$\mathbb{CP}^N$-Rosochatius system, superintegrability, and supersymmetry

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We propose a new superintegrable mechanical system on the complex projective space $\mathbb{CP}^N$ involving a potential term together with coupling to a constant magnetic fields. This system can be viewed as a $\mathbb{CP}^N$-analog of both the flat singular oscillator and its spherical analog known as the "Rosochatius system." We find its constants of motion and calculate their (highly nonlinear) algebra. We also present its classical and quantum solutions. The system belongs to the class of "Kähler oscillators" admitting $SU(2|1)$ supersymmetric extension. We show that, in the absence of magnetic field and with the special choice of the characteristic parameters, one can construct $\mathcal{N} = 4, d = 1$ Poincaré supersymmetric extension of the system considered.

I. INTRODUCTION

The (D-dimensional) isotropic oscillator and the relevant Coulomb problem play a pivotal role among other textbook examples of D-dimensional integrable systems. They are distinguished by the "maximal superintegrability" property, which is the existence of $2D - 1$ functionally independent constants of motion [1]. The rational Calogero model with oscillator potential [2], being a nontrivial generalization of isotropic oscillator, is also maximally superintegrable [3]. Moreover, the Calogero model with Coulomb potential is superintegrable too [4]. All these systems, being originally defined on a plane, admit the maximally superintegrable deformations to the spheres (see Ref. [5] for the spherical generalizations of the oscillator and Coulomb problem, and Ref. [4] for the Calogero-oscillator and Calogero-Coulomb ones). The integrable spherical generalizations of aniso-tropic oscillator [6], Stark-Coulomb, and two-center Coulomb problems [7] are also known.

In contrast to the spherical extensions, the generalizations to other curved spaces have not attracted much attention so far. The only exception seems to be the isotropic oscillator on the complex/quaternionic spaces considered in Ref. [8,9].

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These systems reveal an important feature: they remain superintegrable after coupling to a constant magnetic/Belavin Polyakov Schwarz Tyupkin (BPST) instanton field, though cease to be maximally superintegrable. One may pose a question:

How do we construct the superintegrable generalizations of Calogero-oscillator and Calogero-Coulomb models on complex and quaternionic projective spaces?

In this paper we make the first steps toward the answer. Due to the complexity of the problem we restrict our attention to the simplest particular case. Namely, we construct the superintegrable $\mathbb{CP}^N$-generalization of the N-dimensional singular oscillator (the simplest rational Calogero-oscillator model) which is defined by the Hamiltonian

$$H_{SW} = \sum_{a=1}^N \left(\frac{p_a^2}{2} + \frac{g_a^2}{2x_a^2} + \frac{\omega^2 x_a^2}{2} \right),$$

$$\{p_a, x_b\} = \delta_{ab}, \qquad \{p_a, p_b\} = \{x_a, x_b\} = 0. \quad (1)$$

This model is less trivial than it looks at first sight: it has a variety of hidden constants of motion which form a non-linear symmetry algebra and endow the system with the maximal superintegrability property. Its extensive studies were initiated more than fifty years ago by Smorodinsky with collaborators [10] and are continuing up to now (see, e.g., [11] and references therein). Sometimes this model is referred to as the Smorodinsky-Winternitz system, though it has been known for many years.

The maximally superintegrable spherical counterpart of the Smorodinsky-Winternitz system is defined by the Hamiltonian suggested by Rosochatius in 1877 [12]

$$H_{\text{Ros}} = \frac{1}{2} \sum_{a,b=1}^{N} \left(\delta_{ab} - \frac{x_a x_b}{r_0^2} \right) p_a p_b + \sum_{a=1}^{N} \left(\frac{\omega_a^2 r_0^2}{x_a^2} + \frac{\omega^2 r_0^2 x_a^2}{2x_0^2} \right),$$

$$x_a^2 + x_0^2 = r_0^2. \tag{2}$$

It is a particular case of the integrable systems obtained by restricting the free particle and oscillator systems to a sphere. It was studied by many authors from different viewpoints, including its reinvention as a superintegrable spherical generalization of Smorodinsky-Winternitz system [13–15]. The Rosochatius model, as well as its hybrid with the Neumann model suggested in 1859 [16], attract a stable interest for years due to their relevance to a wide circle of physical and mathematical problems. Recently, the Rosochatius-Neumann system was encountered, while studying strings [17], extreme black hole geodesics [15,18], and the Klein-Gordon equation in curved backgrounds [19].

In this paper we propose a superintegrable generalization of Rosochatius (and Smorodinsky-Winternitz) system on the complex projective space $\mathbb{CP}^N$. It is defined by the Hamiltonian[1]

$$\mathcal{H}_{\text{Ros}} = \frac{1}{r_0^2} (1 + z\bar{z})(\delta^{a\bar{b}} + z^a \bar{z}^b) \pi_a \bar{\pi}_b$$
$$+ r_0^2 (1 + z\bar{z}) \left(\omega_0^2 + \sum_{a=1}^{N} \frac{\omega_a^2}{z^a \bar{z}^a} \right) - r_0^2 \sum_{i=0}^{N} \omega_i^2, \tag{3}$$

and by the Poisson brackets providing the interaction with a constant magnetic field of the magnitude B

$$\{\pi_a, z^b\} = \delta_a^b, \qquad \{\bar{\pi}_a, \bar{z}^b\} = \delta_{\bar{a}}^{\bar{b}},$$

$$\{\pi_a, \bar{\pi}_b\} = \imath B r_0^2 \left(\frac{\delta_{a\bar{b}}}{1 + z\bar{z}} - \frac{\bar{z}^a z^b}{(1 + z\bar{z})^2} \right). \tag{4}$$

We will call it $\mathbb{CP}^N$-Rosochatius system.[2]

Reducing this $2N$-dimensional system by the action of N manifest $U(1)$ symmetries, $z^a \to e^{\imath\kappa_a} z^a$, $\pi_a \to e^{-\imath\kappa_a} \pi_a$, we recover the N-dimensional Rosochatius system (2) (see Sec. III).

On the other hand, rescaling the coordinates and momenta as $r_0 z^a \to z^a$, $\pi_a/r_0 \to \pi_a$ and taking the limit $r_0 \to \infty$, $\omega_a \to 0$ with $r_0^2 \omega_a = g_a$ kept finite, we arrive at the so-called "$\mathbb{C}^N$-Smorodinsky-Winternitz system" [20]

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[1]Hereafter we use the notation $z\bar{z} \equiv \sum_{c=1}^{N} z^c \bar{z}^c$.
[2]Despite the fact that $U(N)$ symmetry is explicitly broken in (3) (down to $U(1)^N$), hereafter we use the $U(N)$ covariant notation, such that π_a and $\pi_{\bar{b}}$ transform, respectively, as $\bar{z}^a$ and z^b and there are three equivalent forms of the $U(N)$ invariant tensor, $\delta_{a\bar{b}}$, δ_a^b, and $\delta_{\bar{b}}^{\bar{a}}$.

$$\mathcal{H}_{\text{SW}} = \sum_{a=1}^{N} \left(\pi_a \bar{\pi}_a + \omega_0^2 z^a \bar{z}^a + \frac{g_a^2}{z^a \bar{z}^a} \right), \qquad \{\pi_a, z^b\} = \delta_a^b,$$

$$\{\bar{\pi}_a, \bar{z}^b\} = \delta_{\bar{a}}^{\bar{b}}, \qquad \{\pi_a, \bar{\pi}_b\} = \imath B \delta_{a\bar{b}}. \tag{5}$$

Since the reductions of the $\mathbb{CP}^N$-Rosochatius system yield superintegrable systems, it is quite natural that it proves to be superintegrable on its own.

We will show that $\mathbb{CP}^N$-Rosochatius system belongs to the class of "Kähler oscillators" [8,21] which admit $SU(2|1)$ supersymmetrization (or a "weak $\mathcal{N} = 4$" supersymmetrization, in the terminology of Smilga [22]). A few years ago it was found that these systems naturally arise within the appropriate $SU(2|1)$, $d = 1$ superspace formalism developed in a series of papers [23]. This research was partly motivated by the study of the field theories with curved rigid analogs of Poincaré supersymmetry [24]. In the absence of the background magnetic field and for the special choice of the parameters ω_i, the $\mathbb{CP}^N$-Rosochatius system admits $\mathcal{N} = 4$, $d = 1$ Poincaré supersymmetric extension.

Finally, note that the $\mathbb{C}^N$-Smorodinsky-Winternitz system (5) can be interpreted as a set of N two-dimensional ring-shaped oscillators interacting with a constant magnetic field orthogonal to the plane. As opposed to (5), the $\mathbb{CP}^N$-Rosochatius system does not split into a set of N two-dimensional decoupled systems. Instead, it can be interpreted as describing *interacting* particles with a position-dependent mass in the two-dimensional quantum rings (along the lines of ref. [25–27]).

To summarize, the $\mathbb{CP}^N$-*Rosochatius system* suggested is of interest from many points of view. Its study is the subject of the remainder of this paper. It is organized as follows.

In Sec. II we review the main properties of the complex projective space $\mathbb{CP}^N$, the simplest related systems like the $\mathbb{CP}^N$-Landau problem and the $\mathbb{CP}^N$-oscillator, and then derive the potential specifying the $\mathbb{CP}^N$-Rosochatius system.

In Sec. III we present the classical $\mathbb{CP}^N$-Rosochatius model in a constant magnetic field and find that, in addition to N manifest $U(1)$ symmetries, this system possesses additional $2N - 1$ functionally-independent second-order constants of motion. The latter property implies the (nonmaximal) superintegrability of the model considered. We present the explicit expressions of the constants of motion and calculate their algebra. We also show that the reduction of the $\mathbb{CP}^N$-Rosochatius model by manifest $U(1)$ symmetries reproduces the original N-dimensional ($\mathbb{S}^N$-) Rosochatius system.

In Sec. IV we separate the variables and find classical solutions of the $\mathbb{CP}^N$-Rosochatius model.

In Sec. V we study the quantum $\mathbb{CP}^N$-Rosochatius system and find its spectrum which depends on $N + 1$ quantum numbers, as well as the relevant wavefunctions.

In Sec. VI we construct $\mathcal{N} = 4$ supersymmetric extensions of the $\mathbb{CP}^N$-Rosochatius system.

In Sec. VII we give an account of open problems and possible generalizations.

In the subsequent consideration we put, for simplicity, $r_0 = 1$.

II. PRELIMINARIES: COMPLEX PROJECTIVE SPACES

In this section we present the basic properties of complex projective space $\mathbb{CP}^N$, briefly describe the Landau problem and the oscillator on this space, and construct the $\mathbb{CP}^N$-analog of the Rosochatius system.

The N-dimensional complex projective space is a space of complex rays in the $(N + 1)$-dimensional complex Euclidian space $(\mathbb{C}^{N+1}, \sum_{i=0}^N du^i d\bar{u}^i)$, with u^i being homogeneous coordinates of the complex projective space. Equivalently, it can be defined as the quotient $\mathbb{S}^{2N+1}/U(1)$, where $\mathbb{S}^{2N+1}$ is the $(2N + 1)$-dimensional sphere embedded in $\mathbb{C}^{N+1}$ by the constraint $\sum_{i=1}^N u^i \bar{u}^i = 1$. One can solve the latter by introducing locally "inhomogeneous" coordinates $z_{(i)}^a$

$$z_{(i)}^a = \frac{u^a}{u^i}, \quad \text{with} \quad a \neq i, u^i \neq 0. \tag{6}$$

Hence, the full complex projective space can be covered by $N + 1$ charts marked by the indices $i = 0, ..., N$, with the following transition functions on the intersection of ith and jth charts:

$$z_{(i)}^a = \frac{z_{(j)}^a}{z_{(j)}^i}. \tag{7}$$

Let us endow $\mathbb{C}^{N+1}$ with the canonical Poisson brackets $\{u^i, \bar{u}^j\} = \imath \delta^{i\bar{j}}$, and define, with respect to them, the $u(N + 1)$ algebra formed by the generators

$$h_{i\bar{j}} = \bar{u}^i u^j. \tag{8}$$

Reducing the manifold $\mathbb{C}^{N+1}$ by the action of the $U(1)$ group with the generator $h_0 = \sum_{i=0}^N u^i \bar{u}^i$, we arrive at the $SU(N + 1)$-invariant Kähler structure defined by the Fubini-Study metrics

$$\sum_{a,b=1}^N g_{a\bar{b}} dz^a d\bar{z}^b = \sum_{a,b=1}^N \frac{\partial^2 \log(1 + z\bar{z})}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b$$
$$= \sum_{a,b=1}^N \left(\frac{\delta_{a\bar{b}}}{1 + z\bar{z}} - \frac{\bar{z}^a z^b}{(1 + z\bar{z})^2} \right) dz^a d\bar{z}^b. \tag{9}$$

This metrics is obviously invariant under the passing from one chart to another. Hence, we can omit the indices

marking charts and assume, without loss of generality, that we are dealing with 0th chart, so that the indices a, b, c run from 1 to N.

Being Kähler manifold, the complex projective space is equipped with the Poisson brackets $\{\bar{z}^a, z^b\}_0 = -\imath g^{\bar{a}b}$, where $g^{\bar{a}b} = (1 + z\bar{z})(\delta^{\bar{a}b} + \bar{z}^a z^b)$ is the inverse Fubini-Study metrics. The $su(N + 1)$ isometry of $\mathbb{CP}^N$ is generated by the holomorphic Hamiltonian vector fields defined as the following momentum maps (Killing potentials)

$$h_{a\bar{b}} = \frac{\bar{z}^a z^b}{1 + z\bar{z}}, \qquad h_a = \frac{2\bar{z}^a}{1 + z\bar{z}}. \tag{10}$$

Now, let us introduce, on the cotangent bundle of $\mathbb{C}^{N+1}$, the canonical Poisson brackets $\{p_i, u^j\} = \delta_{ij}$, and define the $su(N + 1)$ algebra with the generators

$$L_{i\bar{j}} = \imath(p_i u^j - \bar{p}_j \bar{u}^i) - \frac{\delta_{i\bar{j}}}{N} L_0, \quad \text{where } L_0 = \imath \sum_{i=0}^N (p_i u^i - \bar{p}_i \bar{u}^i). \tag{11}$$

Reducing this phase space by the action of generators L_0, $h_0 = \sum_i u^i \bar{u}^i$, and finally fixing their values as $L_0 = 2B$, $h_0 = 1$, we arrive at the Poisson brackets (4) (with $r_0 = 1$). They describe an electrically charged particle on $\mathbb{CP}^N$ interacting with a constant magnetic field of the magnitude B and set the corresponding twisted symplectic structure

$$\Omega_0 = \sum_{a=1}^N (dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a) + B \sum_{a,b=1}^N \imath g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \tag{12}$$

with $g_{a\bar{b}}$ being defined in (9).

The inhomogeneous coordinates and momenta z^a, π_a are related to the homogeneous ones p_i, u^i as [28]

$$z^a = \frac{u^a}{u^0}, \qquad \pi_a = \sum_{b=1}^N g_{a\bar{b}} \left(\frac{p_b}{\bar{u}^0} - \bar{z}^b \frac{p_0}{\bar{u}^0} \right). \tag{13}$$

The $su(N + 1)$ generators (11) are reduced to the following ones

$$J_{a\bar{b}} = \imath(z^b \pi_a - \bar{\pi}_b \bar{z}^a) - B \frac{\bar{z}^a z^b}{1 + z\bar{z}},$$
$$J_a = \pi_a + \bar{z}^a(\bar{z}\,\bar{\pi}) + \imath B \frac{\bar{z}^a}{1 + z\bar{z}}: \tag{14}$$

$$\{J_{\bar{a}b}, J_{\bar{c}d}\} = i\delta_{\bar{a}d} J_{\bar{b}c} - i\delta_{\bar{c}b} J_{\bar{a}d},$$
$$\{J_a, \bar{J}_b\} = -i(J_{a\bar{b}} + J_0 \delta_{a\bar{b}}),$$
$$\{J_a, J_{b\bar{c}}\} = iJ_b \delta_{a\bar{c}}, \tag{15}$$

where $J_0 \equiv \sum_{a=1}^N J_{a\bar{a}} + B$.

With these expressions at hand we can now consider some superintegrable systems on $\mathbb{CP}^N$.

$\mathbb{CP}^N$-*Landau problem.* The $\mathbb{CP}^N$-Landau problem is defined by the symplectic structure (12) and the free-particle Hamiltonian identified with a Casimir of $su(N+1)$ algebra

$$
\begin{aligned}
\mathcal{H}_0 &= \sum_{a,b=1}^{N} (1+z\bar{z})(\delta^{a\bar{b}} + z^a \bar{z}^b)\pi_a \bar{\pi}_b = \frac{1}{2}\sum_{i,j=0}^{N} L_{i\bar{j}} L_{j\bar{i}} - \frac{B^2}{2} \\
&= \sum_{a=1}^{N} J_a \bar{J}_a + \frac{\sum_{a,b=1}^{N} J_{a\bar{b}} J_{b\bar{a}} + J_0^2 - B^2}{2} : \{\mathcal{H}_0, L_{ij}\} = 0.
\end{aligned}
\tag{16}
$$

Its quantization was done, e.g., in [29].

$\mathbb{CP}^N$-*oscillator.* The $\mathbb{CP}^N$-oscillator is defined by the symplectic structure (12) and the Hamiltonian [8]

$$
\mathcal{H}_{osc} = \sum_{a,b=1}^{N} (1+z\bar{z})(\delta^{a\bar{b}} + z^a \bar{z}^b)\pi_a \bar{\pi}_b + \omega^2 \sum_{a=1}^{N} z^a \bar{z}^a. \tag{17}
$$

It respects manifest $U(N)$ symmetry with the generators $J_{a\bar{b}}$ (14), and additional hidden symmetries given by the proper analog of "Fradkin tensor",

$$
I_{a\bar{b}} = J_a \bar{J}_b + \omega^2 \bar{z}^a z^b. \tag{18}
$$

The full symmetry algebra of this system reads

$$
\begin{aligned}
\{J_{\bar{a}b}, J_{\bar{c}d}\} &= \iota\delta_{\bar{a}d} J_{\bar{b}c} - \iota\delta_{\bar{c}b} J_{\bar{a}d}, \\
\{I_{a\bar{b}}, J_{c\bar{d}}\} &= \iota\delta_{a\bar{d}} I_{c\bar{b}} - \iota\delta_{c\bar{b}} I_{a\bar{d}}
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\{I_{a\bar{b}}, I_{c\bar{d}}\} = &\ \iota\omega^2 \delta_{a\bar{d}} J_{c\bar{b}} - \iota\omega^2 \delta_{c\bar{b}} J_{a\bar{d}} - \iota I_{c\bar{b}}(J_{a\bar{d}} + J_0 \delta_{a\bar{d}}) \\
&+ \iota I_{a\bar{d}}(J_{c\bar{b}} + J_0 \delta_{c\bar{b}}),
\end{aligned}
\tag{20}
$$

where $J_0 = \iota \sum_{a=1}^{N}(z^a \pi_a - \bar{\pi}_a \bar{z}^a) + B\frac{1}{1+z\bar{z}}$.

The Hamiltonian (17) is expressed via the symmetry generators as follows

$$
\mathcal{H}_{osc} = \sum_{a=1}^{N} I_{a\bar{a}} + \frac{1}{2}\sum_{a,b=1}^{N} J_{a\bar{b}} J_{b\bar{a}} + \frac{J_0^2 - B^2}{2}. \tag{21}
$$

The quantum mechanics associated with this Hamiltonian was considered in [30]. In the flat limit, the $\mathbb{CP}^N$-oscillator goes over to the $\mathbb{C}^N$-oscillator interacting with a constant magnetic field.

$\mathbb{CP}^N$-*Rosochatius system.* The $\mathbb{CP}^N$-oscillator, being a superintegrable system (for $N > 1$), has an obvious drawback: it lacks covariance under transition from one chart to another. This noncovariance becomes manifest after expressing the Hamiltonian (17) via the $SU(N+1)$ symmetry generators and the homogeneous coordinates u^i,

$$
\mathcal{H}_{osc} = \frac{\sum_{i,j=0}^{N} L_{i\bar{j}} L_{j\bar{i}} - B^2}{2} + \frac{\omega^2}{u^0 \bar{u}^0} - \omega^2. \tag{22}
$$

This expression allows one to immediately construct $(N+1)$-parameter deformation of the $\mathbb{CP}^N$-oscillator, such that it is manifestly form-invariant under passing from one chart to another accompanied by the appropriate change of the parameters ω_i. The relevant potential is

$$
V_{\text{Ros}} = \sum_{i=0}^{N}\left(\frac{\omega_i^2}{u^i \bar{u}^i} - \omega_i^2\right), \quad \text{with} \quad \sum_{i=0}^{N} u^i \bar{u}^i = 1. \tag{23}
$$

In the case when all parameters ω_i are equal, the system is globally defined on the complex projective space with the punctured points $u^i = 0$.

The system with the potential (23) is just the $\mathbb{CP}^N$-Rosochatius system mentioned in Introduction. Now we turn to its investigation as the main subject of the present paper.

III. $\mathbb{CP}^N$-ROSOCHATIUS SYSTEM

We consider the N-parameter deformation of the $\mathbb{CP}^N$-oscillator by the potential (23), in what follows referred to as the "$\mathbb{CP}^N$-Rosochatius system". It is defined by the Hamiltonian (3) and Poisson brackets (4) with $r_0 = 1$. Equivalently, this system can be defined by the symplectic structure (12) and the Hamiltonian

$$
\mathcal{H}_{\text{Ros}} = \sum_{a,b=1}^{N} g^{\bar{a}b} \bar{\pi}_a \pi_b + (1+z\bar{z})\left(\omega_0^2 + \sum_{a=1}^{N}\frac{\omega_a^2}{z^a \bar{z}^a}\right) - \sum_{i=0}^{N} \omega_i^2, \tag{24}
$$

where $g^{\bar{a}b} = (1+z\bar{z})(\delta^{\bar{a}b} + \bar{z}^a z^b)$ is the inverse Fubini-Study metrics.

The model has N manifest (kinematical) $U(1)$ symmetries with the generators

$$
J_{a\bar{a}} = \iota\pi_a z^a - \iota\bar{\pi}_a \bar{z}^a - B\frac{z^a \bar{z}^a}{1+z\bar{z}} : \{J_{a\bar{a}}, \mathcal{H}\} = 0, \tag{25}
$$

and hidden symmetries with the second-order generators $I_{ij} = (I_{0a}, I_{ab})$ defined as

$$
I_{0a} = J_{0a} \bar{J}_{0a} + \omega_0^2 z^a \bar{z}^a + \frac{\omega_a^2}{\bar{z}^a z^a},
$$

$$
I_{ab} = J_{a\bar{b}} J_{b\bar{a}} + \omega_a^2 \frac{z^b \bar{z}^b}{z^a \bar{z}^a} + \omega_b^2 \frac{z^a \bar{z}^a}{z^b \bar{z}^b} : \{I_{ij}, \mathcal{H}\} = 0. \tag{26}
$$

In the homogeneous coordinates, the hidden symmetry generators can be cast in a more succinct form

$$
I_{ij} = J_{i\bar{j}} J_{j\bar{i}} + \omega_i^2 \frac{u^j \bar{u}^j}{u^i \bar{u}^i} + \omega_j^2 \frac{u^i \bar{u}^i}{u^j \bar{u}^j}. \tag{27}
$$

The relevant symmetry algebra is given by the brackets

$$\{J_{a\bar{a}}, I_{ij}\} = 0,$$
$$\{I_{ij}, I_{kl}\} = \delta_{jk}T_{ijl} + \delta_{ik}T_{jkl} - \delta_{jl}T_{ikl} - \delta_{il}T_{ijk}, \quad (28)$$

with

$$(T_{ijk})^2 = 2(I_{ij} - J_{\bar{i}i}J_{j\bar{j}})(I_{jk} - J_{j\bar{j}}J_{k\bar{k}})(I_{ik} - J_{\bar{i}i}J_{k\bar{k}})$$
$$+ 2I_{ij}I_{ik}I_{jk} + J_{\bar{i}i}^2 J_{jj}^2 J_{k\bar{k}}^2$$
$$- (I_{jk}^2 J_{\bar{i}i}^2 + I_{ij}^2 J_{k\bar{k}}^2 + I_{ik}^2 J_{j\bar{j}}^2) - 4(\omega_k^2 I_{ij}(I_{ij} - J_{\bar{i}i}J_{j\bar{j}})$$
$$+ \omega_i^2 I_{jk}(I_{jk} - J_{j\bar{j}}J_{k\bar{k}}) + \omega_j^2 I_{ik}(I_{ik} - J_{\bar{i}i}J_{k\bar{k}}))$$
$$+ 4\omega_j^2 \omega_k^2 J_{\bar{i}i}^2 + 4\omega_i^2 \omega_k^2 J_{j\bar{j}}^2 + 4\omega_i^2 \omega_j^2 J_{k\bar{k}}^2 + 16\omega_i^2 \omega_j^2 \omega_k^2.$$
$$(29)$$

The Hamiltonian is expressed via these generators as follows

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^{N+1} I_{ij} + \sum_{a=1}^{N} \omega_a^2 + \frac{J_0^2 - B^2}{2}$$
$$= \sum_{a=1}^{N} I_{0a} + \sum_{a,b=1}^{N} \frac{I_{ab}}{2} + \sum_{a=1}^{N} \omega_a^2 + \frac{J_0^2 - B^2}{2}. \quad (30)$$

This consideration actually proves the superintegrability of the $\mathbb{CP}^N$-Rosochatius system. The number of the functionally independent constants of motion will be counted in the end of this section.

For sure, the symmetry algebra written above can be found by a direct calculation of the Poisson brackets between the symmetry generators. However, there is a more elegant and simple way to construct it. Namely, one has to consider the symmetry algebra of the $\mathbb{C}^{N+1}$-Smorodinsky-Winternitz system [20] with *vanishing* magnetic field, and to reduce it, by action of the generators $\sum_{i=0}^{N} \iota(p_i u^i - \bar{p}_i \bar{u}^i)$, $\sum_{i=0}^{N} u^i \bar{u}^i$ (see the previous section), to the symmetry algebra of the $\mathbb{CP}^N$-Rosochatius system.

A. Reduction to (spherical) Rosochatius system

In order to understand the relationship with the standard Rosochatius system (defined on the sphere) let us pass to the real canonical variables y_a, φ^a, p_a, p_{φ_a}

$$z^a = y_a e^{\iota \varphi_a},$$
$$\pi_a = \frac{1}{2}\left(p_a - \iota\left(\frac{p_{\varphi_a}}{y_a} + \frac{By_a}{1+y^2}\right)\right)e^{-\iota\varphi_a}:$$
$$\Omega = dp_a \wedge dy_a + dp_{\varphi_a} \wedge d\varphi_a. \quad (31)$$

In these variables the Hamiltonian (24) is rewritten as

$$\mathcal{H}_{\text{Ros}} = \frac{1}{4}\left(1 + \sum_{c=1}^{N} y_c^2\right)\left[\sum_{a,b=1}^{N} (\delta_{ab} + y_a y_b)p_a p_b \right.$$
$$\left. + 4\tilde{\omega}_0^2 + 4\sum_{a=1}^{N} \frac{\tilde{\omega}_a^2}{y_a^2}\right] - E_0, \quad (32)$$

where

$$\tilde{\omega}_a^2 = \omega_a^2 + \frac{1}{4}p_{\varphi_a}^2, \qquad \tilde{\omega}_0^2 = \omega_0^2 + \frac{1}{4}\left(B + \sum_{a=1}^{N} p_{\varphi_a}\right)^2,$$
$$E_0 = \frac{B^2}{4} + \sum_{i=0}^{N} \omega_i^2. \quad (33)$$

Then, performing the reduction by cyclic variables φ^a (i.e., by fixing the momenta p_φ^a), we arrive at the Rosochatius system on the sphere with $y_a = x_a/x_0$, where (x_0, x_a) are ambient Cartesian coordinates, $\sum_{i=0}^{N} x_i^2 = 1$:

$$x_a = \frac{y_a}{\sqrt{1 + \sum_{c=1}^{N} y_c^2}}, \qquad x_0 = \frac{1}{\sqrt{1 + \sum_{c=1}^{N} y_c^2}}. \quad (34)$$

As was already noticed, the $\mathbb{S}^N$-Rosochatius system is maximally superintegrable, i.e., it has $2N - 1$ functionally independent constants of motion. From the above reduction we conclude that the $\mathbb{CP}^N$-Rosochatius system has $2N - 1 + N = 3N - 1$ functionally independent integrals. Hence, it lacks N integrals needed for the maximal superintegrability.

IV. CLASSICAL SOLUTIONS

To obtain the classical solutions of the $\mathbb{CP}^N$-Rosochatius system we introduce the spherical coordinates through the recursion

$$y_N = r\cos\theta_{N-1},$$
$$y_\alpha = r\sin\theta_{N-1}u_\alpha, \quad \text{with} \quad r = \tan\theta_N, \quad \sum_{\alpha=1}^{N-1} u_\alpha^2 = 1, \quad (35)$$

where y_a were defined by (31). In terms of these coordinates the Hamiltonian (32) takes the form

$$\mathcal{H}_{\text{Ros}} \equiv \mathcal{I}_N - E_0$$
$$= \frac{1}{4}(1 + r^2)\left((1 + r^2)p_r^2 + \frac{4\mathcal{I}_{N-1}(\theta)}{r^2} + 4\tilde{\omega}_0^2\right) - E_0,$$
$$\mathcal{I}_a = \frac{p_{\theta_a}^2}{4} + \frac{\mathcal{I}_{a-1}}{\sin^2\theta_a} + \frac{\tilde{\omega}_{a+1}^2}{\cos^2\theta_a}, \quad (36)$$

with E_0, $\omega_N \equiv \tilde{\omega}_0$ defined in (33), $a = 1, \ldots, N$ and $\mathcal{I}_0 = 0$.

Thus we singled out the complete set of Liouville integrals $(\mathcal{H}_{\mathrm{Ros}}, \mathcal{I}_\alpha, p_{\varphi_a})$, and separated the variables. It is by no means the unique choice of Liouville integrals and of the coordinate frame in which the Hamiltonian admits the separation of variables. However, for our purposes it is enough to deal with any particular choice.

With the above expressions at hand, we can derive classical solutions of the system by solving the Hamilton-Jacobi equation

$$
\mathcal{H}\left(p_\mu = \frac{\partial S}{\partial x^\mu}, x^\mu\right) = E, \quad \text{with} \quad x^\mu = (\theta_a, \varphi_a),
$$

$$
p_\mu = (p_a, p_{\varphi_a}). \tag{37}
$$

To this end, we introduce the generating function of the form

$$
S_{\mathrm{tot}} = 2\sum_{a=1}^N S_a(\theta_a) + \sum_{a=1}^N p_{\varphi_a}\varphi_a. \tag{38}
$$

Substituting this ansatz in the Hamilton-Jacobi equation, we immediately separate the variables and arrive at the set of ordinary differential equations:

$$
\left(\frac{dS_a}{d\theta_a}\right)^2 + \frac{c_{a-1}}{\sin^2\theta_a} + \frac{\tilde{\omega}_{a+1}^2}{\cos^2\theta_a} = c_a, \quad a = 1, \ldots, N,
$$

$$
c_N := E + E_0, \quad \tilde{\omega}_{N+1}^2 := \tilde{\omega}_0^2. \tag{39}
$$

Solving these equations, we obtain

$$
S_a = \int d\theta_a \sqrt{c_a - \frac{c_{a-1}}{\sin^2\theta_a} - \frac{\tilde{\omega}_{a+1}^2}{\cos^2\theta_a}}. \tag{40}
$$

Thus we have found the general solution of the Hamilton-Jacobi equation (i.e., the solution depending on $2N$ integration constants c_a, p_{φ_a}).

In order to get the solutions of the classical equations of motion, we should differentiate the generating functions with respect to these integration constants and then equate the resulting functions to some constants t_0, κ_α, and φ_0^a,

$$
\frac{\partial S_{\mathrm{tot}}}{\partial E} = t - t_0, \quad \frac{\partial S_{\mathrm{tot}}}{\partial c_\alpha} = 2\sum_{b=1}^N \frac{\partial S_b}{\partial c_\alpha} = \kappa_\alpha,
$$

$$
\alpha = 1, \ldots, N-1, \quad \frac{\partial S_{\mathrm{tot}}}{\partial p_{\varphi_a}} = \varphi^a + \sum_{b=1}^N 2\frac{\partial S_b}{\partial p_{\varphi_a}} = \varphi_0^a. \tag{41}
$$

Introducing

$$
\xi_a := \sin^2\theta_a, \qquad \mathcal{A}_a := \frac{c_a + c_{a-1} - \tilde{\omega}_{a+1}^2}{2c_a}, \tag{42}
$$

we obtain from (41)

$$
\xi_N - \mathcal{A}_N = \sqrt{\mathcal{A}_N^2 - \frac{c_{N-1}}{c_N}} \sin 2\sqrt{c_N}(t - t_0), \tag{43}
$$

$$
\xi_\alpha = \sqrt{\mathcal{A}_\alpha^2 - \frac{c_{\alpha-1}}{c_\alpha}}\left(\frac{\sin\kappa_\alpha(\xi_{\alpha+1}\mathcal{A}_{\alpha+1} - \frac{c_\alpha}{c_{\alpha+1}}) + \cos\kappa_\alpha\sqrt{-\xi_{\alpha+1}^2 + 2\xi_{\alpha+1}\mathcal{A}_{\alpha+1} - \frac{c_\alpha}{c_{\alpha+1}}}}{\xi_{\alpha+1}\sqrt{\frac{c_{\alpha+1}}{c_\alpha}\mathcal{A}_{\alpha+1}^2 - 1}}\right) + \mathcal{A}_\alpha, \tag{44}
$$

$$
\varphi^a - \varphi_0^a = -\frac{p_{\varphi_a}}{4\tilde{\omega}_{a+1}}\arctan\frac{2\tilde{\omega}_{a+1}\sqrt{c_{a-1}(\xi_a - 1) - \xi_a(c_a(\xi_a - 1) + \tilde{\omega}_{a+1}^2)}}{-c_{a-1}(\xi_a - 1) + c_a(\xi_a - 1)\tilde{\omega}_{a+1}^2(\xi_a + 1)}. \tag{45}
$$

Thereby we have derived the explicit classical solutions of our $\mathbb{CP}^N$-Rosochatius system.

V. QUANTIZATION

In order to quantize the $\mathbb{CP}^N$-Rosochatius system we replace the Poisson brackets (4) by the commutators (with $r_0 = 1$)

$$
[\hat{\pi}_a, z^b] = -\imath\hbar\delta_a^b, \quad [\hat{\pi}_a, \hat{\bar{\pi}}_b] = \hbar B\left(\frac{\delta_{a\bar{b}}}{1+z\bar{z}} - \frac{\bar{z}^a z^b}{(1+z\bar{z})^2}\right). \tag{46}
$$

The appropriate quantum realization of the momenta operators reads

$$
\hat{\pi}_a = -\imath\left(\hbar\frac{\partial}{\partial z^a} + \frac{B}{2}\frac{\bar{z}^a}{1+z\bar{z}}\right),
$$

$$
\hat{\bar{\pi}}_a = -\imath\left(\hbar\frac{\partial}{\partial\bar{z}^a} - \frac{B}{2}\frac{\bar{z}^a}{1+z\bar{z}}\right). \tag{47}
$$

Then we define the quantum Hamiltonian

$$\hat{\mathcal{H}}_{\text{Ros}} = \frac{1}{2} g^{a\bar{b}} (\hat{\pi}_a \hat{\bar{\pi}}_b + \hat{\bar{\pi}}_b \hat{\pi}_a) + \hbar^2 (1 + z\bar{z}) \left(\omega_0^2 + \sum_{a=1}^{N} \frac{\omega_a^2}{z^a \bar{z}^a} \right)$$

$$- \hbar^2 \sum_{i=0}^{N} \omega_i^2. \qquad (48)$$

The kinetic term in this Hamiltonian is written as the Laplacian on Kähler manifold (coupled to a magnetic field) defined with respect to the volume element $dv_{\mathbb{CP}^N} = (1 + z\bar{z})^{-(1+N)}[dz d\bar{z}]$, while in the potential term we have made the replacement $\omega_i \to \hbar\omega_i$.

In terms of the real coordinates $z^a = y_a e^{i\varphi_a}$ this Hamiltonian reads [cf. (32)]

$$\hat{\mathcal{H}}_{\text{Ros}} = \left(1 + \sum_{c=1}^{N} y_c^2 \right) \Bigg[-\frac{\hbar^2}{4} \Bigg(\sum_{a,b=1}^{N} (\delta_{ab} + y_a y_b) \frac{\partial^2}{\partial y_a \partial y_b}$$

$$+ \sum_{a=1}^{N} \left(y_a + \frac{1}{y^a} \right) \partial_{y_a} \Bigg) + \hat{\bar{\omega}}_{N+1}^2 + \sum_{a=1}^{N} \frac{\hat{\bar{\omega}}_a^2}{4y_a^2} \Bigg] - \tilde{E}_0.$$

$$(49)$$

Here we introduced the operators

$$\hat{\bar{\omega}}_{N+1}^2 = \left(\frac{B}{\hbar} + \frac{1}{\hbar} \sum_{a=1}^{N} \hat{p}_{\varphi_a} \right)^2 + 4\omega_0^2, \quad \hat{\bar{\omega}}_a^2 = 4\omega_a^2 + \frac{\hat{p}_{\varphi_a}^2}{\hbar^2} \qquad (50)$$

with

$$\hat{p}_{\varphi_a} = \hat{J}_{a\bar{a}} = -i\hbar \frac{\partial}{\partial\varphi^a} \qquad \tilde{E}_0 = \frac{B^2}{4} + \hbar^2 \sum_{i=0}^{N} \omega_i^2. \qquad (51)$$

Clearly, these operators are quantum analogs of the classical quantities (33). In the spherical coordinates (35) the Hamiltonian (49) takes the form

$$\hat{\mathcal{H}}_{\text{Ros}} = \hat{\mathcal{I}}_N - \tilde{E}_0,$$

$$\hat{\mathcal{I}}_a = -\frac{\hbar^2}{4} \Bigg((\sin\theta_a)^{1-a} \frac{\partial}{\partial\theta_a} \left((\sin\theta_a)^{a-1} \frac{\partial}{\partial\theta_a} \right)$$

$$+ (a\cot\theta_a - \tan\theta_a) \frac{\partial}{\partial\theta_a} \Bigg) + \frac{\hat{\mathcal{I}}_{a-1}}{\sin^2\theta_a} + \frac{\hbar^2 \hat{\bar{\omega}}_{a+1}^2}{4\cos^2\theta_a}, \quad (52)$$

where $a = 1, \ldots, N$ and $\hat{\mathcal{I}}_0 = 0$.

This prompts us to consider the spectral problem[3]

$$\hat{J}_{a\bar{a}} \Psi = \hbar m_a \Psi, \qquad \hat{\mathcal{I}}_a \Psi = \frac{\hbar^2}{4} l_a (l_a + 2a) \Psi, \qquad (53)$$

where l_a are the appropriate "spin" quantum numbers, and separate the variables by the choice of the wave function in

such a way that it resolves first N equations in the above problem,

$$\Psi = \frac{1}{(2\pi)^{N/2}} \prod_{a=1}^{N} \psi_a(\theta_a) e^{im_a\varphi_a}, \qquad m_a = 0, \pm 1, \pm 2, \ldots \quad (54)$$

Then, passing to the variables $\xi_a = \sin^2\theta_a$, we transform the reduced spectral problem to the system of N ordinary differential equations

$$-\xi_a(1 - \xi_a)\psi_a'' + ((a+1)\xi - a)\psi_a'$$

$$+ \frac{1}{4} \left(\frac{l_{a-1}(l_{a-1} + 2a - 2)}{\xi_a} + \frac{\tilde{\omega}_{a+1}^2}{1 - \xi_a} - l_a(l_a + 2a) \right) \psi_a = 0.$$

$$(55)$$

These equations can be cast in the form of a hypergeometric equation through the following substitution

$$\psi(\xi_a) = \xi_a^{\frac{l_{a-1}}{2}} (1 - \xi_a)^{\frac{\omega_{a+1}}{2}} f(\xi_a): \qquad (56)$$

$$\xi_a(1 - \xi_a)f'' + (l_{a-1} + a - \xi(l_{a-1} + a + \tilde{\omega}_{a+1} + 1))f'$$

$$- \frac{1}{4}(l_{a-1} + \tilde{\omega}_{a+1} - l_a)(l_{a-1} + \tilde{\omega}_{a+1} + l_a + 2a))f = 0. \quad (57)$$

The regular solution of this equation is the hypergeometric function [31]

$$f_a(\xi) = C_0 F(-n_a; l_{a-1} + \tilde{\omega}_{a+1} + a + n_a; l_{a-1} + a; \xi_a),$$

$$l_a = 2n_a + l_{a-1} + \tilde{\omega}_{a+1}, \qquad (58)$$

with

$$n_a = 0, 1, 2\ldots \qquad \tilde{\omega}_a = \sqrt{4\omega_a^2 + m_a^2}. \qquad (59)$$

Therefore, $l_N = \sum_{a=1}^{N}(2n_a + \tilde{\omega}_a)$, so that the energy spectrum is given by the expressions

$$E_{n,\{m_a\}} = \frac{\hbar^2}{4} \Bigg(2n + N + \sqrt{\left(B/\hbar + \sum_{a=1}^{N} m_a \right)^2 + 4\omega_0^2}$$

$$+ \sum_{a=1}^{N} \sqrt{4\omega_a^2 + m_a^2} \Bigg)^2 - \frac{B^2 + \hbar^2 N^2}{4} - \hbar^2 \sum_{i=0}^{N} \omega_i^2,$$

$$(60)$$

where $n = \sum_{a=1}^{N} n_a = 0, 1, \ldots$[4]

---

[3]In the classical limit, $\hbar \to 0, m_a, l_a \to \infty$, the eigenvalues $\hbar m_a$ yield p_{φ_a} and $\hbar l_a$ yield $\sqrt{c_a}$.

[4]For the integer parameters n_a the hypergeometric function (58) is reduced to Jacobi polynomials. We thank the referee for this remark.

Thus the spectrum of the quantum $\mathbb{CP}^N$-Rosochatius system depends on $N + 1$ quantum numbers. This is in full agreement with the fact that this system has $3N - 1$ functionally independent constants of motion (let us recall that the spectrum of D-dimensional quantum mechanics with $D + K$ independent integrals of motion depends on $D - K$ quantum numbers. E.g, the spectrum of maximally superintegrable system depends on the single (principal) quantum number).

Let us also write down the explicit expressions for the non-normalized wave functions and the $\mathbb{CP}^N$ volume element

$$\Psi_{\{n_a\},\{m_a\}} = \frac{C_0}{(2\pi)^{N/2}} \prod_{a=1}^{N} \xi_a^{\frac{l_{a-1}}{2}} (1 - \xi_a)^{\frac{\omega_{a+1}}{2}} e^{\imath m_a \varphi_a}$$
$$\times F(-n_a; l_{a-1}, +\tilde{\omega}_{a+1} + a + n_a; l_{a-1} + a; \xi_a)$$

$$dv_{\mathbb{CP}^N} = \frac{1}{(1 + y^2)^{N+1}} \prod_{a=1}^{N} y_a dy_a d\varphi_a, \qquad (61)$$

where

$$\xi_a = \frac{y_a^2}{y_a^2 + y_{a+1}^2}. \qquad (62)$$

A. Reduction to quantum (spherical) Rosochatius system

From the above consideration it is clear that, by fixing the eigenvalues of $\hat{J}_{a\bar{a}} = \hat{p}_{\varphi_a}$, we can reduce the Hamiltonians (48) and (49) to those of the quantum (spherical) Rosochatius system, the classical counterpart of which is defined by Eq. (32).

However, the quantization of (32) through replacing the kinetic term by the Laplacian yields a slightly different expression for the Hamiltonian

$$\hat{H}_{\mathrm{Ros}} = -\frac{\hbar^2}{4} \left(1 + \sum_{c=1}^{N} y_c^2\right) \left[\sum_{a,b=1}^{N} (\delta_{ab} + y_a y_b) \frac{\partial^2}{\partial y_a \partial y_b} \right.$$
$$\left. + \sum_{a=1}^{N} \left(2 y_a \partial_{y_a} + \frac{g_a^2}{y_a^2}\right) + g_0^2 \right]. \qquad (63)$$

This is because the volume element on N-dimensional sphere is different from that reduced from $\mathbb{CP}^N$:

$$dv_{S^N} = \frac{1}{(1 + \sum_{c=1}^{N} y_c^2)^{(N+1)/2}} \prod_{a=1}^{N} dy_a, \qquad (64)$$

and it gives rise to a different Laplacian as compared to that directly obtained by reduction of the Laplacian on $\mathbb{CP}^N$.

As a result, the relation between wave functions of the (spherical) Rosochatius system and those of the $\mathbb{CP}^N$-Rosochatius system is as follows,

$$\Psi_{\mathrm{sph}} = \sqrt{\frac{(1 + \sum_{c=1}^{N} y_c^2)^{(N+1)}}{\prod_{a=1}^{N} y_a}} \Psi. \qquad (65)$$

So in order to transform the reduced $\mathbb{CP}^N$-Rosochatius Hamiltonian to the spherical one (63), we have to redefine the wave functions presented in (61) and perform the respective similarity transformation of the Hamiltonian.

VI. SUPERSYMMETRY

Let us briefly discuss the possibility of supersymmetrization of the $\mathbb{CP}^N$-Rosochatius system, postponing the detailed analysis for a separate study [32]. The $\mathbb{CP}^N$-Rosochatius system belongs to the class of the so-called "Kähler oscillators" [8,21] (up to a constant shift of the Hamiltonian), and therefore, admits $SU(2|1)$ (or, equivalently, "weak $\mathcal{N} = 4$") supersymmetric extension. Namely, its Hamiltonian (24) can be cast in the form

$$\mathcal{H}_{\mathrm{Ros}} = \sum_{a,b=1}^{N} g^{\bar{a}b} (\bar{\pi}_a \pi_b + |\omega|^2 \partial_{\bar{a}} K \partial_b K) - \left| \sum_{i=0}^{N} \omega_i \right|^2$$
$$- \sum_{i=0}^{N} |\omega_i|^2, \qquad (66)$$

with

$$K = \log(1 + z\bar{z}) - \frac{1}{|\omega|} \sum_{a=1}^{N} (\omega_a \log z^a + \bar{\omega}_a \log \bar{z}^a),$$

$$\omega = \omega_0 + \sum_{a=1}^{N} \omega_a. \qquad (67)$$

Here, as opposed to the previous sections, we assume that ω_i are complex numbers, i.e., we replaced

$$\omega_i \to \omega_i e^{\imath \nu_i}, \qquad (68)$$

with ν_i being arbitrary real constants.

The $SU(2|1)$ superextension just mentioned is reduced to that with $\mathcal{N} = 4, d = 1$ Poincaré supersymmetry under the conditions[5]

$$B = 0, \qquad \omega = \sum_{i=0}^{N} \omega_i = 0. \qquad (69)$$

One could expect that the second constraint corresponds to the vanishing potential. However, it is not the case: looking at the explicit expression for the Hamiltonian, one can see that the parameter ω does not appear in denominators anymore. Indeed, the second constraint above leads to the

---

[5]From the viewpoint of $SU(2|1)$ mechanics, B is just the parameter of contraction to $\mathcal{N} = 4, d = 1$ supersymmetry [23].
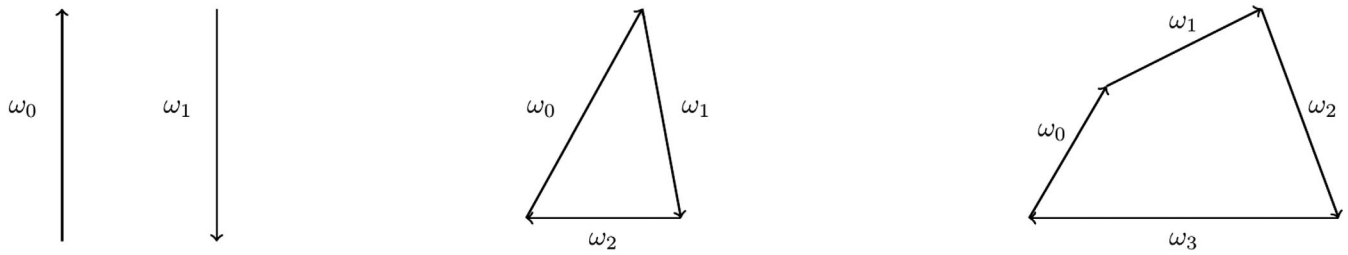
FIG. 1.   Frequencies for $N = 1, 2, 3$.

relation $|\omega_0|^2 = |\sum_{a=1}^{N} \omega_a|^2$, which allows us to represent the Hamiltonian (24) in the following form

$$\mathcal{H}_{\text{Ros}} = \sum_{a,b=1}^{N} g^{\bar{a}b}(\bar{\pi}_a \pi_b + \partial_{\bar{a}} \bar{U} \partial_b U) - \sum_{i=0}^{N} |\omega_i|^2, \quad (70)$$

where $U(z)$ is the holomorphic function ("superpotential")

$$U(z) = \sum_{a=1}^{N} \omega_a \log z^a. \quad (71)$$

It is well known that the systems with such a Hamiltonian admit the $\mathcal{N} = 4$ supersymmetric extension in the absence of magnetic field (see, e.g., [33]). Explicitly, it amounts to the following consideration.

Let us consider a $(2N.4N)_C$-dimensional phase space equipped with the symplectic structure (till the end of this section we assume the summation over repeating indices)

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta_{\alpha}^c \bar{\eta}^{d\alpha} dz^a \wedge d\bar{z}^b$$

$$+ \frac{1}{2} g_{a\bar{b}} D\eta_{\alpha}^a \wedge D\bar{\eta}_{\alpha}^b, \quad (72)$$

where $D\eta_{\alpha}^a = d\eta_{\alpha}^a + \Gamma_{bc}^a \eta_{\alpha}^b dz^c$ with $\Gamma_{bc}^a, R_{a\bar{b}c\bar{d}}$ being, respectively, the components of connection and curvature of the Kähler structure associated with the Fubini-Study metrics (9), $\eta_{\alpha}^a, \bar{\eta}_{\alpha}^a$ are Grassmann variables with additional $SU(2)$ indices $\alpha = 1, 2$. The lower- and upper-case $SU(2)$ indices are related by the antisymmetric matrix $\epsilon_{\alpha\beta}$ and its inverse $\epsilon^{\alpha\beta}$ ($\epsilon_{12} = \epsilon^{21} = 1$).

The Poisson brackets defined by (72) are given by the following nonzero relations and their complex conjugates:

$$\{\pi_a, z^b\} = \delta_a^b, \qquad \{\pi_a, \eta_{\alpha}^b\} = -\Gamma_{ac}^b \eta_{\alpha}^c,$$

$$\{\pi_a, \bar{\pi}_b\} = -R_{a\bar{b}c\bar{d}} \eta_{\alpha}^c \bar{\eta}^{d\alpha}, \qquad \{\eta_{\alpha}^a, \bar{\eta}^{b\beta}\} = g^{a\bar{b}} \delta_{\alpha}^{\beta}. \quad (73)$$

Straightforward calculations show that the following supercharges and Hamiltonian obey the $\mathcal{N} = 4, d = 1$ Poincaré superalgebra

$$Q^{\alpha} = \pi_a \eta^{a\alpha} + \imath \bar{U}_{,\bar{a}} \bar{\eta}^{a\alpha}, \qquad \bar{Q}_{\alpha} = \bar{\pi}_a \bar{\eta}_{\alpha}^a + \imath U_{,a} \eta_{\alpha}^a. \quad (74)$$

$$\mathcal{H}_{\text{SUSY}} = \mathcal{H}_{\text{Ros}} - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}_{\alpha}^b \eta^{c\beta} \bar{\eta}_{\beta}^d + \frac{\imath}{2} U_{,a;b} \eta^{a\alpha} \eta_{\alpha}^b$$

$$+ \frac{\imath}{2} \bar{U}_{,\bar{a};\bar{b}} \bar{\eta}^{a\alpha} \bar{\eta}_{\alpha}^b : \quad (75)$$

$$\{Q^{\alpha}, \bar{Q}_{\beta}\} = \delta_{\beta}^{\alpha} \left(\mathcal{H}_{\text{SUSY}} + \sum_{i=0}^{N} |\omega_i|^2 \right),$$

$$\{Q^{\alpha}, Q^{\beta}\} = \{\bar{Q}_{\alpha}, \bar{Q}_{\beta}\} = \{Q^{\alpha}, \mathcal{H}_{\text{SUSY}}\} = \{\bar{Q}_{\alpha}, \mathcal{H}_{\text{SUSY}}\} = 0, \quad (76)$$

Hence, when the constraints (69) are imposed, we can construct $\mathcal{N} = 4$ supersymmetric extension of the $\mathbb{CP}^N$-Rosochatius system.

An interesting issue is the symmetries of the supersymmetric system constructed. Writing down the explicit expressions for the Hamiltonian and supercharges one can be convinced that they are explicitly invariant under $U(1)$-transformations $z^a \to e^{\imath\kappa} z^a, \pi_a \to e^{-\imath\kappa} \pi_a, \eta^{a\alpha} \to e^{\imath\kappa} \eta^{a\alpha}$ which are obviously canonical transformations. Hence, one can easily construct the "supersymmetric counterpart" of $U(1)$ generators (25). However, it is still unclear whether hidden symmetries of the system one started with can be lifted to its supersymmetric extension. A more detailed analysis of these questions will be a subject of [32].

Let us emphasize that the restriction $\omega = 0$ can be graphically represented as a planar polygon with the edges $|\omega_i|$ (see Fig. 1), which leads to the inequality

$$|\omega_i| \leq \sum_{j \neq i} |\omega_j|. \quad (77)$$

This implies that:
(i) For $N = 1$ the constraint $\omega = 0$ uniquely fixes the values of parameters in the case of $\mathbb{CP}^1$: $\nu_0 = -\nu_1$ and $|\omega_0| = |\omega_1|$. The latter property leads to the appearance of discrete symmetry

$$z \to \frac{1}{z}. \quad (78)$$

(ii) For $N = 2$ the above constraints amount to a triangle, which fixes the parameters ν_a as follows

$$\cos(\nu_2 - \nu_0) = \frac{|\omega_1|^2 - |\omega_0|^2 - |\omega_2|^2}{2|\omega_0||\omega_2|},$$

$$\cos(\nu_1 - \nu_0) = \frac{|\omega_2|^2 - |\omega_0|^2 - |\omega_2|^2}{2|\omega_0||\omega_1|}. \qquad (79)$$

(iii) For $N > 2$ the parameters ν_a are not uniquely fixed, so that we obtain a family of $\mathcal{N} = 4$ supersymmetric Hamiltonians depending on up to $N - 1$ parameters.

We observe that for any value of N at least one parameter ν_i remains unfixed. But this does not affect our consideration since such parameter can be absorbed into a redefinition of fermionic variables.

Finally, note that the constraint $\sum_{i=1}^{N} \omega_i = 0$ also appeared in constructing $\mathcal{N} = 4$ supersymmetric extension of the $\mathbf{S}^N$-Rosochatius system [34], but with ω_i being real numbers. The above trick with complexification of the parameters ω_i is seemingly applicable to the $\mathbf{S}^N$-Rosochatius system as well, hopefully giving rise to a less restrictive form of the $\mathcal{N} = 4$ superextension of the latter.

VII. CONCLUDING REMARKS

In this paper we proposed the superintegrable $\mathbb{CP}^N$-analog of Rosochatius and Smorodinsky-Winternitz systems which is specified by the presence of constant magnetic field and is form-invariant under transition from one chart of $\mathbb{CP}^N$ to others accompanied by the appropriate permutation of the characteristic parameters ω_i. We showed that the system possesses $3N - 1$ functionally independent constants of motion and explicitly constructed its classical and quantum solutions. In the generic case this model admits an extension with $SU(2|1)$ supersymmetry, which is reduced, under the special choice of the characteristic parameters and in the absence of magnetic field, to the "flat" $\mathcal{N} = 4, d = 1$ Poincaré' supersymmetry.

When all constants ω_i are equal, the system is covariant under the above transitions between charts and so becomes globally defined on the whole $\mathbb{CP}^N$ manifold. This covariance implies N discrete symmetries,

$$z^a \to \frac{1}{z^a}, \qquad z^\alpha \to \frac{z^\alpha}{z^a}, \quad \text{with} \quad \alpha \neq a. \qquad (80)$$

Moreover, in this special case the model always admits (in the absence of magnetic field) $\mathcal{N} = 4, d = 1$ Poincaré' supersymmetrization because the inequality (77) is automatically satisfied. The model with equal ω_i can be also interpreted as a model of N *interacting* particles with an effective position-dependent mass located in the quantum ring. This agrees with the property that, in the flat limit, the model under consideration can be interpreted as an ensemble of N free particles in a single quantum ring interacting

with a constant magnetic field orthogonal to the plane (cf. [25–27]). Thus the property of the exact solvability/superintegrability of the suggested model in the presence of constant magnetic field (equally as of the superextended model implying the appropriate inclusion of spin) makes it interesting also from this point of view.

The obvious next tasks are the study of classical and quantum $SU(2|1)$ supersymmetric extension of the $\mathbb{CP}^N$ Rosochatius system [32], as well as the construction of its Lax pair formulation.

Two important possible generalizations of the proposed system are the following ones:

(i) An analog of the $\mathbb{CP}^N$-Rosochatius system on the quaternionic projective space $\mathbb{HP}^N$ in the presence of BPST instanton.

Presumably, it can be defined by the Hamiltonian (3) and the symplectic structure (12), in which π_a, z^a are replaced by quaternionic variables, and the last term in (12) by terms responsible for interaction with BPST instanton [35] (see also [36,37] and [9]). The phase space of this system is expected to be $T^* \mathbb{HP}^N \times \mathbb{CP}^1$, due to the isospin nature of instanton. We can hope that this system is also superintegrable and that an interaction with BPST instanton preserves the superintegrability. On this way we can also expect intriguing links with the recently explored quaternion-Kähler deformations of $\mathcal{N} = 4$ mechanics [38]. These models also admit homogeneous $\mathbb{HP}^N$ backgrounds.

(ii) $\mathbb{CP}^N$-analog of Coulomb problem.

Such an extension could be possible, keeping in mind the existence of superintegrable spherical analog of Coulomb problem with additional $\sum_i g_i^2/x_i^2$ potential, as well as the observation that the (spherical) Rosochatius system is a real section of the $\mathbb{CP}^N$-Rosochatius system.

One of the key motivations of the present study was to derive the superintegrable $\mathbb{CP}^N$- and $\mathbb{C}^N$- generalizations of rational Calogero model. Unfortunately, until now we succeeded in constructing only trivial extensions of such kind. We still hope to reach the general goal just mentioned in the future.

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