{\alpha=1}^{10} A_\mu^\alpha T^\alpha - i g_B Q_X B_\mu \right\} \Phi_{(\mathbf{1},\mathbf{4})}. \tag{3.18}$$

Here $SO(5)_W$ generators $\{T^\alpha\}$ consist of $SU(2)_L$, $SU(2)_R$ generators $\{T^{a_L}, T^{a_R}\}$ $(a = 1,\ 2,\ 3)$ and $SO(5)/SO(4)$ generators $\{T^{\hat p} = T^{p5}/\sqrt{2}\}$ $(p = 1\text{–}4)$. The corresponding canonically normalized gauge fields are

$$\begin{aligned} A_M^{a_L} &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \epsilon^{abc} A_M^{bc} + A_M^{a4} \right), \\ A_M^{a_R} &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \epsilon^{abc} A_M^{bc} - A_M^{a4} \right), \\ A_M^{\hat p} &= A_M^{p5}. \end{aligned} \tag{3.19}$$

B_M represents the $U(1)_X$ gauge field.

The brane scalar field $\Phi_{(\mathbf{1},\mathbf{4})}$ is decomposed as

$$\Phi_{(\mathbf{1},\mathbf{4})} = \begin{pmatrix} \Phi_{[\mathbf{2},\mathbf{1}]} \\ \Phi_{[\mathbf{1},\mathbf{2}]} \end{pmatrix}, \tag{3.20}$$

where $[\mathbf{2},\mathbf{1}]$ and $[\mathbf{1},\mathbf{2}]$ represent $SU(2)_L \times SU(2)_R$ content. $\Phi_{(\mathbf{1},\mathbf{4})}$ develops a nonvanishing VEV:

$$\langle \Phi_{(\mathbf{1},\mathbf{4})} \rangle = \begin{pmatrix} 0_2 \\ v_2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ w \end{pmatrix}. \tag{3.21}$$

The nonvanishing VEV breaks $SU(3)_C \times SO(5) \times U(1)_X$ to $SU(3)_C \times SU(2)_L \times U(1)_Y$. As shown in Appendix A, one can define the conjugate scalar field $\tilde\Phi_{(\mathbf{1},\mathbf{4})}$ in $(\mathbf{1},\mathbf{4})_{-\frac{1}{2}}$ by

$$\tilde\Phi_{(\mathbf{1},\mathbf{4})} = \begin{pmatrix} i\sigma^2 \Phi^*_{[\mathbf{2},\mathbf{1}]} \\ -i\sigma^2 \Phi^*_{[\mathbf{1},\mathbf{2}]} \end{pmatrix}. \tag{3.22}$$

Its VEV is given by

$$\langle \tilde\Phi_{(\mathbf{1},\mathbf{4})} \rangle = \begin{pmatrix} 0_2 \\ \tilde v_2 \end{pmatrix}, \qquad \tilde v_2 = \begin{pmatrix} -w^* \\ 0 \end{pmatrix}. \tag{3.23}$$

The combination of the nonvanishing VEV $\langle \Phi_{(\mathbf{4},\mathbf{1})(3)} \rangle$ on the UV brane (at $y = 0$) and the orbifold BCs P_j $(j = 0,\ 1)$ reduces $SU(3)_C \times SO(5) \times U(1)_X$ to the SM gauge group $G_{\rm SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$.

3. Action for the brane fermion χ^α

The action for the gauge-singlet brane fermion $\chi^\alpha(x)$ is

$$S^\chi_{\rm brane} = \int d^5x \sqrt{-\det G}\, \delta(y) \left\{ \frac{1}{2} \bar\chi^\alpha \gamma^\mu \partial_\mu \chi^\alpha - \frac{1}{2} M^{\alpha\beta} \bar\chi^\alpha \chi^\beta \right\}. \tag{3.24}$$

$\chi^\alpha(x)$ satisfies the Majorana condition $\chi^c = \chi$:

$$\chi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \chi^c = \begin{pmatrix} +\eta^c \\ -\xi^c \end{pmatrix} = e^{i\delta_C} \begin{pmatrix} +\sigma^2 \eta^* \\ -\sigma^2 \xi^* \end{pmatrix}. \tag{3.25}$$

4. Brane interactions and mass terms for fermions

On the UV brane there can be $SU(3)_C \times SO(5) \times U(1)_X$-invariant brane interactions among the bulk fermion, brane fermion, and brane scalar fields. We consider

$$\begin{aligned} S^{\rm int}_{\rm brane} &= \int d^5x \sqrt{-\det G}\, \delta(y) (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3), \\ \mathcal{L}_1 &= -\{ \kappa^{\alpha\beta} \bar\Psi^{\ \alpha}_{(\mathbf{3},\mathbf{4})} \Phi_{(\mathbf{1},\mathbf{4})} \cdot \Psi^{+\beta}_{(\mathbf{3},\mathbf{1})} + {\rm H.c.} \}, \\ \mathcal{L}_2 &= -\{ \tilde\kappa'^{\alpha\beta} \bar\Psi^{\ \alpha}_{(\mathbf{1},\mathbf{4})} \Gamma^a \tilde\Phi_{(\mathbf{1},\mathbf{4})} \cdot (\Psi^{-\beta}_{(\mathbf{1},\mathbf{5})})_a + {\rm H.c.} \}, \\ \mathcal{L}_3 &= -\{ \tilde\kappa_{\mathbf{1}}^{\alpha\beta} \bar\chi^\beta \tilde\Phi^\dagger_{(\mathbf{1},\mathbf{4})} \Psi^\alpha_{(\mathbf{1},\mathbf{4})} + {\rm H.c.} \}, \end{aligned} \tag{3.26}$$

where κ's are coupling constants.

$\langle \Phi_{(\mathbf{1},\mathbf{4})} \rangle \neq 0$ generates mass terms on the UV brane from the interaction in (3.26). Together with the inherent Majorana masses in (3.24) brane fermion masses are given by

$$\begin{aligned} S^{\rm fermion}_{\rm brane\ mass} &= \int d^5x \sqrt{-\det G}\, \delta(y) (\mathcal{L}_1^m + \mathcal{L}_2^m + \mathcal{L}_3^m + \mathcal{L}_\chi^m), \\ \mathcal{L}_1^m &= 2\mu_1^{\alpha\beta} \bar{\check d}'^\alpha_R \check D_L^{+\beta} + {\rm H.c.}, \\ \mathcal{L}_2^m &= -\tilde\mu_2^{\alpha\beta} \{ i2 (\overline{\check e}'^\alpha_L \check E_R^{-\beta} + \overline{\check\nu}'^\alpha_L \check N_R^{-\beta}) + \sqrt{2} \overline{\check\nu}'^\alpha_R \check S_L^{-\beta} \} + {\rm H.c.}, \\ \mathcal{L}_3^m &= -\frac{m_B^{\alpha\beta}}{\sqrt{k}} (\bar\chi^\beta \check\nu'^\alpha_R + \bar{\check\nu}'^\alpha_R \chi^\beta), \\ \mathcal{L}_{\chi 1}^m &= -\frac{1}{2} M^{\alpha\beta} \bar\chi^\alpha \chi^\beta. \end{aligned} \tag{3.27}$$

Here $2\mu_1^{\alpha\beta} = \sqrt{2} \kappa^{\alpha\beta} w$, $2\tilde\mu_2^{\alpha\beta} = \sqrt{2} \tilde\kappa'^{\alpha\beta} w$, and $m_B^{\alpha\beta} = \tilde\kappa_{\mathbf{1}}^{\alpha\beta} w \sqrt{k}$. $\mu_1^{\alpha\beta}$ and $\tilde\mu_2^{\alpha\beta}$ are dimensionless, whereas $m_B^{\alpha\beta}$ and $M^{\alpha\beta}$ have a dimension of mass.

5. Brane mass terms for gauge bosons

$\langle \Phi_{(\mathbf{1},\mathbf{4})} \rangle \neq 0$ also yields additional brane mass terms for the 4D components of the $SO(5) \times U(1)_X$ gauge fields. It follows from (3.17) that

$$S_{\rm brane}^{\rm gauge}=\int d^5x\sqrt{-\det G}\delta(y)\times\left\{-\frac{g_A^2|w|^2}{4}(A_\mu^{1_R}A^{1_R\mu}+A_\mu^{2_R}A^{2_R\mu})-\frac{(g_A^2+g_B^2)|w|^2}{4}A_\mu^{3'_R}A^{3'_R\mu}\right\}, \tag{3.28}$$

where

$$\begin{pmatrix}A_M^{3'_R}\\ B_M^Y\end{pmatrix}=\begin{pmatrix}c_\phi & -s_\phi\\ s_\phi & c_\phi\end{pmatrix}\begin{pmatrix}A_M^{3_R}\\ B_M\end{pmatrix},\quad c_\phi=\cos\phi=\frac{g_A}{\sqrt{g_A^2+g_B^2}},\quad s_\phi=\sin\phi=\frac{g_B}{\sqrt{g_A^2+g_B^2}}. \tag{3.29}$$

The 5D gauge coupling $g_Y^{\rm 5D}$ of $U(1)_Y$ is given by

$$g_Y^{\rm 5D}=\frac{g_Ag_B}{\sqrt{g_A^2+g_B^2}}=g_As_\phi. \tag{3.30}$$

$A_\mu^{1_R}$, $A_\mu^{2_R}$, and $A_\mu^{3'_R}$ obtain large brane masses, which effectively change the BCs on the UV brane for the corresponding fields.

Note that the 4D $SU(2)_L$ gauge coupling constant is related to g_A by

$$g_w=\frac{g_A}{\sqrt{L}}. \tag{3.31}$$

The three 4D SM gauge coupling constants g_s, g_w, g_Y of $SU(3)_C$, $SU(2)_L$, $U(1)_Y$ at the m_Z scale are $\alpha_s=g_s^2/4\pi=0.1184\pm0.0007$, $\alpha_w=g_s^2/4\pi=\alpha_{\rm EM}/\sin^2\theta_W$, and $\alpha_Y=g_Y^2/4\pi=\alpha_{\rm EM}/\cos^2\theta_W$, where $\alpha_{\rm EM}^{-1}=127.916\pm0.015$ and $\sin^2\theta_W=0.23116\pm0.00013$ [39]. In the $SU(3)_C\times SO(5)\times U(1)_X$ GHU, the $SU(2)_R$ gauge coupling constants are the same as the SM $SU(2)_L$ gauge coupling constants. With the relation (3.30) one finds that

$$\frac{4\pi L}{g_A^2}=\alpha_w^{-1}\simeq29.56,\qquad \frac{4\pi L}{g_B^2}=\alpha_Y^{-1}-\alpha_w^{-1}\simeq68.78, \tag{3.32}$$

at the m_Z scale.

D. Higgs boson and the twisted gauge

A 4D Higgs boson is contained in the $(\mathbf{1},\mathbf{2},\mathbf{2})$ component of $A_y^{SO(5)}$ as tabulated in Table I. In the z coordinate $A_z=(kz)^{-1}A_y$ $(1\le z\le z_L)$ and

$$A_z^{(j5)}(x,z)=\frac{1}{\sqrt{k}}\phi_j(x)u_H(z)+\cdots,\quad u_H(z)=\sqrt{\frac{2}{z_L^2-1}}z,\quad \Phi(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2+i\phi_1\\ \phi_4-i\phi_3\end{pmatrix}. \tag{3.33}$$

$\Phi(x)$ corresponds to the doublet Higgs field in the SM.

At the quantum level, Φ develops a nonvanishing expectation value. Without loss of generality we assume $\langle\phi_1\rangle,\langle\phi_2\rangle,\langle\phi_3\rangle=0$ and $\langle\phi_4\rangle\neq0$, which is related to the AB phase θ_H in the fifth dimension. Eigenvalues of

$$\hat{W}=P\exp\left\{ig_A\int_{-L}^{L}dyA_y\right\}\cdot P_1P_0 \tag{3.34}$$

are gauge invariant. For $A_y=(2k)^{-1/2}\phi_4(x)v_H(y)T^{(45)}$, where $v_H(y)=ke^{ky}u_H(z)$ for $0\le y\le L$ and $v_H(-y)=v_H(y)=v_H(y+2L)$, one finds

$$\hat{W}=\exp\{i\hat{\theta}_H(x)\cdot 2T^{(45)}\},\quad \hat{\theta}_H(x)=\frac{g_A}{2}\sqrt{\frac{z_L^2-1}{k}}\phi_4(x). \tag{3.35}$$

The eigenvalues of $2T^{(45)}$ in the spinor representation are ±1, and $\hat{\theta}_H(x)$ is the AB phase. We denote $\langle\hat{\theta}_H\rangle=\theta_H$. The 4D neutral Higgs field $H(x)$ is the fluctuation mode of $\phi_4(x)$ around $\langle\phi_4\rangle$. Hence one finds

$$A_z^{(45)}(x,z)=\frac{1}{\sqrt{k}}\{\theta_Hf_H+H(x)\}u_H(z)+\cdots,\quad f_H=\frac{2}{g_A}\sqrt{\frac{k}{z_L^2-1}}=\frac{2}{g_w}\sqrt{\frac{k}{L(z_L^2-1)}}. \tag{3.36}$$

Under an $SO(5)$ gauge transformation

$$\Omega(y;\alpha)=\exp\left\{-i\frac{g_A\alpha}{\sqrt{2k}}\int_y^L dyv_H(y)T^{(45)}\right\}, \tag{3.37}$$

orbifold boundary conditions $\{P_0,P_1\}$ are changed to

$$P_0'=\Omega(0;2\alpha)P_0=\exp\left\{-i\frac{\alpha}{f_H}\cdot 2T^{(45)}\right\}\cdot P_0,\qquad P_1'=P_1, \tag{3.38}$$

and $\hat{\theta}_H(x)$ is transformed to $\hat{\theta}'_H(x)=\hat{\theta}_H(x)+(\alpha/f_H)$. For $\alpha/f_H=2\pi n$ (where n is an integer), the boundary conditions remain unchanged, whereas θ_H changes to $\theta'_H=\theta_H+2\pi n$. This property reflects the gauge-invariant nature of the AB phase $e^{i\theta_H}$.

Now we go to a new gauge by adopting $\alpha = -\theta_H f_H$ so that $\langle \hat{\theta}'_H \rangle = \theta'_H = 0$, which is called the twisted gauge. It is most convenient to evaluate various physical quantities in this gauge. The twisted gauge was originally introduced in Refs. [40,41] and has been extensively employed in the analysis of GHU. (See, e.g., Refs. [10,33].) Note that the gauge transformation in (3.37) becomes, for $0 \le y \le L$,

$$\Omega(z) = \Omega(y; -\theta_H f_H) = \exp\{i\theta(z) T^{(45)}\},$$
$$\theta(z) = \theta_H \frac{z_L^2 - z^2}{z_L^2 - 1}. \tag{3.39}$$

Quantities in the twisted gauge are denoted with tildes below. In the twisted gauge, the background field vanishes ($\tilde{\theta}_H = 0$), whereas the boundary conditions change as (3.38). For the $SO(5)$ vector representation **5**, the boundary condition matrices $\tilde{P}_j^{\rm vec}$ ($j = 0, 1$) are

$$\tilde{P}_0^{SO(5)} = \Omega(0)^2 P_0^{SO(5)} = e^{2i\theta_H T^{(45)}} P_0^{SO(5)},$$
$$\tilde{P}_1^{SO(5)} = P_1^{SO(5)}. \tag{3.40}$$

For the $SO(5)$ vector representation **5**, the boundary condition matrices $\tilde{P}_j^{\rm vec}$ ($j = 0, 1$) become

$$\tilde{P}_0^{\rm vec} = \begin{pmatrix} I_3 & & \\ & \cos 2\theta_H & -\sin 2\theta_H \\ & -\sin 2\theta_H & -\cos 2\theta_H \end{pmatrix},$$
$$\tilde{P}_1^{\rm vec} = \begin{pmatrix} I_4 & \\ & -1 \end{pmatrix}, \tag{3.41}$$

and for the $SO(5)$ spinor representation **4**,

$$\tilde{P}_0^{\rm sp} = \sigma^0 \otimes \begin{pmatrix} \cos\theta_H & -i\sin\theta_H \\ i\sin\theta_H & -\cos\theta_H \end{pmatrix},$$
$$\tilde{P}_1^{\rm sp} = \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}. \tag{3.42}$$

Here $T_{\rm sp}^{(45)} = \frac{1}{2}\sigma^0 \otimes \sigma^1$ has been used.

IV. SPECTRUM OF GAUGE FIELDS

The spectrum of gauge fields in the present model (type B) is the same as the spectrum in the previous model (type A). We here quote the result for completeness. The bilinear part of the action of gauge fields in (3.10) takes the form

$$S' = \int d^4x \frac{dz}{kz} \sum_{j<k} \bigg[\frac{1}{2} A_\mu^{(jk)} \{ \eta^{\mu\nu} (\Box + k^2 \mathcal{P}_4) - (1 - \xi^{-1}) \partial^\mu \partial^\nu \} A_\nu^{(jk)} + \frac{1}{2} k^2 A_z^{(jk)} (\Box + \xi k^2 \mathcal{P}_z) A_z^{(jk)} + \bar{c}^{(jk)} (\Box + \xi k^2 \mathcal{P}_4) c^{(jk)} \bigg],$$
$$\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu, \qquad \mathcal{P}_4 = z \frac{\partial}{\partial z} \frac{1}{z} \frac{\partial}{\partial z}, \qquad \mathcal{P}_z = \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \frac{1}{z}. \tag{4.1}$$

Additional brane mass terms in (3.28) arise for the A_μ components of $(SU(2)_R \times U(1)_X)/U(1)_Y$.

Boundary conditions in the original gauge are given, in the absence of brane interactions, by

$$\begin{cases} N: \ \frac{\partial}{\partial z} A_\mu = 0 & \text{for parity+} \\ D: \ A_\mu = 0 & \text{for parity}- \end{cases}$$
$$\begin{cases} N: \ \frac{\partial}{\partial z} (\frac{1}{z} A_z) = 0 & \text{for parity+} \\ D: \ A_z = 0 & \text{for parity}- \end{cases} \tag{4.2}$$

at $z = 1$ ($y = 0$) and $z = z_L$ ($y = L$). The parity of each field is summarized in Table I. Because of the brane interaction (3.28), boundary conditions of $A_\mu^{1_R, 2_R, 3'_R}$ at $z = 1$ become

$$D_{\rm eff}(\omega): \left(\frac{\partial}{\partial z} - \omega \right) A_\mu^{1_R, 2_R} = 0, \quad \omega = \frac{g_A^2 w^2}{4k},$$
$$D_{\rm eff}(\omega'): \left(\frac{\partial}{\partial z} - \omega' \right) A_\mu^{3'_R} = 0, \quad \omega' = \frac{(g_A^2 + g_B^2) w^2}{4k}. \tag{4.3}$$

For sufficiently large w, boundary conditions of $A_\mu^{1_R, 2_R, 3'_R}$ at $z = 1$ are modified from the Neumann condition to the Dirichlet condition for low-lying modes in their KK towers. Boundary conditions of gauge fields are summarized in Table IV.

In the twisted gauge, all fields obey free equations in the bulk $1 < z < z_L$, whereas boundary conditions at $z = 1$ become θ_H dependent and nontrivial. $SO(5)$ gauge fields in

TABLE IV. The boundary conditions for the gauge fields at $z = 1$, z_L are summarized. N and D stand for Neumann and Dirichlet conditions, respectively. $D_{\rm eff}$ stands for the effective Dirichlet condition specified in (4.3).

		Number of generators	A_μ	A_z
(1)	$SU(3)_C$	8	(N, N)	(D, D)
(2)	$SU(2)_L$	3	(N, N)	(D, D)
(3)	$U(1)_Y$	1	(N, N)	(D, D)
(4)	$(SU(2)_R \cup U(1)_X)/U(1)_Y$	3	$(D_{\rm eff}, N)$	(D, D)
(5)	$SO(5)_W/(SU(2)_L \cup SU(2)_R)$	4	(D, D)	(N, N)

the twisted gauge are given by $\tilde{A}_M = \Omega(z)A_M\Omega(z)^{-1} + (i/g_A)\Omega(z)\partial_M\Omega(z)^{-1}$, where $\Omega(z)$ is given by (3.39). In particular one finds that

$$\begin{aligned} A_M^{a4} &= \cos\theta(z)\tilde{A}_M^{a4} - \sin\theta(z)\tilde{A}_M^{a5}, \quad (a=1,2,3),\\ A_M^{a5} &= \sin\theta(z)\tilde{A}_M^{a4} + \cos\theta(z)\tilde{A}_M^{a5},\\ A_z^{45} &= \tilde{A}_z^{45} - \frac{\sqrt{2}}{g_A}\theta'(z) = \tilde{A}_z^{45} + \frac{2\sqrt{2}}{g_A}\theta_H\frac{z}{z_L^2-1}, \end{aligned} \tag{4.4}$$

while the other components are unchanged.

At $z=z_L$, $\theta(z_L)=0$, and $\tilde{A}_M$ satisfies the same boundary condition as A_M at $z=z_L$. Consequently, wave functions for $\tilde{A}_\mu$ and $\tilde{A}_z$ are given by the functions tabulated in Table V. The basis functions $C(z;\lambda)$ and $S(z;\lambda)$ there are defined in, e.g., Refs. [10,33], and they are listed in Appendix B.

A. A_μ components

The mass spectra of A_μ components are the following.

(i) $(\tilde{A}_\mu^{a_L}, \tilde{A}_\mu^{a_R}, \tilde{A}_\mu^{\hat{a}})$ $(a=1,\,2)$: W and W_R towers

The boundary conditions at $z=1$ are

$$\frac{\partial}{\partial z}A_\mu^{a_L}=0, \qquad \left(\frac{\partial}{\partial z}-\omega\right)A_\mu^{a_R}=0, \qquad A_\mu^{\hat{a}}=0. \tag{4.5}$$

$\partial A_\mu^{a_R}/\partial z$ is evaluated at $z=1^+$. These conditions with (4.4) lead to the equation which determines the mass spectrum $\{m_n = k\lambda_n\}$:

$$2C'(SC'+\lambda\sin^2\theta_H) - \omega C(2SC'+\lambda\sin^2\theta_H) = 0. \tag{4.6}$$

Here $C=C(1;\lambda)$, $S=S(1;\lambda)$, $C'=C'(1;\lambda)$, and $S'=S'(1;\lambda)$.

For sufficiently large ω, the second term in Eq. (4.6) approximately determines the spectra of low-lying KK modes. This approximation is justified for $w \gg m_{\rm KK}$. In this approximation, the spectra of W and W_R towers are determined by

$$\begin{aligned} &W \text{ tower}: 2S(1;\lambda)C'(1;\lambda) + \lambda\sin^2\theta_H = 0,\\ &W_R \text{ tower}: C(1;\lambda)=0. \end{aligned} \tag{4.7}$$

It follows that the mass of the W boson $m_W = m_{W^{(0)}}$ is given by

$$m_W \simeq \sqrt{\frac{k}{L}}z_L^{-1}\sin\theta_H \simeq \frac{\sin\theta_H}{\pi\sqrt{kL}}m_{\rm KK}, \tag{4.8}$$

where $m_{\rm KK} = \pi k/(z_L-1) \simeq \pi k z_L^{-1}$.

(ii) $(\tilde{A}_\mu^{3_L}, \tilde{A}_\mu^{3'_R}, \tilde{A}_\mu^{\hat{3}}, B_\mu^Y)$: γ, Z, and Z_R towers

The boundary conditions at $z=1$ are

$$\begin{aligned} &\frac{\partial}{\partial z}A_\mu^{3_L}=0, \qquad \left(\frac{\partial}{\partial z}-\omega'\right)A_\mu^{3'_R}=0,\\ &A_\mu^{\hat{3}}=0, \qquad \frac{\partial}{\partial z}B_\mu^Y=0. \end{aligned} \tag{4.9}$$

The spectrum is determined by

$$\begin{aligned} &C'[2C'(SC'+\lambda\sin^2\theta_H)\\ &\quad - \omega' C\{2SC' + (1+s_\phi^2)\lambda\sin^2\theta_H\}] = 0. \end{aligned} \tag{4.10}$$

For sufficiently large ω', the spectrum of low-lying KK modes is approximately determined by the second term. One finds that

$$\begin{aligned} &\gamma \text{ tower}: C'(1;\lambda)=0,\\ &Z \text{ tower}: 2S(1;\lambda)C'(1;\lambda) + (1+s_\phi^2)\lambda\sin^2\theta_H = 0,\\ &Z_R \text{ tower}: C(1;\lambda)=0. \end{aligned} \tag{4.11}$$

The mass of the Z boson $m_Z = m_{Z^{(0)}}$ is given by

$$\begin{aligned} m_Z &\simeq \sqrt{1+s_\phi^2}\sqrt{\frac{k}{L}}z_L^{-1}\sin\theta_H\\ &\simeq \sqrt{1+s_\phi^2}\frac{\sin\theta_H}{\pi\sqrt{kL}}m_{\rm KK}. \end{aligned} \tag{4.12}$$

We recall the relation [12]

$$\frac{1}{\sqrt{1+s_\phi^2}} \simeq \cos\theta_W, \qquad \sin\theta_W \simeq \frac{g'_B}{\sqrt{g_A^2+2g_B'^2}}. \tag{4.13}$$

It follows from (4.8) and (4.12) that

$$m_Z \simeq \frac{m_W}{\cos\theta_W}, \tag{4.14}$$

which coincides with the relation in the SM.

(iii) $\tilde{A}_\mu^{\hat{4}}$: $A^{\hat{4}}$ tower

$A_\mu^{\hat{4}}$ obeys the (D,D) boundary condition and there is no zero mode. Its spectrum is determined by

$$\hat{A}^4 \text{ tower}: S(1;\lambda)=0. \tag{4.15}$$

(iv) $SU(3)_C$ gluons

The boundary condition is (N,N) so that

$$\text{gluon tower}: C'(1;\lambda)=0. \tag{4.16}$$

TABLE V. Wave functions of the gauge fields in the twisted gauge. N and D stand for Neumann and Dirichlet conditions at $z=z_L$. The basis functions $C(z;\lambda)$ and $S(z;\lambda)$ are given in Appendix B.

BC at $z=z_L$	N	D
$\tilde{A}_\mu$	$C(z;\lambda)$	$S(z;\lambda)$
$\tilde{A}_z$	$S'(z;\lambda)$	$C'(z;\lambda)$

B. A_z components

The mass spectra of A_z components are the following. Except for the zero modes, masses are given by $\{m_n = \xi k \lambda_n\}$.

(i) $A_z^{ab} (1 \le a < b \le 3), B_z$

These components satisfy boundary conditions (D, D) so that

$$C'(1;\lambda) = 0. \tag{4.17}$$

(ii) A_z^{a4}, A_z^{a5} $(a = 1, 2, 3)$

The boundary conditions at $z = 1$ are

$$A_z^{a4} = 0, \qquad \frac{\partial}{\partial z}\left(\frac{1}{z} A_z^{a5}\right) = 0. \tag{4.18}$$

The spectrum is determined by

$$S(1;\lambda) C'(1;\lambda) + \lambda \sin^2\theta_H = 0. \tag{4.19}$$

(iii) A_z^{45}: Higgs tower

The boundary conditions of A_z^{45} are (N, N) and the spectrum is determined by

$$\text{Higgs tower}: S(1;\lambda) = 0. \tag{4.20}$$

There is a zero mode, which will acquire a mass at the one-loop level.

(iv) $SU(3)_C A_z$

There are no zero modes. Their components satisfy boundary conditions (D, D). The mass spectrum is determined by

$$C'(1;\lambda) = 0. \tag{4.21}$$

V. SPECTRUM OF FERMION FIELDS

We determine the mass spectra of fermion fields. It will be seen that the mass spectrum of quarks and leptons in three generations is reproduced except for the down-quark mass which turns out smaller than the up-quark mass ($m_d < m_u$). To evaluate the effective potential $V_{\rm eff}(\theta_H)$ for the AB phase θ_H, one needs to know the mass spectra of the dark fermion fields in (3.7) and (3.8) as well. We summarize the result for dark fermions in Appendix D for completeness.

In the original gauge, the background gauge field in $SO(5)$ is

$$g A_z^{cl} = \frac{g_A}{\sqrt{2}} A_z^{(45)} T^{45} = -\theta'(z) T^{45}, \tag{5.1}$$

where $\theta(z)$ is defined in (3.39). We introduce the following derivatives:

$$D_\pm(c) = \pm \frac{\partial}{\partial z} + \frac{c}{z},$$
$$\hat{D}_\pm(c) = D_\pm(c) \pm i\theta'(z) T^{45}. \tag{5.2}$$

To simplify the notation, the bulk mass parameters of various fields are denoted as

$$c_Q = c_{\Psi^\alpha_{(\mathbf{3},\mathbf{4})}}, \qquad c_L = c_{\Psi^\alpha_{(\mathbf{1},\mathbf{4})}},$$
$$c_{D^\pm} = c_{\Psi^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}}, \qquad c_{V^\pm} = c_{\Psi^{\pm\beta}_{(\mathbf{1},\mathbf{5})}}. \tag{5.3}$$

We have suppressed generation indices α, β. In this paper we consider the cases $c_{D^+} = \pm c_{D^-}$ and $c_{V^+} = \pm c_{V^-}$, for which exact solutions are available.

The components of $SO(5)$ spinor fermions $\Psi_{(\mathbf{3},\mathbf{4})}$ and $\Psi_{(\mathbf{1},\mathbf{4})}$ in the original and twisted gauges are related to each other by

$$\chi = \begin{pmatrix} \cos\frac{1}{2}\theta(z) & -i\sin\frac{1}{2}\theta(z) \\ -i\sin\frac{1}{2}\theta(z) & \cos\frac{1}{2}\theta(z) \end{pmatrix} \tilde{\chi}, \tag{5.4}$$

where χ is given by

$$\chi = \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} d \\ d' \end{pmatrix}, \begin{pmatrix} e \\ e' \end{pmatrix}, \begin{pmatrix} \nu \\ \nu' \end{pmatrix}. \tag{5.5}$$

$T^{45} = \frac{1}{2}\sigma^1$ for these χ's.

A. Up-type quarks

$Q_{\rm EM} = +\frac{2}{3}$: u, u' $(\Psi_{(\mathbf{3},\mathbf{4})})$

There are no brane mass terms. The boundary conditions are given by $D_+ \check{u}_L = 0$, $\check{u}_R = 0$, $\check{u}'_L = 0$, and $D_- \check{u}'_R = 0$ at $z = 1$, z_L. The equations of motion in the twisted gauge are

$$-i\delta \begin{pmatrix} u_L^\dagger \\ u_L'^\dagger \end{pmatrix}: -kD_-(c_Q) \begin{pmatrix} \tilde{\check{u}}_R \\ \tilde{\check{u}}'_R \end{pmatrix} + \sigma^\mu \partial_\mu \begin{pmatrix} \tilde{\check{u}}_L \\ \tilde{\check{u}}'_L \end{pmatrix} = 0,$$
$$i\delta \begin{pmatrix} u_R^\dagger \\ u_R'^\dagger \end{pmatrix}: -kD_+(c_Q) \begin{pmatrix} \tilde{\check{u}}_L \\ \tilde{\check{u}}'_L \end{pmatrix} + \bar{\sigma}^\mu \partial_\mu \begin{pmatrix} \tilde{\check{u}}_R \\ \tilde{\check{u}}'_R \end{pmatrix} = 0. \tag{5.6}$$

$(\tilde{\check{u}}, \tilde{\check{u}}')$ satisfy the same boundary conditions at $z = z_L$ as $(\check{u}, \check{u}')$ so that one can write, in terms of basis functions summarized in Appendix B,

$$\begin{pmatrix} \tilde{\check{u}}_R \\ \tilde{\check{u}}'_R \end{pmatrix} = \begin{pmatrix} \alpha_u S_R^Q \\ \alpha'_u C_R^Q \end{pmatrix} f_R(x), \qquad \begin{pmatrix} \tilde{\check{u}}_L \\ \tilde{\check{u}}'_L \end{pmatrix} = \begin{pmatrix} \alpha_u C_L^Q \\ \alpha'_u S_L^Q \end{pmatrix} f_L(x), \tag{5.7}$$

where $C^Q_{L/R} = C_{L/R}(z, \lambda, c_Q)$; $S^Q_{L/R} = S_{L/R}(z, \lambda, c_Q)$; $\bar{\sigma}\partial f_R(x) = k\lambda f_L(x)$; and $\sigma\partial f_L(x) = k\lambda f_R(x)$. Both right- and left-handed modes have the same coefficients α_u and $\alpha_{u'}$ as a consequence of Eqs. (5.6).

By making use of (5.4), the boundary conditions at $z = 1$ for the right-handed components $\check{u}_R = 0$ and $D_- \check{u}'_R = 0$ become

$$K_u \begin{pmatrix} \alpha_u \\ \alpha_{u'} \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta_H S_R^Q & -i\sin\frac{1}{2}\theta_H C_R^Q \\ -i\sin\frac{1}{2}\theta_H C_L^Q & \cos\frac{1}{2}\theta_H S_L^Q \end{pmatrix} \begin{pmatrix} \alpha_u \\ \alpha_{u'} \end{pmatrix} = 0. \tag{5.8}$$

Here $S^Q_{L/R} = S_{L/R}(1, \lambda, c_Q)$, etc. $\det K_u = 0$ leads to the equation determining the spectrum:

$$S_L^Q S_R^Q + \sin^2\frac{\theta_H}{2} = 0. \tag{5.9}$$

The mass of the lowest mode (up-type quark) $m = k\lambda$ is given by

$$m_u = \begin{cases} \pi^{-1}\sqrt{1-4c_Q^2}\,\sin\frac{1}{2}\theta_H m_{\rm KK} & \text{for } |c_Q| < \frac{1}{2}, \\ \pi^{-1}\sqrt{4c_Q^2-1}\,z_L^{-|c_Q|+0.5}\sin\frac{1}{2}\theta_H m_{\rm KK} & \text{for } |c_Q| > \frac{1}{2}. \end{cases} \tag{5.10}$$

Note that $S_L(z;\lambda,-c) = -S_R(z;\lambda,c)$ and $C_L(z;\lambda,-c) = C_R(z;\lambda,c)$. With a given m_u, there are two solutions to (5.9): $c_Q > 0$ and $c_Q < 0$.

B. Down-type quarks

$Q_{\rm EM} = -\frac{1}{3}$: d, d', $D^\pm$ $(\Psi_{(\mathbf{3},\mathbf{4})}, \Psi^\pm_{(\mathbf{3},\mathbf{1})})$

As seen from Table III, parity even modes at $y = 0$ with $(P_0, P_1) = (+,+)$ are d_L, d'_R, D_L^+, and D_R^-. From the action (3.16) and the $\mathcal{L}_1^m$ term in (3.27), the equations of motion in the original gauge are given by

$$\begin{aligned}
&\begin{matrix}(a)\\(b)\end{matrix}: \ -i\delta\begin{pmatrix} d_L^\dagger \\ d_L'^\dagger\end{pmatrix}: \ -k\hat{D}_-(c_Q)\begin{pmatrix}\check{d}_R\\ \check{d}'_R\end{pmatrix} + \sigma^\mu\partial_\mu\begin{pmatrix}\check{d}_L\\ \check{d}'_L\end{pmatrix} = 0,\\
&\begin{matrix}(c)\\(d)\end{matrix}: \ i\delta\begin{pmatrix} d_R^\dagger \\ d_R'^\dagger\end{pmatrix}: \ \bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{d}_R\\ \check{d}'_R\end{pmatrix} - k\hat{D}_+(c_Q)\begin{pmatrix}\check{d}_L\\ \check{d}'_L\end{pmatrix} = 2\mu_1\delta(y)\begin{pmatrix}0\\ \check{D}_L^+\end{pmatrix},\\
&(e): \ -i\delta D_L^{+\dagger}: \ -k\hat{D}_-(c_{D+})\check{D}_R^+ + \sigma^\mu\partial_\mu\check{D}_L^+ - \frac{m_D^*}{z}\check{D}_R^- = 2\mu_1^*\delta(y)\check{d}'_R,\\
&(f): \ i\delta D_R^{+\dagger}: \ \bar{\sigma}^\mu\partial_\mu\check{D}_R^+ - k\hat{D}_+(c_{D+})\check{D}_L^+ - \frac{m_D}{z}\check{D}_L^- = 0,\\
&(g): \ -i\delta D_L^{-\dagger}: \ -k\hat{D}_-(c_{D-})\check{D}_R^- + \sigma^\mu\partial_\mu\check{D}_L^- - \frac{m_D^*}{z}\check{D}_R^+ = 0,\\
&(h): \ i\delta D_R^{-\dagger}: \ \bar{\sigma}^\mu\partial_\mu\check{D}_R^- - k\hat{D}_+(c_{D-})\check{D}_L^- - \frac{m_D}{z}\check{D}_L^+ = 0.
\end{aligned} \tag{5.11}$$

Note that the mass dimension of each coupling constant and field is, e.g., $[\check{d}_{R/L}] = 2$, $[k] = [m_D] = 1$, and $[\mu_1] = 0$.

The following arguments are parallel to those in Ref. [33]. Under the parity transformation around $y = 0$, $\Psi_+ = d_L$, d'_R, D_L^+, and D_R^- are parity even, whereas $\Psi_- = d_R$, d'_L, D_R^+, and D_L^- are parity odd. Note that $\Psi_-(y)|_{-\epsilon}^{+\epsilon} = 2\Psi_-(+\epsilon)$ and

$$D_\pm(c) = \frac{e^{-\sigma(y)}}{k}\left\{\pm\frac{\partial}{\partial y} + c\sigma'(y)\right\} \tag{5.12}$$

in the y coordinate. We integrate the equations for parity odd fields, (a), (d), (e), and (h) in (5.11), from $y = -\epsilon$ to $+\epsilon$ to find

$$\begin{aligned}
&(a) \Rightarrow \check{d}_R(\epsilon) = 0,\\
&(d) \Rightarrow -2\check{d}'_L(\epsilon) - 2\mu_1\check{D}_L^+(0) = 0,\\
&(e) \Rightarrow 2\check{D}_R^+(\epsilon) - 2\mu_1^*\check{d}'_R(0) = 0,\\
&(h) \Rightarrow \check{D}_L^-(\epsilon) = 0.
\end{aligned} \tag{5.13}$$

For parity-even fields, we evaluate the equations at $y = +\epsilon$ by using the relations (5.13):

$$\begin{aligned}
&(c) \Rightarrow \hat{D}_+(c_Q)\check{d}_L = 0,\\
&(b) \Rightarrow \mu_1[\hat{D}_-(c_{D+})\check{D}_R^+ + \tilde{m}_D^*\check{D}_R^-] + \hat{D}_-(c_Q)\check{d}'_R = 0,\\
&(f) \Rightarrow \mu_1^*\hat{D}_+(c_Q)\check{d}'_L - \hat{D}_+(c_{D+})\check{D}_L^+ = 0,\\
&(g) \Rightarrow \hat{D}_-(c_{D-})\check{D}_R^- + \tilde{m}_D^*\mu_1^*\check{d}'_R = 0,
\end{aligned} \tag{5.14}$$

where the equations of motion (e) and (d) at $y = +\epsilon$ have been made use of. Relations (5.13) and (5.14) specify the boundary conditions at $z = 1^+$. We examine the spectrum in two cases, $c_{D^+} = c_{D^-}$ and $c_{D^+} = -c_{D^-}$ below.

1. Case I: $c_{D^+} = c_{D^-} = c_D$

The BCs at $z = z_L$ are given by

$$\begin{cases} d_R = 0,\\ D_+(c_Q)d_L = 0,\\ D_-(c_Q)d'_R = 0,\\ d'_L = 0, \end{cases} \qquad \begin{cases} D_R^+ = 0,\\ D_+(c_D)D_L^+ = 0,\\ D_-(c_D)D_R^- = 0,\\ D_L^- = 0. \end{cases} \tag{5.15}$$

In the twisted gauge, the BCs in (5.15) are satisfied by mode functions in (B6) and (B23) so that one can write as

$$\begin{pmatrix} \tilde{\check{d}}_R \\ \tilde{\check{d}}'_R \\ \tilde{\check{D}}^+_R \\ \tilde{\check{D}}^-_R \end{pmatrix} = \begin{pmatrix} \alpha_d S_R(z;\lambda,c_Q) \\ \alpha_{d'} C_R(z;\lambda,c_Q) \\ a_d \mathcal{S}_{R2}(z;\lambda,c_D,\tilde{m}_D) + b_d \mathcal{S}_{R1}(z;\lambda,c_D,\tilde{m}_D) \\ a_d \mathcal{C}_{R1}(z;\lambda,c_D,\tilde{m}_D) + b_d \mathcal{C}_{R2}(z;\lambda,c_D,\tilde{m}_D) \end{pmatrix},$$

$$\begin{pmatrix} \tilde{\check{d}}_L \\ \tilde{\check{d}}'_L \\ \tilde{\check{D}}^+_L \\ \tilde{\check{D}}^-_L \end{pmatrix} = \begin{pmatrix} \alpha_d C_L(z;\lambda,c_Q) \\ \alpha_{d'} S_L(z;\lambda,c_Q) \\ a_d \mathcal{C}_{L2}(z;\lambda,c_D,\tilde{m}_D) + b_d \mathcal{C}_{L1}(z;\lambda,c_D,\tilde{m}_D) \\ a_d \mathcal{S}_{L1}(z;\lambda,c_D,\tilde{m}_D) + b_d \mathcal{S}_{L2}(z;\lambda,c_D,\tilde{m}_D) \end{pmatrix},$$

$$\tag{5.16}$$

where α_d, $\alpha_{d'}$, a_d, and b_d are parameters.

Boundary conditions at $z = 1^+$ for the left-handed fields $\check{d}_L$, $\check{d}'_L$, $\check{D}^+_L$, and $\check{D}^-_L$ are found from Eqs. (5.13) and (5.14) to be

$$\begin{aligned} &(c):\ \lambda\left(\cos\frac{\theta_H}{2}\alpha_d S^Q_R - i\sin\frac{\theta_H}{2}\alpha_{d'} C^Q_R\right) = 0, \\ &(d):\ -i\sin\frac{\theta}{2}\alpha_d C^Q_L + \cos\frac{\theta}{2}\alpha_{d'} S^Q_L + \mu_1(a_d \mathcal{C}^D_{L2} + b_d \mathcal{C}^D_{L1}) = 0, \\ &(f):\ \lambda\mu_1^*\left(-i\sin\frac{\theta_H}{2}\alpha_d S^Q_R + \cos\frac{\theta_H}{2}\alpha_{d'} C^Q_R\right) \\ &\qquad -\lambda(a_d \mathcal{S}^D_{R2} + b_d \mathcal{S}^D_{R1}) + \tilde{m}_D(a_d \mathcal{S}^D_{L1} + b_d \mathcal{S}^D_{L2}) = 0, \\ &(h):\ a_d \mathcal{S}^D_{L1} + b_d \mathcal{S}^D_{L2} = 0, \end{aligned} \tag{5.17}$$

where $S^Q_{L/R} \coloneqq S_{L/R}(z=1;\lambda,c_Q)$; $\mathcal{S}^D_{L/Rj} \coloneqq \mathcal{S}_{L/Rj}(z=1;\lambda,c_D,\tilde{m}_D)$; etc. Conditions in (5.17) are summarized as

$$\mathcal{M}^D_L V^D = \begin{pmatrix} \cos\frac{\theta_H}{2} S^Q_R & -i\sin\frac{\theta_H}{2} C^Q_R & 0 & 0 \\ -i\sin\frac{\theta_H}{2} C^Q_L & \cos\frac{\theta_H}{2} S^Q_L & \mu_1 \mathcal{C}^D_{L2} & \mu_1 \mathcal{C}^D_{L1} \\ -i\mu_1^* \sin\frac{\theta_H}{2} S^Q_R & \mu_1^*\cos\frac{\theta_H}{2} C^Q_R & -\mathcal{S}^D_{R2} & -\mathcal{S}^D_{R1} \\ 0 & 0 & \mathcal{S}^D_{L1} & \mathcal{S}^D_{L2} \end{pmatrix} \times \begin{pmatrix} \alpha_d \\ \alpha_{d'} \\ a_d \\ b_d \end{pmatrix} = 0. \tag{5.18}$$

The mass spectrum is determined by

$$\begin{aligned} \det \mathcal{M}^D_L = {}& \left(S^Q_L S^Q_R + \sin^2\frac{\theta_H}{2}\right)(\mathcal{S}^D_{L1}\mathcal{S}^D_{R1} - \mathcal{S}^D_{L2}\mathcal{S}^D_{R2}) \\ & + |\mu_1|^2 C^Q_R S^Q_R (\mathcal{S}^D_{L1}\mathcal{C}^D_{L1} - \mathcal{S}^D_{L2}\mathcal{C}^D_{L2}) = 0. \end{aligned} \tag{5.19}$$

Note the relations (B21).

To lift the degeneracy between the up-type and down-type quark masses, the μ_1 term in (5.19) is necessary. Its coefficient contains the factor $C^Q_R = C_R(1;\lambda,c_Q)$. For the first and second generations, $|c_Q| = |c_u|, |c_c| > \frac{1}{2}$. For $\lambda z_L \ll 1$, $C_R(1;\lambda,c) \sim z_L^{-c} \ll 1$ for $c > \frac{1}{2}$ and $C_R(1;\lambda,c) \gg 1$ for $c < -\frac{1}{2}$. The detailed study shows that with $c > \frac{1}{2}$, Eq. (5.19) necessarily yields the first KK mode with a mass much less than $m_{\rm KK}$, which contradicts observation. One needs to take c_u, $c_c < 0$. For the third generation, $|c_t| < \frac{1}{2}$, and this problem does not show up.

Consider case I, $c_{D^+} = c_{D^-} = c_D > 0$, with $\tilde{m}_D > 1/2$ and $c_D - \tilde{m}_D > 1/2$. The up-type quark mass m_u for $|c_Q| > \frac{1}{2}$ is approximately given by

$$m_u = \lambda_u z_L \simeq \sqrt{4c_Q^2 - 1}\, z_L^{-|c_Q|+\frac{1}{2}} \sin\frac{\theta_H}{2} \tag{5.20}$$

from Eq. (5.9). Substituting

$$\begin{aligned} S^Q_L S^Q_R + \sin^2\frac{\theta_H}{2} &\simeq -\frac{\lambda^2 z_L^{2|c_Q|+1}}{4c_Q^2 - 1} + \sin^2\frac{\theta_H}{2} \simeq -\frac{(\lambda^2 - \lambda_u^2) z_L^{2|c_Q|+1}}{4c_Q^2 - 1}, \\ \mathcal{S}^Q_{L1}\mathcal{S}^Q_{R1} - \mathcal{S}^Q_{L2}\mathcal{S}^Q_{R2} &\simeq z_L^{2\tilde{m}_D} - \lambda^2 z_L^{2c_D+1}\left(\frac{1}{4c_D^2 - (2\tilde{m}_D+1)^2} + \frac{1}{4c_D^2 - (2\tilde{m}_D-1)^2}\right), \\ C^Q_R S^Q_R &\simeq \begin{cases} \frac{\lambda}{2c_Q - 1} & \text{for } c_Q > 0, \\ \frac{\lambda z_L^{2|c_Q|+1}}{2|c_Q|+1} & \text{for } c_Q < 0, \end{cases} \\ \mathcal{S}^Q_{L1}\mathcal{C}^Q_{L1} - \mathcal{S}^Q_{L2}\mathcal{C}^Q_{L2} &\simeq -2\lambda z_L^{2c_D+1}\left(\frac{1}{2(c_D+\tilde{m}_D)+1} + \frac{1}{2(c_D-\tilde{m}_D)+1}\right) \end{aligned} \tag{5.21}$$

into $\mathcal{M}_L^D$ in Eq. (5.19), we find

$$\det\mathcal{M}_L^D = -\frac{(\lambda^2-\lambda_u^2)z_L^{2|c_Q|+1}}{4c_Q^2-1}(z_L^{2\tilde{m}_D}-\lambda^2 z_L^{2c_D+1}A) + |\mu_1|^2\begin{Bmatrix}\frac{1}{2c_Q-1}\\ \frac{z_L^{2|c_Q|+1}}{2|c_Q|+1}\end{Bmatrix}(-2\lambda^2 z_L^{2c_D+1})B = 0 \ \text{for} \begin{cases} c_Q>0\\ c_Q<0\end{cases}, \tag{5.22}$$

where

$$A = \frac{1}{4c_D^2-(2\tilde{m}_D+1)^2}+\frac{1}{4c_D^2-(2\tilde{m}_D-1)^2} > 0, \tag{5.23}$$

$$B = \frac{1}{2(c_D+\tilde{m}_D)+1}+\frac{1}{2(c_D-\tilde{m}_D)+1} > 0. \tag{5.24}$$

Both A and B are $O(1)$. If $z_L^{2\tilde{m}_D} \gg \lambda^2 z_L^{2c_D+1}$, then it follows from (5.22) that

$$\lambda^2 \simeq \begin{cases} \frac{\lambda_u^2}{1+2|\mu_1|^2(2c_Q+1)z_L^{-2c_Q+2c_D-2\tilde{m}_D}B} < \lambda_u^2 & \text{for } c_Q > \frac{1}{2},\\ \frac{\lambda_u^2}{1+2|\mu_1|^2(2|c_Q|-1)z_L^{2c_D-2\tilde{m}_D+1}B} < \lambda_u^2 & \text{for } c_Q < -\frac{1}{2}.\end{cases} \tag{5.25}$$

In other words, the spectrum for the second generation $m_s < m_c$ can be reproduced with appropriate μ_1, c_Q, and $\tilde{m}_D$.

Indeed, one can show that the smallest value of λ^2 determined from Eq. (5.19) necessarily becomes smaller than λ_u^2 with general $\mu_1 \neq 0$, c_Q, and $\tilde{m}_D$. For $\lambda z_L \ll 1$, Eq. (5.19) reduces to the form $(\lambda^2-\lambda_u^2)(\lambda^2-a) - b|\mu_1|^2\lambda^2 = 0$, where $a \gg \lambda_u^2$ and $b > 0$. Consequently the two roots $\lambda^2 = \lambda_\pm^2$ satisfy $\lambda_-^2 < \lambda_u^2$ and $\lambda_+^2 > a$. This implies that the spectrum $m_d > m_u$ cannot be realized at the tree level in the current scheme. It is left for future investigation to find a solution to this problem.

Typical values of the parameters reproducing the quark mass spectrum (except for m_d) are tabulated in Table VI. $\det\mathcal{M}_L^D$ in Eq. (5.19) for the second generation is plotted as a function of λ for $\tilde{m}_D = 1.0$ and various values of μ_1 in Fig. 1.

2. Case II: $c_{D^+} = -c_{D^-} = c_D$

The BCs at $z = z_L$ are given by

$$\begin{cases} d_R = 0,\\ D_+(c_Q)d_L = 0,\\ D_-(c_Q)d'_R = 0,\\ d'_L = 0,\end{cases} \qquad \begin{cases} D_R^+ = 0,\\ D_+(c_D)D_L^+ = 0,\\ D_+(c_D)D_R^- = 0,\\ D_L^- = 0.\end{cases} \tag{5.26}$$

In the twisted gauge, the BCs in Eq. (5.26) are satisfied by mode functions in (B6) and (B46) so that one can write

$$\begin{pmatrix}\tilde{\check{d}}_R\\ \tilde{\check{d}}'_R\\ \tilde{\check{D}}_R^+\\ \tilde{\check{D}}_R^-\end{pmatrix} = \begin{pmatrix}\alpha_d S_R(z;\lambda,c_Q)\\ \alpha_{d'}C_R(z;\lambda,c_Q)\\ a_d\hat{\mathcal{S}}_{R2}(z;\lambda,c_D,\tilde{m}_D)+b_d\hat{\mathcal{S}}_{R1}(z;\lambda,c_D,\tilde{m}_D)\\ a_d\hat{\mathcal{C}}_{L1}(z;\lambda,c_D,\tilde{m}_D)+b_d\hat{\mathcal{C}}_{L2}(z;\lambda,c_D,\tilde{m}_D)\end{pmatrix},$$
$$\begin{pmatrix}\tilde{\check{d}}_L\\ \tilde{\check{d}}'_L\\ \tilde{\check{D}}_L^+\\ \tilde{\check{D}}_L^-\end{pmatrix} = \begin{pmatrix}\alpha_d C_L(z;\lambda,c_Q)\\ \alpha_{d'}S_L(z;\lambda,c_Q)\\ a_d\hat{\mathcal{C}}_{L2}(z;\lambda,c_D,\tilde{m}_D)+b_d\hat{\mathcal{C}}_{L1}(z;\lambda,c_D,\tilde{m}_D)\\ -a_d\hat{\mathcal{S}}_{R1}(z;\lambda,c_D,\tilde{m}_D)-b_d\hat{\mathcal{S}}_{R2}(z;\lambda,c_D,\tilde{m}_D)\end{pmatrix}, \tag{5.27}$$

where α_d, $\alpha_{d'}$, a_d, and b_d are parameters.

From Eqs. (5.13) and (5.14), we find the boundary conditions at $z = 1$ for the left-handed fields. The manipulation is similar to that in case I. The difference appears only for terms involving $D_{L/R}^-$. It is straightforward to see

$$\mathcal{M}_L^D V^D = \begin{pmatrix}\cos\frac{\theta_H}{2}S_R^Q & -i\sin\frac{\theta_H}{2}C_R^Q & 0 & 0\\ -i\sin\frac{\theta_H}{2}C_L^Q & \cos\frac{\theta_H}{2}S_L^Q & \mu_1\hat{\mathcal{C}}_{L2}^D & \mu_1\hat{\mathcal{C}}_{L1}^D\\ -i\mu_1^*\sin\frac{\theta_H}{2}S_R^Q & \mu_1^*\cos\frac{\theta_H}{2}C_R^Q & -\hat{\mathcal{S}}_{R2}^D & -\hat{\mathcal{S}}_{R1}^D\\ 0 & 0 & \hat{\mathcal{S}}_{R1}^D & \hat{\mathcal{S}}_{R2}^D\end{pmatrix} \times \begin{pmatrix}\alpha_d\\ \alpha_{d'}\\ a_d\\ b_d\end{pmatrix} = 0, \tag{5.28}$$

TABLE VI. Parameters which reproduce the spectrum of quarks for $\theta_H = 0.15$, $z_L = 10^{10}$. $m_{\rm KK} = 8.062$ TeV. The masses of the first KK modes of up-type and down-type quarks are also shown. c_u, $c_c < 0$ for the reason described below Eq. (5.19). The values $m_u = 1.27$ MeV, $m_s = 55$ MeV, $m_c = 619$ MeV, $m_b = 2.89$ GeV, and $m_t = 171.17$ GeV have been used. $m_d = 0.9m_u$ has been used for the first generation.

Quarks	c_Q	μ_1	c_D	$\tilde{m}_D$	$m_{d^{(1)}}$ (TeV)	$m_{u^{(1)}}$ (TeV)
(u,d)	−1.044	0.01	0.6194	1.0	4.59	8.23
		0.1	0.4612	1.0	4.80	
(c,s)	−0.7546	0.1	0.6808	1.0	5.40	7.16
		10.	0.0949	1.0	5.22	
(t,b)	+0.2287	0.1	0.5838	0.1	2.84	7.20
		10.	0.3791	0.1	2.84	
	−0.2287	0.1	1.044	1.0	5.06	
		10.	0.8352	1.0	5.06	

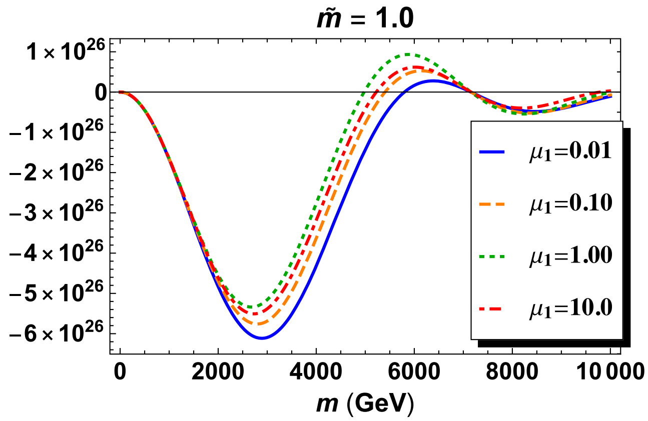

FIG. 1. Spectrum of strange quark tower. $\det \mathcal{M}_L^D$ in Eq. (5.19) is plotted as a function of $m = k\lambda$ for $\tilde{m}_D = 1.0$ and various values of μ_1. The mass spectrum $\{m_n = k\lambda_n\}$ is determined by roots of $\det \mathcal{M}_L^D = 0$. $m_{\rm KK} = 8062$ GeV.

where $S^Q_{L/R} \coloneqq S_{L/R}(z=1;\lambda, c_Q)$; $\hat{\mathcal{S}}^D_{L/Rj} = \hat{\mathcal{S}}_{L/Rj}(z=1; \lambda, c_D, \tilde{m}_D)$; etc. The spectrum is determined by

$$\det \mathcal{M}_L^D = \left(S^Q_L S^Q_R + \sin^2\frac{\theta_H}{2}\right)\{(\hat{\mathcal{S}}^D_{R1})^2 - (\hat{\mathcal{S}}^D_{R2})^2\} + |\mu_1|^2 C^Q_R S^Q_R(\hat{\mathcal{S}}^D_{R1}\hat{\mathcal{C}}^D_{L1} - \hat{\mathcal{S}}^D_{R2}\hat{\mathcal{C}}^D_{L2}) = 0. \quad (5.29)$$

Note the relation (B35).

For $|c_Q|, \hat{c} > \frac{1}{2}$, $c_D > 0$, and $\lambda z_L \ll 1$, we have

$$S^Q_L S^Q_R + \sin^2\frac{\theta_H}{2} \simeq -\frac{(\lambda^2 - \lambda_u^2) z_L^{2|c_Q|+1}}{4c_Q^2 - 1},$$
$$(\hat{\mathcal{S}}^D_{R1})^2 - (\hat{\mathcal{S}}^D_{L2})^2 \sim -\alpha_+^2 z_L^{2\hat{c}},$$
$$\hat{\mathcal{S}}^D_{R1}\mathcal{C}^D_{L1} - \hat{\mathcal{S}}^D_{R2}\mathcal{C}^D_{L2} \sim (1+\alpha_+^2)\frac{\lambda z_L^{2\hat{c}}}{2\hat{c}-1}, \quad (5.30)$$

so that

$$\det \mathcal{M}_L^D \simeq -\frac{(\lambda^2 - \lambda_u^2) z_L^{2|c_Q|+1}}{4c_Q^2 - 1}\cdot(-\alpha_+^2 z_L^{2\hat{c}}) + |\mu_1|^2 \left\{ \begin{array}{c} \frac{1}{2c_Q - 1} \\ \frac{z_L^{2|c_Q|+1}}{2|c_Q|+1} \end{array} \right\} (1+\alpha_+^2)\frac{\lambda^2 z_L^{2\hat{c}}}{2\hat{c}+1} = 0$$
$$\text{for} \left\{ \begin{array}{l} c_Q > \frac{1}{2} \\ c_Q < -\frac{1}{2} \end{array} \right. . \quad (5.31)$$

Thus we find

$$\lambda^2\left[1 + \frac{|\mu_1|^2}{2\hat{c}-1}\left\{ \begin{array}{c} (2c_Q+1) z_L^{-2|c_Q|-1} \\ 2|c_Q| - 1 \end{array} \right\}\frac{1+\alpha_+^2}{\alpha_+^2}\right] = \lambda_u^2 \quad \text{for} \left\{ \begin{array}{l} c_Q > \frac{1}{2} \\ c_Q < -\frac{1}{2} \end{array} \right. . \quad (5.32)$$

We observe that $\lambda^2 < \lambda_u^2$ so that $m_d > m_u$ cannot be realized with this parametrization, as in case I.

C. Charged lepton

$Q_{\rm EM} = -1$: e, e' $(\Psi_{(\mathbf{1},\mathbf{4})})$

In general, $\Psi_{(\mathbf{1},\mathbf{4})}$ may couple with $\Psi^{\pm}_{(\mathbf{1},\mathbf{5})}$ through the brane interaction $\mathcal{L}_2^m$ in (3.27). We suppose that $\tilde{\mu}_2$ there is sufficiently small so that the effect of $\mathcal{L}_2^m$ can be ignored. In this case the equations and boundary conditions for e, e' take the same form as those for u, u'. Mode functions and boundary conditions are summarized as

$$\begin{pmatrix} \check{\tilde{e}}_R \\ \check{\tilde{e}}'_R \end{pmatrix} = \begin{pmatrix} \alpha_e S_R(z,\lambda,c_L) \\ \alpha_{e'} C_R(z,\lambda,c_L) \end{pmatrix},$$
$$\begin{pmatrix} \check{\tilde{e}}_L \\ \check{\tilde{e}}'_L \end{pmatrix} = \begin{pmatrix} \alpha_e C_L(z,\lambda,c_L) \\ \alpha_{e'} S_L(z,\lambda,c_L) \end{pmatrix},$$
$$\begin{pmatrix} \cos\frac{1}{2}\theta_H S^L_R & -i\sin\frac{1}{2}\theta_H C^L_R \\ -i\sin\frac{1}{2}\theta_H C^L_L & \cos\frac{1}{2}\theta_H S^L_L \end{pmatrix}\begin{pmatrix} \alpha_e \\ \alpha_{e'} \end{pmatrix} = 0, \quad (5.33)$$

where $S^L_{L/R} = S_{L/R}(1,\lambda,c_L)$, etc., in the last equation. The mass spectrum is determined by

$$S^L_L S^L_R + \sin^2\frac{\theta_H}{2} = 0. \quad (5.34)$$

The mass of the lowest mode (charged lepton) $m = k\lambda$ is given by

$$m_e = \pi^{-1}\sqrt{4c_L^2 - 1}\, z_L^{-|c_L|+0.5}\sin\frac{1}{2}\theta_H m_{\rm KK}. \quad (5.35)$$

Note $|c_L| > \frac{1}{2}$.

D. Neutrino

$Q_{\rm EM} = 0$: $\nu, \nu', \chi(\Psi_{(\mathbf{1},\mathbf{4})(-3)}, \chi)$

As mentioned above, we assume that $\mathcal{L}_2^m$ can be ignored. The brane interaction $\mathcal{L}_3$ in (3.26) yields the coupling between ν' and χ, $\mathcal{L}_3^m$ in (3.27). It leads to the gauge-Higgs seesaw mechanism [35]. In the present paper we treat the case in which all brane interactions are diagonal in generations. In particular we set $M^{\alpha\beta} = -M_\alpha \delta^{\alpha\beta}$ in (3.27).

Equations of motion are given by

$$\begin{array}{l}(a)\\(b)\end{array}: -i\delta\begin{pmatrix}\nu_L^\dagger\\ \nu_L'^\dagger\end{pmatrix}: -k\hat{D}_-(c_L)\begin{pmatrix}\check{\nu}_R\\ \check{\nu}'_R\end{pmatrix}+\sigma^\mu\partial_\mu\begin{pmatrix}\check{\nu}_L\\ \check{\nu}'_L\end{pmatrix}=0,$$

$$\begin{array}{l}(c)\\(d)\end{array}: i\delta\begin{pmatrix}\nu_R^\dagger\\ \nu_R'^\dagger\end{pmatrix}: \bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{\nu}_R\\ \check{\nu}'_R\end{pmatrix}-k\hat{D}_+(c_L)\begin{pmatrix}\check{\nu}_L\\ \check{\nu}'_L\end{pmatrix}=\frac{2m_B}{\sqrt{k}}\delta(y)\begin{pmatrix}0\\ \eta\end{pmatrix},$$

$$(e): i\delta\eta^\dagger: \left\{\sigma^\mu\partial_\mu\eta-\frac{m_B}{\sqrt{k}}\nu'_R+M\eta^c\right\}\delta(y)=0. \tag{5.36}$$

ν_R and ν'_L are parity-odd at $y=0$, whereas ν_L and ν'_R are parity-even. We integrate the equations (a) and (d) in the vicinity of $y=0$ and evaluate the equations (b) and (c) at $y=+\epsilon$ to find boundary conditions at $y=+\epsilon$ as

$$\begin{aligned}&(a)\Rightarrow \check{\nu}_R(x,\epsilon)=0,\\ &(d)\Rightarrow -\check{\nu}'_L(x,\epsilon)=+\frac{m_B}{\sqrt{k}}\eta(x),\\ &(b)\Rightarrow -\hat{D}_-(c_L)\check{\nu}'_R-\frac{m_B^2}{k^2}\check{\nu}'_R+\frac{m_BM}{k^{3/2}}\eta^c=0,\\ &(c)\Rightarrow \hat{D}_+(c_L)\check{\nu}_L=0.\end{aligned} \tag{5.37}$$

Boundary conditions at $z=z_L$ are given by $D_+(c_L)\check{\nu}_L=\check{\nu}_R=0$ and $\check{\nu}'_L=D_-(c_L)\check{\nu}'_R=0$.

Mode functions of these fields in the twisted gauge can be written as

$$\begin{aligned}&\begin{pmatrix}\tilde{\check{\nu}}_R\\ \tilde{\check{\nu}}'_R\\ \eta^c\end{pmatrix}=\begin{pmatrix}\alpha_\nu S_R^L\\ i\alpha_{\nu'}C_R^L\\ \mp i\alpha_\eta^*/\sqrt{k}\end{pmatrix}f_{\pm R}(x),\\ &\begin{pmatrix}\tilde{\check{\nu}}_L\\ \tilde{\check{\nu}}'_L\\ \eta\end{pmatrix}=\begin{pmatrix}\alpha_\nu C_L^L\\ i\alpha_{\nu'}S_L^L\\ i\alpha_\eta/\sqrt{k}\end{pmatrix}f_{\pm L}(x),\\ &\bar{\sigma}^\mu\partial_\mu f_{\pm R}(x)=k\lambda f_{\pm L}(x),\qquad \sigma^\mu\partial_\mu f_{\pm L}(x)=k\lambda f_{\pm R}(x),\\ &f_{\pm L}(x)^c=e^{i\delta_C}\sigma^2 f_{\pm L}(x)^*=\pm f_{\pm R}(x),\end{aligned} \tag{5.38}$$

where $S_{L/R}^L=S_{L/R}(z;\lambda,c_L)$ and $C_{L/R}^L=C_{L/R}(z;\lambda,c_L)$, and δ_C is defined in Eq. (3.25). Explicit forms of $f_{\pm L/R}$ are given in Appendix C. One can take α_ν, $\alpha_{\nu'}$, α_η to be real. In this case $\sigma^\mu\partial_\mu\eta=\mp k\lambda\eta^c$ is satisfied so that the equation (e) in Eq. (5.36) implies that

$$\frac{m_B}{\sqrt{k}}\check{\nu}'_R\Big|_{y=0}-(M\mp k\lambda)\eta^c=0. \tag{5.39}$$

With this identity, the third relation in Eq. (5.37) can be rewritten as

$$\hat{D}_-(c_L)\check{\nu}'_R\mp\frac{m_B\lambda}{\sqrt{k}}\eta^c=0. \tag{5.40}$$

Substituting (5.38) into (5.37), one finds

$$K_\nu\begin{pmatrix}\alpha_\nu\\ \alpha_{\nu'}\\ \alpha_\eta\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta_H}{2}S_R^L & \sin\frac{\theta_H}{2}C_R^L & 0\\ -\sin\frac{\theta_H}{2}C_L^L & \cos\frac{\theta_H}{2}S_L^L & \frac{m_B}{k}\\ m_B\sin\frac{\theta_H}{2}S_R^L & -m_B\cos\frac{\theta_H}{2}C_R^L & k\lambda\mp M\end{pmatrix}\times\begin{pmatrix}\alpha_\nu\\ \alpha_{\nu'}\\ \alpha_\eta\end{pmatrix}=0, \tag{5.41}$$

where $S_{L/R}^L=S_{L/R}(1;\lambda,c_L)$, etc. From $\det K_\nu=0$, we find the mass spectrum formula for the neutrino sector[1]:

$$\det K_\nu=(k\lambda\pm M)\left\{S_L^LS_R^L+\sin^2\frac{\theta_H}{2}\right\}+\frac{m_B^2}{k}S_R^LC_R^L=0. \tag{5.42}$$

One of the solutions with $f_{+R/L}(x)$ or $f_{-R/L}(x)$ allows a small mass eigenvalue $m_\nu=k\lambda_\nu>0$. For $M>0$, the neutrino mode is obtained with $f_{+R/L}(x)$. Noting that $\lambda z_L\ll 1$ and $k\lambda\ll M$, one finds the neutrino mass given by

$$m_\nu\simeq\begin{cases}\frac{m_e^2Mz_L^{2c_L+1}}{(2c_L+1)m_B^2} & \text{for } c_L>\frac{1}{2},\\ \frac{m_e^2M}{(2|c_L|-1)m_B^2} & \text{for } c_L<-\frac{1}{2}.\end{cases} \tag{5.43}$$

The gauge-Higgs seesaw mechanism [35,42,43] is characterized by a 3×3 mass matrix

$$\frac{i}{2}(\nu_{0L}^{c\dagger},\nu_{0R}'^\dagger,\eta^{c\dagger})\begin{pmatrix}0 & m_e & 0\\ m_e & 0 & \tilde{m}_B\\ 0 & \tilde{m}_B & M\end{pmatrix}\begin{pmatrix}\nu_{0L}\\ \nu_{0R}'^c\\ \eta\end{pmatrix}+\text{H.c.}, \tag{5.44}$$

where m_e is its corresponding charged lepton mass. The structure takes the same form as the inverse seesaw mechanism in Ref. [43] and yields very light neutrino mass $m_\nu\sim m_e^2M/\tilde{m}_B^2$. The Majorana mass M may take a moderate value. In particular, for $c_L<-\frac{1}{2}$, $m_\nu\sim 1$ meV is obtained with $m_B\sim 1$ TeV and $M\sim 50$ GeV. For $c_L>\frac{1}{2}$, m_B has to take a rather large value, larger than the Planck mass.

Typical parameters in the lepton sector are summarized in Table VII. $|c_L|$ and $m_{e^{(1)}}$ are fixed by m_e. The value of M can be varied. The spectrum does not depend on M very much. As is seen in the table, a very light neutrino excited

[1]There was an error of a factor 2 in the right side of equation (d) in (5.36) in the previous papers [35,36]. The formulas (5.42) and (5.43) reflect this correction.

TABLE VII. Parameters which reproduce the spectrum of leptons for $\theta_H = 0.15$, $z_L = 10^{10}$. $m_{\rm KK} = 8.062$ TeV. The masses of the first KK mode leptons are also shown in units of TeV. For $c_L > 0$, there appear light neutrino excitation modes, ν_s. The values $m_e = 0.511$ MeV, $m_\mu = 105.7$ MeV, $m_\tau = 1.776$ GeV, and $m_\nu = 1$ meV have been used.

Leptons	c_L	M (GeV)	m_B (GeV)	m_{ν_s}	$m_{\nu^{(1)}}$ (TeV)	$m_{e^{(1)}}$ (TeV)
(ν_e, e)	1.086	10^3	6.6×10^{19}	6.8 MeV	8.38	8.38
		1	2.1×10^{18}	6.8 MeV	8.38	8.38
	-1.086	10^3	1.5×10^4	…	8.38	8.38
		1	4.7×10^2	…	0.51	8.38
(ν_μ, μ)	0.839	10^3	5.0×10^{19}	1.4 GeV	7.47	7.47
	-0.839	10^3	1.2×10^7	…	7.47	7.47
(ν_τ, τ)	0.703	10^3	3.9×10^{19}	24. GeV	6.96	6.96
	-0.703	10^3	8.8×10^8	…	6.96	6.96

mode ν_s appears for positive c_L. This does not necessarily mean inconsistency with the observation. The ν_s mode may become a candidate for warm dark matter [44], though more detailed investigation of gauge couplings is necessary to see the feasibility. For negative c_L, a very light neutrino excited mode appears only when M becomes very small. The spectrum of the neutrino towers are shown in Fig. 2 for $c_L > 0$ and in Fig. 3 for $c_L < 0$.

E. *W* couplings of quarks and leptons

As has been shown above, the quark and lepton mass spectrum can be reproduced except that the down-quark mass turns out lighter than the up-quark mass. At this stage, one might worry about the W couplings of quarks and leptons in the current scheme. In the gauge-Higgs unification, the W boson at $\theta_H \neq 0$ necessarily contains the original $SU(2)_R$ component as seen in Sec. IV A. If quarks and leptons originated from only spinor representation multiplets in $SO(5)$, right-handed components of quarks and leptons also would have had nonvanishing couplings to W, which contradicts the observation.

The left-handed quark and lepton doublets are mainly in the spinor representation of $SO(5)$, which have nominal W

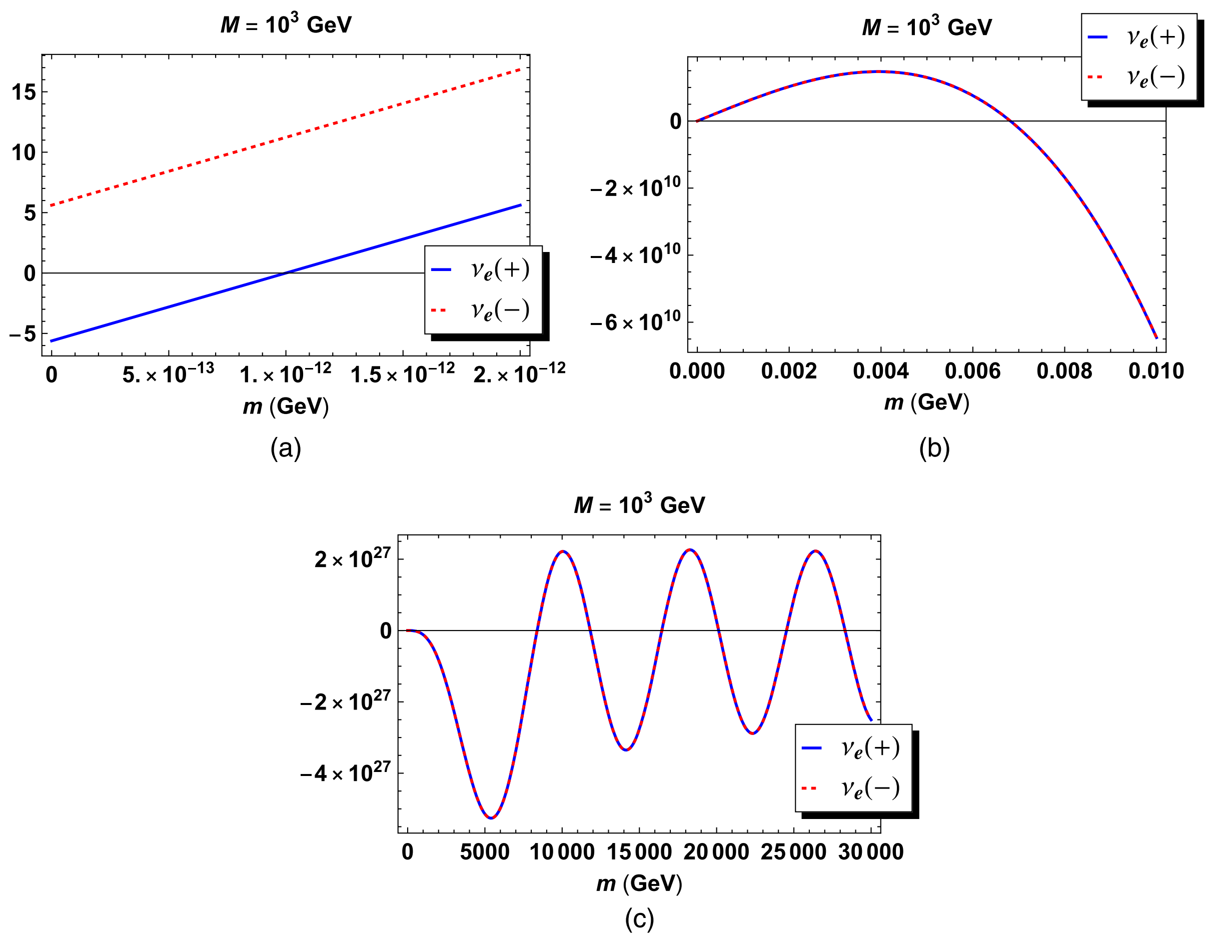

FIG. 2. Spectrum of electron neutrino tower for $c_e > 0$. $\det K_\nu$ in (5.42) is plotted as a function of $m = k\lambda$ in various mass ranges for $\theta_H = 0.15$, $z_L = 10^{10}$, $m_{\rm KK} = 8.062$ TeV, and $M = 1$ TeV. The mass spectrum $\{m_n = k\lambda_n\}$ is determined by roots of $\det K_\nu = 0$. $\nu_e(\pm)$ indicates the case of $f_{\pm L/R}(x)$ in (5.38). Only $\nu_e(+)$ has a solution corresponding to ν_e with $m_{\nu_e} = 1$ meV. In (b) and (c), the curves for $\nu_e(+)$ and $\nu_e(-)$ almost overlap with each other at this scale.

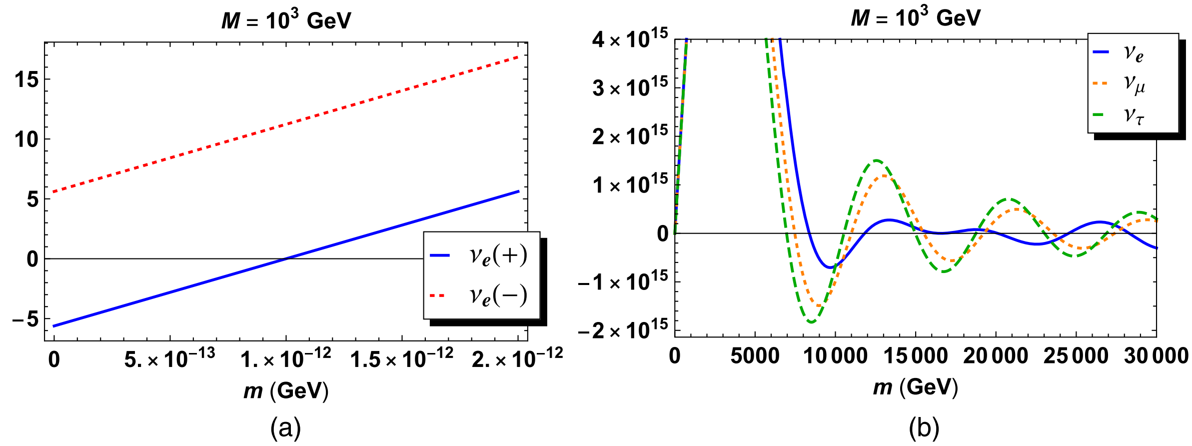

FIG. 3. Spectrum of neutrino towers for $c_L < 0$. As in Fig. 2, $\det K_\nu$ in (5.42) is plotted as a function of $m = k\lambda$ in two mass ranges for $\theta_H = 0.15$, $z_L = 10^{10}$, $m_{\rm KK} = 8.062$ TeV, and $M = 1$ TeV. The mass spectrum $\{m_n = k\lambda_n\}$ is determined by roots of $\det K_\nu = 0$. (a) Only $\nu_e(+)$ has a solution corresponding to ν_e with $m_{\nu_e} = 1$ meV. (b) The spectra of ν_e, ν_μ, ν_τ towers are shown. $\nu(+)$ and $\nu(-)$ towers almost overlap in this figure. For the ν_e tower, the masses of the third and fourth KK modes are 16.46 and 16.67 TeV, respectively.

couplings. The mechanism in the current model for making right-handed quarks and leptons having almost vanishing W couplings is the following. The up-type quarks are contained solely in the spinor multiplets. The down-type quarks are contained in both the spinor and singlet representations of $SO(5)$. Left-handed down-type quarks are mostly in the spinor representation multiplets, whereas right-handed down-type quarks are mostly in the singlet representation multiplets so that right-handed up-type quarks have almost vanishing W couplings to right-handed down-type quarks.

The mechanism in the lepton sector is different. With the presence of brane fermions χ, the gauge-Higgs seesaw mechanism functions in the neutrino sector. Right-handed neutrinos become heavy, acquiring $O(m_{\rm KK})$ masses, and decouple from right-handed charged leptons.

Indeed, one can evaluate the W couplings of quarks and leptons by determining wave functions of quarks and leptons from the mass-determining matrices explained above and inserting them into the original action. The result is shown in Table VIII. It is seen that the μ-e universality in the charged current interactions holds to high accuracy, provided the same sign of c_L is adopted. It is also confirmed that the W couplings of right-handed quarks and leptons are strongly suppressed. A more detailed study of gauge couplings, including Z and Z' couplings, will be given separately.

TABLE VIII. W couplings of quarks and leptons for $\theta_H = 0.15$, $z_L = 10^{10}$. $m_{\rm KK} = 8.062$ TeV. The couplings are defined by $\mathcal{L} = W_\mu(g_L^W \bar{u}_L \gamma^\mu d_L + g_R^W \bar{u}_R \gamma^\mu d_R)$ for the (u,d) doublet. In the SM, $g_L^W = g_w/\sqrt{2}$ and $g_R^W = 0$. For the (u,d) doublet, we set $m_d = 0.9 m_u$.

Leptons	c_L	M	$\frac{g_L^W}{g_w/\sqrt{2}} - 1$	$\frac{g_R^W}{g_w/\sqrt{2}}$
(ν_e, e)	1.086	1 TeV	-2.64×10^{-3}	$O(10^{-11})$
	-1.086	1 TeV	-5.24×10^{-3}	$O(10^{-23})$
(ν_μ, μ)	0.839	1 TeV	-2.64×10^{-3}	$O(10^{-14})$
	-0.839	1 TeV	-5.25×10^{-3}	$O(10^{-21})$
(ν_τ, τ)	0.703	1 TeV	-2.64×10^{-3}	$O(10^{-15})$
	-0.703	1 TeV	-5.25×10^{-3}	$O(10^{-19})$

Quarks	c_Q	μ_1	$\tilde{m}_D$	$\frac{g_L^W}{g_w/\sqrt{2}} - 1$	$\frac{g_R^W}{g_w/\sqrt{2}}$
(u,d)	-1.044	0.1	1.0	-5.24×10^{-3}	$O(10^{-14})$
(c,s)	-0.7546	0.1	1.0	-5.25×10^{-3}	$O(10^{-9})$
(t,b)	0.2287	0.1	0.1	-3.43×10^{-3}	$O(10^{-4})$
	-0.2287	0.1	1.0	-4.41×10^{-3}	$O(10^{-5})$

VI. SUMMARY AND DISCUSSIONS

In this paper we have presented a new model of the $SO(5) \times U(1) \times SU(3)$ gauge-Higgs unification in which quark and lepton multiplets are introduced in the spinor, vector, and singlet representations of $SO(5)$ such that they can be implemented in the $SO(11)$ gauge-Higgs grand unification scheme. This should be contrasted to the previous model in which all quark and lepton multiplets are introduced in the vector representation of $SO(5)$. The up-type quarks are contained solely in the spinor representation. The right-handed down-type quarks are mainly contained in the singlet representation of $SO(5)$. $SO(5) \times U(1) \times SU(3)$ singlet brane Majorana fermions are introduced on the UV brane. The coupling of these brane fermions to bulk fermion multiplets induces the gauge-Higgs seesaw mechanism in the neutrino sector, which takes the same form as the inverse seesaw mechanism in four-dimensional GUT theories.

With $SO(5) \times U(1) \times SU(3)$ gauge-invariant brane interactions taken into account, the quark-lepton mass spectrum has been reproduced with the exception that the down-quark mass (m_d) becomes lighter than the up-quark mass (m_u). A solution to this problem has yet to be found. The compatibility with grand unification severely restricts matter content and interactions in the gauge-Higgs unification. Nevertheless it is very encouraging that the model yields almost the same W couplings of quarks and leptons.

The present model serves as a viable alternative to the standard model. If it is the case, phenomenological consequences of the model need to be clarified. As in the previous model, Z' bosons (the first KK modes of γ, Z, and Z_R) are predicted around the 7 to 10 TeV range. We have seen in Sec. V that the bulk mass parameters (c_u, c_c) of quark multiplets $\Psi_{(\mathbf{3},\mathbf{4})}$ in the first and second generations must be negative to avoid exotic light excitation modes of down-quark type. The bulk mass parameters c_L of lepton multiplets can be either positive or negative. The sign of the bulk mass parameters is critically important to determine the behavior of wave functions. For $c > +\frac{1}{2}$ ($c < -\frac{1}{2}$), left-handed quarks/leptons are localized near the UV (IR) brane, whereas right-handed ones are localized near the IR (UV) brane. As Z' bosons are localized near the IR brane, right-handed (left-handed) quarks/leptons have larger couplings to Z' bosons for $c > +\frac{1}{2}$ ($c < -\frac{1}{2}$). The effect of the large parity violation can be seen in the e^+e^- collisions through interference terms. In particular, cross sections of various fermion-pair production processes should reveal distinct dependence on the e^- polarization. [14]

With the mass spectra of all fields having been determined, one can investigate the effective potential $V_{\rm eff}(\theta_H)$ to show that EW symmetry is dynamically broken. The flavor mixing in the quark and lepton sectors and the dark matter are also among the problems to be solved in the gauge-Higgs unification scenario. We shall come back to these issues in the near future.

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APPENDIX A: $SO(5)$

The generators of $SO(5)$, $T_{jk} = -T_{kj} = T_{jk}^\dagger$ ($j, k = 1, 2, 3, 4, 5$), satisfy the algebra

$$[T_{ij}, T_{kl}] = i(\delta_{ik}T_{jl} - \delta_{il}T_{jk} + \delta_{jl}T_{ik} - \delta_{jk}T_{il}). \tag{A1}$$

In the adjoint representation,

$$\begin{aligned}&(T_{ij})_{pq} = -i(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}),\\ &\mathrm{tr}(T_{jk}T_{lm}) = 2(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}), \qquad \mathrm{tr}(T_{jk})^2 = 2.\end{aligned} \tag{A2}$$

We take the following basis of $SO(5)$ Clifford algebra:

$$\begin{aligned}&\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}I_4,\\ &\Gamma_a = \sigma^a \otimes \sigma^1 \quad (a = 1, 2, 3),\\ &\Gamma_4 = \sigma^0 \otimes \sigma^2,\\ &\Gamma_5 = \sigma^0 \otimes \sigma^3 = -\Gamma_1\Gamma_2\Gamma_3\Gamma_4,\end{aligned} \tag{A3}$$

where $\sigma^0 = I_2$ and $\{\sigma^a\}$ are Pauli matrices. In terms of Γ_j, the $SO(5)$ generators in the spinor representation are given by

$$\begin{aligned}&T_{jk} = -\frac{i}{4}[\Gamma_j, \Gamma_k] \quad \left(= -\frac{i}{2}\Gamma_j\Gamma_k \quad \text{for } j \neq k\right),\\ &(T_{jk})^2 = \frac{1}{4}I_4, \qquad \mathrm{tr}(T_{jk})^2 = 1.\end{aligned} \tag{A4}$$

The orbifold boundary conditions P_0, P_1 in Eqs. (3.4) break $SO(5)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$. The generators of the corresponding $SO(4) \simeq SU(2)_L \times SU(2)_R$ in the spinor representation are given by

$$\begin{aligned}&\vec{T}_L = \frac{1}{2}\begin{pmatrix} T_{23} + T_{14} \\ T_{31} + T_{24} \\ T_{12} + T_{34} \end{pmatrix} = \frac{1}{2}\vec{\sigma} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\\ &\vec{T}_R = \frac{1}{2}\begin{pmatrix} T_{23} - T_{14} \\ T_{31} - T_{24} \\ T_{12} - T_{34} \end{pmatrix} = \frac{1}{2}\vec{\sigma} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned} \tag{A5}$$

These generators become block-diagonal so that an $SO(5)$ spinor representation $\mathbf{4}$ can be decomposed into $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ of $SO(4) \simeq SU(2)_L \times SU(2)_R$:

$$\Psi_{\mathbf{4}} = \begin{pmatrix} \Psi_{(\mathbf{2},\mathbf{1})} \\ \Psi_{(\mathbf{1},\mathbf{2})} \end{pmatrix}. \tag{A6}$$

In the representation (A3) one finds that

$$\begin{aligned}&\Gamma_j^* = (-1)^{j+1}\Gamma_j,\\ &R := -i\Gamma_2\Gamma_4 = R^\dagger = R^{-1} = \sigma^2 \otimes \sigma^3,\\ &R\Gamma_j R = (-1)^{j+1}\Gamma_j, \qquad R\Gamma_j^* R = \Gamma_j,\\ &RT_{jk}^* R = -T_{jk}.\end{aligned} \tag{A7}$$

It follows that for an $SO(5)$ spinor $\Psi_{\mathbf{4}}$, the R-transformed one also transforms as $\mathbf{4}$:

$$\begin{aligned}\tilde{\Psi}_{\mathbf{4}} &\coloneqq iR\Psi_{\mathbf{4}}^{*},\\ \Psi_{\mathbf{4}}' &= \left(1+\frac{i}{2}\epsilon_{jk}T_{jk}\right)\Psi_{\mathbf{4}}\\ \Rightarrow \tilde{\Psi}_{\mathbf{4}}' &= \left(1+\frac{i}{2}\epsilon_{jk}T_{jk}\right)\tilde{\Psi}_{\mathbf{4}}.\end{aligned} \tag{A8}$$

Its $SO(5)$ content is given by

$$\tilde{\Psi}_{\mathbf{4}} = \begin{pmatrix}\tilde{\Psi}_{(\mathbf{2},\mathbf{1})}\\ \tilde{\Psi}_{(\mathbf{1},\mathbf{2})}\end{pmatrix} = \begin{pmatrix} i\sigma^2\Psi^{*}_{(\mathbf{2},\mathbf{1})}\\ -i\sigma^2\Psi^{*}_{(\mathbf{1},\mathbf{2})}\end{pmatrix}. \tag{A9}$$

APPENDIX B: BASIS FUNCTIONS

We summarize basis functions in the RS space.

1. Gauge fields

We define

$$F_{\alpha,\beta}(u,v) \equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v), \tag{B1}$$

where $J_\alpha(x)$ and $Y_\alpha(x)$ are Bessel functions of the first and second kind, respectively. For gauge bosons, $C = C(z;\lambda)$ and $S = S(z;\lambda)$ are defined as solutions of

$$-\mathcal{P}_4\begin{pmatrix}C\\ S\end{pmatrix} = \left(-\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}\right)\begin{pmatrix}C\\ S\end{pmatrix} = \lambda^2\begin{pmatrix}C\\ S\end{pmatrix}, \tag{B2}$$

with boundary conditions $C = z_L$, $S = 0$, $C' = 0$, and $S' = \lambda$ at $z = z_L$. They are given by

$$\begin{aligned} C(z;\lambda) &= +\frac{\pi}{2}\lambda z z_L F_{1,0}(\lambda z,\lambda z_L),\\ C'(z;\lambda) &= +\frac{\pi}{2}\lambda^2 z z_L F_{0,0}(\lambda z,\lambda z_L),\\ S(z;\lambda) &= -\frac{\pi}{2}\lambda z F_{1,1}(\lambda z,\lambda z_L),\\ S'(z;\lambda) &= -\frac{\pi}{2}\lambda^2 z F_{0,1}(\lambda z,\lambda z_L).\end{aligned} \tag{B3}$$

We note that

$$\begin{aligned} -\mathcal{P}_z\begin{pmatrix}C'\\ S'\end{pmatrix} &= \lambda^2\begin{pmatrix}C'\\ S'\end{pmatrix},\\ CS' - SC' &= \lambda z.\end{aligned} \tag{B4}$$

2. Massless fermion fields

For massless fermions in five dimensions, we define

$$\begin{aligned}\begin{pmatrix}C_L\\ S_L\end{pmatrix}(z;\lambda,c) &= \pm\frac{\pi}{2}\lambda\sqrt{z z_L}F_{c+\frac{1}{2},c\mp\frac{1}{2}}(\lambda z,\lambda z_L),\\ \begin{pmatrix}C_R\\ S_R\end{pmatrix}(z;\lambda,c) &= \mp\frac{\pi}{2}\lambda\sqrt{z z_L}F_{c-\frac{1}{2},c\pm\frac{1}{2}}(\lambda z,\lambda z_L),\end{aligned} \tag{B5}$$

which satisfy

$$\begin{aligned} &D_+\begin{pmatrix}C_L\\ S_L\end{pmatrix} = \lambda\begin{pmatrix}S_R\\ C_R\end{pmatrix}, \qquad D_-\begin{pmatrix}C_R\\ S_R\end{pmatrix} = \lambda\begin{pmatrix}S_L\\ C_L\end{pmatrix},\\ &C_LC_R - S_LS_R = 1,\\ &\qquad C_R = C_L = 1, \quad S_R = S_L = 0, \quad \text{at } z = z_L.\end{aligned} \tag{B6}$$

They also satisfy

$$\begin{aligned} C_L(z;\lambda,-c) &= C_R(z;\lambda,c),\\ S_L(z;\lambda,-c) &= -S_R(z;\lambda,c).\end{aligned} \tag{B7}$$

3. Massive fermion fields

As seen in (3.16), $\check{\Psi}^{\pm\alpha}_{(\mathbf{3},\mathbf{1})}$ and $\check{\Psi}^{\pm\beta}_{(\mathbf{1},\mathbf{5})}$ have additional pseudo-Dirac bulk mass terms in the action. To find basis functions for these massive fermions, we consider the action for $N^{\pm}$ fields given by

$$\begin{aligned}&\int d^4x\int_1^{z_L}\frac{dz}{k}\left\{\bar{\check{N}}^{+}\mathcal{D}_0(c_+)\check{N}^{+} + \bar{\check{N}}^{-}\mathcal{D}_0(c_-)\check{N}^{-}\right.\\ &\quad\left. -\frac{k\tilde{m}}{z}(\bar{\check{N}}^{+}\check{N}^{-} + \bar{\check{N}}^{-}\check{N}^{+})\right\},\\ &\text{where } \mathcal{D}_0(c) = \begin{pmatrix}-kD_-(c) & \sigma^\mu\partial_\mu\\ \bar{\sigma}^\mu\partial_\mu & -kD_+(c)\end{pmatrix}.\end{aligned} \tag{B8}$$

$\tilde{m}$ is dimensionless, and $k\tilde{m}$ corresponds to m_D^α and m_V^β in (3.16).

To find eigenmodes with four-dimensional mass $k\lambda$, we write $\check{N}^{\pm}_R(x,z) = N_{\pm R}(z)f_R(x)$ and $\check{N}^{\pm}_L(x,z) = N_{\pm L}(z)f_L(x)$ as described below Eq. (5.7). Then $N_{\pm R}(z)$ and $N_{\pm L}(z)$ must satisfy

$$\begin{aligned} D_-(c_\pm)N_{\pm R} - \lambda N_{\pm L} + \frac{\tilde{m}}{z}N_{\mp R} &= 0,\\ D_+(c_\pm)N_{\pm L} - \lambda N_{\pm R} + \frac{\tilde{m}}{z}N_{\mp L} &= 0.\end{aligned} \tag{B9}$$

Note that

$$D_\pm(c)D_\mp(c) = -\frac{d^2}{dz^2} + \frac{c(c\mp 1)}{z^2}. \tag{B10}$$

We consider two cases: $c_+ = c_-$ and $c_+ = -c_-$.

a. Case I. $c_+ = c_- = c$

It follows immediately from (B9) that

$$\begin{aligned} &D_-(c\pm\tilde{m})(N_{+R}\pm N_{-R}) = \lambda(N_{+L}\pm N_{-L}), \\ &D_+(c\pm\tilde{m})(N_{+L}\pm N_{-L}) = \lambda(N_{+R}\pm N_{-R}). \end{aligned} \tag{B11}$$

General solutions are given by

$$\begin{aligned} \begin{pmatrix} N_{\pm R} \\ N_{\pm L} \end{pmatrix} &= a\begin{pmatrix} C_R^{c+\tilde{m}} \\ S_L^{c+\tilde{m}} \end{pmatrix} + b\begin{pmatrix} S_R^{c+\tilde{m}} \\ C_L^{c+\tilde{m}} \end{pmatrix} \\ &\pm a'\begin{pmatrix} C_R^{c-\tilde{m}} \\ S_L^{c-\tilde{m}} \end{pmatrix} \pm b'\begin{pmatrix} S_R^{c-\tilde{m}} \\ C_L^{c-\tilde{m}} \end{pmatrix}. \end{aligned} \tag{B12}$$

Here $C_{L/R}^{c\pm\tilde{m}} = C_{L/R}(z;\lambda,c\pm\tilde{m})$ and $S_{L/R}^{c\pm\tilde{m}} = S_{L/R}(z;\lambda,c\pm\tilde{m})$.

At this stage we define basis functions by

$$\begin{aligned} \mathcal{C}_{R1}(z;\lambda,c,\tilde{m}) &= C_R(z;\lambda,c+\tilde{m}) + C_R(z;\lambda,c-\tilde{m}), \\ \mathcal{C}_{R2}(z;\lambda,c,\tilde{m}) &= S_R(z;\lambda,c+\tilde{m}) - S_R(z;\lambda,c-\tilde{m}), \\ \mathcal{S}_{L1}(z;\lambda,c,\tilde{m}) &= S_L(z;\lambda,c+\tilde{m}) + S_L(z;\lambda,c-\tilde{m}), \\ \mathcal{S}_{L2}(z;\lambda,c,\tilde{m}) &= C_L(z;\lambda,c+\tilde{m}) - C_L(z;\lambda,c-\tilde{m}), \\ \mathcal{C}_{L1}(z;\lambda,c,\tilde{m}) &= C_L(z;\lambda,c+\tilde{m}) + C_L(z;\lambda,c-\tilde{m}), \\ \mathcal{C}_{L2}(z;\lambda,c,\tilde{m}) &= S_L(z;\lambda,c+\tilde{m}) - S_L(z;\lambda,c-\tilde{m}), \\ \mathcal{S}_{R1}(z;\lambda,c,\tilde{m}) &= S_R(z;\lambda,c+\tilde{m}) + S_R(z;\lambda,c-\tilde{m}), \\ \mathcal{S}_{R2}(z;\lambda,c,\tilde{m}) &= C_R(z;\lambda,c+\tilde{m}) - C_R(z;\lambda,c-\tilde{m}), \end{aligned} \tag{B13}$$

which satisfy the equations and boundary conditions

$$\begin{aligned} &D_-(c)\begin{pmatrix} \mathcal{C}_{R1} \\ \mathcal{C}_{R2} \end{pmatrix} = \lambda\begin{pmatrix} \mathcal{S}_{L1} \\ \mathcal{S}_{L2} \end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix} \mathcal{S}_{R2} \\ \mathcal{S}_{R1} \end{pmatrix}, \\ &D_-(c)\begin{pmatrix} \mathcal{S}_{R1} \\ \mathcal{S}_{R2} \end{pmatrix} = \lambda\begin{pmatrix} \mathcal{C}_{L1} \\ \mathcal{C}_{L2} \end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix} \mathcal{C}_{R2} \\ \mathcal{C}_{R1} \end{pmatrix}, \\ &D_+(c)\begin{pmatrix} \mathcal{C}_{L1} \\ \mathcal{C}_{L2} \end{pmatrix} = \lambda\begin{pmatrix} \mathcal{S}_{R1} \\ \mathcal{S}_{R2} \end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix} \mathcal{S}_{L2} \\ \mathcal{S}_{L1} \end{pmatrix}, \\ &D_+(c)\begin{pmatrix} \mathcal{S}_{L1} \\ \mathcal{S}_{L2} \end{pmatrix} = \lambda\begin{pmatrix} \mathcal{C}_{R1} \\ \mathcal{C}_{R2} \end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix} \mathcal{C}_{L2} \\ \mathcal{C}_{L1} \end{pmatrix}, \\ &\qquad \mathcal{S}_{Rj} = \mathcal{S}_{Lj} = D_-(c)\mathcal{C}_{Rj} = D_+(c)\mathcal{C}_{Lj} = 0 \\ &\qquad\qquad \text{at } z = z_L. \end{aligned} \tag{B14}$$

Note also

$$\begin{aligned} &\mathcal{C}_{Rj}(z;\lambda,-c,\tilde{m}) = \mathcal{C}_{Lj}(z;\lambda,c,\tilde{m}), \\ &\mathcal{S}_{Rj}(z;\lambda,-c,\tilde{m}) = -\mathcal{S}_{Lj}(z;\lambda,c,\tilde{m}), \\ &\mathcal{C}_{R/Lj}(z;\lambda,c,-\tilde{m}) = (-1)^{j-1}\mathcal{C}_{R/Lj}(z;\lambda,c,\tilde{m}), \\ &\mathcal{S}_{R/Lj}(z;\lambda,c,-\tilde{m}) = (-1)^{j-1}\mathcal{S}_{R/Lj}(z;\lambda,c,\tilde{m}). \end{aligned} \tag{B15}$$

In the $\tilde{m}\to 0$ limit,

$$\begin{aligned} &\mathcal{C}_{R1}\to 2C_R, \quad \mathcal{S}_{R1}\to 2S_R, \quad \mathcal{C}_{L1}\to 2C_L, \quad \mathcal{S}_{L1}\to 2S_L, \\ &\quad \mathcal{C}_{R2}, \quad \mathcal{S}_{R2}, \quad \mathcal{C}_{L2}, \quad \mathcal{S}_{L2}\to 0. \end{aligned} \tag{B16}$$

Two types of boundary conditions appear at $z = z_L$.

Type A: $(N_{+R}, N_{-R}, N_{+L}, N_{-L}) = (+,-,-,+)$

When parity assignment at $y = L$ for $(N_{+R}, N_{-R}, N_{+L}, N_{-L})$ is $(+,-,-,+)$, boundary conditions at $z = z_L$ become

$$\begin{aligned} D_-(c)N_{+R} &= 0, \qquad N_{+L} = 0, \\ N_{-R} &= 0, \qquad D_+(c)N_{-L} = 0. \end{aligned} \tag{B17}$$

In this case $a = a'$ and $b = -b'$ in (B12) and solutions can be written as

$$\begin{pmatrix} N_{+R} \\ N_{+L} \\ N_{-R} \\ N_{-L} \end{pmatrix} = a\begin{pmatrix} \mathcal{C}_{R1}(z;\lambda,c,\tilde{m}) \\ \mathcal{S}_{L1}(z;\lambda,c,\tilde{m}) \\ \mathcal{S}_{R2}(z;\lambda,c,\tilde{m}) \\ \mathcal{C}_{L2}(z;\lambda,c,\tilde{m}) \end{pmatrix} + b\begin{pmatrix} \mathcal{C}_{R2}(z;\lambda,c,\tilde{m}) \\ \mathcal{S}_{L2}(z;\lambda,c,\tilde{m}) \\ \mathcal{S}_{R1}(z;\lambda,c,\tilde{m}) \\ \mathcal{C}_{L1}(z;\lambda,c,\tilde{m}) \end{pmatrix}, \tag{B18}$$

where a, b are arbitrary constants.

If N's have the same parity assignment at $y = 0$ as that at $y = L$, then (B17) must be satisfied at $z = 1$ as well. Substituting (B18) into (B17) and evaluating the conditions at $z = 1$, one finds

$$\begin{pmatrix} \mathcal{S}_{L1} & \mathcal{S}_{L2} \\ \mathcal{S}_{R2} & \mathcal{S}_{R1} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0, \tag{B19}$$

where $\mathcal{S}_{L1} = \mathcal{S}_{L1}(1;\lambda,c,\tilde{m})$, etc. The mass spectrum is determined by

$$\mathcal{S}_{L1}\mathcal{S}_{R1} - \mathcal{S}_{L2}\mathcal{S}_{R2} = 0. \tag{B20}$$

Note that

$$\begin{aligned}\mathcal{S}_{L1}\mathcal{S}_{R1}-\mathcal{S}_{L2}\mathcal{S}_{R2}+2&=\mathcal{C}_{L1}\mathcal{C}_{R1}-\mathcal{C}_{L2}\mathcal{C}_{R2}-2\\&=S_L^{c+\tilde{m}}S_R^{c-\tilde{m}}+S_L^{c-\tilde{m}}S_R^{c+\tilde{m}}\\&\quad+C_L^{c+\tilde{m}}C_R^{c-\tilde{m}}+C_L^{c-\tilde{m}}C_R^{c+\tilde{m}},\\ \mathcal{S}_{L1}\mathcal{C}_{L1}-\mathcal{S}_{L2}\mathcal{C}_{L2}&=2(S_L^{c+\tilde{m}}C_L^{c-\tilde{m}}+S_L^{c-\tilde{m}}C_L^{c+\tilde{m}}).\end{aligned}\tag{B21}$$

Type B: $(N_{+R},N_{-R},N_{+L},N_{-L})=(-,+,+,-)$

When parity assignment at $y=L$ for $(N_{+R},N_{-R},N_{+L},N_{-L})$ is $(-,+,+,-)$, boundary conditions at $z=z_L$ become

$$\begin{aligned}N_{+R}&=0, &\quad D_+(c)N_{+L}&=0,\\ D_-(c)N_{-R}&=0, &\quad N_{-L}&=0.\end{aligned}\tag{B22}$$

In this case $a=-a'$ and $b=b'$ in (B12) and solutions can be written as

$$\begin{pmatrix}N_{+R}\\N_{+L}\\N_{-R}\\N_{-L}\end{pmatrix}=a\begin{pmatrix}\mathcal{S}_{R2}(z;\lambda,c,\tilde{m})\\\mathcal{C}_{L2}(z;\lambda,c,\tilde{m})\\\mathcal{C}_{R1}(z;\lambda,c,\tilde{m})\\\mathcal{S}_{L1}(z;\lambda,c,\tilde{m})\end{pmatrix}+b\begin{pmatrix}\mathcal{S}_{R1}(z;\lambda,c,\tilde{m})\\\mathcal{C}_{L1}(z;\lambda,c,\tilde{m})\\\mathcal{C}_{R2}(z;\lambda,c,\tilde{m})\\\mathcal{S}_{L2}(z;\lambda,c,\tilde{m})\end{pmatrix},\tag{B23}$$

where a and b are arbitrary constants.

If N's have the same parity assignment at $y=0$ as that at $y=L$, then (B22) must be satisfied at $z=1$ as well. Substituting (B23) into (B22) and evaluating the conditions at $z=1$, one finds

$$\begin{pmatrix}\mathcal{S}_{R2}&\mathcal{S}_{R1}\\\mathcal{S}_{L1}&\mathcal{S}_{L2}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0.\tag{B24}$$

The mass spectrum is determined by

$$\mathcal{S}_{L1}\mathcal{S}_{R1}-\mathcal{S}_{L2}\mathcal{S}_{R2}=0.\tag{B25}$$

b. Case II. $c_+=-c_-=c$

The special case $c_+=-c_-=c$ naturally emerges in the context of six-dimensional gauge-Higgs grand unification [35]. The bulk (vector) mass parameter c appears there as a coefficient in the vector component γ^6, which becomes the bulk mass parameter in the RS space, $\pm c$, for 6D Weyl ($\gamma^7=\pm$) components. In this case, Eq. (B9) becomes

$$\begin{aligned}D_-(c)N_{+R}-\lambda N_{+L}+\frac{\tilde{m}}{z}N_{-R}&=0,\\ D_+(c)N_{+L}-\lambda N_{+R}+\frac{\tilde{m}}{z}N_{-L}&=0,\\ -D_+(c)N_{-R}-\lambda N_{-L}+\frac{\tilde{m}}{z}N_{+R}&=0,\\ -D_-(c)N_{-L}-\lambda N_{-R}+\frac{\tilde{m}}{z}N_{+L}&=0.\end{aligned}\tag{B26}$$

To find solutions to Eqs. (B26), we note that

$$\left\{-\frac{d^2}{dz^2}+\frac{c(c\mp1)}{z^2}+\frac{\tilde{m}^2}{z^2}-\lambda^2\right\}N_{\pm R}-\frac{\tilde{m}}{z^2}N_{\mp R}=0.\tag{B27}$$

We seek solutions in the form $N_{+R}=f(z)$ and $N_{-R}=\alpha f(z)$. Solutions exist provided $-c-\alpha\tilde{m}=c-\tilde{m}/\alpha$ is satisfied, or $\alpha=\alpha_\pm$, where

$$\alpha_\pm=\frac{1}{\tilde{m}}(-c\pm\hat{c}),\quad \alpha_+\alpha_-=-1,\quad \hat{c}=\sqrt{c^2+\tilde{m}^2}.\tag{B28}$$

With $\alpha=\alpha_\pm$, $f(z)$ satisfies

$$\{D_\pm(\hat{c})D_\mp(\hat{c})-\lambda^2\}f(z)=0.\tag{B29}$$

Hence general solutions are given by

$$\begin{aligned}\begin{pmatrix}N_{+R}\\N_{-R}\end{pmatrix}&=a\begin{pmatrix}C_R^{\hat{c}}\\\alpha_+C_R^{\hat{c}}\end{pmatrix}+b\begin{pmatrix}S_R^{\hat{c}}\\\alpha_+S_R^{\hat{c}}\end{pmatrix}\\&\quad+a'\begin{pmatrix}C_L^{\hat{c}}\\\alpha_-C_L^{\hat{c}}\end{pmatrix}+b'\begin{pmatrix}S_L^{\hat{c}}\\\alpha_-S_L^{\hat{c}}\end{pmatrix},\end{aligned}\tag{B30}$$

where $C_{L/R}^{\hat{c}}=C_{L/R}(z;\lambda,\hat{c})$ and $S_{L/R}^{\hat{c}}=S_{L/R}(z;\lambda,\hat{c})$.

To find the corresponding solutions for $N_{\pm L}$, we make use of the identities

$$\begin{aligned}D_-(c)&=+D_-(\hat{c})-\frac{\tilde{m}\alpha_+}{z}=-D_+(\hat{c})-\frac{\tilde{m}\alpha_-}{z},\\ D_+(c)&=+D_+(\hat{c})-\frac{\tilde{m}\alpha_+}{z}=-D_-(\hat{c})-\frac{\tilde{m}\alpha_-}{z}\end{aligned}\tag{B31}$$

to find

$$\begin{aligned}\begin{pmatrix}N_{+L}\\N_{-L}\end{pmatrix}&=a\begin{pmatrix}S_L^{\hat{c}}\\\alpha_+S_L^{\hat{c}}\end{pmatrix}+b\begin{pmatrix}C_L^{\hat{c}}\\\alpha_+C_L^{\hat{c}}\end{pmatrix}\\&\quad-a'\begin{pmatrix}S_R^{\hat{c}}\\\alpha_-S_R^{\hat{c}}\end{pmatrix}-b'\begin{pmatrix}C_R^{\hat{c}}\\\alpha_-C_R^{\hat{c}}\end{pmatrix}.\end{aligned}\tag{B32}$$

Basis functions for case II are defined as follows:

$$\begin{aligned}
\hat{\mathcal{C}}_{R1}(z;\lambda,c,\tilde{m}) &= C_R(z;\lambda,\hat{c}) + \alpha_+^2 C_L(z;\lambda,\hat{c}),\\
\hat{\mathcal{C}}_{R2}(z;\lambda,c,\tilde{m}) &= \alpha_+\{S_L(z;\lambda,\hat{c}) + S_R(z;\lambda,\hat{c})\},\\
\hat{\mathcal{S}}_{L1}(z;\lambda,c,\tilde{m}) &= S_L(z;\lambda,\hat{c}) - \alpha_+^2 S_R(z;\lambda,\hat{c}),\\
\hat{\mathcal{S}}_{L2}(z;\lambda,c,\tilde{m}) &= \alpha_+\{C_L(z;\lambda,\hat{c}) - C_R(z;\lambda,\hat{c})\},\\
\hat{\mathcal{C}}_{L1}(z;\lambda,c,\tilde{m}) &= C_L(z;\lambda,\hat{c}) + \alpha_+^2 C_R(z;\lambda,\hat{c}),\\
\hat{\mathcal{C}}_{L2}(z;\lambda,c,\tilde{m}) &= \alpha_+\{S_R(z;\lambda,\hat{c}) + S_L(z;\lambda,\hat{c})\},\\
\hat{\mathcal{S}}_{R1}(z;\lambda,c,\tilde{m}) &= S_R(z;\lambda,\hat{c}) - \alpha_+^2 S_L(z;\lambda,\hat{c}),\\
\hat{\mathcal{S}}_{R2}(z;\lambda,c,\tilde{m}) &= \alpha_+\{C_R(z;\lambda,\hat{c}) - C_L(z;\lambda,\hat{c})\}.
\end{aligned}\tag{B33}$$

We note that $\hat{\mathcal{C}}_{L2}(z;\lambda,c,\tilde{m}) = \hat{\mathcal{C}}_{R2}(z;\lambda,c,\tilde{m})$ and $\hat{\mathcal{S}}_{L2}(z;\lambda,c,\tilde{m}) = -\hat{\mathcal{S}}_{R2}(z;\lambda,c,\tilde{m})$. With the aid of (B28) and (B31), one finds

$$\begin{aligned}
&D_-(c)\begin{pmatrix}\hat{\mathcal{C}}_{R1}\\ \hat{\mathcal{C}}_{R2}\end{pmatrix} = \lambda\begin{pmatrix}\hat{\mathcal{S}}_{L1}\\ \hat{\mathcal{S}}_{L2}\end{pmatrix} + \frac{\tilde{m}}{z}\begin{pmatrix}\hat{\mathcal{S}}_{L2}\\ \hat{\mathcal{S}}_{L1}\end{pmatrix},\\
&D_+(c)\begin{pmatrix}\hat{\mathcal{S}}_{L1}\\ \hat{\mathcal{S}}_{L2}\end{pmatrix} = \lambda\begin{pmatrix}\hat{\mathcal{C}}_{R1}\\ \hat{\mathcal{C}}_{R2}\end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix}\hat{\mathcal{C}}_{R2}\\ \hat{\mathcal{C}}_{R1}\end{pmatrix},\\
&D_-(c)\begin{pmatrix}\hat{\mathcal{S}}_{R1}\\ \hat{\mathcal{S}}_{R2}\end{pmatrix} = \lambda\begin{pmatrix}\hat{\mathcal{C}}_{L1}\\ \hat{\mathcal{C}}_{L2}\end{pmatrix} - \frac{\tilde{m}}{z}\begin{pmatrix}\hat{\mathcal{C}}_{L2}\\ \hat{\mathcal{C}}_{L1}\end{pmatrix},\\
&D_+(c)\begin{pmatrix}\hat{\mathcal{C}}_{L1}\\ \hat{\mathcal{C}}_{L2}\end{pmatrix} = \lambda\begin{pmatrix}\hat{\mathcal{S}}_{R1}\\ \hat{\mathcal{S}}_{R2}\end{pmatrix} + \frac{\tilde{m}}{z}\begin{pmatrix}\hat{\mathcal{S}}_{R2}\\ \hat{\mathcal{S}}_{R1}\end{pmatrix},\\
&\hat{\mathcal{S}}_{Rj} = \hat{\mathcal{S}}_{Lj} = D_-(c)\hat{\mathcal{C}}_{Rj} = D_+(c)\hat{\mathcal{C}}_{Lj} = 0 \quad \text{at } z = z_L.
\end{aligned}\tag{B34}$$

Note that

$$\hat{\mathcal{S}}_{R1}\hat{\mathcal{C}}_{L1} - \hat{\mathcal{S}}_{R2}\hat{\mathcal{C}}_{L2} = (1+\alpha_+^2)(S_R^{\hat{c}} C_L^{\hat{c}} - \alpha_+^2 S_L^{\hat{c}} C_R^{\hat{c}}).\tag{B35}$$

As $c \to -c$, $\alpha_\pm \to -\alpha_\mp$ so that

$$\begin{aligned}
\hat{\mathcal{C}}_{Rj}(z;\lambda,-c,\tilde{m}) &= \alpha_-^2 \hat{\mathcal{C}}_{Lj}(z;\lambda,c,\tilde{m}),\\
\hat{\mathcal{C}}_{Lj}(z;\lambda,-c,\tilde{m}) &= \alpha_-^2 \hat{\mathcal{C}}_{Rj}(z;\lambda,c,\tilde{m}),\\
\hat{\mathcal{S}}_{Rj}(z;\lambda,-c,\tilde{m}) &= -\alpha_-^2 \hat{\mathcal{S}}_{Lj}(z;\lambda,c,\tilde{m}),\\
\hat{\mathcal{S}}_{Lj}(z;\lambda,-c,\tilde{m}) &= -\alpha_-^2 \hat{\mathcal{S}}_{Rj}(z;\lambda,c,\tilde{m}).
\end{aligned}\tag{B36}$$

Further, as $\tilde{m} \to -\tilde{m}$, $\alpha_\pm \to -\alpha_\pm$ and

$$\begin{aligned}
\hat{\mathcal{C}}_{R/Lj}(z;\lambda,c,-\tilde{m}) &= (-1)^{j-1}\hat{\mathcal{C}}_{R/Lj}(z;\lambda,c,\tilde{m}),\\
\hat{\mathcal{S}}_{R/Lj}(z;\lambda,c,-\tilde{m}) &= (-1)^{j-1}\hat{\mathcal{S}}_{R/Lj}(z;\lambda,c,\tilde{m}).
\end{aligned}\tag{B37}$$

In the $\tilde{m} \to 0$ limit

$$\begin{aligned}
&\hat{\mathcal{C}}_{R/L1}(z;\lambda,c,0) = C_{R/L1}(z;\lambda,c),\\
&\hat{\mathcal{S}}_{R/L1}(z;\lambda,c,0) = S_{R/L1}(z;\lambda,c),\\
&\hat{\mathcal{C}}_{R/L2}(z;\lambda,c,0) = \hat{\mathcal{S}}_{R/L2}(z;\lambda,c,0) = 0.
\end{aligned}\tag{B38}$$

Two types of boundary conditions appear at $z = z_L$.

Type A: $(N_{+R}, N_{-R}, N_{+L}, N_{-L}) = (+,-,-,+)$

When parity assignment at $y = L$ for $(N_{+R}, N_{-R}, N_{+L}, N_{-L})$ is $(+,-,-,+)$, boundary conditions at $z = z_L$ become

$$\begin{aligned}
D_-(c)N_{+R} &= 0, \qquad N_{+L} = 0,\\
N_{-R} &= 0, \qquad D_-(c)N_{-L} = 0,
\end{aligned}\tag{B39}$$

which leads to the conditions for the parameters in (B30) and (B32):

$$\begin{cases} a\alpha_+ + a'\alpha_- = 0,\\ b - b' = 0.\end{cases}\tag{B40}$$

It follows that solutions can be written as

$$\begin{pmatrix}N_{+R}\\ N_{+L}\\ N_{-R}\\ N_{-L}\end{pmatrix} = \tilde{a}\begin{pmatrix}\hat{\mathcal{C}}_{R1}(z;\lambda,c,\tilde{m})\\ \hat{\mathcal{S}}_{L1}(z;\lambda,c,\tilde{m})\\ -\hat{\mathcal{S}}_{L2}(z;\lambda,c,\tilde{m})\\ \hat{\mathcal{C}}_{R2}(z;\lambda,c,\tilde{m})\end{pmatrix} + \tilde{b}\begin{pmatrix}\hat{\mathcal{C}}_{R2}(z;\lambda,c,\tilde{m})\\ \hat{\mathcal{S}}_{L2}(z;\lambda,c,\tilde{m})\\ -\hat{\mathcal{S}}_{L1}(z;\lambda,c,\tilde{m})\\ \hat{\mathcal{C}}_{R1}(z;\lambda,c,\tilde{m})\end{pmatrix},\tag{B41}$$

where $\tilde{a} = a$ and $\tilde{b} = b/\alpha_+$ are arbitrary constants.

If N's have the same parity assignment at $y = 0$ as that at $y = L$, then (B39) must be satisfied at $z = 1$ as well. Substituting (B41) into (B39) and evaluating the conditions at $z = 1$, one finds

$$\begin{pmatrix}\hat{\mathcal{S}}_{L1} & \hat{\mathcal{S}}_{L2}\\ \hat{\mathcal{S}}_{L2} & \hat{\mathcal{S}}_{L1}\end{pmatrix}\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix} = 0,\tag{B42}$$

where $\hat{\mathcal{S}}_{L1} = \hat{\mathcal{S}}_{L1}(1;\lambda,c,\tilde{m})$, etc. The mass spectrum is determined by

$$\hat{\mathcal{S}}_{L1}^2 - \hat{\mathcal{S}}_{L2}^2 = 0.\tag{B43}$$

Type B: $(N_{+R}, N_{-R}, N_{+L}, N_{-L}) = (-,+,+,-)$

When parity assignment at $y = L$ for $(N_{+R}, N_{-R}, N_{+L}, N_{-L})$ is $(-,+,+,-)$, boundary conditions at $z = z_L$ become

$$\begin{aligned} N_{+R} &= 0, \qquad D_+(c)N_{+L} = 0, \\ D_+(c)N_{-R} &= 0, \qquad N_{-L} = 0. \end{aligned} \tag{B44}$$

This leads to

$$\begin{cases} a + a' = 0, \\ b\alpha_+ - b'\alpha_- = 0. \end{cases} \tag{B45}$$

It follows that solutions can be written as

$$\begin{pmatrix} N_{+R} \\ N_{+L} \\ N_{-R} \\ N_{-L} \end{pmatrix} = \tilde{a} \begin{pmatrix} \hat{\mathcal{S}}_{R2}(z;\lambda,c,\tilde{m}) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c,\tilde{m}) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c,\tilde{m}) \\ -\hat{\mathcal{S}}_{R1}(z;\lambda,c,\tilde{m}) \end{pmatrix} + \tilde{b} \begin{pmatrix} \hat{\mathcal{S}}_{R1}(z;\lambda,c,\tilde{m}) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c,\tilde{m}) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c,\tilde{m}) \\ -\hat{\mathcal{S}}_{R2}(z;\lambda,c,\tilde{m}) \end{pmatrix}, \tag{B46}$$

where $\tilde{a} = a/\alpha_+$ and $\tilde{b} = b$ are arbitrary constants.

If N's have the same parity assignment at $y = 0$ as that at $y = L$, then (B44) must be satisfied at $z = 1$ as well. Substituting (B46) into (B44) and evaluating the conditions at $z = 1$, one finds

$$\begin{pmatrix} \hat{\mathcal{S}}_{R2} & \hat{\mathcal{S}}_{R1} \\ \hat{\mathcal{S}}_{R1} & \hat{\mathcal{S}}_{R2} \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = 0. \tag{B47}$$

The mass spectrum is determined by

$$\hat{\mathcal{S}}_{R1}^2 - \hat{\mathcal{S}}_{R2}^2 = 0. \tag{B48}$$

APPENDIX C: MAJORANA FERMIONS

We summarize the notation adopted in the present paper concerning Majorana fermions in four dimensions. Dirac matrices are

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1), \\ &\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} = (\pm I_2, \vec{\sigma}), \\ &\gamma^5 = \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}. \end{aligned} \tag{C1}$$

We define $\bar{\psi} = i\psi^\dagger\gamma^0$. Charge conjugation is given by $\psi^C = U_C(\bar{\psi})^t$, where $U_C\gamma^{\mu t}U_C^\dagger = -\gamma^\mu$. In our representation

$$\begin{aligned} &U_C = ie^{i\delta_c}\begin{pmatrix} \sigma^2 & \\ & \sigma^2 \end{pmatrix}, \\ &\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \psi^C = \begin{pmatrix} \eta^c \\ -\xi^c \end{pmatrix} = e^{i\delta_c}\begin{pmatrix} \sigma^2\eta^* \\ -\sigma^2\xi^* \end{pmatrix}. \end{aligned} \tag{C2}$$

Note that $(\psi^C)^C = \psi$, whereas $(\eta^c)^c = -\eta$ and $(\xi^c)^c = -\xi$. It follows that

$$\begin{aligned} \overline{\psi_1}\psi_2 &= -i\eta_1^\dagger\xi_2 + i\xi_1^\dagger\eta_2 = \overline{\psi_2^C}\psi_1^C, \\ \overline{\psi_1}\gamma^\mu\partial_\mu\psi_2 &= -i\eta_1^\dagger\sigma^\mu\partial_\mu\eta_2 + i\xi_1^\dagger\bar{\sigma}^\mu\partial_\mu\xi_2, \\ -i\eta_1^\dagger\sigma^\mu\partial_\mu\eta_2 &= -i\partial_\mu\eta_2^{c\,\dagger}\bar{\sigma}^\mu\eta_1^c \sim i\eta_2^{c\dagger}\bar{\sigma}^\mu\partial_\mu\eta_1^c, \end{aligned} \tag{C3}$$

and so on.

In (5.38) we have introduced wave functions of mass eigenstates satisfying

$$\begin{aligned} &\bar{\sigma}^\mu\partial_\mu f_{\pm R}(x) = m f_{\pm L}(x), \qquad \sigma^\mu\partial_\mu f_{\pm L}(x) = m f_{\pm R}(x), \\ &f_{\pm L}(x)^c = e^{i\delta_c}\sigma^2 f_{\pm L}(x)^* = \pm f_{\pm R}(x). \end{aligned} \tag{C4}$$

Explicit forms of $f_{\pm L/R}(x)$ are given, for modes propagating in the x_3-direction with $\vec{p} = (0,0,p)$, by

$$\begin{aligned} f_{+L}^{(1)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E+p}\,e^{-iEt+ipx_3} \\ e^{i\delta_c}\sqrt{E-p}\,e^{iEt-ipx_3} \end{pmatrix}, \\ f_{+R}^{(1)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} -i\sqrt{E-p}\,e^{-iEt+ipx_3} \\ ie^{i\delta_c}\sqrt{E+p}\,e^{iEt-ipx_3} \end{pmatrix}, \\ f_{+L}^{(2)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E+p}\,e^{iEt-ipx_3} \\ -e^{i\delta_c}\sqrt{E-p}\,e^{-iEt+ipx_3} \end{pmatrix}, \\ f_{+R}^{(2)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} i\sqrt{E-p}\,e^{iEt-ipx_3} \\ ie^{i\delta_c}\sqrt{E+p}\,e^{-iEt+ipx_3} \end{pmatrix}, \\ f_{-L}^{(1)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E+p}\,e^{-iEt+ipx_3} \\ -e^{i\delta_c}\sqrt{E-p}\,e^{iEt-ipx_3} \end{pmatrix}, \\ f_{-R}^{(1)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} -i\sqrt{E-p}\,e^{-iEt+ipx_3} \\ -ie^{i\delta_c}\sqrt{E+p}\,e^{iEt-ipx_3} \end{pmatrix}, \\ f_{-L}^{(2)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E+p}\,e^{iEt-ipx_3} \\ e^{i\delta_c}\sqrt{E-p}\,e^{-iEt+ipx_3} \end{pmatrix}, \\ f_{-R}^{(2)} &= \frac{1}{\sqrt{2E}}\begin{pmatrix} i\sqrt{E-p}\,e^{iEt-ipx_3} \\ -ie^{i\delta_c}\sqrt{E+p}\,e^{-iEt+ipx_3} \end{pmatrix}. \end{aligned} \tag{C5}$$

Here $E = \sqrt{p^2 + m^2}$.

APPENDIX D: DARK FERMIONS

In addition to the quark and lepton multiplets, we introduce dark fermion multiplets in the bulk, which give relevant contributions to the effective potential $V_{\rm eff}(\theta_H)$ to induce the electroweak symmetry breaking by the Hosotani mechanism. They naturally appear from grand unified theory.

1. $Q_{\rm EM} = \frac{2}{3}, -\frac{1}{3}$: ($\Psi_{(3,4)} \equiv \Psi_F$)

The bulk mass parameter of this multiplet, c_F, is assumed to satisfy $|c_F| < \frac{1}{2}$. Ψ_F satisfies boundary condition (3.8). There are no zero modes. The spectrum is vectorlike. (F_1, F_1') in Table III forms a pair analogous to the (u, u') pair, whereas (F_2, F_2') is analogous to the (d, d') pair. Both pairs satisfy, in the twisted gauge, the equations similar to Eq. (5.6) with c_Q replaced by c_F.

With the boundary conditions at $y = L$ taken into account, mode functions can be written as

$$\begin{pmatrix} \check{\tilde{F}}_{1R} \\ \check{\tilde{F}}'_{1R} \end{pmatrix} = \begin{pmatrix} \alpha_F S_R(z,\lambda,c_F) \\ \alpha_{F'} C_R(z,\lambda,c_F) \end{pmatrix} f_R(x),$$
$$\begin{pmatrix} \check{\tilde{F}}_{1L} \\ \check{\tilde{F}}'_{1L} \end{pmatrix} = \begin{pmatrix} \alpha_F C_L(z,\lambda,c_F) \\ \alpha_{F'} S_L(z,\lambda,c_F) \end{pmatrix} f_L(x). \quad \text{(D1)}$$

The boundary conditions at $z = 1$ are flipped, however, and we have $D_-\check{F}_{1R} = 0$ and $\check{F}'_{1R} = 0$ there to find

$$K_F \begin{pmatrix} \alpha_F \\ \alpha_{F'} \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta_H C_L^F & -i\sin\frac{1}{2}\theta_H S_L^F \\ -i\sin\frac{1}{2}\theta_H S_R^F & \cos\frac{1}{2}\theta_H C_R^F \end{pmatrix} \begin{pmatrix} \alpha_F \\ \alpha_{F'} \end{pmatrix} = 0. \quad \text{(D2)}$$

Here $S_{L/R}^F = S_{L/R}(1,\lambda,c_F)$, etc. $\det K_F = 0$ leads to the equation determining the spectrum:

$$S_L^F S_R^F + \cos^2\frac{\theta_H}{2} = 0. \quad \text{(D3)}$$

There are no light modes for $|c_F| < \frac{1}{2}$ and small θ_H. The spectrum of the (F_2, F_2') pair is also given by (D3).

2. $Q_{\rm EM} = \pm 1$: $E^\pm$, $\hat{E}^\pm$ ($\Psi^\pm_{(1,5)}$)

In general $\Psi^+_{(1,5)}$ and $\Psi^-_{(1,5)}$ may have different bulk mass parameters c_{V^+} and c_{V^-}. For charged particles $E^\pm$, equations of motion are given by

$$-kD_-(c_{V^+})\check{E}^+_R + \sigma^\mu\partial_\mu \check{E}^+_L - \frac{m_V^*}{z}\check{E}^-_R = 0,$$
$$\bar{\sigma}^\mu\partial_\mu \check{E}^+_R - kD_+(c_{V^+})\check{E}^+_L - \frac{m_V}{z}\check{E}^-_L = 0,$$
$$-kD_-(c_{V^-})\check{E}^-_R + \sigma^\mu\partial_\mu \check{E}^-_L - \frac{m_V^*}{z}\check{E}^+_R = 0,$$
$$\bar{\sigma}^\mu\partial_\mu \check{E}^-_R - kD_+(c_{V^-})\check{E}^-_L - \frac{m_V}{z}\check{E}^+_L = 0. \quad \text{(D4)}$$

E^+ and E^- couple with each other through the mass m_V. Boundary conditions are given by $\check{E}^+_R = D_+(c_{V^+})\check{E}^+_L = 0$ and $D_-(c_{V^-})\check{E}^-_R = \check{E}^-_L = 0$ at $z = 1$, z_L.

Mode functions can easily be found for $c_{V^+} = \pm c_{V^-}$. They are summarized in Appendix B 3. We quote the results there. We note that the same result is obtained for $\hat{E}^\pm$ as for $E^\pm$.

a. Case I: $c_{V^+} = c_{V^-} = c_V$

We denote $\tilde{m}_V = m_V/k$. The boundary condition is type B. Mode functions are given by (B23):

$$\begin{pmatrix} \check{E}^+_R \\ \check{E}^+_L \\ \check{E}^-_R \\ \check{E}^-_L \end{pmatrix} = a \begin{pmatrix} \mathcal{S}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{C}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{C}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{S}_{L1}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix} + b \begin{pmatrix} \mathcal{S}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{C}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{C}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \mathcal{S}_{L2}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix}, \quad \text{(D5)}$$

where a and b are arbitrary constants. The expression is valid both in the original gauge and in the twisted gauge, as these fields do not couple to θ_H at the tree level. The spectrum is determined by (B25):

$$\mathcal{S}^V_{L1}\mathcal{S}^V_{R1} - \mathcal{S}^V_{L2}\mathcal{S}^V_{R2} = 0, \quad \text{(D6)}$$

where $\mathcal{S}^V_{L1} = \mathcal{S}_{L1}(1;\lambda,c_V,\tilde{m}_V)$, etc.

b. Case II: $c_{V^+} = -c_{V^-} = c_V$

In this case, mode functions are given by (B46):

$$\begin{pmatrix} \check{E}^+_R \\ \check{E}^+_L \\ \check{E}^-_R \\ \check{E}^-_L \end{pmatrix} = a \begin{pmatrix} \hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix} + b \begin{pmatrix} \hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix}, \quad \text{(D7)}$$

where a and b are arbitrary constants. The spectrum is determined by (B48):

$$(\hat{\mathcal{S}}^V_{R1})^2 - (\hat{\mathcal{S}}^V_{R2})^2 = 0, \quad \text{(D8)}$$

where $\hat{\mathcal{S}}^V_{R1} = \hat{\mathcal{S}}_{R1}(1;\lambda,c_V,\tilde{m}_V)$, etc.

3. $Q_{\rm EM} = 0$: $N^\pm$, $\hat{N}^\pm$, $S^\pm$ ($\Psi^\pm_{(1,5)}$)

$N^\pm$, $\hat{N}^\pm$, and $S^\pm$ couple with each other through θ_H. Equations of motion in the original gauges are

$$-k\hat{D}_-(c_{V^\pm})\begin{pmatrix}\check{\hat{N}}{}^\pm_R\\ \check{N}{}^\pm_R\\ \check{S}{}^\pm_R\end{pmatrix}+\sigma^\mu\partial_\mu\begin{pmatrix}\check{\hat{N}}{}^\pm_L\\ \check{N}{}^\pm_L\\ \check{S}{}^\pm_L\end{pmatrix}-\frac{m_V^*}{z}\begin{pmatrix}\check{\hat{N}}{}^\mp_R\\ \check{N}{}^\mp_R\\ \check{S}{}^\mp_R\end{pmatrix}=0,$$

$$\bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\check{\hat{N}}{}^\pm_R\\ \check{N}{}^\pm_R\\ \check{S}{}^\pm_R\end{pmatrix}-k\hat{D}_+(c_{V^\pm})\begin{pmatrix}\check{\hat{N}}{}^\pm_L\\ \check{N}{}^\pm_L\\ \check{S}{}^\pm_L\end{pmatrix}-\frac{m_V}{z}\begin{pmatrix}\check{\hat{N}}{}^\mp_L\\ \check{N}{}^\mp_L\\ \check{S}{}^\mp_L\end{pmatrix}=0. \tag{D9}$$

Note that $\hat{D}_\pm(c)$ is given by (5.2).

The relation between the original and twisted gauges is given by $\Psi^\pm_{(\mathbf{1},\mathbf{5})}=\Omega(z)\tilde{\Psi}^\pm_{(\mathbf{1},\mathbf{5})}$, where $\Omega(z)=e^{i\theta(z)T_{45}}$, so that

$$\psi_3=\tilde{\psi}_3,\qquad \begin{pmatrix}\psi_4\\ \psi_5\end{pmatrix}=\begin{pmatrix}\cos\theta(z) & \sin\theta(z)\\ -\sin\theta(z) & \cos\theta(z)\end{pmatrix},$$
$$\psi_3^\pm=\frac{i}{\sqrt{2}}(\hat{N}^\pm+N^\pm),\qquad \psi_4^\pm=\frac{1}{\sqrt{2}}(\hat{N}^\pm-N^\pm),$$
$$\psi_5^\pm=S^\pm, \tag{D10}$$

and therefore

$$\begin{pmatrix}\check{\hat{N}}{}^\pm\\ \check{N}{}^\pm\\ \check{S}{}^\pm\end{pmatrix}=\bar{\Omega}(z)\begin{pmatrix}\tilde{\check{\hat{N}}}{}^\pm\\ \tilde{\check{N}}{}^\pm\\ \tilde{\check{S}}{}^\pm\end{pmatrix},$$
$$\bar{\Omega}(z)=V\begin{pmatrix}1&0&0\\ 0&\cos\theta(z)&\sin\theta(z)\\ 0&-\sin\theta(z)&\cos\theta(z)\end{pmatrix}V^{-1},$$
$$V=V^{-1}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ \frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\ 0&0&1\end{pmatrix}. \tag{D11}$$

It follows that

$$\hat{D}_-(c_{V^\pm})\begin{pmatrix}\check{\hat{N}}{}^\pm_R\\ \check{N}{}^\pm_R\\ \check{S}{}^\pm_R\end{pmatrix}=\bar{\Omega}(z)D_-(c_{V^\pm})\begin{pmatrix}\tilde{\check{\hat{N}}}{}^\pm_R\\ \tilde{\check{N}}{}^\pm_R\\ \tilde{\check{S}}{}^\pm_R\end{pmatrix}, \tag{D12}$$

and so on. Boundary conditions in the original gauge are

$$\check{\hat{N}}{}^+_R=\hat{D}_+(c_{V^+})\check{\hat{N}}{}^+_L=\hat{D}_-(c_{V^-})\check{\hat{N}}{}^-_R=\check{\hat{N}}{}^-_L=0,$$
$$\check{N}{}^+_R=\hat{D}_+(c_{V^+})\check{N}{}^+_L=\hat{D}_-(c_{V^-})\check{N}{}^-_R=\check{N}{}^-_L=0,$$
$$\hat{D}_-(c_{V^+})\check{S}{}^+_R=\check{S}{}^+_L=\check{S}{}^-_R=\hat{D}_+(c_{V^-})\check{S}{}^-_L=0, \tag{D13}$$

at both $z=1$ and $z=z_L$.

a. Case I: $c_{V^+}=c_{V^-}=c_V$

The boundary conditions in the twisted gauge at $z=z_L$ are obtained from (D13) by replacing $\hat{D}_\pm(c)$ by $D_\pm(c)$. Mode functions of the N and $\hat{N}$ fields are given by (B23), whereas those of the S field are given by (B18):

$$\begin{pmatrix}\tilde{\check{\hat{N}}}{}^+_R\\ \tilde{\check{\hat{N}}}{}^+_L\\ \tilde{\check{\hat{N}}}{}^-_R\\ \tilde{\check{\hat{N}}}{}^-_L\end{pmatrix}=a_{\hat{N}}\begin{pmatrix}\mathcal{S}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L1}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix}+b_{\hat{N}}\begin{pmatrix}\mathcal{S}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L2}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix},$$

$$\begin{pmatrix}\tilde{\check{N}}{}^+_R\\ \tilde{\check{N}}{}^+_L\\ \tilde{\check{N}}{}^-_R\\ \tilde{\check{N}}{}^-_L\end{pmatrix}=a_N\begin{pmatrix}\mathcal{S}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L1}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix}+b_N\begin{pmatrix}\mathcal{S}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L2}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix},$$

$$\begin{pmatrix}\tilde{\check{S}}{}^+_R\\ \tilde{\check{S}}{}^+_L\\ \tilde{\check{S}}{}^-_R\\ \tilde{\check{S}}{}^-_L\end{pmatrix}=a_S\begin{pmatrix}\mathcal{C}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L2}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix}+b_S\begin{pmatrix}\mathcal{C}_{R2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{L2}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{S}_{R1}(z;\lambda,c_V,\tilde{m}_V)\\ \mathcal{C}_{L1}(z;\lambda,c_V,\tilde{m}_V)\end{pmatrix}, \tag{D14}$$

where $\tilde{m}_V=m_V/k$ and $a_{\hat{N}}$, $b_{\hat{N}}$, a_N, b_N, a_S, and b_S are arbitrary parameters.

We insert (D14) into the boundary conditions (D13) at $z=1$. With the aid of (D11) and (D12), one finds that

$$K_N \begin{pmatrix} a_{\hat{N}} \\ a_N \\ a_S \\ b_{\hat{N}} \\ b_N \\ b_S \end{pmatrix} = 0, \quad K_N = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix},$$

$$A = \begin{pmatrix} \mathcal{S}^V_{R2} & 0 & 0 \\ 0 & c_H \mathcal{S}^V_{R2} & s_H \mathcal{C}^V_{R1} \\ 0 & -s_H \mathcal{C}^V_{L2} & c_H \mathcal{S}^V_{L1} \end{pmatrix}, \quad B = \begin{pmatrix} \mathcal{S}^V_{R1} & 0 & 0 \\ 0 & c_H \mathcal{S}^V_{R1} & s_H \mathcal{C}^V_{R2} \\ 0 & -s_H \mathcal{C}^V_{L1} & c_H \mathcal{S}^V_{L2} \end{pmatrix},$$
$$C = \begin{pmatrix} \mathcal{S}^V_{L1} & 0 & 0 \\ 0 & c_H \mathcal{S}^V_{L1} & s_H \mathcal{C}^V_{L2} \\ 0 & -s_H \mathcal{C}^V_{R1} & c_H \mathcal{S}^V_{R2} \end{pmatrix}, \quad D = \begin{pmatrix} \mathcal{S}^V_{L2} & 0 & 0 \\ 0 & c_H \mathcal{S}^V_{L2} & s_H \mathcal{C}^V_{L1} \\ 0 & -s_H \mathcal{C}^V_{R2} & c_H \mathcal{S}^V_{R1} \end{pmatrix}, \tag{D15}$$

where $c_H = \cos\theta_H$, $s_H = \sin\theta_H$, and $\mathcal{S}^V_{L1} = \mathcal{S}_{L1}(1;\lambda, c_V, \tilde{m}_V)$, etc. The spectrum is determined by $\det K_N = 0$:

$$\det K_N = \det \begin{pmatrix} \mathcal{S}^V_{R2} & \mathcal{S}^V_{R1} \\ \mathcal{S}^V_{L1} & \mathcal{S}^V_{L2} \end{pmatrix} \det \begin{pmatrix} c_H \mathcal{S}^V_{R2} & s_H \mathcal{C}^V_{R1} & c_H \mathcal{S}^V_{R1} & s_H \mathcal{C}^V_{R2} \\ -s_H \mathcal{C}^V_{L2} & c_H \mathcal{S}^V_{L1} & -s_H \mathcal{C}^V_{L1} & c_H \mathcal{S}^V_{L2} \\ c_H \mathcal{S}^V_{L1} & s_H \mathcal{C}^V_{L2} & c_H \mathcal{S}^V_{L2} & s_H \mathcal{C}^V_{L1} \\ -s_H \mathcal{C}^V_{R1} & c_H \mathcal{S}^V_{R2} & -s_H \mathcal{C}^V_{R2} & c_H \mathcal{S}^V_{R1} \end{pmatrix}$$
$$= (\mathcal{S}^V_{L1}\mathcal{S}^V_{R1} - \mathcal{S}^V_{L2}\mathcal{S}^V_{R2})\{c_H^4(\mathcal{S}^V_{L1}\mathcal{S}^V_{R1} - \mathcal{S}^V_{L2}\mathcal{S}^V_{R2})^2 + s_H^4(\mathcal{C}^V_{L1}\mathcal{C}^V_{R1} - \mathcal{C}^V_{L2}\mathcal{C}^V_{R2})^2 + s_H^2 c_H^2(\mathcal{C}^V_{R1}\mathcal{C}^V_{L2} - \mathcal{C}^V_{R2}\mathcal{S}^V_{L1})^2$$
$$+ s_H^2 c_H^2(\mathcal{C}^V_{L1}\mathcal{S}^V_{R2} - \mathcal{C}^V_{L2}\mathcal{S}^V_{R1})^2 + 2 s_H^2 c_H^2(\mathcal{C}^V_{L1}\mathcal{S}^V_{L1} - \mathcal{C}^V_{L2}\mathcal{S}^V_{L2})(\mathcal{C}^V_{R1}\mathcal{S}^V_{R1} - \mathcal{C}^V_{R2}\mathcal{S}^V_{R2})\}. \tag{D16}$$

b. Case II: $c_{V^+} = -c_{V^-} = c_V$

The boundary conditions (D13) become

$$\begin{aligned} \check{\hat{N}}{}^+_R = \hat{D}_+(c_V)\check{\hat{N}}{}^+_L = \hat{D}_+(c_V)\check{\hat{N}}{}^-_R = \check{\hat{N}}{}^-_L = 0, \\ \check{N}^+_R = \hat{D}_+(c_V)\check{N}^+_L = \hat{D}_+(c_V)\check{N}^-_R = \check{N}^-_L = 0, \\ \hat{D}_-(c_V)\check{S}^+_R = \check{S}^+_L = \check{S}^-_R = \hat{D}_-(c_V)\check{S}^-_L = 0, \end{aligned} \tag{D17}$$

at both $z = 1$ and $z = z_L$. Mode functions of the N and $\hat{N}$ fields are given by (B46), whereas those of the S field are given by (B41):

$$\begin{pmatrix} \tilde{\check{N}}_R^+ \\ \tilde{\check{N}}_L^+ \\ \tilde{\check{N}}_R^- \\ \tilde{\check{N}}_L^- \end{pmatrix} = a_{\hat{N}} \begin{pmatrix} \hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix} + b_{\hat{N}} \begin{pmatrix} \hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix},$$

$$\begin{pmatrix} \tilde{\check{N}}_R^+ \\ \tilde{\check{N}}_L^+ \\ \tilde{\check{N}}_R^- \\ \tilde{\check{N}}_L^- \end{pmatrix} = a_N \begin{pmatrix} \hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix} + b_N \begin{pmatrix} \hat{\mathcal{S}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix},$$

$$\begin{pmatrix} \tilde{\check{S}}_R^+ \\ \tilde{\check{S}}_L^+ \\ \tilde{\check{S}}_R^- \\ \tilde{\check{S}}_L^- \end{pmatrix} = a_S \begin{pmatrix} \hat{\mathcal{C}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{S}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix} + b_S \begin{pmatrix} \hat{\mathcal{C}}_{R2}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{S}}_{L2}(z;\lambda,c_V,\tilde{m}_V) \\ -\hat{\mathcal{S}}_{L1}(z;\lambda,c_V,\tilde{m}_V) \\ \hat{\mathcal{C}}_{R1}(z;\lambda,c_V,\tilde{m}_V) \end{pmatrix}, \tag{D18}$$

where $a_{\hat{N}}$, $b_{\hat{N}}$, a_N, b_N, a_S, and b_S are arbitrary parameters.

We insert (D18) into the boundary conditions (D17) at $z=1$. This time we have, instead of (D15),

$$K_N = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix},$$
$$\hat{A} = \begin{pmatrix} \hat{\mathcal{S}}_{R2}^V & 0 & 0 \\ 0 & c_H\hat{\mathcal{S}}_{R2}^V & s_H\hat{\mathcal{C}}_{R1}^V \\ 0 & -s_H\hat{\mathcal{C}}_{L2}^V & c_H\hat{\mathcal{S}}_{L1}^V \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} \hat{\mathcal{S}}_{R1}^V & 0 & 0 \\ 0 & c_H\hat{\mathcal{S}}_{R1}^V & s_H\hat{\mathcal{C}}_{R2}^V \\ 0 & -s_H\hat{\mathcal{C}}_{L1}^V & c_H\hat{\mathcal{S}}_{L2}^V \end{pmatrix},$$
$$\hat{C} = \begin{pmatrix} \hat{\mathcal{S}}_{R1}^V & 0 & 0 \\ 0 & c_H\hat{\mathcal{S}}_{R1}^V & -s_H\hat{\mathcal{C}}_{R2}^V \\ 0 & -s_H\hat{\mathcal{C}}_{L1}^V & -c_H\hat{\mathcal{S}}_{L2}^V \end{pmatrix}, \qquad \hat{D} = \begin{pmatrix} \hat{\mathcal{S}}_{R2}^V & 0 & 0 \\ 0 & c_H\hat{\mathcal{S}}_{R2}^V & -s_H\hat{\mathcal{C}}_{R1}^V \\ 0 & -s_H\hat{\mathcal{C}}_{L2}^V & -c_H\hat{\mathcal{S}}_{L1}^V \end{pmatrix}, \tag{D19}$$

where $\hat{\mathcal{S}}_{L1}^V = \hat{\mathcal{S}}_{L1}(1;\lambda,c_V,\tilde{m}_V)$, etc. The spectrum is determined by

$$\begin{aligned} \det K_N = {}& \{(\hat{\mathcal{S}}_{R1}^V)^2 - (\hat{\mathcal{S}}_{R2}^V)^2\} \\ & \times \{\cos^4\theta_H[(\hat{\mathcal{S}}_{L1}^V)^2 - (\hat{\mathcal{S}}_{L2}^V)^2][(\hat{\mathcal{S}}_{R1}^V)^2 - (\hat{\mathcal{S}}_{R2}^V)^2] + \sin^4\theta_H[(\hat{\mathcal{C}}_{L1}^V)^2 - (\hat{\mathcal{C}}_{L2}^V)^2][(\hat{\mathcal{C}}_{R1}^V)^2 - (\hat{\mathcal{C}}_{R2}^V)^2] \\ & + 2\sin^2\theta_H\cos^2\theta_H(\hat{\mathcal{S}}_{L1}^V\hat{\mathcal{S}}_{R1}^V - \hat{\mathcal{S}}_{L2}^V\hat{\mathcal{S}}_{R2}^V)(\hat{\mathcal{C}}_{L1}^V\hat{\mathcal{C}}_{R1}^V - \hat{\mathcal{C}}_{L2}^V\hat{\mathcal{C}}_{R2}^V) \\ & - 2\sin^2\theta_H\cos^2\theta_H(\hat{\mathcal{S}}_{L1}^V\hat{\mathcal{S}}_{R2}^V - \hat{\mathcal{S}}_{L2}^V\hat{\mathcal{S}}_{R1}^V)(\hat{\mathcal{C}}_{L1}^V\hat{\mathcal{C}}_{R2}^V - \hat{\mathcal{C}}_{L2}^V\hat{\mathcal{C}}_{R1}^V)\} = 0. \end{aligned} \tag{D20}$$

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