Small-x evolution of $2n$-tuple Wilson line correlator revisited: The nonsingular kernels

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The Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner equation tells how gauge invariant higher order Wilson line correlators would evolve at high energy. In this article, we have revisited the equation and presented a convenient integrodifferential form of this equation that is carrying identical-looking generalized kernels for all explicit real and virtual terms. In the equation, the "real" terms correspond to splitting (say at position z) of this $2n$-tuple correlator to various pairs of $2m$-tuple and $(2n+2-2m)$-tuple correlators, whereas "virtual" terms correspond to splitting into pairs of $2m$-tuple and $(2n-2m)$-tuple correlators. The generalized kernels of virtual terms with $m=0$ (no splitting) and of real terms with $m=1$ (splitting with at least one dipole) have poles, and when integrated over z, they do generate ultraviolet logarithmic divergences, separately for real and virtual terms. However, we have shown that, except these two cases in all other terms, the corresponding kernels, separately for real and virtual terms, have rather softened ultraviolet singularity and when integrated over z do not generate ultraviolet logarithmic divergences. We have also studied implication of this in the strong scattering regime where only virtual terms are effective.

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I. INTRODUCTION

High energy scattering in QCD [1] can be most conveniently addressed using the color dipole degrees of freedom by Mueller [2,3]. In the study of high energy scattering of a projectile parton and a target nucleus, the small-x evolution can be introduced either in the wave function of the projectile (the parton) or in the wave function of the target (the nucleus). The Balitsky-Kovchegov (BK) evolution equation/Balitsky hierarchy [4,5] accomplishes the first, while the other equivalent approach is realized by Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) [6–8] evolution equation. In this context, when estimating the color averaged expectation value of certain operator, one generally is in need of an appropriate weight function for the color field. In the color glass condensate (CGC) [9–12] effective theory, the spatial distribution of the color sources that produce classical color field inside the large target nucleus is taken to be Gaussian. The JIMWLK formalism generalizes this Gaussian weight of a classical gluon field to a rapidity-dependent weight functional $\mathcal{W}_Y[\alpha]$, which no longer remains Gaussian as it evolve across the energy or rapidity. Unlike the McLerran-Venugopalan (MV) model where the weight function is Gaussian always, here the weight function has to be determined from the JIMWLK equation itself for evaluation at certain rapidity Y. All physical measurable quantities are expressed as gauge invariant operators $\mathcal{O}$ built with the color field α, and corresponding expectation values are obtained after averaging over the stochastic color field α:

$$\langle \mathcal{O} \rangle \equiv \int \mathcal{D}\alpha\, \mathcal{O}[\alpha]\, \mathcal{W}_Y[\alpha]. \tag{1}$$

The functional differential evolution equation for $\mathcal{W}_Y[\alpha]$ is the JIMWLK equation and reads

$$\frac{\partial}{\partial Y}\mathcal{W}_Y[\alpha] = \mathcal{H}\,\mathcal{W}_Y[\alpha], \tag{2}$$

where $Y \equiv \ln(1/x)$ and $\mathcal{H}$ is the JIMWLK Hamiltonian,

$$\mathcal{H} \equiv \frac{1}{2}\int_{xy} \frac{\delta}{\delta\alpha^a_Y(x)}\chi^{ab}(x,y)\frac{\delta}{\delta\alpha^b_Y(y)}. \tag{3}$$

The integral sign with subscript xy, in the Hamiltonian, denotes integration over the transverse coordinates x and y. The kernel $\chi^{ab}(x,y)$ is a functional of α upon which it depends through the Wilson lines e.g., $\tilde{U}(x)$ and $\tilde{U}^\dagger(x)$ (as Wilson lines are build with $\alpha \equiv \alpha^a T^a$ in the adjoint representation), as,

$$\chi^{ab}(x,y) = \frac{1}{\pi}\int \frac{d^2z}{(2\pi)^2}\mathcal{K}(x,y,z)(1-\tilde{U}^\dagger_x\tilde{U}_z)^{fa}(1-\tilde{U}^\dagger_z\tilde{U}_y)^{fb}, \tag{4}$$

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with the transverse kernel,

$$\mathcal{K}(x,y,z) \equiv \frac{(x-z).(y-z)}{(x-z)^2(z-y)^2}, \tag{5}$$

and, e.g.,

$$\tilde{U}_x^\dagger \equiv \tilde{U}^\dagger(x) = \mathcal{P}\exp\left(ig\int dx^- \alpha^a(x^-,x)T^a\right). \tag{6}$$

Here, $\mathcal{P}$ denotes path ordering in x^-, and integration over x^- runs over the longitudinal extent of the hadron which increases with Y. Action of the functional derivative on Wilson lines in the adjoint representation reads as

$$\frac{\delta}{\delta\alpha_\tau^a(y)}\tilde{U}^\dagger(x) = ig\delta^{(2)}(x-y)T^a\tilde{U}_x^\dagger$$
$$\frac{\delta}{\delta\alpha_\tau^a(y)}\tilde{U}(x) = -ig\delta^{(2)}(x-y)\tilde{U}_xT^a. \tag{7}$$

Within the leading logarithmic accuracy, the CGC effective theory prescribes following energy evolution for general gauge invariant operator $\mathcal{O}$:

$$\frac{\partial}{\partial Y}\langle\hat{\mathcal{O}}\rangle_Y = \langle\mathcal{H}\hat{\mathcal{O}}\rangle_Y. \tag{8}$$

The brackets refer to average over color fields in the target nucleus, properly accompanied by the rapidity-dependent CGC weight function as mentioned before, and $\mathcal{H}$ is the JIMWLK Hamiltonian,

$$\mathcal{H} \equiv -\frac{1}{16\pi^3}\int_z \mathcal{M}_{xyz}(1+\tilde{U}_{\boldsymbol{x}}^\dagger\tilde{U}_{\boldsymbol{y}} - \tilde{U}_{\boldsymbol{x}}^\dagger\tilde{U}_{\boldsymbol{z}} - \tilde{U}_{\boldsymbol{z}}^\dagger\tilde{U}_{\boldsymbol{y}})^{ab} \times \frac{\delta}{\delta\alpha_x^a}\frac{\delta}{\delta\alpha_y^b}, \tag{9}$$

where $\mathcal{M}_{xyz}$ is the dipole kernel,

$$\mathcal{M}_{xyz} \equiv \frac{(x-y)^2}{(x-z)^2(z-y)^2}. \tag{10}$$

In Muller's dipole model, the color dipole is a quark-antiquark pair in an overall color singlet state. The operator for the color dipole contains two Wilson lines in their fundamental representation,

$$\mathcal{S}^{(2)} \equiv \frac{1}{N_c}\mathrm{Tr}[U(x_1)U^\dagger(x_2)]. \tag{11}$$

The evolution equation for the dipole, in the large-N_c limit, known as Balitsky-Kovchegov equation. This is rather simple equation that contains only two terms. Both the terms have identical kernels. This is however not true for higher order color correlators. Next, higher point correlators are a color quadrupole [13] and color sextupole [14] that contain four and six Wilson lines, respectively, as

$$\mathcal{S}^{(4)} \equiv \frac{1}{N_c}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)], \tag{12}$$

and

$$\mathcal{S}^{(6)} \equiv \frac{1}{N_c}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]. \tag{13}$$

The evolution equation for the quadrupole was first derived by Dominguez *et al.* [13] and for the sextupole was first derived by Iancu and Triantafyllopoulos in Ref. [14]. The evolution equation of the general $2n$-point constructed from the Wilson lines in the fundamental representation

$$\mathcal{S}^{(2n)} \equiv \frac{1}{N_c}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)\ldots U(x_{2n-1}) \times U^\dagger(x_{2n})]. \tag{14}$$

has been derived by Ayala *et al.* [15].[1] Recently, Shi *et al.* derived a general expression of the $2n$-point correlator within the approximation of the MV model [18].

In this article, we have revisited the JIMWLK equation for the $2n$-tuple Wilson line correlator. We presented a convenient integrodifferential form of this equation that is carrying identical-looking generalized kernels for all explicit real and virtual terms. In the equation, the "real" terms correspond to splitting (say at position z) of this $2n$-tuple correlator to various pairs of $2m$-tuple and $(2n+2-2m)$-tuple correlators, whereas "virtual" terms correspond to splitting into pairs of $2m$-tuple and $(2n-2m)$-tuple correlators. The generalized kernels of virtual terms with $m=0$ (no splitting) and of real terms with $m=1$ (splitting with at least one dipole) have poles, and when integrated over z, they do generate ultraviolet logarithmic divergences, separately for real and virtual terms. However, we have shown that, except these two cases in all other terms, the corresponding kernels, separately for real and virtual terms, have rather softened ultraviolet singularity and when integrated over z do not generate ultraviolet logarithmic divergences. This is key result of this work. We have also studied implication of this

[1]The KLWMIJ evolution equation corresponds to the evolution of the projectile weight functional in the scattering of a dilute projectile on a dense target. The KLWMIJ equation is dual to JIMWLK evolution of the same object in the scattering of a dense projectile on a dilute target [16]. Computation of the KLWMIJ evolution for correlators of an arbitrary number of Wilson lines have been done in Ref. [17].

in the strong scattering regime where only virtual terms are effective.

This paper is organized as follows. In Sec. II, we present the integrodifferential form of the evolution equation for the $2n$-tuple correlator. A few special cases, e.g., quadrupole, sextupole, and octupole, have been jotted down in the Appendix. In Sec. III, we analyze the generic kernel of the equation and demonstrate that it is ultraviolet safe when the splitting does not involve any dipole. In Sec. IV, we study the unitarity asymptotic. Finally, we conclude in Sec. V.

II. HIGH ENERGY EVOLUTION OF COLOR 2n-TUPLE CORRELATOR

Here, we present the explicit integrodifferential form of the JIMWLK evolution equation for the $2n$-tuple Wilson line correlator. This is derived by operating the JIMWLK Hamiltonian in Eq. (9) on a general gauge invariant operator with $2n$-Wilson lines in their fundamental representation. All virtual terms are generated by the first two terms of the Hamiltonian in Eq. (9), whereas the last two terms produce the real terms. All the terms in the above equation are leading in N_c, while all the subleading terms of the order of $1/N_c^2$, generated at the intermediate steps, cancel after summing up all the contributions in the final equation. The procedure to derive the equation has been suggested by Iancu and Triantafyllopoulos in Ref. [14]. The same equation in a somewhat different form was derived earlier by Ayala *et al.* in Ref. [15]. Here, we note that in the equation, whenever the transverse position index notation is greater than $2n$, it should be realized with its modulo, e.g., $x_{2n+k} \equiv x_k$,

$$
\begin{aligned}
&\frac{\partial}{\partial Y}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)\ldots U(x_{2n-1})U^\dagger(x_{2n})] = \frac{\bar{\alpha}_s}{4\pi}\left(\frac{1}{1+\delta_{n,1}}\right)\\
&\times\int_z \sum_{k=0}^{\lfloor n/2\rfloor-1}\sum_{l=0}^{n-1}\mathcal{K}^{(2l;2l+2k+2)}_{(2l+1;2l+2k+1)}\mathrm{Tr}[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U(x_{2l+1+2k})U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_{2l+2k+2})\ldots U(x_{2l-1})U^\dagger(x_{2l})]\\
&+\sum_{k=0}^{\lfloor n/2\rfloor-1}\sum_{l=0}^{n-1}\mathcal{K}^{(2l+1;2l+2k+3)}_{(2l+2;2l+2k+2)}\mathrm{Tr}[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U^\dagger(x_{2l+2+2k})U(z)]\mathrm{Tr}[U^\dagger(z)U(x_{2l+2k+3})\ldots U^\dagger(x_{2l})U(x_{2l+1})]\\
&+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathcal{K}^{(2l;2l+2k+3)}_{(2l+1;2l+2k+2)}\mathrm{Tr}[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U^\dagger(x_{2l+1+2k+1})]\mathrm{Tr}[U(x_{2l+2k+3})U^\dagger(x_{2l+2k+4})\ldots\ldots U^\dagger(x_{2l})]\\
&+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathcal{K}^{(2l+1;2l+2k+4)}_{(2l+2;2l+2k+3)}\mathrm{Tr}[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U(x_{2l+2k+3})]\mathrm{Tr}[U^\dagger(x_{2l+2k+4})U(x_{2l+2k+5})\ldots U(x_{2l+1})]\\
&+\delta_{1,n\,\mathrm{mod}2}\sum_{l=0}^{\lceil n/2\rceil-1}\mathcal{K}^{(2l;2l+n+1)}_{(2l+1;2l+n)}\mathrm{Tr}[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U(x_{2l+n})U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_{2l+n+1})\ldots U(x_{2l-1})U^\dagger(x_{2l})]\\
&+\delta_{1,n\,\mathrm{mod}2}\sum_{l=0}^{\lceil n/2\rceil-2}\mathcal{K}^{(2l+1;2l+n+2)}_{(2l+2;2l+n+1)}\mathrm{Tr}[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U^\dagger(x_{2l+n+1})U(z)]\mathrm{Tr}[U^\dagger(z)U(x_{2l+n+2})\ldots U^\dagger(x_{2l})U(x_{2l+1})]\\
&+\delta_{0,n\,\mathrm{mod}2}\sum_{l=0}^{n/2-1}\mathcal{K}^{(2l;2l+n+1)}_{(2l+1;2l+n)}\mathrm{Tr}[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U^\dagger(x_{2l+n})]\mathrm{Tr}[U(x_{2l+n+1})U^\dagger(x_{2l+n+2})\ldots\ldots U^\dagger(x_{2l})]\\
&+\delta_{0,n\,\mathrm{mod}2}\sum_{l=0}^{n/2-1}\mathcal{K}^{(2l+1;2l+n+2)}_{(2l+2;2l+n+1)}\mathrm{Tr}[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U(x_{2l+n+1})]\mathrm{Tr}[U^\dagger(x_{2l+n+2})U(x_{2l+n+3})\ldots U(x_{2l+1})]\\
&-\mathcal{P}_{2n}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)\ldots U(x_{2n-1})U^\dagger(x_{2n})], \qquad (15)
\end{aligned}
$$

where the kernels are of two types: first one is

$$
\mathcal{K}^{(c;d)}_{(a;b)} \equiv \frac{(x_a-x_d)^2}{(x_a-z)^2(z-x_d)^2}+\frac{(x_b-x_c)^2}{(x_b-z)^2(z-x_c)^2}-\frac{(x_a-x_b)^2}{(x_a-z)^2(z-x_b)^2}-\frac{(x_c-x_d)^2}{(x_c-z)^2(z-x_d)^2}, \qquad (16)
$$

the stand alone second kernel is the last term,

$$
\mathcal{P}_{2n} \equiv \sum_{j=1}^{2n}\frac{(x_j-x_{j+1})^2}{(x_j-z)^2(z-x_{j+1})^2}. \qquad (17)
$$

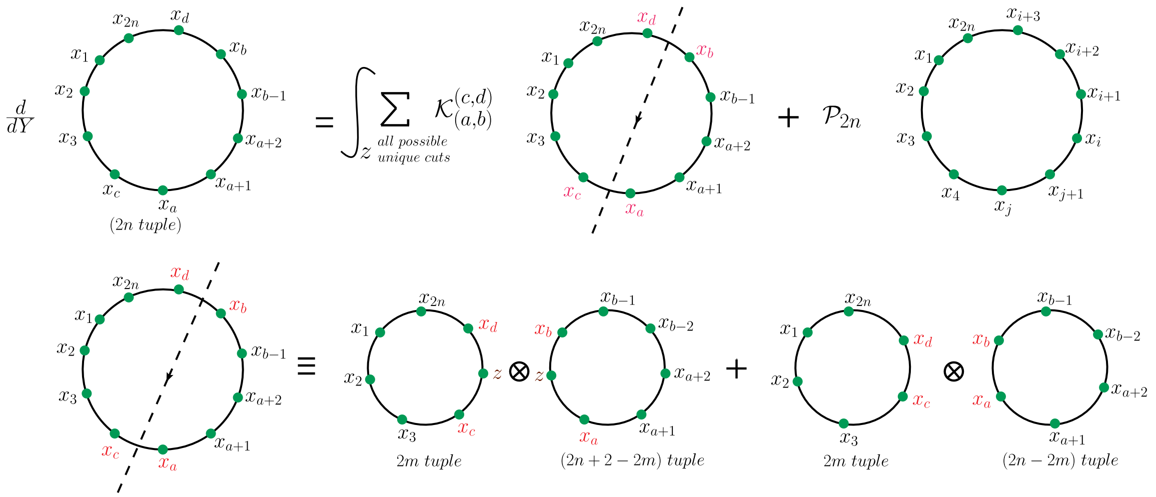

FIG. 1. Schematic representation of the JIMWLK evolution for the $2n$-tuple Wilson line correlator as given in Eq. (15). Circles corresponds to the general multiorder Wilson line correlator; the circle with a cut line refers to terms corresponding to both real and virtual splitting, while $\mathcal{K}$ and $\mathcal{P}$ are associated kernels defined in Eqs. (16) and (17).

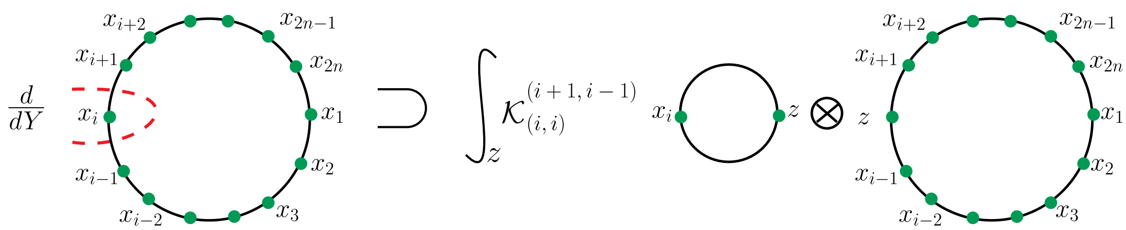

FIG. 2. Real splitting of $2n$-tuple into dipole and another $2n$-tuple correlator.

The terms involving $U(z)$ or $U^\dagger(z)$ are the real terms which describe splitting, at position z, of this $2n$-tuple correlator to a pair of $2m$-tuple and $(2n+2-2m)$-tuple correlators. These terms have been generated by the last two terms of the Hamiltonian. The virtual terms correspond to splitting into pairs of $2m$-tuple and $(2n-2m)$-tuple correlators and are generated by the first two terms of the Hamiltonian and are necessary for the probability conservation and unitarity restoration. All the possible ultraviolet (i.e., short distance) divergences in the dipole kernel get canceled out between the virtual and real terms because of the probability conservation together with the property of color transparency.

The above equation suffers from the problem in the sense that it is not a closed equation because the right-hand side includes higher-point correlations. The way to deal with this difficulty is the same as for the Balitsky-Kovchegov equation assuming that, for a large nucleus, these correlators can be factored as products of correlators involving only one trace at a time when the large-N_c limit is taken.

A careful look at the equation reveals that the terms in the equations are broadly of two types: terms that corresponds to splitting (either real or virtual) and the $2n$-tuple term itself. While the class of kernels $\mathcal{K}^{(c;d)}_{(a;b)}$ is of the former, the $\mathcal{P}_{2n}$ is associated with the last term, i.e., the $2n$-tuple term. This is schematically represented in Fig. 1: Circles corresponds to a general multiorder Wilson line correlator, and the circle with a cut line refers to terms corresponding to both real and virtual splitting. As is evident from figure, the kernel $\mathcal{K}^{(c;d)}_{(a;b)}$ is defined by set of four points (x_a, x_b, x_c, x_d) where the actual splitting happened for that particular term. When a dipole is produced in a real splitting two of the four position coordinate would be identical (See Fig. 2).

III. KERNEL FOR 2n-TUPLE CORRELATOR

A. Dipole evolution

For dipole $n=1$, the evolution equation becomes

$$\frac{\partial}{\partial Y}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\rangle_Y = \frac{\bar{\alpha}_s}{4\pi}\frac{1}{2}\int_z \mathcal{K}^{(2;2)}_{(1;1)}\langle \mathrm{Tr}[U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)]\rangle_Y - (\mathcal{P}_{(1,2)}+\mathcal{P}_{(2,1)})\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\rangle_Y. \tag{18}$$

Now, the kernels are

$$\mathcal{K}^{(2;2)}_{(1;1)} = \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2} + \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2} - \frac{(x_1-x_1)^2}{(x_1-z)^2(z-x_1)^2} - \frac{(x_2-x_2)^2}{(x_2-z)^2(z-x_2)^2} = 2\frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2} \quad (19)$$

and

$$\mathcal{P}_{(1,2)} = \mathcal{P}_{(2,1)} = \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2}. \quad (20)$$

Important simplifications and factorizations occur in the large-N_c limit that leads to the BK equation for color dipole $S(x_1,x_2) \equiv (1/N_c)\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\rangle_Y$,

$$\frac{\partial}{\partial Y} S(x_1,x_2) = \frac{\bar{\alpha}_s}{4\pi}\int_z \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2} \times [S(x_1,z)S(z,x_2) - S(x_1,x_2)], \quad (21)$$

within a mean field approximation. Now, the kernel of the dipole evolution equation is of the following form,

$$\mathcal{K} \equiv \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2}, \quad (22)$$

and this kernel together with the measure d^2z is $SL(2,C)$ invariant. The kernel is singular at $z=x_1$ and $z=x_2$ as well. These two poles when integrated over z, in the strong scattering regime where all transverse distances are much larger than inverse saturation momentum, generate logarithmic divergences as

$$\int_\rho d^2z \frac{(x_1-x_2)^2}{(x_1-z)^2(z-x_2)^2} = 2\pi \ln\frac{|x_1-x_2|^2}{\rho^2}, \quad (23)$$

where $1/\rho^2$ is some ultraviolet cutoff which is usually taken to be the saturation scale $Q_s^2(Y)$. In the limit $\rho \to 0$ [or $Q_s^2(Y) \to \infty$], the integral is logarithmic divergent. For this particular case of dipole equation, the kernels for both real and virtual terms are identical, and this short distance, i.e., ultraviolet singularities coming from real and virtual terms would cancel each other in the overall solution.

B. Splitting with at least one dipole: Real terms

When a general $2n$-tuple splits into two lower ordered color multipoles of which one is a dipole then also the kernel as well as the integral over the kernel both are ultraviolet divergent. However, it is evident from Eqs. (16) and (17) that the kernels for a general real splitting term where atleast one daughter is a dipole (e.g., $\langle \mathrm{Tr}[U(x_i)U^\dagger(z)]\rangle$), can be written as (note for this particular case $x_a = x_b = x_i$),

$$\begin{aligned}\mathcal{K}^{(i+1;i-1)}_{(i;i)} &\equiv \frac{(x_i-x_{i-1})^2}{(x_i-z)^2(z-x_{i-1})^2} + \frac{(x_i-x_{i+1})^2}{(x_i-z)^2(z-x_{i+1})^2} - \frac{(x_i-x_i)^2}{(x_i-z)^2(z-x_i)^2} - \frac{(x_{i+1}-x_{i-1})^2}{(x_{i+1}-z)^2(z-x_{i-1})^2},\\ &= \frac{(x_i-x_{i-1})^2}{(x_i-z)^2(z-x_{i-1})^2} + \frac{(x_i-x_{i+1})^2}{(x_i-z)^2(z-x_{i+1})^2} - \frac{(x_{i+1}-x_{i-1})^2}{(x_{i+1}-z)^2(z-x_{i-1})^2}. \end{aligned} \quad (24)$$

When integrated over z, in the strong scattering regime where all transverse distances are much larger than inverse saturation momentum, this kernel again generates logarithmic divergences as,

$$\begin{aligned}\int_\rho d^2z\mathcal{K}^{(i+1;i-1)}_{(i;i)} &= \int d^2z \frac{(x_i-x_{i-1})^2}{(x_i-z)^2(z-x_{i-1})^2} + \frac{(x_i-x_{i+1})^2}{(x_i-z)^2(z-x_{i+1})^2} - \frac{(x_{i+1}-x_{i-1})^2}{(x_{i+1}-z)^2(z-x_{i-1})^2},\\ &= 2\pi\ln\frac{|x_i-x_{i-1}|^2}{\rho^2} + 2\pi\ln\frac{|x_i-x_{i+1}|^2}{\rho^2} - 2\pi\ln\frac{|x_{i+1}-x_{i-1}|^2}{\rho^2},\\ &= 2\pi\ln\frac{|x_i-x_{i-1}|^2|x_i-x_{i+1}|^2}{|x_{i+1}-x_{i-1}|^2\rho^2}. \end{aligned} \quad (25)$$

Clearly, in the limit $\rho \to 0$, Eq. (25) is divergent. This is also true for the virtual term for which the $2n$-tuple correlator is not splitting into daughters, i.e., [the last term in Eq. (15)]

$$\int_\rho d^2z\,\mathcal{P}_{(j;j+1)} = 2\pi\ln\prod_1^{2n}\frac{|x_j-x_{j+1}|^2}{\rho^2}. \quad (26)$$

This particular term is generated from the first term (identity) and second term of the Hamiltonian in Eq. (9) and is also clearly divergent in the ultraviolet limit, i.e., in the limit $\rho \to 0$.

C. Real splitting without a daughter dipole

As shown in Fig. 1, (x_a, x_b) and (x_c, x_d) are the pair of transverse positions through which the splitting occurs either for real terms or for virtual terms. Interestingly, when the real splitting does not involve any dipole or is not a virtual splitting of two daughters of identical order ($2n$-tuple splits into two n-tuples), generally the kernel would not show any logarithmic divergence after the integration due to cancellation between terms,

$$\begin{aligned}\int_\rho d^2z\,\mathcal{K}^{(c;d)}_{(a;b)} &= \int d^2z\frac{(x_a-x_d)^2}{(x_a-z)^2(z-x_d)^2}+\frac{(x_b-x_c)^2}{(x_b-z)^2(z-x_c)^2}-\frac{(x_a-x_b)^2}{(x_a-z)^2(z-x_b)^2}-\frac{(x_c-x_d)^2}{(x_c-z)^2(z-x_d)^2},\\ &=2\pi\ln\frac{|x_a-x_d|^2}{\rho^2}+2\pi\ln\frac{|x_b-x_c|^2}{\rho^2}-2\pi\ln\frac{|x_a-x_b|^2}{\rho^2}-2\pi\ln\frac{|x_c-x_d|^2}{\rho^2},\\ &=2\pi\ln\frac{|x_a-x_d|^2|x_b-x_c|^2}{|x_a-x_b|^2|x_c-x_d|^2}.\end{aligned}\tag{27}$$

Equation (27) is independent of the UV cutoff ρ and hence is ultraviolet finite.

IV. $2n$-TUPLE CORRELATOR IN THE UNITARY LIMIT

In the strong regime, one may drop all the real terms in Eq. (15) because they are one order higher (contains two Wilson lines more) than their counter virtual terms. The equation can be written as

$$\begin{aligned}&\frac{\partial}{\partial Y}\mathcal{S}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n})\\ &=\frac{\bar{\alpha}_s}{4\pi}\left(\frac{1}{1+\delta_{n,1}}\right)+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathcal{Q}^{(2l;2l+2k+3)}_{(2l+1;2l+2k+2)}\mathcal{S}^{(2k+2)}(x_{2l+1},x_{2l+2}\ldots x_{2l+1+2k+1})\mathcal{S}^{(2n-2k-2)}(x_{2l+2k+3},x_{2l+2k+4}\ldots x_{2l})\\ &\quad+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathcal{Q}^{(2l+1;2l+2k+4)}_{(2l+2;2l+2k+3)}\mathcal{S}^{(2k+2)}(x_{2l+2},x_{2l+3}\ldots x_{2l+2k+3})\mathcal{S}^{(2n-2k-2)}(x_{2l+2k+4},x_{2l+2k+5}\ldots x_{2l+1})\\ &\quad+\delta_{0,n\,\mathrm{mod}2}\sum_{l=0}^{n/2-1}\mathcal{Q}^{(2l;2l+n+1)}_{(2l+1;2l+n)}\mathcal{S}^{(n)}(x_{2l+1},x_{2l+2}\ldots x_{2l+n})\mathcal{S}^{(n)}(x_{2l+n+1},x_{2l+n+2}\ldots x_{2l})\\ &\quad+\delta_{0,n\,\mathrm{mod}2}\sum_{l=0}^{n/2-1}\mathcal{Q}^{(2l+1;2l+n+2)}_{(2l+2;2l+n+1)}\mathcal{S}^{(n)}(x_{2l+2},x_{2l+3}\ldots x_{2l+n+1})\mathcal{S}^{(n)}(x_{2l+n+2},x_{2l+n+3}\ldots x_{2l+1})\\ &\quad-\mathcal{R}\,\mathcal{S}^{(2n)}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n}),\end{aligned}\tag{28}$$

where $\mathcal{Q}$'s are defined as

$$\mathcal{Q}^{c;d}_{a;b}\equiv\int_\rho d^2z\,\mathcal{K}^{(c;d)}_{(a;b)}=2\pi\ln\frac{|x_a-x_d|^2|x_b-x_c|^2}{|x_a-x_b|^2|x_c-x_d|^2}\tag{29}$$

and do not explicitly depends on any infrared cut, whereas the factor $\mathcal{R}$ is defined as

$$\mathcal{R}\equiv\int_\rho d^2z\,\mathcal{P}_{(j;j+1)}=2\pi\ln\prod_1^{2n}\frac{|x_j-x_{j+1}|^2}{\rho^2}.\tag{30}$$

Equation (28) can further be simplified to

$$
\begin{aligned}
&\frac{\partial}{\partial Y}\mathcal{S}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n})\\
&=\frac{\bar{\alpha}_s}{4\pi}\left(\frac{1}{1+\delta_{n,1}}\right)+\sum_{k=0}^{\lceil n/2\rceil-1}\sum_{l=1}^{2n}2\pi\ln\frac{|x_l-x_{l+2k}|^2|x_{l+2k-1}-x_{l-1}|^2}{|x_l-x_{l+2k-1}|^2|x_{l-1}-x_{l+2k}|^2}\mathcal{S}^{(2k)}(x_l,x_{2l+2}\ldots x_{l+2k-1})\mathcal{S}^{(2n-2k)}(x_{l+2k},x_{l+2k+1}\ldots x_{x_{l-1}})\\
&\quad+\delta_{0,n\,\mathrm{mod}2}\sum_{l=1}^{n}2\pi\ln\frac{|x_l-x_{l+n}|^2|x_{l+n-1}-x_{l-1}|^2}{|x_l-x_{l+n-1}|^2|x_{l-1}-x_{l+n}|^2}\mathcal{S}^{(n)}(x_l,x_{l+1}\ldots x_{l+n-1})\mathcal{S}^{(n)}(x_{l+n},x_{l+n+2}\ldots x_{l-1})\\
&\quad-2\pi\mathcal{S}^{(2n)}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n})\ln\prod_{j=1}^{2n}|x_j-x_{j+1}|^2Q_s^2(Y),
\end{aligned}
\tag{31}
$$

where in the last term we take the cutoff ρ to be the inverse saturation momentum $Q_s(Y)$ that explicitly depends on the rapidity. In the limit $Y\to\infty$, only the last term would survive,

$$
\frac{\partial}{\partial Y}\ln\mathcal{S}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n})=-\frac{\bar{\alpha}_s}{2}\left(\frac{1}{1+\delta_{n,1}}\right)\ln\prod_{j=1}^{2n}|x_j-x_{j+1}|^2Q_s^2(Y),
\tag{32}
$$

and this equation can be solved to get the Levin-Tuchin asymptotic solution for the $2n$-tuple Wilson line correlator in the unitarity limit,

$$
\mathcal{S}(x_1,x_2,x_3,x_4\ldots x_{2n-1},x_{2n})=S_0^{(2n)}\exp\left[-\frac{1+2i\nu_0}{2(1+\delta_{n,1})\chi(0,\nu_0)}\ln^2\left(\prod_{j=1}^{2n}|x_j-x_{j+1}|^2Q_s^2(Y)\right)\right].
\tag{33}
$$

Here, we have used following leading order expression for the saturation momentum [1],

$$
Q_s(Y)=Q_{s0}\exp\left(\bar{\alpha}_s\frac{\chi(0,\nu_0)}{1+2i\nu_0}Y\right)\approx Q_{s0}e^{2.44\bar{\alpha}_sY},
\tag{34}
$$

where

$$
\chi(0,\nu)=2\psi(1)-\psi\left(\frac{1}{2}+i\nu\right)-\psi\left(\frac{1}{2}-i\nu\right),
\tag{35}
$$

and ψ is the digamma function with $\chi(0,\nu_0)/(1+2i\nu_0)\approx 2.44$ and $\nu_0\approx-0.1275i$. This specific value stems from the saddle point condition along the saturation line.

V. CONCLUSION

In this article, we revisited the evolution equation for the general $2n$-tuple correlators in their fundamental representation. We start from the JIMWLK Hamiltonian and act on a single trace of $2n$-Wilson lines in order to derive its evolution equation in the small-x limit and subsequently study the solution of such an equation in the unitarity limit. This is accordance with the hierarchy of evolution of Wilson-line operators suggested by Balitsky [4] and conform the equivalence of the Balitsky hierarchy of evolution and the JIMWLK evolution for this general $2n$-tuple color correlator. The real terms, that correspond to splitting of this $2n$-tuple correlator to various pairs of $2m$-tuple and $(2n+2-2m)$-tuple correlators, and the virtual terms, that correspond to splitting into pairs of $2m$-tuple and $(2n-2m)$-tuple correlators, are explicit in this integrodifferential equation. In this paper, we have also shown that, except the two special cases, the generalized kernels, separately for real and virtual terms, have no ultraviolet singularity and therefore do not generate ultraviolet logarithmic divergences. This is the key result of this work. Even though there exist approximation methods to evaluate higher multipole operators, e.g., one either uses the Langevin form of the JIMWLK [19] equation or relies on the Gaussian approximation [20], the result presented here could be convenient to calculate multipole color correlators numerically.

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APPENDIX: EVOLUTION EQUATIONS FOR QUADRUPOLE, SEXTUPOLE AND OCTUPOLE

Since there are four different terms inside the square brackets in the definition of the JIMWLK Hamiltonian, we compute the contributions of these four terms separately. The coefficient of the non-leading N_c terms vanishes in intermediate steps of the calculations. In the end, we find the sum of all four of these contributions leads to the final results without finite N_c corrections. Here, we present the results of the quadrupole, sextupole, and octupole evolution.

1. Quadrupole

The quadrupole evolution equation ($n = 2$) is

$$\begin{aligned}
&\frac{\partial}{\partial Y}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&= \frac{\bar{\alpha}_s}{4\pi}\int_z \mathcal{K}^{(4;2)}_{(1;1)}\langle \mathrm{Tr}[U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&\quad + \mathcal{K}^{(1;3)}_{(2;2)}\langle \mathrm{Tr}[U^\dagger(x_2)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_3)U^\dagger(x_4)U(x_1)]\rangle_Y \\
&\quad + \mathcal{K}^{(2;4)}_{(3;3)}\langle \mathrm{Tr}[U(x_3)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_4)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad + \mathcal{K}^{(3;1)}_{(4;4)}\langle \mathrm{Tr}[U^\dagger(x_4)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y \\
&\quad + \mathcal{K}^{(4;3)}_{(1;2)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\mathrm{Tr}[U(x_3)U^\dagger(x_4)] \\
&\quad + \mathcal{K}^{(4;1)}_{(3;2)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_2)]\mathrm{Tr}[U(x_1)U^\dagger(x_4)] \\
&\quad - \mathcal{P}_4\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)].
\end{aligned} \tag{A1}$$

2. Sextupole

The sextupole evolution ($n = 3$) equation is

$$\begin{aligned}
&\frac{\partial}{\partial Y}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)] \\
&= \frac{\bar{\alpha}_s}{4\pi}\int_z \mathcal{K}^{(6;2)}_{(1;1)}\langle \mathrm{Tr}[U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y \\
&\quad + \mathcal{K}^{(1;3)}_{(2;2)}\langle \mathrm{Tr}[U^\dagger(x_2)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_1)]\rangle_Y \\
&\quad + \mathcal{K}^{(2;4)}_{(3;3)}\langle \mathrm{Tr}[U(x_3)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad + \mathcal{K}^{(3;5)}_{(4;4)}\langle \mathrm{Tr}[U^\dagger(x_4)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_5)U^\dagger(x_6)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y \\
&\quad + \mathcal{K}^{(4;6)}_{(5;5)}\langle \mathrm{Tr}[U(x_5)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_6)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&\quad + \mathcal{K}^{(5;1)}_{(6;6)}\langle \mathrm{Tr}[U^\dagger(x_6)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)]\rangle_Y \\
&\quad + \mathcal{K}^{(6;4)}_{(1;3)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y \\
&\quad + \mathcal{K}^{(2;6)}_{(3;5)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_6)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad + \mathcal{K}^{(4;2)}_{(5;1)}\langle \mathrm{Tr}[U(x_5)U^\dagger(x_6)U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&\quad + \mathcal{K}^{(6;3)}_{(1;2)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\mathrm{Tr}[U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y \\
&\quad + \mathcal{K}^{(1;4)}_{(2;3)}\langle \mathrm{Tr}[U^\dagger(x_2)U(x_3)]\mathrm{Tr}[U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_1)]\rangle_Y \\
&\quad + \mathcal{K}^{(2;5)}_{(3;4)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_4)]\mathrm{Tr}[U(x_5)U^\dagger(x_6)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad + \mathcal{K}^{(3;6)}_{(4;5)}\langle \mathrm{Tr}[U^\dagger(x_4)U(x_5)]\mathrm{Tr}[U^\dagger(x_6)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{K}_{(5;6)}^{(4;1)}\langle \mathrm{Tr}[U(x_5)U^\dagger(x_6)]\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&+\mathcal{K}_{(6;1)}^{(5;2)}\langle \mathrm{Tr}[U^\dagger(x_6)U(x_1)]\mathrm{Tr}[U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)]\rangle_Y \\
&-\mathcal{P}_6\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y.
\end{aligned} \tag{A2}
$$

3. Octupole

The octupole ($n=4$) evolution equation is

$$
\begin{aligned}
&\frac{\partial}{\partial Y}\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U(x_8)^\dagger] \\
&=\frac{\bar{\alpha}_s}{4\pi}\int_z \mathcal{K}_{(1;1)}^{(8;2)}\langle \mathrm{Tr}[U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)]\rangle_Y \\
&\quad+\mathcal{K}_{(2;2)}^{(1;3)}\langle \mathrm{Tr}[U^\dagger(x_2)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(6)U(x_7)U^\dagger(x_8)U(x_1)]\rangle_Y \\
&\quad+\mathcal{K}_{(3;3)}^{(2;4)}\langle \mathrm{Tr}[U(x_3)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad+\mathcal{K}_{(4;4)}^{(3;5)}\langle \mathrm{Tr}[U^\dagger(x_4)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(8)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y \\
&\quad+\mathcal{K}_{(5;5)}^{(4;6)}\langle \mathrm{Tr}[U(x_5)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&\quad+\mathcal{K}_{(6;6)}^{(5;7)}\langle \mathrm{Tr}[U^\dagger(x_6)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(2)U(x_3)U^\dagger(x_4)U(x_5)]\rangle_Y \\
&\quad+\mathcal{K}_{(7;7)}^{(6;8)}\langle \mathrm{Tr}[U(x_7)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y \\
&\quad+\mathcal{K}_{(8;8)}^{(7;1)}\langle \mathrm{Tr}[U^\dagger(x_8)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)]\rangle_Y \\
&\quad+\mathcal{K}_{(1;5)}^{(8;6)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_6)U(x_7)U^\dagger(x_8)]\rangle_Y \\
&\quad+\mathcal{K}_{(2;6)}^{(7;1)}\langle \mathrm{Tr}[U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_7)U^\dagger(x_8)U(x_1)]\rangle_Y \\
&\quad+\mathcal{K}_{(3;7)}^{(2;8)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_8)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad+\mathcal{K}_{(4;8)}^{(3;1)}\langle \mathrm{Tr}[U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y \\
&\quad+\mathcal{K}_{(5;1)}^{(4;2)}\langle \mathrm{Tr}[U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y \\
&\quad+\mathcal{K}_{(6;2)}^{(5;3)}\langle \mathrm{Tr}[U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_3)U^\dagger(x_4)U(x_5)]\rangle_Y \\
&\quad+\mathcal{K}_{(7;3)}^{(6;4)}\langle \mathrm{Tr}[U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(z)]\mathrm{Tr}[U(z)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y \\
&\quad+\mathcal{K}_{(8;4)}^{(7;5)}\langle \mathrm{Tr}[U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(z)]\mathrm{Tr}[U^\dagger(z)U(x_5)U^\dagger(x_6)U(x_7)]\rangle_Y \\
&\quad+\mathcal{K}_{(1;4)}^{(8;5)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\mathrm{Tr}[U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)]\rangle_Y \\
&\quad+\mathcal{K}_{(2;5)}^{(1;6)}\langle \mathrm{Tr}[U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)]\mathrm{Tr}[U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)]\rangle_Y \\
&\quad+\mathcal{K}_{(3;6)}^{(2;7)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\mathrm{Tr}[U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)]\rangle_Y \\
&\quad+\mathcal{K}_{(4;7)}^{(3;8)}\langle \mathrm{Tr}[U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)]\mathrm{Tr}[U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y \\
&\quad+\mathcal{K}_{(1;2)}^{(8;3)}\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)]\mathrm{Tr}[U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)]\rangle_Y \\
&\quad+\mathcal{K}_{(2;3)}^{(1;4)}\langle \mathrm{Tr}[U^\dagger(x_2)U(x_3)]\mathrm{Tr}[U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)]\rangle_Y
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{K}^{(2;5)}_{(3;4)}\langle \mathrm{Tr}[U(x_3)U^\dagger(x_4)]\mathrm{Tr}[U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)]\rangle_Y\\
&+\mathcal{K}^{(3;6)}_{(4;5)}\langle \mathrm{Tr}[U^\dagger(x_4)U(x_5)]\mathrm{Tr}[U^\dagger(x_6)U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)]\rangle_Y\\
&+\mathcal{K}^{(4;7)}_{(5;6)}\langle \mathrm{Tr}[U(x_5)U^\dagger(x_6)]\mathrm{Tr}[U(x_7)U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)]\rangle_Y\\
&+\mathcal{K}^{(5;8)}_{(6;7)}\langle \mathrm{Tr}[U(x_6)U^\dagger(x_7)]\mathrm{Tr}[U^\dagger(x_8)U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)]\rangle_Y\\
&+\mathcal{K}^{(6;1)}_{(7;8)}\langle \mathrm{Tr}[U^\dagger(x_7)U(x_8)]\mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)]\rangle_Y\\
&+\mathcal{K}^{(7;2)}_{(8;1)}\langle \mathrm{Tr}[U(x_8)U^\dagger(x_1)]\mathrm{Tr}[U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)]\rangle_Y\\
&-\mathcal{P}_8\langle \mathrm{Tr}[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)U(x_7)U^\dagger(x_8)]\rangle_Y. \qquad \text{(A3)}
\end{aligned}
$$

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