Asymptotic analysis of spin foam amplitude with timelike triangles

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The large-j asymptotic behavior of the four-dimensional spin foam amplitude is investigated for the extended spin foam model (Conrady-Hnybida extension) on a simplicial complex. We study the most general situation in which timelike tetrahedra with timelike triangles are taken into account. The large-j asymptotic behavior is determined by the critical configurations of the amplitude. We identify the critical configurations that correspond to the Lorentzian simplicial geometries with timelike tetrahedra and triangles. Their contributions to the amplitude are asymptotic phases, whose exponents equal the Regge action of gravity. The amplitude may also contains critical configurations corresponding to nondegenerate split signature 4-simplices and degenerate vector geometries. But vertex amplitudes containing at least one timelike and one spacelike tetrahedra only give Lorentzian 4-simplices, while the split signature or degenerate 4-simplex does not appear.

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I. INTRODUCTION

Spin foam models arise as a covariant formulation of loop quantum gravity (LQG); for a review, see [1–5]. A spin foam can be regraded as a Feynmann diagram with 5-valent vertices, corresponding to quantum 4-simplices, as building blocks of the discrete quantum spacetime. The boundary of a 4-simplex contains five tetrahedra. As one of the popular spin foam models, the Lorentzian Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK) model comes with a gauge fixing within each tetrahedron such that in the local frame the timelike normal vector of the tetrahedron reads $u = (1, 0, 0, 0)$ in a 4D Minkowski spacetime with signature $(-1, 1, 1, 1)$, known as the "time gauge." As a result, this model is subject to the restriction that tetrahedra and triangles are all spacelike [6], such that the tetrahedra lives in a Euclidean subspace. As a result, such spin foam models only correspond to a special class of 4D Lorentzian triangulations. However, in the extended spin foam model by Conrady and Hnybida, some tetrahedron normal vectors are chosen to be spacelike $u = (0, 0, 0, 1)$. As a result, the model contains timelike tetrahedra and triangles which live in 3D Minkowski subspaces [7–9].

The semiclassical behavior of spin foam models is determined by its large-j asymptotics. Recently there have been many investigations of large-j spin foams, in particular, the asymptotics of EPRL/FK model [10–18], and models with the cosmological constant [19,20]. It has been shown that, in large-j asymptotics, the spin foam amplitude is dominated by the contributions from critical configurations, which gives the simplicial geometries and discrete Regge action on a simplicial complex. The resulting geometries from the above analysis only have spacelike tetrahedra and spacelike triangles. Recently, the asymptotics of the Hnybida-Conrady extended model with a timelike tetrahedron was investigated in [21]. The critical configurations of the extended model give simplicial geometries containing timelike tetrahedra. But the limitation is that all the triangles are still spacelike within each timelike tetrahedron.

In this paper, we extend the semiclassical analysis of the extended model to general situations, in which we take into account both timelike tetrahedra and timelike triangles. Our work is motivated by the examples of geometries in classical Lorentzian Regge calculus, and their convergence to smooth geometries [22–24]. In all examples, the Regge geometries contain timelike triangles. In order to have the Regge geometries emerge as critical configurations from spin foam model, we have to extend the semiclassical analysis to contain timelike triangles.

In our analysis, we first derive the large-j integral form of the extended spin foam model with coherent states for timelike triangles. The large-j asymptotic analysis is based

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on the stationary phase approximation of the integral. The asymptotics of the integral is a sum of contributions from critical configurations.

Before coming to our main result, we would like to mention some key assumptions for the validity of the result: The following results are valid when we assume every timelike tetrahedron contains at least one spacelike and one timelike triangle. This is the case in all Regge geometry examples mentioned above. Our results also apply to some special cases when all triangles in a tetrahedron are timelike. Moreover, all tetrahedra in our discussion are assumed to be nondegenerate. Here we do not consider the critical configurations with a degenerate tetrahedron. Finally, the Hessian evaluated at every critical configuration is assumed to be a nondegenerate matrix.

The main result is summarized as follows: First, for a single 4-simplex and its vertex amplitude, it is important to have boundary data satisfy the length matching condition and orientation matching condition. Namely, (1) among the five tetrahedra reconstructed by the boundary data (by the Minkowski theorem), each pair of them are glued with their common triangles matching in shape (matching their three edge lengths), and (2) all tetrahedra have the same orientation. The amplitude has critical configurations only if these two conditions are satisfied, otherwise the amplitude is suppressed asymptotically. The critical configurations have geometrical interpretations as geometrical 4-simplices, which may generally have one of three possible signatures: Lorentzian, split, or degnerate.

(i) When the 4-simplex has Lorentzian signatures: The contribution at the critical configuration is given by a phase, whose exponent is the Regge action with a sign related to orientations, i.e., the vertex amplitude gives asymptotically

$$A_v \sim N_+ e^{iS_\Delta} + N_- e^{-iS_\Delta} \tag{1.1}$$

up to an overall phase depending on the boundary coherent state. The Regge action in the 4-simplex reads $S_\Delta = \sum_f A_f \theta_f$ with A_f the area of triangle f. θ_f relates to the dihedral angle Θ_f by $\theta_f = \pi - \Theta_f$. The area spectrum is different between timelike and spacelike triangles in a timelike tetrahedron.

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma j_f & \text{spacelike triangle} \end{cases}. \tag{1.2}$$

$n_f \in \mathbb{Z}_+$ satisfies the simplicity constraint $n_f = \gamma s_f$ where $s_f \in \mathbb{R}_+$ labels the continuous series irreps of SU(1,1). $j_f \in \mathbb{Z}_+/2$ labels the discrete series irreps of SU(1,1). $N_\pm$ are geometric factors that depend on the lengths and orientations of the reconstructed 4-simplex.

(ii) The reconstructed 4-simplices have split signatures: The vertex amplitude gives asymptotically

$$A_v \sim N_+ e^{i\gamma^{-1} S_\Delta} + N_- e^{-i\gamma^{-1} S_\Delta} \tag{1.3}$$

up to an overall phase. Here $S_\Delta = \sum_f A_f \theta_f$ where θ_f is a boost dihedral angle.

(iii) The reconstructed 4-simplices are degenerate (vector geometry) and there is a single critical point. The asymptotical vertex amplitude is given by a phase depending on the boundary coherent states.

It is important to remark that for a vertex amplitude containing at least one timelike and one spacelike tetrahedron, critical configurations only give Lorentzian 4-simplices, while the split signature and degenerate 4-simplex do not appear. The last two cases only appear when all tetrahedra are timelike in a vertex amplitude. The situation is similar to the Lorentzian EPRL/FK model, where the Euclidean signature and degenerate 4-simplex appear because all tetrahedra are spacelike.

Our analysis is generalized to the spin foam amplitude on a simplicial complex $\mathcal{K}$ with many 4-simplices. We identify the critical configurations corresponding to simplicial geometries with all 4-simplices being Lorentzian and globally oriented. The configurations come in pairs, corresponding to opposite global orientations. Each pair gives the following asymptotic contribution to the spin foam amplitude (up to an overall phase)

$$N_+ e^{iS_\mathcal{K}} + N_- e^{-iS_\mathcal{K}} \tag{1.4}$$

where

$$S_\mathcal{K} = \sum_{f\,\text{bulk}} A_f \varepsilon_f + \sum_{f\,\text{boundary}} A_f(\theta_f + p_f \pi) \tag{1.5}$$

is the Regge action on the simplicial complex, up to a boundary term with $p_f \in \mathbb{Z}$ (p_f is the number of 4-simplices sharing f minus 1). The additional boundary term $p_f A_f \pi$ does not affect the Regge equation of motion. Here the simplicial geometries and Regge action generally contain timelike tetrahedra and timelike triangles. ε_f is the deficit angle. ε_f and θ_f at timelike triangles are given by

$$\varepsilon_f = 2\pi - \sum_f \Theta_f(v), \qquad \theta_f = \pi - \sum_f \Theta_f(v). \tag{1.6}$$

$\Theta_f(v)$ is the dihedral angle within the 4-simplex at v. It is a rotation angle between spacelike normals of tetrahedra, because the tetrahedra sharing a timelike triangle are all timelike.

To obtain (1.4), we have assumed each bulk triangle is shared by an even number of 4-simplices. This assumption is true in many important examples of classical Regge calculus.

This paper is organized as follows. In Sec. II, we write the coherent states for timelike triangles in large-j approximation and express the spin foam amplitude in terms of the coherent states. In Sec. III, we derive and analyze the critical equations. The critical equations are reformulated in a geometrical form for a timelike tetrahedron containing both spacelike and timelike triangles. Then in Sec. IV, we reconstruct nondegenerate simplicial geometries from critical configurations. In Sec. V, the critical configurations for degenerate geometries are analyzed. Finally, in Sec. VII, we derive the difference between phases evaluated at pairs of critical configurations corresponding to the oppositely orientated simplicial geometries.

II. SPIN FOAM AMPLITUDE IN TERMS OF SU(1, 1) CONTINUOUS COHERENT STATES

The spin foam models are defined as a state sum model over the simplicial manifold $\mathcal{K}$ and its dual, which consists of simplices σ_v, tetrahedra τ_e, triangles f, edges, and vertices (v, e, and f are labels for vertices, edges, and faces on the dual graph, respectively). A triangulation is obtained by gluing simplices σ with pairs of their boundaries (tetrahedrons τ). The phase space associated with manifold $\mathcal{K}$ is

$$P_{\mathcal{K}} = T^*\mathrm{SL}(2,\mathbb{C})^L, \qquad (\Sigma_f^{IJ}, h_f) \in T^*\mathrm{SL}(2,\mathbb{C}) \tag{2.1}$$

for a Lorentzian model, where L is the number of triangles, $h_f \in \mathrm{SL}(2,\mathbb{C})$ is the holonomy along the edges, and $\Sigma_f^{IJ} \in \mathfrak{sl}(2,\mathbb{C})$ is its conjugate momenta. h_f can be decomposed as

$$h_f = \prod_{v\subset\partial f} g_{ev} g_{ve'} \tag{2.2}$$

where $g_{ve} \in \mathrm{SL}(2,\mathbb{C})$ and $g_{ev} = g_{ve}{}^{-1}$. Σ_f^{IJ} is subject to the simplicity constraint

$$\frac{\gamma}{1+\gamma^2}(u_e)^I((1-\gamma*)\Sigma_{fIJ}) = 0 \tag{2.3}$$

where u_e is a 4-normal vector associated to each tetrahedron t_e, γ is a real number known as the Immirizi parameter, and $*$ is the Hodge dual operator. Geometrically, the simplicity constraint implies that each triangle f in tetrahedron t_e is associated with a simple bivector

$$B_f = \frac{\gamma}{1+\gamma^2}(1-\gamma*)\Sigma_f. \tag{2.4}$$

The state sum is defined over all the quantum states of the physical Hilbert space on a given $\mathcal{K}$, given as

$$Z(\mathcal{K}) = \sum_J \prod_f \mu_f(J_f) \prod_v A_v(J_f, i_e). \tag{2.5}$$

Here, $J = \vec{j}_f$ represents the combination of labels of the $\mathrm{SL}(2,\mathbb{C})$ irreps associated to each triangle. i_e is the intertwiner associated with each tetrahedron

$$i_e \in \mathrm{Inv}_G[V_{J_1} \otimes \ldots \otimes V_{J_4}] \tag{2.6}$$

which impose the gauge invariance. The vertex amplitude $A_v(J_f, i_e)$ associated with each 4-simplex σ_v captures the dynamics of the model, while the face amplitude $\mu_f(J_f)$ is a weight for the J sum.

Usually a partial gauge fixing is taken to the above models, which correspond to pick a special normal u for all of the tetrahedra $\forall_e, u_e = u$. As a result, the intertwiners associated with each tetrahedron defined above are replaced by the intertwiners of the stabilizer group $H \in G$. There are two different gauge fixings:

(i) $u = (1,0,0,0)$, $H = SU(2)$, EPRL/FK models.
(ii) $u = (0,0,0,1)$, $H = SU(1,1)$, the Conrady-Hnybida extension.

which, after imposing the quantum simplicity constraint (2.3), lead to the following conditions [6,7,25]:

(i) $u = (1,0,0,0)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{2.7}$$

(ii) $u = (0,0,0,1)$, spacelike triangles

$$\rho = \gamma n, \qquad n = j \tag{2.8}$$

(iii) $u = (0,0,0,1)$, timelike triangles

$$\rho = -n/\gamma, \qquad s = \frac{1}{2}\sqrt{n^2/\gamma^2 - 1}. \tag{2.9}$$

Here ($\rho \in \mathbb{R}, n \in \mathbb{Z}/2$) are the labels of $\mathrm{SL}(2,\mathbb{C})$ irreps, $j \in \mathbb{N}/2$ is the label of SU(2) irreps or the SU(1,1) discrete series, and $s \in \mathbb{R}$ is the label of the $\mathrm{SU}(1,1)$ continous series, and we will give a brief introduction of SU(1,1) and $\mathrm{SL}(2,\mathbb{C})$ representation theory later. As a result, the area spectrum is given by

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma j_f & \text{spacelike triangle} \end{cases}. \tag{2.10}$$

The spin foam vertex amplitude can be expressed in the coherent state representation:

$$A_v(K) = \sum_{j_f} \prod_f \mu(j_f) \int_{\mathbb{SL}(2,\mathbb{C})} \prod_e dg_{ve} \prod_{(e,f)} \int_{S^2} dN_{ef} \langle \Psi_{\rho_f n_f}(N_{ef}) | D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) | \Psi_{\rho_f n_f}(N_{e'f}) \rangle. \tag{2.11}$$

Here N is the unit vector in a sphere or hyperboloid which labels the coherent states $|\Psi_{\rho n}\rangle$ of SL(2, $\mathbb{C}$) in the unitary irrep $\mathcal{H}_{(\rho,n)}$. By SU(1,1) the decomposition of the SL(2, $\mathbb{C}$) unitary irrep, the SL(2, $\mathbb{C}$) irrep is isomorphic to a direct sum of irreps of SU(1,1). The area of timelike triangles is related to SU(1,1) spin s and the Immirzi parameter γ by $A_f = \gamma\sqrt{s^2 + 1/4}$, which is consistent with the spectrum from a canonical approach [7,26]. However, the solution of the quantum simplicity constraint (2.3) on timelike triangles that induced a Y map where the physical Hilbert space $\mathcal{H} \in \mathcal{H}_{(\rho,n)}$ is isomorphic to the continuous series of SU(1,1) with spin s fixed by (2.9). As a result, the area spectrum is now given by

$$A_f = \gamma\sqrt{s^2 + 1/4} = \frac{n_f}{2} \tag{2.12}$$

which is quantized.

In the following, we first give a brief introduction of the SU(1,1) and SL(2, $\mathbb{C}$) representation theory. Then we write the SL(2, $\mathbb{C}$) states explicitly using continuous SU(1, 1) coherent states in terms of spinor variables. Finally, we derive the integral from of the spin foam amplitude on timelike triangles with a spin foam action.

A. Representation theory of the SL(2, $\mathbb{C}$) and SU(1, 1) groups

The SL(2, $\mathbb{C}$) group has six generators J^i and K^i with commutation relation

$$[J^i, J^i] = \epsilon_k^{ij} J^k, [J^i, K^j] = \epsilon_k^{ij} K^k,$$
$$[K^i, K^j] = -\epsilon_k^{ij} J^k. \tag{2.13}$$

The unitary representations of the group are labeled by pairs of numbers ($\rho \in \mathbb{R}, n \in \mathbb{Z}_+$) from the two Casimirs

$$C_1 = 2(\vec{J}^2 - \vec{K}^2) = \frac{1}{2}(n^2 - \rho^2 - 4)$$
$$C_2 = -4\vec{J}\cdot\vec{K} = n\rho. \tag{2.14}$$

The Hilbert space $\mathcal{H}_{(\rho,n)}$ of unitary irrep of SL(2, $\mathbb{C}$) can be represented as a space of homogeneous functions $F: \mathbb{C}^2\backslash\{0\} \to \mathbb{C}$ with the homogeneity property

$$F(\beta z_1, \beta z_2) = \beta^{i\rho/2+n/2-1}\beta^{*i\rho/2-n/2-1} F(z_1, z_2). \tag{2.15}$$

The inner product in $\mathcal{H}_{(\rho,n)}$ is given by

$$\langle F_1|F_2\rangle = \int_{\mathbb{CP}_1} \pi((F_1)^* F_2 \omega) \tag{2.16}$$

where $\pi: \mathbb{C}^2\backslash\{0\} \to \mathbb{CP}_1$. ω is the SL(2, $\mathbb{C}$) invariant 2-form defined by

$$\omega = \frac{\mathrm{i}}{2}(z_2 \mathrm{d}z_1 - z_1 \mathrm{d}z_2) \wedge (\bar{z}_2 \mathrm{d}\bar{z}_1 - \bar{z}_1 \mathrm{d}\bar{z}_2). \tag{2.17}$$

SU(1,1) group is a subgroup of SL(2, $\mathbb{C}$) with generators $\vec{F} = (J^3, K^1, K^2)$. $\vec{F}$ and $\vec{G} = \mathrm{i}\vec{F} = (K^3, -J^1, -J^2)$ transform as Minkowski vectors under SU(1,1). The Casimir reads $Q = (J^3)^2 - (K^1)^2 - (K_2)^2$. The unitary representation of the SU(1,1) group is usually built from the eigenstates of J^3 which are labeled by j, m:

$$\langle jm|jm'\rangle = \delta_{mm'} \tag{2.18}$$

where m is the eigenvalue of J^3 and j is related to the eigenvalues of the Casimir Q.

The unitary irrep of SU(1,1) contains two series: the discrete series and continuous series. For the discrete series, one has

$$Q|jm\rangle = j(j+1)|jm\rangle, \quad \text{with} \quad j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots. \tag{2.19}$$

The eigenvalue m of J^3 takes the values

$$m = -j, -j+1, -j+2\ldots. \quad \text{or} \quad m = j, j-1, j-2\ldots. \tag{2.20}$$

The Hilbert spaces of spin j are denoted by $\mathcal{D}_j^{\pm}$ with $m \gtrless 0$. For the continuous series, Q takes the continuous value

$$Q|jm\rangle = j(j+1)|jm\rangle \tag{2.21}$$

where $j = -1/2 + \mathrm{i}s$ and s is a real number $s \in \mathbb{R}_+$. Thus, in the continuous case, we can use s instead of j to represent the spin. The eigenvalues m takes the values

$$m = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots. \tag{2.22}$$

The irreps of this series are denoted by C_s^{ϵ} where $\epsilon = 0, 1/2$ corresponds to the integer m and half-integer m, respectively.

Instead of $|jm\rangle$, one may also choose the generalized continuous eigenstates $|j\lambda\sigma\rangle$ of K^1 as the basis of the irrep Hilbert space [27]:

$$\langle j\lambda'\sigma'|j\lambda\sigma\rangle = \delta(\lambda - \lambda')\delta_{\sigma\sigma'} \tag{2.23}$$

where $\sigma = 0, 1$ distinguish the twofold degeneracy of the spectrum and λ here is a real number. For continuous series irreps, Casimir Q takes

$$Q|j\lambda\sigma\rangle = j(j+1)|j\lambda\sigma\rangle = -\left(s^2 + \frac{1}{4}\right)|j\lambda\sigma\rangle. \tag{2.24}$$

B. Unitary irreps of SL(2, ℂ) and the decomposition into the SU(1, 1) continuous state

The Hilbert space $\mathcal{H}_{(\rho,n)}$ can be decomposed as a direct sum of irreps of SU(1,1). The decomposition can be derived from the homogeneity property and the Plancherel decomposition of SU(1,1). As shown in [28], the functions F in the $\mathrm{SL}(2,\mathbb{C})$ Hilbert space satisfying (2.15) can be described by pairs of functions $f^\alpha\colon \mathrm{SU}(1,1) \to \mathbb{C}, \alpha = \pm 1$ via

$$F(z_1, z_2) = \sqrt{\pi}(\alpha\langle z, z\rangle)^{i\rho/2-1} f^\alpha(v^\alpha(z_1, z_2)), \tag{2.25}$$

where v^α is the induced SU(1,1) matrix

$$v^\alpha = \begin{cases} \frac{1}{\sqrt{\langle z,z\rangle}}\begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix}, & \alpha = 1 \\ \frac{1}{\sqrt{-\langle z,z\rangle}}\begin{pmatrix} \bar{z}_2 & \bar{z}_1 \\ z_1 & z_2 \end{pmatrix}, & \alpha = -1 \end{cases} \tag{2.26}$$

with $\langle z, z\rangle = z^\dagger \sigma_3 z = \bar{z}_1 z_1 - \bar{z}_2 z_2$ being the SU(1,1) invariant inner product. Here α is a signature

$$\alpha = \begin{cases} 1, & |z_1| > |z_2| \\ -1, & |z_1| < |z_2| \end{cases}. \tag{2.27}$$

Then $\mathcal{H}_{(\rho,n)}$ is isomorphic to the Hilbert space $L^2(\mathrm{SU}(1,1)) \oplus L^2(\mathrm{SU}(1,1))$ with the inner product

$$\langle (f_1^+, f_1^-)|(f_2^+, f_2^-)\rangle = \sum_\alpha \int dv (f_1^\alpha(v))^* f_2^\alpha(v) \tag{2.28}$$

where dv is the SU(1,1) measure.

The function f in SU(1,1) continuous series representations with a continuous basis reads

$$f_{j\lambda}^\alpha(z) = \begin{cases} \sqrt{2j+1}(D^j_{n/2,\lambda}(v(z)), 0), & \alpha = 1 \\ \sqrt{2j+1}(0, D^j_{-n/2,\lambda}(v(z))), & \alpha = -1 \end{cases}. \tag{2.29}$$

Notice that here we assume $s \neq 0$. $D^j_{m\lambda}$ is the Wigner matrix with mixed bases (2.18) and (2.23)

$$D^j_{m\lambda\sigma}(v) = \langle j, m|v(z)|j, \lambda, \sigma\rangle. \tag{2.30}$$

Recall the quantum simplicity constraint (2.9),

$$\rho = -n/\gamma, \qquad s = \frac{1}{2}\sqrt{n^2/\gamma^2 - 1}. \tag{2.31}$$

Asymptotically, when $s \gg 1$, we have

$$\rho \sim -2s \sim -\frac{n}{\gamma}. \tag{2.32}$$

Since n is discrete, s and ρ are also discrete. Using the representation matrix of the continuous series of SU(1,1), and some transformations of the hypergeometric function and asymptotic analysis, we prove that when $n \gg 1$ and $\lambda = -s$ (the detailed derivation is shown in Appendix A),

$$\begin{aligned} D^j_{\frac{n}{2},-s}(v) = {} & \frac{1}{\sqrt{s|\gamma + \Im(\bar{v}_1 v_2)|}} \\ & \times \left(\tilde{T}^j_{+\sigma}\left(\frac{v_1 - v_2}{\sqrt{2}}\right)^{\frac{n}{2}-\mathrm{i}s}\left(\overline{\frac{v_1 - v_2}{\sqrt{2}}}\right)^{-\frac{n}{2}-\mathrm{i}s}\right. \\ & \left. -\tilde{T}^j_{-\sigma}\left(\frac{v_1 + v_2}{\sqrt{2}}\right)^{\frac{n}{2}+\mathrm{i}s}\left(\overline{\frac{v_1 + v_2}{\sqrt{2}}}\right)^{-\frac{n}{2}+\mathrm{i}s}\right) \end{aligned} \tag{2.33}$$

where $\sqrt{2}\tilde{T}^j_{\pm\sigma} = \sqrt{2}S^j_{n/2,-s,\sigma}/T^j_{\pm\sigma}$ are some phases: $\tilde{T}_\pm \overline{\tilde{T}_\pm} = 1/2$.[1] The detailed definitions of $S^j_{n/2,-s,\sigma}$ and $T^j_{\pm\sigma}$ are given in (A8) and (A46).

The $m = -n/2$ case in (2.29) can be obtained by the relation

$$D^{\sigma j}_{-m,\lambda}(v) = -(-1)^\sigma \mathrm{e}^{-\mathrm{i}\pi m} D^{\sigma j}_{m,\lambda}(\bar{v}). \tag{2.34}$$

When $\alpha = 1$, we would like to write elements of $v^\alpha \in \mathrm{SU}(1,1)$ introduced in (2.26) as

$$\frac{v_1 - v_2}{\sqrt{2}} = \frac{\langle \bar{z}, l_0^+\rangle}{\sqrt{\langle z, z\rangle}}, \qquad \frac{v_1 + v_2}{\sqrt{2}} = \frac{\langle \bar{z}, l_0^-\rangle}{\sqrt{\langle z, z\rangle}}, \tag{2.35}$$

where

$$l_0^\pm = \frac{1}{\sqrt{2}}(n_1 \pm n_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{2.36}$$

Notice that $\langle l_0^+, l_0^+\rangle = \langle l_0^-, l_0^-\rangle = 0$, $\langle l_0^-, l_0^+\rangle = 1$; thus, they form a null basis in $\mathbb{C}^2$. Similarly, for $\alpha = -1$, we have

$$\frac{v_1 - v_2}{\sqrt{2}} = -\frac{\langle l_0^+, \bar{z}\rangle}{\sqrt{-\langle z, z\rangle}}, \qquad \frac{v_1 + v_2}{\sqrt{2}} = \frac{\langle l_0^-, \bar{z}\rangle}{\sqrt{-\langle z, z\rangle}}. \tag{2.37}$$

[1] Here we ignore the regulator in (A43) for the zero points of $|\gamma + \Im(\bar{v}_1 v_2)|$ since it will appear naturally as the integration contribution from this $1/2$ singularity in the inner product. One can check Appendix A for details.

With this notation, we finally obtain

$$F^{(\rho,n)}_{-s,\sigma,\alpha}(z)=\frac{\sqrt{\pi}\alpha^{n/2+\sigma+1}}{\sqrt{s}\sqrt{\alpha\langle z,z\rangle}\sqrt{|\alpha(\gamma-\mathrm{i})\langle z,z\rangle+2\mathrm{i}\alpha\langle\bar z,l_0^-\rangle\langle l_0^+,\bar z\rangle|}}\times\left(\tilde T^j_{+\sigma}(\alpha\langle z,z\rangle)^{i\rho/2+\mathrm{i}s}(\langle l_0^+,\bar z\rangle\langle\bar z,l_0^+\rangle)^{-\mathrm{i}s}\left(\frac{\langle\bar z,l_0^+\rangle}{\langle l_0^+,\bar z\rangle}\right)^{\frac{n}{2}}-\tilde T^j_{-\sigma}(\alpha\langle z,z\rangle)^{i\rho/2-\mathrm{i}s}(\langle l_0^-,\bar z\rangle\langle\bar z,l_0^-\rangle)^{\mathrm{i}s}\left(\frac{\langle\bar z,l_0^-\rangle}{\langle l_0^-,\bar z\rangle}\right)^{\frac{n}{2}}\right). \tag{2.38}$$

One can check the homogeneity property (2.15):

$$F(\lambda z)=\lambda^{m+\mathrm{i}\rho/2-1}\bar\lambda^{-m+\mathrm{i}\rho/2-1}F(z). \tag{2.39}$$

The coherent state is built from the reference state $\lambda=-s$, and we choose $\sigma=1$, according to [8]

$$\Psi^{(\rho,n)}_{\tilde g,\alpha}(z)=D^{(\rho,n)}(\tilde g)F^{(\rho,n)}_{-s,1,\alpha}(z)=\frac{\sqrt{\mathrm{i}\pi}\tilde S^j_{m,-s,\sigma}\alpha^{-2\mathrm{i}s+m}}{\sqrt{|\langle z,z\rangle|}\sqrt{|(\gamma-\mathrm{i})\langle z,z\rangle+2\mathrm{i}\langle\bar z,l^-\rangle\langle l^+,\bar z\rangle|}}\times\left(\tilde T^j_{+1}\langle z,z\rangle^{i\rho/2+\mathrm{i}s}(\langle l^+,\bar z\rangle\langle\bar z,l^+\rangle)^{-\mathrm{i}s}\left(\frac{\langle\bar z,l^+\rangle}{\langle l^+,\bar z\rangle}\right)^{\frac{n}{2}}-\tilde T^j_{-1}\langle z,z\rangle^{i\rho/2-\mathrm{i}s}(\langle l^-,\bar z\rangle\langle\bar z,l^-\rangle)^{\mathrm{i}s}\left(\frac{\langle\bar z,l^-\rangle}{\langle l^-,\bar z\rangle}\right)^{\frac{n}{2}}\right) \tag{2.40}$$

where $\tilde g\in SU(1,1)$, and $l^\pm=\tilde g^{-1\dagger}l_0^\pm$ is defined though

$$\langle l_0^\pm,\overline{\tilde g^t z}\rangle=\langle\tilde g^{-1\dagger}l_0^\pm,\bar z\rangle=\langle l^\pm,\bar z\rangle. \tag{2.41}$$

C. Spin foam amplitude

Now we can write down explicitly the inner product between the coherent states appearing in the amplitude (2.11) by inserting (2.40) and using (2.16):

$$\begin{aligned}\langle\Psi^{(\rho_f,n_f)}_{\tilde g_{e'f}\delta}|D^{(\rho_f,n_f)}(g_{ve'}g_{ev})|\Psi^{(\rho_f,n_f)}_{\tilde g_{ef}\delta}\rangle&=\sum_\alpha\int_{CP_1}\omega_{z_{vf}}\Psi^{(\rho_f,n_f)}_{\tilde g_{e'f}\delta\alpha}(g^t_{ve'}z_{vf})\overline{\Psi^{(\rho_f,n_f)}_{\tilde g_{ef}\delta\alpha}}(g^t_{ev}z_{vf})\\&=\int_{CP_1/\langle Z,Z\rangle=0}\frac{\omega_{z_{vf}}}{h_{vef}h_{ve'f}}(N_{f+}\mathrm{e}^{S_{vf+}}+N_{f-}\mathrm{e}^{S_{vf-}}+N_{fx+}\mathrm{e}^{S_{vfx+}}+N_{fx-}\mathrm{e}^{S_{vfx-}})\end{aligned} \tag{2.42}$$

where N are some normalization factors, and ω is the $\mathrm{SL}(2,\mathbb{C})$ invariant measure defined in (2.17). The exponents read

$$S_{vf\pm}=S_{ve'f\pm}-S_{vef\pm},\qquad S_{vfx\pm}=S_{ve'f\pm}-S_{vef\mp} \tag{2.43}$$

with

$$S_{vef\pm}=s_f\left[\gamma\ln\frac{\langle Z_{vef},l^\pm_{ef}\rangle}{\langle l^\pm_{ef},Z_{vef}\rangle}\mp\mathrm{i}\ln\langle Z_{vef},l^\pm_{ef}\rangle\langle l^\pm_{ef},Z_{vef}\rangle+\mathrm{i}(-1\pm1)\ln\langle Z_{vef},Z_{vef}\rangle\right] \tag{2.44}$$

where $Z_{vef}=g^\dagger_{ve}\bar z_{vf}$. $l^\pm_{ef}$ here is defined as $l^\pm=v(N_{ef})^{-1\dagger}l_0^\pm$ with $l_0^\pm$ defined in (2.36), and $v(N_{ef})\in\mathrm{SU}(1,1)$ which encodes the unit normal. $\langle Z_{ve'f},Z_{ve'f}\rangle$ has the same sign as $\langle Z_{vef},Z_{vef}\rangle$. The integrand is invariant under the following gauge transformations:

$$g_{ve}\to g_vg_{ve},\qquad z_{vf}\to\lambda_{vf}(g_v^T)^{-1}z_{vf} \tag{2.45}$$

$$g_{ve}\to s_{ve}g_{ve},\qquad s_{ve}=\pm1 \tag{2.46}$$

$$g_{ve}\to g_{ve}v_e,\qquad l^\pm_{ef}\to v_el^\pm_{ef}, \tag{2.47}$$

where $g_v\in\mathrm{SL}(2,\mathbb{C})$, $v_e\in\mathrm{SU}(1,1)$, and $\lambda_{vf}\in\mathbb{C}\backslash\{0\}$.

It is worth pointing out that both $S_{vf\pm}$ and $S_{vfx\pm}$ are purely imaginary, and they are all proportional to s_f which will be uniform scaled later to derive the asymptotics. The real valued function h is given by

$$h_{vef} = |\langle Z_{vef}, Z_{vef}\rangle| \sqrt{\left|\gamma - \mathrm{i} + \frac{2\langle l^-_{ef}, Z_{vef}\rangle\langle Z_{vef}, l^+_{ef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle}\right|}. \tag{2.48}$$

h_{vef} can be 0 when we integrate over z on $\mathbb{CP}_1$ and $\mathrm{SL}(2,\mathbb{C})$ group elements g in (2.11), and the zeros of h are exactly the points where we define the principle value, i.e., at $\langle Z, Z\rangle = 0$. However, as shown in Appendix B, the singularities due to h are of half order; thus, the final integral remains finite at these points.

III. ANALYSIS OF CRITICAL POINTS

As we show above, the actions $S_{vf\pm}$ and $S_{vfx\pm}$ are purely imaginary, and they are proportional to s_f. Thus, we can use stationary phase approximation to evaluate the amplitude in the semiclassical limit where s is uniformly scaled by a factor $\Lambda \to \infty$. Note that the denominator h defined by (2.48) in (2.42) contains a 1/2 order singular point at $\langle Z, Z\rangle = 0$, as shown in Appendix B. Then the integral is of the following type:

$$I = \int \mathrm{d}x \frac{1}{\sqrt{x - x_0}} g(x) \mathrm{e}^{\Lambda S(x)}. \tag{3.1}$$

Here g is an analytic function which does not scale with Λ. There are two different asymptotic equations for such a type of integral according to the critical point x_c located exactly at the branch point x_0 or away from it. According to [29], if x_c is located exactly at x_0, the leading order contribution will locate at the critical points (which is also the branch points), and the asymptotic expansion is given by

$$I \sim g(x_c) \frac{\pi \mathrm{e}^{\mathrm{i}\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda|\det H(x_c)|}\right)^{1/4} \mathrm{e}^{\Lambda S(x_c)} \tag{3.2}$$

where $H(x_c)$ is the Hessian matrix at x_c, and $\mu = \mathrm{sgn}\det H(x_c)$.

As we explain in the following sections, the critical points of Eq. (2.42) are always located at the branch points, when every tetrahedron containing the timelike triangle f also contains at least one spacelike triangle. It is quite generic to have every tetrahedron contain both timelike and spacelike triangles in a simplicial geometry. In addition, in case that we consider tetrahedra with all triangles timelike, for a single vertex amplitude, the critical point is again located at the branch points, when the boundary data give the closed geometrical boundary of a 4-simplex (i.e., the tetrahedra at the boundary are glued with shape matching). We do not consider the possibility other than (3.2).

A. Equation of motion

Since both $S_{vf\pm}$ and $S_{vfx\pm}$ are purely imaginary, their critical points or, namely, critical configurations, are solutions of equations of motion. The equations of motion are given by the variations of S with respect to spinors z, SU(1,1) group elements v, and $\mathrm{SL}(2,\mathbb{C})$ group elements g.

Before calculating the variation, we would like to introduce a decomposition of spinor Z. We first introduce following lemmas:

Lemma III.1: Given a specific l^+ satisfying $\langle l^+, l^+\rangle = 0$, there exists $\tilde{l}^-$, s.t. $\langle l^+, \tilde{l}^-\rangle = 1, \langle \tilde{l}^-, \tilde{l}^-\rangle = 0$. For two elements $\tilde{l}^-_1$ and $\tilde{l}^-_2$ satisfying the condition, they are related by

$$\tilde{l}^-_1 = \tilde{l}^-_2 + \mathrm{i}\eta l^+, \qquad \eta \in R. \tag{3.3}$$

This is easy to proof since $\langle \tilde{l}^- + \mathrm{i}\eta l^+, \tilde{l}^- + \mathrm{i}\eta l^+\rangle = \eta^2\langle l^+, l^+\rangle + \langle \tilde{l}^-_2, \tilde{l}^-_2\rangle - \mathrm{i}\eta\langle l^+, \tilde{l}^-\rangle + \mathrm{i}\eta\langle \tilde{l}^-, l^+\rangle$ and $\langle l^+, \tilde{l}^- + \mathrm{i}\eta l^+\rangle = \langle l^+, \tilde{l}^-\rangle + \mathrm{i}\eta\langle l^+, l^+\rangle$.

Lemma III.2: For a given l^+ and $\tilde{l}^-$ defined by Lemma III.1, l^+ and $\tilde{l}^-$ form a null basis in two-dimensional spinor space.

This lemma is proved by using the fact that given l^+ and $\tilde{l}^-$, there exists a SU(1,1) element $\tilde{g}$, such that $l^+ = \tilde{g} l^+_0$ and $\tilde{l}^- = \tilde{g} l^-_0$, and the fact that l^+_0 and l^-_0 forms a null basis.

With Lemma III.2, for a given l^+ or l^-, we have

Theorem III.3: For a given l^+ and $\tilde{l}^-$ defined by Lemma III.1, spinor Z_{vef} always can be decomposed as

$$Z_{vef} = \zeta_{vef}(\tilde{l}^{\mp}_{ef} + \alpha_{vef} l^{\pm}_{ef}) \tag{3.4}$$

where $\zeta_{vef} \in \mathbb{C}$ and $\alpha_{vef} \in \mathbb{C}$.

At the vertex v, from the action S_{vef+} (S_{vef-}), we only have l^+ (l^-) entering the action; thus, we can choose arbitrarily $\tilde{l}^{\mp}_{vef}$ to form a basis. By Lemma III.1, we can always write $\tilde{l}'^{\mp}_{vef} = \tilde{l}'^{\mp}_{vef} + \mathrm{i}\Im(\alpha_{vef}) l^{\pm}_{ef}$ s.t.,

$$Z_{vef} = \zeta_{vef}(\tilde{l}^{\mp}_{vef} + \Re(\alpha_{vef}) l^{\pm}_{ef}). \tag{3.5}$$

$\Im(\alpha)$ is basis dependent. It is easy to check that if we replace Z inside the action (2.43) by the decomposition (3.4), the action is independent of $\Im(\alpha)$, which means that $\Im(\alpha)$ is a gauge freedom.

We will drop the tilde on $\tilde{l}$ in the following. One should keep in mind that we have the freedom to choose the l^- (l^+) such that for some vertices v, $\Im(\alpha_{vef}) = 0$.

From the decomposition of Z_{vef}, there is naturally a constraint. By the fact of $Z_{vef} = g^\dagger_{ve}\bar{z}_{vf}$, we have

$$\bar{z}_{vf} = g^{-1\dagger}_{ve} Z_{vef} = g^{-1\dagger}_{ve'} Z_{ve'f}. \tag{3.6}$$

In terms of the decomposition of Z_{vef}

$$g_{ve}^{-1\dagger}(l^{\pm}_{ef}+\alpha_{vef}l^{\mp}_{ef})=\frac{\zeta_{ve'f}}{\zeta_{vef}}g_{ve'}^{-1\dagger}(l^{\pm}_{e'f}+\alpha_{ve'f}l^{\mp}_{vef}). \tag{3.7}$$

This can be written as

$$g_{ve}J(l^{\pm}_{ef}+\alpha_{vef}l^{\mp}_{ef})=\frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}}g_{ve'}J(l^{\pm}_{e'f}+\alpha_{ve'f}l^{\mp}_{ef}) \tag{3.8}$$

where we used the antilinear map J:

$$J(a,b)^T=(-\bar{b},\bar{a}),\qquad JgJ^{-1}=-JgJ=g^{-1\dagger}. \tag{3.9}$$

1. Variation with respect to z

From the definition of the SU(1,1) inner product, for arbitrary spinor u we have

$$\delta_{\bar{z}}\langle u,Z\rangle=\delta_{\bar{z}}(u^\dagger\eta g^\dagger\bar{z})=(g\eta u)^\dagger\delta\bar{z},$$
$$\delta_z\langle Z,u\rangle=\delta_z((g^\dagger\bar{z})^\dagger\eta u)=(\delta z)^T(g\eta u). \tag{3.10}$$

Then it is straightforward to see the variation of S_{vef} leading to

$$\delta_{\bar{z}}S_{vef\pm}=\left(\frac{n_f}{2}\pm\mathrm{i}s_f\right)\frac{(g_{ve}\eta l^{\pm}_{ef})^\dagger}{\langle l^{\pm}_{ef},Z_{vef}\rangle}-\mathrm{i}(\rho_f\pm s_f)\frac{(g_{ve}\eta Z_{vef})^\dagger}{\langle Z_{vef},Z_{vef}\rangle} \tag{3.11}$$

and

$$\delta_z S=-\overline{\delta_{\bar{z}}S} \tag{3.12}$$

which comes from the fact that S is purely imaginary. With the definition of S_{vf} in (2.43), after inserting the decomposition, we obtain the following equations:

$$\delta S_{vf+}=(\gamma-\mathrm{i})s_f\left(\frac{g_{ve}\eta l^{+}_{ef}}{\bar{\zeta}_{vef}}-\frac{g_{ve'}\eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}}\right)=0\quad\text{with}\quad Z=\zeta(l^-+\alpha l^+) \tag{3.13}$$

$$\delta S_{vf-}=-\mathrm{i}s_f\left(\frac{g_{ve}\eta n_{vef}}{\Re(\alpha_{vef})\bar{\zeta}_{vef}}-\frac{g_{ve'}\eta n_{ve'f}}{\Re(\alpha_{ve'f})\bar{\zeta}_{ve'f}}\right)=0\quad\text{with}\quad Z=\zeta(l^++\alpha l^-) \tag{3.14}$$

$$\delta S_{vfx+}=-(\gamma-\mathrm{i})s_f\frac{g_{ve'}\eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}}-\mathrm{i}s_f\frac{g_{ve}\eta n_{vef}}{\Re(\alpha_{vef})\bar{\zeta}_{vef}}=0\quad\text{with}\quad Z_{e'}=\zeta(l^-+\alpha l^+)\ \&\ Z_e=\zeta(l^++\alpha l^-) \tag{3.15}$$

$$\delta S_{vfx-}=(\gamma-\mathrm{i})s_f\frac{g_{ve}\eta l^{+}_{vef}}{\bar{\zeta}_{vef}}+\mathrm{i}s_f\frac{g_{ve'}\eta n_{ve'f}}{\Re(\alpha_{ve'f})\bar{\zeta}_{ve'f}}=0\quad\text{with}\quad Z_e=\zeta(l^-+\alpha l^+)\ \&\ Z_{e'}=\zeta(l^++\alpha l^-) \tag{3.16}$$

where

$$n_{vef}\coloneqq l^{+}_{ef}+\mathrm{i}(\gamma\Re(\alpha_{vef})+\Im(\alpha_{vef}))l^{-}_{ef}. \tag{3.17}$$

Note that n_{vef} here satisfies Lemma III.2 and can form a basis with l^-_{ef} given in S_{vef-}.

2. Variation with respect to SU(1,1) group elements v_{ef}

Since $l^{\pm}=v^{-1\dagger}l^{\pm}_0$ with $v\in\mathrm{SU}(1,1)$, the variation with respect to $l^{\pm}$ is the variation with respect to the SU(1,1) group element v. If we consider a small perturbation of v which is given by $v'=v\mathrm{e}^{-\epsilon_iF^i}$, where F^i are generators of SU(1,1) group, we have $v'^{-1}=\mathrm{e}^{\epsilon_iF^i}v^{-1}$. The variation is then given by

$$\delta v^{-1}=\epsilon_iF^iv^{-1},\qquad \delta v^{-1\dagger}=\epsilon_iv^{-1\dagger}(F^i)^\dagger. \tag{3.18}$$

Thus, for arbitrary spinor u, we have

$$\delta\langle u,m\rangle=\delta\langle u,v^{-1\dagger}m_0\rangle=\epsilon^i\langle u,v^{-1\dagger}F_i^\dagger m_0\rangle$$
$$\delta\langle m,u\rangle=\delta\langle v^{-1\dagger}m_0,u\rangle=\epsilon^i\langle v^{-1\dagger}F_i^\dagger m_0,u\rangle. \tag{3.19}$$

When $S_{ef}=S_{vef\pm}-S_{v'ef\pm}$, the variation reads

$$\delta S=\epsilon^i\left(\frac{n_f}{2}\mp\mathrm{i}s_f\right)\left(\frac{\langle Z_{v'ef},v_{ef}^{-1\dagger}F_i^\dagger l_0^{\pm}\rangle}{\langle Z_{v'ef},l^{\pm}_{ef}\rangle}-\frac{\langle Z_{vef},v_{ef}^{-1\dagger}F_i^\dagger l_0^{\pm}\rangle}{\langle Z_{vef},l^{\pm}_{ef}\rangle}\right)$$
$$+\epsilon^i\left(\frac{n_f}{2}\pm\mathrm{i}s_f\right)\left(\frac{\langle v_{ef}^{-1\dagger}F_i^\dagger l_0^{\pm},Z_{vef}\rangle}{\langle l^{\pm}_{ef},Z_{vef}\rangle}-\frac{\langle v_{ef}^{-1\dagger}F_i^\dagger l_0^{\pm},Z_{v'ef}\rangle}{\langle l^{\pm}_{ef},Z_{v'ef}\rangle}\right). \tag{3.20}$$

While $S_{ef} = S_{vef\pm} - S_{v'ef\mp}$, we have

$$\delta S = \epsilon^i \left(\frac{n_f}{2}\right)\left(\frac{\langle Z_{vef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle} - \frac{\langle Z_{v'ef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} + \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{vef}\rangle}{\langle l_{ef}^\pm, Z_{vef}\rangle} - \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp, Z_{v'ef}\rangle}{\langle l_{ef}^\mp, Z_{v'ef}\rangle}\right)$$
$$+ \epsilon^i s_f \left(\frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{vef}\rangle}{\langle l_{ef}^\pm, Z_{vef}\rangle} + \frac{\langle v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm, Z_{v'ef}\rangle}{\langle l_{ef}^\pm, Z_{v'ef}\rangle} + \frac{\langle Z_{v'ef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\mp\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} + \frac{\langle Z_{vef}, v_{ef}^{-1\dagger} F_i^\dagger l_0^\pm\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right). \tag{3.21}$$

Since $F^i = 1/2(\mathrm{i}\sigma_3, \sigma_1, \sigma_2,)$ are SU(1,1) generators, we have

$$(F^0)^\dagger l_0^\pm = \frac{\mathrm{i}}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ \pm1\end{pmatrix} = \frac{\mathrm{i}}{2} l_0^\mp \tag{3.22}$$

$$(F^1)^\dagger l_0^\pm = \frac{1}{2\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \pm1\end{pmatrix} = \pm\frac{1}{2} l_0^\pm \tag{3.23}$$

$$(F^2)^\dagger l_0^\pm = \frac{1}{2\sqrt{2}}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}1\\ \pm1\end{pmatrix} = \mp\frac{1}{2} l_0^\mp. \tag{3.24}$$

Then in the first case we are only left with one equation, which reads

$$0 = \left(\frac{n_f}{2} \mp \mathrm{i}s_f\right)\left(\frac{\langle Z_{v'ef}, \mathrm{i}l_{ef}^\mp\rangle}{\langle Z_{v'ef}, l_{ef}^\pm\rangle} - \frac{\langle Z_{vef}, \mathrm{i}l_{ef}^\mp\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right) + \left(\frac{n_f}{2} \pm \mathrm{i}s_f\right)\left(\frac{\langle \mathrm{i}l_{ef}^\mp, Z_{vef}\rangle}{\langle l_{ef}^\pm, Z_{vef}\rangle} - \frac{\langle \mathrm{i}l_{ef}^\mp, Z_{v'ef}\rangle}{\langle l_{ef}^\pm, Z_{v'ef}\rangle}\right). \tag{3.25}$$

After inserting the decomposition $Z = \zeta(l^\mp + \alpha l^\pm)$, correspondingly, we get

$$0 = \left(\frac{n_f}{2} \mp \mathrm{i}s_f\right)(\bar\alpha_{v'ef} - \bar\alpha_{vef}) + \left(\frac{n_f}{2} \pm \mathrm{i}s_f\right)(\alpha_{v'ef} - \alpha_{vef})$$
$$= 2\mathrm{i}s_f\gamma\Re(\alpha_{v'ef} - \alpha_{vef}) \pm 2\mathrm{i}s_f\Im(\alpha_{vef} - \alpha_{v'ef}). \tag{3.26}$$

The solution reads

$$\gamma\Re(\alpha_{vef}) \mp \Im(\alpha_{vef}) = \gamma\Re(\alpha_{v'ef}) \mp \Im(\alpha_{v'ef}). \tag{3.27}$$

Here $\Im(\alpha)$ is the decomposition of Z with respect to $l_{ef}^\mp$ specified by v_{ef}. Note that in this case, we only have $l_{ef}^+(l_{ef}^-)$ in the action; thus, there is an ambiguity of v_{ef}. However, changing v_{ef} corresponds to adding the same constant to both $\Im(\alpha_v)$ and $\Im(\alpha'_v)$; thus, the relation is kept unchanged. After absorbing $\Im(\alpha)$ into $\tilde l$ by a redefinition, the equation actually tells us that

$$\tilde l_{vef}^\mp - \tilde l_{v'ef}^\mp = \pm\gamma(\Re(\alpha_{vef}) - \Re(\alpha_{v'ef}))l_{ef}^\pm \tag{3.28}$$

which fixes the transformation of $\tilde l_{vef}$ between vertices and removes the ambiguity between different vertices v in the bulk. With this redefinition, it is easy to see that n_{vef} defined in (3.17) satisfies $n_{vef} = n_{ve'f}$; thus, we ignore the v variable and define

$$n_{ef} \coloneqq n_{vef} = n_{v'ef}. \tag{3.29}$$

In the mixing case there will be two different equations for F_2 and F_3, which lead to

$$0 = \frac{n_f}{2}\left(\Re\frac{\langle Z_{v'ef}, l^\pm\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} - \Re\frac{\langle Z_{vef}, l^\mp\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right) \pm \mathrm{i}s_f\left(\mathrm{i}\Im\frac{\langle Z_{v'ef}, l^\pm\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} + \mathrm{i}\Im\frac{\langle Z_{vef}, l^\mp\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right) \tag{3.30}$$

$$0 = \frac{n_f}{2}\left(\Re\frac{\langle Z_{v'ef}, l^\pm\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} + \Re\frac{\langle Z_{vef}, l^\mp\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right) \pm \mathrm{i}s_f\left(\mathrm{i}\Im\frac{\langle Z_{v'ef}, l^\pm\rangle}{\langle Z_{v'ef}, l_{ef}^\mp\rangle} - \mathrm{i}\Im\frac{\langle Z_{vef}, l^\mp\rangle}{\langle Z_{vef}, l_{ef}^\pm\rangle}\right). \tag{3.31}$$

The equations give the solutions

$$\gamma\Re(\alpha_{v'ef}) \pm \Im(\alpha_{v'ef}) = 0, \quad \text{with} \quad Z_{v'ef} = \zeta_{v'ef}(l_{ef}^\pm + \alpha_{v'ef} l_{ef}^\mp)$$
$$\gamma\Re(\alpha_{vef}) \mp \Im(\alpha_{vef}) = 0, \quad \text{with} \quad Z_{vef} = \zeta_{vef}(l_{ef}^\mp + \alpha_{vef} l_{ef}^\pm).$$

Here l^+ and l^- completely fix the group element v. α corresponds to the decomposition of Z with these l^+ and l^-. The n_{vef} in this case is simply $n_{vef} = l^+_{ef}$.

3. Variation with respect to $\mathbf{SL}(2,\mathbb{C})$ elements g

With the small perturbation of g, which is given by $g' = g\mathrm{e}^L$, the variation ofthe $\mathrm{SL}(2,\mathbb{C})$ group element g is given by

$$\delta g = gL, \qquad \delta g^\dagger = -L^\dagger g^\dagger \tag{3.32}$$

where L is a linear combination of $\mathrm{SL}(2,\mathbb{C})$ generators, $L = \epsilon_i F^i + \tilde{\epsilon}_i G^i = (\epsilon_i + \mathrm{i}\tilde{\epsilon}_i)F^i$. Here the F's are the SU(1,1) lie algebra generators defined as above, and we use the fact that in spin-1/2 representation $G = \mathrm{i}F$. Then for arbitrary u, we have

$$\begin{aligned}&\delta\langle u, Z\rangle = \delta\langle u, g^\dagger \bar{z}\rangle = \langle u, L^\dagger g^\dagger \bar{z}\rangle = \langle u, L^\dagger Z\rangle\\ &\delta\langle Z, u\rangle = \delta\langle g^\dagger \bar{z}, u\rangle = (L^\dagger g^\dagger \bar{z})^\dagger \eta u = \langle L^\dagger Z, u\rangle.\end{aligned} \tag{3.33}$$

The variation leads to

$$\begin{aligned}\delta S = \sum_f \epsilon_{ef}(v)\Bigg(&-\left(\frac{n_f}{2} \mp \mathrm{i}s_f\right)\left(\frac{\langle L^\dagger Z_{vef}, l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle}\right) + \left(\frac{n_f}{2} \pm \mathrm{i}s_f\right)\left(\frac{\langle l^\pm_{ef}, L^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle}\right)\\ &- \mathrm{i}(\rho_f \pm s_f)\left(\frac{\langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle}\right)\Bigg)\end{aligned} \tag{3.34}$$

where $\epsilon_{ef}(v) = \pm 1$ is determined according to the face orientation that is consistent with the edge e or the opposite (up to a global sign). We have

$$\epsilon_{ef}(v) = -\epsilon_{e'f}(v), \qquad \epsilon_{ef}(v) = -\epsilon_{ef}(v'). \tag{3.35}$$

We write $\epsilon_{ef}(v) = +1$ in the following for simplicity, and recover general ϵ at the end of the derivation.

From the property of the SU(1,1) generator,

$$\eta F \eta = -F^\dagger \tag{3.36}$$

we have

$$\langle F^\dagger Z, u\rangle = -Z^\dagger F \eta u = -Z^\dagger \eta F^\dagger u = -\langle Z, F^\dagger u\rangle. \tag{3.37}$$

Then (3.34) can be written as

$$\begin{aligned}&\sum_f \left(\frac{n_f}{2} \mp \mathrm{i}s_f\right)\left(\frac{\langle Z_{vef}, F^\dagger l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle}\right)\\ &\quad + \left(\frac{n_f}{2} \pm \mathrm{i}s_f\right)\left(\frac{\langle l^\pm_{ef}, F^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle}\right) = 0\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\sum_f -\left(\frac{n_f}{2} \mp \mathrm{i}s_f\right)\frac{\langle Z_{vef}, F^\dagger l^\pm_{ef}\rangle}{\langle Z_{vef}, l^\pm_{ef}\rangle} + \left(\frac{n_f}{2} \pm \mathrm{i}s_f\right)\frac{\langle l^\pm_{ef}, F^\dagger Z_{vef}\rangle}{\langle l^\pm_{ef}, Z_{vef}\rangle}\\ &\quad - 2\mathrm{i}(\rho_f \pm s_f)\frac{\langle Z_{vef}, F^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} = 0.\end{aligned} \tag{3.39}$$

After inserting the decomposition of Z and the solution of the simplicity constraint, we have the following equations: For both $S_\pm$, (3.38) becomes

$$\begin{aligned}0 &= \delta_F S_\pm\\ &= \mp 2\mathrm{i}\sum_f s_f \langle l^\mp_{ef} \mp \mathrm{i}(\gamma \mathrm{Re}(\alpha_{vef}) \mp \Im(\alpha_{vef}))l^\pm_{ef}, F^\dagger l^\pm_{ef}\rangle.\end{aligned} \tag{3.40}$$

(3.39) will leads to different equations for different actions $S_\pm$ due to the appearance of the $\langle Z_{vef}, F^\dagger Z_{vef}\rangle$ term. The variation of S_+ reads

$$\begin{aligned}0 &= \delta_G S_+\\ &= -2\gamma \sum_f s_f \left\langle l^-_{ef} - \mathrm{i}\left(\frac{1}{\gamma}\Re(\alpha_{vef}) - \Im(\alpha_{vef})\right) l^+_{ef}, F^\dagger l^+_{ef}\right\rangle,\end{aligned} \tag{3.41}$$

while the variation of S_- reads

$$\delta_G S_- = 2\mathrm{i}\sum_f s_f \frac{\langle n_{vef}, F^\dagger n_{vef}\rangle}{\mathrm{Re}(\alpha_{vef})} + 2\gamma \sum_f s_f \langle n_{vef}, F^\dagger l^-_{ef}\rangle. \tag{3.42}$$

4. Summary

As a summary, after we introduce the decomposition of Z as (3.4),

$$Z_{vef} = \zeta_{vef}(\tilde{l}^\mp_{ef} + \alpha_{vef} l^\pm_{ef}) \tag{3.43}$$

and a spinor n as (3.17)

$$n_{vef} \coloneqq l^+_{ef} + \mathrm{i}(\gamma \Re(\alpha_{vef}) + \Im(\alpha_{vef}))l^-_{ef} \tag{3.44}$$

the equation of motion is given by the following equations:

(i) parallel transport equations

$$S_{vf+}\colon\ \frac{g_{ve}\eta l^{+}_{ef}}{\bar{\zeta}_{vef}}=\frac{g_{ve'}\eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}},\qquad g^{-1\dagger}_{ve}(l^{-}_{ef}+\alpha_{vef}l^{+}_{ef})=\frac{\zeta_{ve'f}}{\zeta_{vef}}g^{-1\dagger}_{ve'}(l^{-}_{e'f}+\alpha_{ve'f}l^{+}_{vef}) \tag{3.45}$$

$$S_{vf-}\colon\ \frac{g_{ve}\eta n_{vef}}{\Re(\alpha_{vef})\bar{\zeta}_{vef}}=\frac{g_{ve'}\eta n_{ve'f}}{\Re(\alpha_{ve'f})\bar{\zeta}_{ve'f}},\qquad g^{-1\dagger}_{ve}(l^{+}_{ef}+\alpha_{vef}l^{-}_{ef})=\frac{\zeta_{ve'f}}{\zeta_{vef}}g^{-1\dagger}_{ve'}(l^{+}_{e'f}+\alpha_{ve'f}l^{-}_{vef}) \tag{3.46}$$

$$S_{vfx+}\colon\ \frac{g_{ve}\eta n_{vef}}{\Re(\alpha_{vef})\bar{\zeta}_{vef}}=-(1+\mathrm{i}\gamma)\frac{g_{ve'}\eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}},\qquad g^{-1\dagger}_{ve}(l^{+}_{ef}+\alpha_{vef}l^{-}_{ef})=\frac{\zeta_{ve'f}}{\zeta_{vef}}g^{-1\dagger}_{ve'}(l^{-}_{e'f}+\alpha_{ve'f}l^{+}_{vef}) \tag{3.47}$$

$$S_{vfx-}\colon\ -(1+\mathrm{i}\gamma)\frac{g_{ve}\eta l^{+}_{vef}}{\bar{\zeta}_{vef}}=\frac{g_{ve'}\eta n_{ve'f}}{\Re(\alpha_{ve'f})\bar{\zeta}_{ve'f}},\qquad g^{-1\dagger}_{ve}(l^{-}_{ef}+\alpha_{vef}l^{+}_{ef})=\frac{\zeta_{ve'f}}{\zeta_{vef}}g^{-1\dagger}_{ve'}(l^{+}_{e'f}+\alpha_{ve'f}l^{-}_{vef}). \tag{3.48}$$

Here $S_{vf\pm}=S_{ve'f\pm}-S_{vef\pm}$, $S_{vfx\pm}=S_{ve'f\pm}-S_{vef\mp}$ with $S_{vef\pm}$ is the action given in (2.44), the same for $S_{ef\pm}$ and $S_{efx\pm}$.

(ii) vertices relations

$$S_{ef\pm}\colon\ \gamma\Re(\alpha_{vef})\mp\Im(\alpha_{vef})=\gamma\Re(\alpha_{v'ef})\mp\Im(\alpha_{v'ef}) \tag{3.49}$$

$$S_{ef\pm x}\colon\ \gamma\Re(\alpha_{vef})\mp\Im(\alpha_{vef})=\gamma\Re(\alpha_{v'ef})\pm\Im(\alpha_{v'ef})=0 \tag{3.50}$$

(iii) closure constraints

$$0=-2\mathrm{i}\sum_{f/\mathrm{w}S_{+(x)}}s_f\langle l^{-}_{ef}-\mathrm{i}(\gamma\Re(\alpha_{vef})-\Im(\alpha_{vef}))l^{+}_{ef},F^{\dagger}l^{+}_{ef}\rangle+2\mathrm{i}\sum_{f/\mathrm{w}S_{-(x)}}s_f\langle n_{ef},F^{\dagger}l^{-}_{ef}\rangle \tag{3.51}$$

$$0=-2\gamma\sum_{f/\mathrm{w}S_{+(x)}}s_f\left\langle l^{-}_{ef}-\mathrm{i}\left(\frac{1}{\gamma}\Re(\alpha_{vef})-\Im(\alpha_{vef})\right)l^{+}_{ef},F^{\dagger}l^{+}_{ef}\right\rangle+2\sum_{f/\mathrm{w}S_{-(x)}}\mathrm{i}s_f\frac{\langle n_{ef},F^{\dagger}n_{ef}\rangle}{\Re(\alpha_{vef})}+\gamma s_f\langle n_{vef},F^{\dagger}l^{-}_{ef}\rangle \tag{3.52}$$

B. Bivector representation

For given spinors l^- and l^+, there is a 3-vector v^i associated to them

$$v^i=2\langle l^+,F^i l^-\rangle. \tag{3.53}$$

From which we can define a SU(1,1) valued bivector in spin-1/2 representation

$$V=2\langle l^+,F^i l^-\rangle F^i=-2(l^+)^{\dagger}(F^i)^{\dagger}\eta l^- F^i=-\frac{1}{2}(l^+)^{\dagger}\sigma_i\eta l^-\sigma_i=-\eta l^-\otimes(l^+)^{\dagger}+\frac{1}{2}\langle l^+,l^-\rangle I_2 \tag{3.54}$$

where we use the fact that $\eta F\eta=-F^{\dagger}$ and is the completeness of the Pauli matrix. Since $\langle l^-,Fl^+\rangle=-\langle l^+,Fl^-\rangle$,

$$V=-2\langle l^-,F^i l^+\rangle F_i=\eta l^+\otimes(l^-)^{\dagger}-\frac{1}{2}\langle l^+,l^-\rangle I_2. \tag{3.55}$$

From the fact

$$K^i=-K_i=J^{0i},\qquad J^i=J_i=\frac{1}{2}\epsilon^{0i}{}_{jk}J^{jk} \tag{3.56}$$

where $J^i=*K^i$. We have in the spin-1/2 representation, $*\to\mathrm{i}$ and $J^i=\mathrm{i}K^i$. The bivector can be encoded into a $\mathrm{SL}(2,\mathbb{C})$ bivector that in spin-1 representation reads

$$V^{IJ}=\begin{pmatrix}0&-v^1&-v^2&0\\ v^1&0&v^0&0\\ v^2&-v^0&0&0\\ 0&0&0&0\end{pmatrix}. \tag{3.57}$$

Then $(*V)^{IJ}$ reads

$$(*V)^{IJ} = \begin{pmatrix} 0 & 0 & 0 & v^0 \\ 0 & 0 & 0 & -v^2 \\ 0 & 0 & 0 & v^1 \\ -v^0 & v^2 & -v^1 & 0 \end{pmatrix} = (v^I_{ef} \wedge u^J) \tag{3.58}$$

where the encoded 4-vector $v^I_{ef} \coloneqq (v^0, -v^2, v^1, 0)$, $u^I = (0,0,0,1)$. Clearly one can see that

$$v^I = \mathrm{i}(\langle l^- | \hat{\sigma}^I | l^+ \rangle + u^I) \tag{3.59}$$

where $\hat{\sigma} = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)$.

Since $\langle l^+, F^i l^- \rangle = \langle l_0^+, v^\dagger F^i v^{-1\dagger} l_0^- \rangle$, in this sense, v_i is nothing else but the SO(1,2) rotation of 3-vector $v_0 = (0,0,1)$ with group element $v^{-1\dagger}$.

Similarly, we can define

$$W^\pm = 2\mathrm{i}\langle l^\pm, F^i l^\pm \rangle F^i = -\mathrm{i}\eta l^\pm \otimes (l^\pm)^\dagger \tag{3.60}$$

with

$$W^{\pm IJ} = w^{\pm I}_{ef} \wedge u^J, \qquad w^{\pm I} \coloneqq \langle l^\pm | \hat{\sigma}^I | l^\pm \rangle. \tag{3.61}$$

Here $w^{\pm I}_{ef}$ is a null vector $w^{\pm I}_{ef} w^\pm_{ef\,I} = 0$.

We introduce SO(1,3) group elements G given by

$$G_{ve} = \pi(g_{ve}) \tag{3.62}$$

where $\pi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$. Since the action (2.43) is invariant under the transformation $g_{ve} \to \pm g_{ve}$, two group elements related to g_{ve} are gauge equivalent if they satisfy

$$\tilde{G}_{ve} = G_{ve} I^{s_{ve}}, \qquad s_{ve} = \{0,1\} \tag{3.63}$$

where I is the inversion operator. With this gauge transformation, we can always assume $G_{ve} \in \mathrm{SO}_+(1,3)$.

We can write the critical equations in terms of bivectors. The detailed analysis is in Appendix C. Given any solution to the critical equations, we can define a bivector as

$$\begin{aligned} X_{vef} &= -2\mathrm{i}\langle l^-, F^i l^+ \rangle F_i - \mathrm{i}\bar{\alpha}_{vef} \langle l^+, F^i l^+ \rangle F_i \\ &= V_{ef} - (\Im(\alpha_{vef}) + \Re(\alpha_{vef})*) W^+_{ef} \end{aligned} \tag{3.64}$$

or

$$\begin{aligned} X_{vef} &= -2\mathrm{i}\langle n, F^{\dagger i} l^- \rangle F_i - \frac{\mathrm{i} + \gamma}{(1+\gamma^2)\Re(\alpha_{vef})} \langle n, F^{\dagger i} n \rangle \\ F_i &= -V_{ef} - \frac{1 - \gamma *}{(1+\gamma^2)\Re(\alpha_{vef})} W^+_{ef} \end{aligned} \tag{3.65}$$

corresponding to their action, which is composited by S_{vef+} or S_{vef-}. Here V_{ef} is a spacelike bivector and W_{ef} is a null bivector. In spin-1 representation, we can express the above bivector as

$$X^{IJ}_{ef} = (*)(\tilde{v}^I_{vef} \wedge \tilde{u}^J_{vef}) \tag{3.66}$$

where

$$\tilde{v}_{vef} = \begin{cases} v_{ef} - \Im(\alpha_{vef}) w^+_{ef}, & S_{vef+} \\ v_{ef} - \frac{\gamma}{(1+\gamma^2)\Re(\alpha_{vef})} w^+_{ef}, & S_{vef-} \end{cases} \tag{3.67}$$

$$\tilde{u}_{vef} = \begin{cases} u + \Re(\alpha_{vef}) w^+_{ef}, & S_{vef+} \\ u + \frac{1}{(1+\gamma^2)\Re(\alpha_{vef})} w^+_{ef}, & S_{vef-} \end{cases} \tag{3.68}$$

with

$$v_{ef} = \begin{cases} -2\mathrm{i}\langle l^-_{ef}, F^i l^+_{ef} \rangle, & S_{vef+} \\ -2\mathrm{i}\langle n_{ef}, F^i l^-_{ef} \rangle & S_{vef-} \end{cases}, \tag{3.69}$$

$$w^+_{ef} = \begin{cases} 2\langle l^+_{ef}, F^i l^+_{ef} \rangle, & S_{vef+} \\ 2\langle n_{ef}, F^i n_{ef} \rangle & S_{vef-} \end{cases}. \tag{3.70}$$

The bivector X_{vef} satisfies the parallel transport equation:

$$g_{ve} X_{vef} g^{-1}_{ve} = g_{ve'} X_{ve'f} g^{-1}_{ve'}. \tag{3.71}$$

This corresponds to

$$X_f(v) \coloneqq g_{ve} X_{vef} g_{ev} = v^I_{ef}(v) \wedge N^I_e(v) \tag{3.72}$$

where

$$v^I_{ef}(v) \coloneqq G_{ve} \tilde{v}_{vef}, \qquad N^I_e(v) = G_{ve} \tilde{u}_{vef}. \tag{3.73}$$

The closure constraint in terms of the bivector variable then reads

$$2\sum_f \gamma \epsilon_{ef}(v) s_f X_f(v) = \sum_f \epsilon_{ef}(v) B_f(v) = 0 \tag{3.74}$$

where $B_f = 2\gamma s_f X_f = n_f X_f$ with $B_f^2 = -n_f^2$. Note that the closure constraint is composed by two independent equations enrolling $\tilde{v}$ and w^+

$$\begin{aligned} &\sum_f \epsilon_{ef}(v) \tilde{v}_{vef} = 0, \\ &\begin{cases} \sum_f \epsilon_{ef}(v) \Re(\alpha_{vef}) w^+_{ef} = 0, & S_{vef+} \\ \sum_f \epsilon_{ef}(v) (\Re(\alpha_{vef})^{-1} w^+_{ef} = 0, & S_{vef-} \end{cases}. \end{aligned} \tag{3.75}$$

C. Timelike tetrahedron containing both spacelike and timelike triangles

The timelike tetrahedron in a generic simplicial geometry contains both spacelike and timelike triangles. For spacelike triangles, the irreps of SU(1,1) are in the discrete series, in contrast to the continuous series used in timelike triangles. The simplicity constraint is also different from (2.9). This leads to different face actions on triangles with a different signature, and the total action is expressed by the sum of these actions. The action on the spacelike triangle and corresponding critical point equations have already been derived in [21]. The results are reviewed in Appendix D.

The variations with respect to z_{vf} and v_{ef} give equations of motions (3.71) for timelike triangles and (D20) for spacelike triangles, respectively. In addition, for timelike triangles, solutions should satisfy (3.27), (3.32), or (3.32).

The variation with respect to the $\mathrm{SL}(2,\mathbb{C})$ group element g_{ve} involves all faces connected to e, which may include both spacelike and timelike triangles. In general, from (3.40)–(3.42) and (D13)–(D14), the action including different types of triangles gives

$$\begin{aligned}\delta_F S = &-2\mathrm{i}\sum_{f/\mathrm{w}S_{+(x)}} s_f\langle l^-_{ef} - \mathrm{i}(\gamma\Re(\alpha_{vef}) - \Im(\alpha_{vef}))l^+_{ef}, F^\dagger l^+_{ef}\rangle \\ &+ 2\mathrm{i}\sum_{f/\mathrm{w}S_{-(x)}} s_f\langle n_{ef}, F^\dagger l^-_{ef}\rangle - 2\sum_{f/\mathrm{w}S_{\mathrm{sp}}} j_f\langle \xi^\pm_{ef}, F^\dagger \xi^\pm_{ef}\rangle = 0 \end{aligned} \tag{3.76}$$

$$\begin{aligned}&-2\gamma\sum_{f/\mathrm{w}S_{+(x)}} s_f\left\langle l^-_{ef} - \mathrm{i}\left(\frac{1}{\gamma}\Re(\alpha_{vef}) - \Im(\alpha_{vef})\right)l^+_{ef}, F^\dagger l^+_{ef}\right\rangle \\ &+ 2\sum_{f/\mathrm{w}S_{-(x)}} \mathrm{i}s_f\frac{\langle n_{ef}, F^\dagger n_{ef}\rangle}{\Re(\alpha_{vef})} + \gamma s_f\langle n_{vef}, F^\dagger l^-_{ef}\rangle + 2\mathrm{i}\gamma\sum_{f/\mathrm{w}S_{\mathrm{sp}}} j_f\langle \xi^\pm_{ef}, F^\dagger \xi^\pm_{ef}\rangle = 0.\end{aligned} \tag{3.77}$$

The summation of the two equations leads to

$$(1+\gamma^2)\sum_{f/\mathrm{w}S_{+(x)}} s_f\Re(\alpha_{vef})\langle l^+_{ef}, F^i l^+_{ef}\rangle + \sum_{f/\mathrm{w}S_{-(x)}} s_f\frac{\langle n_{ef}, F^i n_{ef}\rangle}{\Re(\alpha_{vef})} = 0. \tag{3.78}$$

This equation only involves timelike triangles. Since ${w^{+i}}_{ef} = \langle l^+_{ef}, F^i l^+_{ef}\rangle$ (or $w^{+i}_{ef} = \langle n_{ef}, F^i n_{ef}\rangle$ in the $S_{-(x)}$ case) are null vectors, the above equation implies summing over null vectors equal to 0. In a tetrahedron that contains both timelike and spacelike triangles, the number of timelike triangles, which is also the number of null vectors here, is less than 4. If one has less than 4 null vectors summed to 0 in four-dimensional Minkowski space, then they are either trivial or collinear. The only possibility to have a nondegenerate tetrahedron from (3.78) is for all the timelike faces to be in the action S_+ and set $\Re(\alpha) = 0$. The solution reads

$$\Re(\alpha_{vef}) = 0 \quad \& \quad \forall_{f\in t_e}, S_f = S_{+(x)}. \tag{3.79}$$

It means that in order to have a critical point, the action associated to each triangle f of the tetrahedron t_e must be S_+ or S_{+x}; other actions do not have stationary point. The closure constraint is now given by (3.76) minus (3.77)

$$\begin{aligned}&-2\mathrm{i}\sum_{f/\mathrm{w}S_{+(x)}} s_f\langle l^-_{ef} + \mathrm{i}\Im(\alpha_{vef})l^+_{ef}, F^i l^+_{ef}\rangle = 0 \\ &-2\sum_{f/\mathrm{w}S_{sp}} j_f\langle \xi^\pm_{ef}, F^i \xi^\pm_{ef}\rangle = 0.\end{aligned} \tag{3.80}$$

The parallel transport equations for timelike triangles still keep the same form as (3.13)–(3.15). After we impose condition (3.79), the parallel transport equation becomes

$$\begin{aligned}&g_{ve}l^+_{ef}\otimes(l^-_{ef} + \mathrm{i}\Im(\alpha_{vef})l^+_{ef})^\dagger g_{ev} \\ &\quad = g_{ve'}l^+_{e'f}\otimes(l^-_{e'f} + \mathrm{i}\Im(\alpha_{ve'f})l^+_{e'f})^\dagger g_{e'v}.\end{aligned} \tag{3.81}$$

One recognizes the same composition of spinors $l^-_{ef} + \mathrm{i}\Im(\alpha_{vef})l^+_{ef}$ in (3.80) and (3.81). This is exactly the spinor satisfying Lemma III.1. Recall (3.27), coming from the variation with respect to SU(1,1) group elements v_{ef}, we have

$$\Im(\alpha_{vef}) = \Im(\alpha_{v'ef}) \tag{3.82}$$

in the S_+ case or $\Im(\alpha_{vef}) = 0$ in the S_{x+} case, respectively. However, recall that for the S_+ case, there is an ambiguity in defining $\tilde{l}^-$ and $\Im(\alpha)$ from Lemma III.1. This ambiguity does not change the action, and gives the same vector $v^i = \langle \tilde{l}^-_{ef}, F^i l^+_{ef}\rangle$. Thus, we can always remove the $\Im(\alpha_{vef})$ by a redefinition of l^-_{ef}, which does not change the geometric form of the critical equations. With (3.82), this redefinition will extended to both end points of the edge e. Thus, we always make the choice that $\Im(\alpha_{vef}) = 0$ and drop all $\Im(\alpha_{vef})$ terms in (3.80) and (3.81).

In the bivector representation, we can build bivectors for timelike triangles,

$$X_{ef} = *(v_{ef} \wedge u), \tag{3.83}$$

with v_{ef} a normalized vector defined by $v^I_{ef} = \mathrm{i}(\langle l^+_{ef} \times |\hat{\sigma}^I|l^-_{ef}\rangle - u^I)$. The parallel transportation equation implies we can define a bivector $X_f(v)$ independent of e

$$X_f(v) = G_{ve}X_{ef}G_{ev}. \tag{3.84}$$

Clearly in this case we have

$$N_e \cdot X_f(v) = 0, \quad \text{with} \quad N_e = G_{ve}u. \tag{3.85}$$

For spacelike triangles, the bivector is defined in (D18). One can see they have exactly the same form as in the timelike case and follow the same condition, except now $v^I_{ef} = \langle\xi^\pm_{ef}|\hat{\sigma}^I|\xi^\pm_{ef}\rangle - \langle\xi^\pm_{ef}||\xi^\pm_{ef}\rangle u^I$ instead. With bivectors X_{ef} and X_f, (3.80) becomes [after recover the sign factor $\epsilon_{ef}(v)$]

$$\sum_{f/\mathrm{w}S_{+(x)}} \epsilon_{ef}(v)s_f X_f(v) - \sum_{f/\mathrm{w}S_{sp}} \epsilon_{ef}(v)j_f X_f(v) = 0. \tag{3.86}$$

In summary, the critical equations for a timelike tetrahedron with both timelike and spacelike triangles implies a nondegenerate tetrahedron geometry only when timelike triangles have the action $S_{+(x)}$. Suppose we have a solution (j_f, g_{ve}, z_{vf}), one can define bivectors

$$B_{ef} = 2A_f X_{ef} = 2A_f * (v_{ef} \wedge u) \tag{3.87}$$

where

$$v^I_{ef} = \begin{cases} -\mathrm{i}(\langle l^+_{ef}|\sigma^I|l^-_{ef}\rangle - u^I) & \text{for timelike triangle} \\ \langle\xi^\pm_{ef}|\sigma^I|\xi^\pm_{ef}\rangle - \langle\xi^\pm, \xi^\pm\rangle u^I & \text{for spacelike case} \end{cases}, \tag{3.88}$$

and

$$A_f = \begin{cases} \gamma s_f = n_f/2 & \text{for timelike triangle} \\ \gamma j_f = \gamma n_f/2 & \text{for spacelike triangle} \end{cases}. \tag{3.89}$$

We define $B_{ef}(v)$ as

$$B_f(v) \coloneqq G_{ve}B_{ef}G_{ev}. \tag{3.90}$$

The critical point equations imply

$$B_{ef}(v) = B_{e'f}(v) = B_f(v) \tag{3.91}$$

$$N_e \cdot B_f(v) = 0 \tag{3.92}$$

$$\sum_{f\in t_e} \epsilon_{ef}(v)B_f(v) = 0 \tag{3.93}$$

where $N^I_e = G_{ve}u^I$, $\epsilon_{ef}(v) = \pm 1$ and changes its sign when exchanging vertex and edge variables.

D. Tetrahedron containing only timelike triangles

Starting from the critical equations derived above, we can see what happens when all faces that appear inside the closure constrain are timelike. For simplicity, we will use the S_+ action as an example, and the other cases will follow similar properties as they can be written in similar forms as S_+.

Suppose we have a solution to critical equations with all the face actions being S_+. As we have shown above, the solution satisfies two closure constraints,

$$\sum_f s_f(v_{ef} + \Im(\alpha_{vef})w^+_{ef}) = 0, \tag{3.94}$$

$$\sum_f s_f \Re(\alpha_{vef})w^+_{ef} = 0. \tag{3.95}$$

Clearly here we have family of solutions generated by the continuous transformations

$$\begin{aligned} &\Re(\alpha_{vef}) \to \tilde{C}_{ve}\Re(\alpha_{vef}), \\ &\Im(\alpha_{vef}) \to \Im(\alpha_{vef}) + C_{ve}\Re(\alpha_{vef}). \end{aligned} \tag{3.96}$$

In other words, the closure constraint only fixes α up to C_{ve} and $\tilde{C}_{ve}$.

Back to the bivectors inside the parallel transportation equation, it is easy to see that the bivector can be rewritten as

$$X = V + (\Im(\alpha) + \Re(\alpha)*)W^+ = X^0 + \Re(\alpha)(C + \tilde{C}*)W^+ \tag{3.97}$$

where $X_0 = V + \Im(\alpha^0_{vef})$ for some given $\Im(\alpha^0_{vef})$. Suppose we have a solution to some fixed C and $\tilde{C}$, the parallel transported bivector then reads

$$G_{ve}X_{ef}G_{ev} = G_{ve}X^0_{ef}G_{ev} + \Re(\alpha)(C + \tilde{C}*)G_{ve}W^+_{ef}G_{ev} = *((G_{ve}\tilde{v}_{vef}) \wedge (G_{ve}\tilde{u}_{vef})). \tag{3.98}$$

From the fact that in spin-1/2 representation $* \to \mathrm{i}$, we define $c \coloneqq C + \mathrm{i}\tilde{C}$.

From the parallel transported vector $\tilde{v}_f \coloneqq G_{ve}\tilde{v}_{vef}$ and $\tilde{u}_f \coloneqq G_{ve}\tilde{u}_{vef}$, one can determine a null vector $\tilde{w}_f$ related to face $f = (e, e')$ uniquely up to a scale by

$$\tilde{w}_f.\tilde{v}_f = \tilde{w}_f.\tilde{u}_f = 0. \tag{3.99}$$

From the definitions of $\tilde{v}$ and $\tilde{u}$, we see that $w_{ef}.\tilde{u}_{vef} = w_{ef}.\tilde{v}_{vef} = 0$ and the same relation for e'. Since $G \in \mathrm{SO}_+(1,3)$, which preserves the inner product, we then have

$$\tilde{w}_f \propto G_{ve}w_{ef} \propto G_{ve'}w_{e'f}. \tag{3.100}$$

Suppose a solution to critical equations determines a geometrical 4-simplex up to scaling and reflection with normals $N_e(v) = G_{ve}u$. (See Appendix E for the geometrical interpretation of the critical solution. We suppose the solution is nondegenerate here. The degenerate case will be discussed in Sec. V.) From this 4-simplex, we can get its boundary tetrahedron with faces normals $v^g_{ef}(v) = G_{ve}v^s_{ef}$. For the two edges e and e' that belong to the same face f, N_e and $N_{e'}$ determine uniquely a null vector (up to scaling), which is perpendicular to N_e and $N_{e'}$. Then from (3.99) and (3.100), the vector is proportional to $\tilde{w}_f$. Then it implies that

$$v^s_{ef} = \tilde{v}_{ef} + d_{ef}w_{ef}. \tag{3.101}$$

The tetrahedra determined by v^s_{ef} (by the Minkowski theorem) satisfy the length matching condition, which further constrains d_{ef}. Ten d_{ef}'s are overconstrained by 20 length matching conditions. $d_{ef} = 0$ corresponds to a solution if the boundary data (relating to $\tilde{v}_{ef}$) also satisfies the length matching condition. We have the parallel transportation equation:

$$g_{ve}X^0_{ef}g_{ev} + d_{ef}g_{ve}W^+_{ef}g_{ev} = g_{ve'}X^0_{e'f}g_{e'v} + d_{e'f}g_{ve'}W^+_{e'f}g_{e'v}. \tag{3.102}$$

However, from (3.98) we know that

$$g_{ve}X^0_{ef}g_{ev} + \Re(\alpha_{ef})c_{ve}g_{ve}W^+_{ef}g_{ev} = g_{ve'}X^0_{e'f}g_{e'v} + \Re(\alpha_{e'f})c_{ve'}g_{ve'}W^+_{e'f}g_{e'v} \tag{3.103}$$

which means

$$(\Re(\alpha_{vef})c_{ve} - d_{ef})g_{ve}W^+_{ef}g_{ev} = (\Re(\alpha_{ve'f})c_{ve'} - d_{e'f})g_{ve'}W^+_{e'f}g_{e'v}. \tag{3.104}$$

They are 10 complex equations, with five complex c_{ve}, thus again giving an overconstrained system.

A special case is that the boundary data itself satisfy the length matching condition. In this case, $d_{ef} = 0$ correspond to a critical solution. It can be further proved that (3.104) with $d_{ef} = 0$ implies

$$\forall_e c_{ve} = 0. \tag{3.105}$$

The condition is nothing else but (3.79), and it is easy to see that in this case the critical equations reduce to (3.87)–(3.91).

IV. GEOMETRIC INTERPRETATION AND RECONSTRUCTION

The critical solutions of the spin foam action are shown to satisfy certain geometrical bivector equations, and we would like to compare them with a discrete Lorentzian geometry. The general construction of a discrete Lorentzian geometry and the relation with critical solutions for spacelike triangles were discussed in detail in [14] and [21]. We will see that our solutions, which include timelike triangles, can be applied to a similar reconstruction procedure. We demonstrate the detailed analysis in Appendix E. The main result is summarized here. The result is valid when every timelike tetrahedron contains both spacelike and timelike triangles. It is also valid for tetrahedra containing only timelike triangles in the special case with Eq. (3.105).

The following condition at a vertex v implies the nondegenerate 4-simplex geometry:

$$\prod_{e1,e2,e3,e4=1}^{5} \det(N_{e1}, N_{e2}, N_{e3}, N_{e4}) \neq 0 \tag{4.1}$$

which means any four out of five normals are linearly independent. Since $N_e = G_{ve}u$, the above nondegeneracy condition is a constraint on G_{ve}. Here $u = (0,0,0,1)$ or $u = (1,0,0,0)$ for a timelike or spacelike tetrahedron.

Then we can prove that satisfying the nondegeneracy condition, each solution $B_{ef}(v)$ at a vertex v determines a geometrical 4-simplex uniquely up to the shift and inversion. The bivectors $B^\Delta_{ef}(v)$ of the reconstructed 4-simplex satisfy

$$B^{\Delta}_{ef}(v) = r(v) B_{ef}(v) \tag{4.2}$$

where $r(v) = \pm 1$ relates to the 4-simplex (topological) orientation defined by an ordering of tetrahedra. The reconstructed normals are determined up to a sign

$$N^{\Delta}_{ve} = (-1)^{s_{ve}} N_{ve}. \tag{4.3}$$

We can prove that for a vertex amplitude, the solution exists only when the boundary data determine tetrahedra that are glued with length matching (the pair of glued triangles have their edge lengths matched).

Given the boundary data, we can determines geometric group elements $G^{\Delta} \in O(1,3)$ from reconstructed normals N^{Δ}. Then it can be shown that, after one chooses s_v and s_{ve}, such that

$$\forall_e \ \det G^{\Delta}_{ve} = (-1)^{s_v} = r(v) \tag{4.4}$$

G^{Δ}_{ve} relates to G_{ve} by

$$G_{ve} = G^{\Delta}_{ve} I^{s_{ve}} (I R_u)^{s_v} \tag{4.5}$$

where R_N is the reflection respecting to normalized vector N defined as

$$(R_N)^I_J = \mathbb{I}^I_J - \frac{2 N^I N_J}{N \cdot N}. \tag{4.6}$$

The choice of $s_{ve} = \pm 1$ corresponds to a gauge freedom and is arbitrary here. Condition (4.4) is called the orientation matching condition, which essentially means that the orientations of five boundary tetrahedra determined by the boundary condition are required to be the same.

For a vertex amplitude, the nondegenerate geometric critical solutions exist if and only if the length matching condition and orientation matching condition are satisfied. Up to gauge transformations, there are two gauge inequivalent solutions which are related to each other by a reflection with respect to any normalized 4-vector e_α (this reflection is referred to as the parity transformation in, e.g., [12–15])

$$\tilde{B}_{ef}(v) = R_{e_\alpha}(B_{ef}(v)), \qquad \tilde{s}_v = s_v + 1 \tag{4.7}$$

which means

$$\tilde{G}_{ve} = R_{e_\alpha} G_{ve} (I R_N). \tag{4.8}$$

Geometrically the second one corresponds to the reflected simplex. These two critical solutions correspond to the same 4-simplex geometry, but are associated to a different sign of the oriented 4-simplex volume $V(v)$. $\mathrm{sgn}(V(v))$ is referred to as the (geometrical) orientation of the 4-simplex,[2] which should not be confused with $r(v)$. This result generalizes [21] to the spin foam vertex amplitude containing timelike triangles.

The reconstruction can be extended to simplicial complex $\mathcal{K}$ with many 4-simplices, in which some critical solutions of the full amplitude correspond to nondegenerate Lorentzian simplicial geometries on $\mathcal{K}$ (see Appendix E). But similar to the situation in [14,15], 4-simplices in $\mathcal{K}$ may have different $\mathrm{sgn}(V(v))$. We may divide the complex $\mathcal{K}$ into subcomplexes, such that each subcomplex is globally orientated; i.e., the sign of the orientated volume $\mathrm{sgn}(V)$ is a constant. Then we have the following result.

For critical solutions corresponding to simplicial geometries with all 4-simplices globally oriented, picking up a pair of them corresponding to opposite global orientations, they satisfy

$$\tilde{G}_f = \begin{cases} R_{u_e} G_f(e) R_{u_e} & \text{internal faces} \\ I^{r_{e1}+r_{e0}} R_{u_{e1}} G_f(e_1, e_0) R_{u_{e0}} & \text{boundary faces} \end{cases} \tag{4.9}$$

where $G_f = \prod_{v \subset \partial f} G_{e'v} G_{ve}$ is the face holonomy. We will use this result to derive the phase difference of their asymptotical contributions to the spin foam amplitude. Note that, the asymptotic formula of the spin foam amplitude is given by summing over all possible configurations of orientations.

V. SPLIT SIGNATURE AND DEGENERATE 4-SIMPLEX

This section discusses the critical solutions that violate the nondegeneracy condition (4.1). We refer to these solutions as degenerate solutions. If the nondegeneracy condition is violated, then in each 4-simplex, all five normals N_e of tetrahedra t_e are parallel, since we only consider nondegenerate tetrahedra [21]. When it happens with all t_e timelike (or spacelike), with the help of gauge transformation $G_{ve} \to G G_{ve}$, we can write $N_e(v) = G_{ve} u, u = (0,0,0,1)$, where all the group variables $G_{ve} \in \mathrm{SO}_+(1,2)$. However, when the vertex amplitude contains at least one timelike and one spacelike tetrahedron, the nondegeneracy condition (4.1) cannot be violated since timelike and spacelike normals certainly cannot be parallel. Therefore, the solutions discussed in this section only appear in the vertex amplitude with all tetrahedra timelike. Moreover, these degenerate solutions appear when the boundary data are special, i.e., they correspond to the boundary of a split signature 4-simplex or a degenerate 4-simplex, as we see in a moment.

When the tetrahedron contains both timelike and spacelike triangles, the closure constraint (3.78) concerning w

[2] $\mathrm{sgn}(V(v))$ is a discrete analog of the volume element compatible to the metric in smooth pseudo-Riemannian geometry.

involves at most 3 null vectors, which directly leads to $\Re(\alpha_{vef}) = 0$ as the only solution. For degenerate solutions, the bivector $X_f(v) = g_{ve}X_{ef}g_{ev}$ in (3.84) becomes

$$X_f(v) = *G_{ve}(v_{ef} \wedge u)G_{ev} = G_{ve}v_{ef} \wedge u = v^g_{vef} \wedge u. \tag{5.1}$$

The parallel transportation equation (3.91) becomes

$$v^g_f(v) = v^g_{ve} = v^g_{ve'} = 2A_f G_{ve}v_{ef}. \tag{5.2}$$

Thus, the degenerate critical solutions satisfy

$$v^g_f(v) = v^g_{ve} = v^g_{ve'}, \qquad \sum_f \epsilon_{ef}(v)v^g_f(v) = 0 \tag{5.3}$$

and the collection of vectors $v^g_f(v)$ is referred to as a vector geometry in [12].

In the case that all triangles in a tetrahedron are timelike, we use S_{vf+} as an example. The degeneracy implies $G_{ve}u = G_{ve'}u = u$. The parallel transportation equation (3.98) becomes

$$(G_{ve}\tilde{v}_{vef} - G_{ve'}\tilde{v}_{ve'f}) \wedge u = c_{ve}\Re(\alpha_{vef})G_{ve}w^+_{ef} \wedge u - c_{ve'}\Re(\alpha_{ve'f})G_{ve'}w^+_{e'f} \wedge u. \tag{5.4}$$

$c_{ve} = C_{ve} + \mathrm{i}\tilde{C}_{ve}$ is the factor which solves the closure constraint with a given normalization of $\Re(\alpha_{vef})$, e.g., $\sum_f \Re(\alpha_{vef}) = 1$ as shown in (3.96). (5.4) directly leads to

$$G_{ve}(\tilde{v}_{vef} + C_{ve}\Re(\alpha_{vef})w_{ef}) = G_{ve'}(\tilde{v}_{ve'f} + C_{ve}\Re(\alpha_{vef})w_{ef}) \tag{5.5}$$

$$\tilde{C}_{ve}\Re(\alpha_{vef})G_{ve}w_{ef} = \tilde{C}_{ve'}\Re(\alpha_{ve'f})G_{ve'}w_{e'f}. \tag{5.6}$$

Notice that from (5.5), since w_{ef} is null and $w_{ef} \cdot v_{ef} = 0$, we have

$$G_{ve}w_{ef} \propto G_{ve'}w_{e'f}. \tag{5.7}$$

It implies that (5.6) is only a function of $\tilde{C}$. However, at a vertex v, there are only five independent $\tilde{C}$ variables out of 10 equations. Thus (5.6) are overconstrained equations and give five consistency conditions for G_{ve} unless $\tilde{C} = 0$.

Actually, one can show that there is no solution when $\tilde{C} \neq 0$. We give the proof here. For simplicity, we only focus on a single 4-simplex.

Suppose we have solutions to above equations with $\tilde{C} \neq 0$; then, the following equations hold according to (5.5), (5.6), and the closure constraint (C14):

$$v^g_f(v) = v^g_{ef}(v) = v^g_{e'f}(v), \qquad \sum_{f\subset t_e}\epsilon_{ef}(v)v^g_{ef}(v) = 0,$$
$$w^g_f(v) = w^g_{ef}(v) = w^g_{e'f}(v), \qquad \sum_{f\subset t_e}\epsilon_{ef}(v)w^g_{ef}(v) = 0, \tag{5.8}$$

where

$$v^g_{ef}(v) = G_{ve}\tilde{v}_{ef} + C_i\Re(\alpha_{vef})G_{ve}w_{ef}$$
$$w^g_{ef}(v) = \tilde{C}_i\Re(\alpha_{vef})G_{ve}w_{ef}. \tag{5.9}$$

Suppose v^g satisfy the length matching condition. From the above equations, $\tilde{v}^g_{ef} = v^g_{ef} + aw^g_{ef}$ with arbitrary real number a are also solutions. This means $\tilde{v}^g$ should also satisfy the length matching condition. However, the transformation from v to $v + aw$ changes the edge lengths of the tetrahedron, and the length matching condition gives constraint to a. This conflicts with the fact that a is arbitrary to form the solution. It means that we cannot have a solution with $\tilde{C} \neq 0$ and the length matching condition satisfied.

Thus, when boundary data satisfies the length matching condition, the only possible solution of (5.6) is $\tilde{C}_{ve} = 0$. This corresponds to $\Re(\alpha) = 0$, which is thus only possible with the action S_+. One recognizes that this is the same condition as in the case of the tetrahedron with both timelike and spacelike triangles, e.g., (3.79). In this case C_{ve} thus $\Im(\alpha)$ can be uniquely determined by the closure and length matching condition. The critical point equations again become (5.2) and (5.3).

In the end of this section, we introduce some relations between the vector geometry and nondegenerate split signature 4-simplex. As shown in Appendix E 6, the vector geometries in three-dimensional subspace V can be mapped to the split signature space M' with signature $(-,+,+,-)$ [flip the signature of $u = (0,0,0,1)$], with the map $\Phi^\pm\colon \wedge^2 M^{4\prime} \to V$ for bivectors B,

$$\Phi^\pm(B) = (\mp B - *'B)\cdot' u. \tag{5.10}$$

$\Phi^\pm$ naturally induced a map from $g \in \mathrm{SO}(2,2)$ to the subgroup $h \in \mathrm{SO}(1,2)$, defined by

$$\Phi^\pm(gBg^{-1}) = \Phi^\pm(g)\Phi^\pm(B). \tag{5.11}$$

If the vertex amplitude has the critical solutions being a pair of non-gauge-equivalent vector geometries $\{G^\pm_{ve}\}$, they are equivalent to a pair of non-gauge-equivalent $\{G_{ve} \in SO(M')\}$ satisfying the nondegenerate condition. One of the nondegenerate $\{G_{ve}\}$ satisfies $G^{\pm}_{ve} = \Phi^{\pm}(G_{ve})$, while the other $\{\tilde{G}_{ve}\}$ satisfies

$$\Phi^{\pm}(\tilde{G}) = \Phi^{\pm}(R_u G R_u) = \Phi^{\mp}(G). \tag{5.12}$$

When the vector geometries are gauge equivalent, the corresponding geometric $SO(M')$ solution is degenerate. In this case the reconstructed 4-simplex is degenerate and the 4-volume is 0.

VI. SUMMARY OF GEOMETRIES

We summarize all possible reconstructed geometries corresponding to critical configurations of the Conrady-Hnybida extended spin foam model (including the EPRL model) here. We first introduce the length matching condition and orientation matching condition for the boundary data. Namely, (1) among the five tetrahedra reconstructed by the boundary data (by the Minkowski theorem), each pair of them are glued with their common triangles matching in shape (matching their three edge lengths), and (2) all tetrahedra have the same orientation. The amplitude will be suppressed asymptotically if the orientation matching condition is not satisfied.

For any given boundary data that satisfies the length matching condition and orientation matching condition, we may have the following reconstructed 4-simplex geometries corresponding to critical configurations of the Conrady-Hynbida model:

(1) Lorentzian $(-+++)$ 4-simplex geometry: reconstructed by boundary data which may contain
 (a) both timelike and spacelike tetrahedra,
 (b) all tetrahedra being timelike,
 (c) all tetrahedra being spacelike.
(2) Split signature $(-++-)$ 4 simplex geometry: This case is only possible when every boundary tetrahedron are timelike.
(3) Euclidean signature $(++++)$ 4-simplex geometry: This case is only possible when every boundary tetrahedron are spacelike.
(4) Degenerate 4 simplex geometry: This case is only possible when all boundary tetrahedron are timelike or all of them are spacelike.

When the length matching condition is not satisfied, we might still have one gauge equivalence class of solutions which determines a single vector geometry. This solution exists again only when all boundary tetrahedron are timelike or all of them are spacelike.

Our analysis is generalized to a simplicial complex $\mathcal{K}$ with many 4-simplices. A most general critical configuration of the Conrady-Hnybida model may mix all the types of geometries on the entire $\mathcal{K}$. One can always make a partition of $\mathcal{K}$ into subregions such that in each region we have a single type of reconstructed geometry with the boundary. However, this may introduce nontrivial transitions between different types of geometries through the boundary shared by them as suggested in [14]. It is important to remark that, if we take the boundary data of each 4-simplex to contain at least one timelike and one spacelike tetrahedron, critical configurations will only give Lorentzian 4-simplices.

VII. PHASE DIFFERENCE

In this section, we compare the difference of the phases given by a pair of critical solutions with opposite (global) $\mathrm{sgn}(V)$ orientations on a simplicial complex $\mathcal{K}$. Recall that the amplitude is defined with SU(1,1) and SU(2) coherent states at the timelike and spacelike boundaries. When we define the coherent state, we have a phase ambiguity from the K_1 direction in SU(1,1) [or the J_3 direction in SU(2)]; thus, the action is determined up to this phase. Thus, the phase difference ΔS is the essential result in the asymptotic analysis of the spin foam vertex amplitude. The phase difference at a spacelike triangle has already been discussed in [21]; we only focus on timelike triangles here.

Given a timelike triangle f, in the Lorentzian signature, the normals N_e and $N_{e'}$ are spacelike and span a spacelike plane, while in the split signature they form a timelike surface. The dihedral angles Θ_f at f are defined as follows: In the Lorentzian signature, the dihedral angle is $\Theta_f = \pi - \theta_f$ where

$$\cos\theta_f = N^{\Delta}_e \cdot N^{\Delta}_{e'}, \qquad \theta_f \in (0,\pi). \tag{7.1}$$

While in the split signature, the boost dihedral angle θ_f is defined by

$$\cosh\theta_f = |N^{\Delta}_e \cdot' N^{\Delta}_{e'}|, \qquad \theta_f \gtrless 0 \quad \text{while } N^{\Delta}_e \cdot' N^{\Delta}_{e'} \gtrless 0. \tag{7.2}$$

A. Lorentzian signature solutions

As we showed before, when every tetrahedron has both timelike and spacelike triangles, the critical solutions only come from S_+. So we focus on the S_+ action.

From the action (2.43), after inserting the decomposition (3.4), we find

$$\begin{aligned} S_{vf+} &= \frac{n_f}{2}\ln\frac{\zeta_{vef}\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}\zeta_{ve'f}} - \mathrm{i}s_f \ln\frac{\zeta_{ve'f}\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}\zeta_{vef}} = -2\mathrm{i}\gamma s_f(\arg(\zeta_{ve'f}) - \arg(\zeta_{vef}) - 2\mathrm{i}s\ln\frac{|\zeta_{ve'f}|}{|\zeta_{vef}|} \\ &= -2\mathrm{i}s_f(\theta_{e'vef} + \gamma\phi_{e'vef}) \end{aligned} \tag{7.3}$$

where θ and ϕ are defined by

$$\theta_{e'vef} \coloneqq \ln\frac{|\zeta_{ve'f}|}{|\zeta_{vef}|},$$
$$\phi_{e'vef} \coloneqq \arg(\zeta_{ve'f}) - \arg(\zeta_{vef}). \tag{7.4}$$

The face action at a triangle dual to a face f then reads

$$S_f = \sum_{v\in\partial f} S_{vf} = -2\mathrm{i}s_f\left(\sum_{v\in\partial f}\theta_{e'vef} + \gamma\sum_{v\in\partial f}\phi_{e'vef}\right). \tag{7.5}$$

We start the analysis from faces dual to boundary triangles (boundary faces) and then going to internal faces.

1. Boundary faces

For critical configurations solving critical equations [we keep $\Im(\alpha) = 0$ by redefinition of l^-_{ef}], they satisfy

$$g_{ve}\eta l^+_{ef} = \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'}\eta l^+_{e'f} \tag{7.6}$$

$$g_{ve}Jl^-_{ef} = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g^{-1\dagger}_{ve'} Jl^-_{e'f}. \tag{7.7}$$

We then have

$$G_f(e_1, e_0)\eta l^+_{e_0 f} = \mathrm{e}^{-\sum_{v\in p_{e_1 e_0}}\theta_{e'vef} + \mathrm{i}\sum_{v\in p_{e_1 e_0}}\phi_{e'vef}}\eta l^+_{e_1 f} \tag{7.8}$$

$$G_f(e_1, e_0)Jl^-_{e_0 f} = \mathrm{e}^{\sum_{v\in p_{e_1 e_0}}\theta_{e'vef} - \mathrm{i}\sum_{v\in p_{e_1 e_0}}\phi_{e'vef}}Jl^-_{e_1 f} \tag{7.9}$$

where $G_f(e_1, e_0)$ is the product of the edge holonomy along the path $p_{e_0 e_1}$

$$G_f(e_1, e_0) \coloneqq g_{e_1 v_1} \ldots g_{e' v_0} g_{v_0 e_0}. \tag{7.10}$$

Suppose we have holonomies G and $\tilde{G}$ from the pair of critical solutions with the global $\mathrm{sgn}(V)$ orientation, then one can see

$$\tilde{G}^{-1}G\eta l^+_{e_0 f} = \mathrm{e}^{-\sum_{v\in p_{e_1 e_0}}\Delta\theta_{e'vef} + \mathrm{i}\sum_{v\in p_{e_1 e_0}}\Delta\phi_{e'vef}}\eta l^+_{e_0 f} \tag{7.11}$$

$$\tilde{G}^{-1}GJl^-_{e_0 f} = \mathrm{e}^{\sum_{v\in p_{e_1 e_0}}\Delta\theta_{e'vef} - \mathrm{i}\sum_{v\in p_{e_1 e_0}}\Delta\phi_{e'vef}}Jl^-_{e_0 f}. \tag{7.12}$$

For a single 4-simplex, the above equations read

$$(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}(g_{e'v}g_{ve})\eta l^+_{e_0 f} = \frac{\bar{\zeta}'_{vef}}{\bar{\zeta}'_{ve'f}}\frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}}\eta l^+_{ef} = \mathrm{e}^{-\Delta\theta_{e'vef} + \mathrm{i}\Delta\phi_{e'vef}}\eta l^+_{ef} \tag{7.13}$$

$$(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}(g_{e'v}g_{ve})Jl^-_{e_0 f} = \frac{\bar{\zeta}'_{ve'f}}{\bar{\zeta}'_{vef}}\frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}}Jl^-_{e_0 f} = \mathrm{e}^{\Delta\theta_{e'vef} - \mathrm{i}\Delta\phi_{e'vef}}Jl^-_{e_0 f} \tag{7.14}$$

which leads to

$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v}g_{ve}\eta l^+_{e_0 f} = \mathrm{e}^{-\Delta\theta_{e'vef} + \mathrm{i}\Delta\phi_{e'vef}}g_{ve}\eta l^+_{e_0 f} \tag{7.15}$$

$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v}g_{ve}Jl_{e_0 f} = \mathrm{e}^{\Delta\theta_{e'vef} - \mathrm{i}\Delta\phi_{e'vef}}g_{ve}Jl^-_{e_0 f}. \tag{7.16}$$

We can define an operator T_{ef} by

$$T_{ef} \coloneqq \eta l^+_{ef} \otimes (l^-_{ef})^\dagger = |\eta l^+_{ef}\rangle\langle l^-_{ef}|. \tag{7.17}$$

From the facts $\langle l^-_{ef}|\eta l^+_{ef}\rangle = \langle l^-_{ef}, l^+_{ef}\rangle = 1, \langle l^-_{ef}|Jl^-_{ef}\rangle = 0$, the action of this operator leads to

$$T_{ef}|\eta l^+_{ef}\rangle = |\eta l^+_{ef}\rangle\langle l^-_{ef}|\eta l^+_{ef}\rangle = |\eta l^+_{ef}\rangle$$
$$T_{ef}|Jl^-_{ef}\rangle = 0. \tag{7.18}$$

From the definition of (3.64) (with $\alpha = 0$), by using (3.55) and (3.60), one can then see

$$X_{ef}|\eta l^+_{ef}\rangle = \frac{1}{2}|\eta l^+_{ef}\rangle, \qquad X_{ef}|Jl^-_{ef}\rangle = -\frac{1}{2}|Jl^-_{ef}\rangle. \tag{7.19}$$

Then we have

$$2X_f g_{ve}|\eta l^+_{ef}\rangle = 2g_{ve}X_{ef}g_{ev}g_{ve}|\eta l^+_{ef}\rangle = g_{ve}2X_{ef}|\eta l^+_{ef}\rangle = g_{ve}|\eta l^+_{ef}\rangle \tag{7.20}$$

$$2X_f g_{ve}|Jl^-_{ef}\rangle = 2g_{ve}X_{ef}g_{ev}g_{ve}|Jl^-_{ef}\rangle = g_{ve}2X_{ef}|Jl^-_{ef}\rangle = -g_{ve}|Jl^-_{ef}\rangle. \tag{7.21}$$

From (7.15) and (7.16), it is easy to see

$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef}X_f + 2\mathrm{i}\Delta\phi_{e'vef}X_f}. \tag{7.22}$$

For a general simplicial complex with the boundary, given a boundary face f with two edges e_0 and e_1 connecting to the boundary, and v is the bulk end point of e_0 if we define

$$G_f(e_1, e_0) = G_f(v, e_1)^{-1} g_{ve_0}. \tag{7.23}$$

It can be proved that

$$G_f(v, e_1) X_{e_1 f} G_f(v, e_1)^{-1} = g_{ve_0} X_{e_0 f} g_{e_0 v} \tag{7.24}$$

which is the generalization of the parallel transportation equation within a single 4-simplex. Then we can apply the same derivation as the single-simplex case by replacing $g_{ve'} \to G(v, e_1)$, which leads to

$$g_{ve} \tilde{G}_f(e_1, e_0)^{-1} G_f(e_1, e_0) g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f} \Delta\theta_{e'vef} X_f + 2\mathrm{i} \sum_{v\in\partial f} \Delta\phi_{e'vef} X_f}. \tag{7.25}$$

2. Internal faces

The discussion of the internal face f is similar to the boundary case. We have

$$G_f \eta l^+_{ef} = \mathrm{e}^{-\sum_{v\in\partial f} \theta_{e'vef} + \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef}} \eta l^+_{ef} \tag{7.26}$$

$$G_f J l^-_{ef} = \mathrm{e}^{\sum_{v\in\partial f} \theta_{e'vef} - \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef}} J l^-_{ef} \tag{7.27}$$

where G_f is the face holonomy

$$G_f \coloneqq \overleftarrow{\prod_{v\in\partial f}} g_{e'v} g_{ve}. \tag{7.28}$$

By the action of the bivector X_{ef} in (7.19),

$$\mathrm{e}^{-\sum_{v\in\partial f} \theta_{e'vef} 2X_{ef} + \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef} 2X_{ef}} |\eta l^+_{ef}\rangle = \mathrm{e}^{-\sum_{v\in\partial f} \theta_{e'vef} + \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef}} |\eta l^+_{ef}\rangle \tag{7.29}$$

$$\mathrm{e}^{-\sum_{v\in\partial f} \theta_{e'vef} 2X_{ef} + \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef} 2X_{ef}} |J l^-_{ef}\rangle = \mathrm{e}^{\sum_{v\in\partial f} \theta_{e'vef} - \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef}} |J l^-_{ef}\rangle. \tag{7.30}$$

Compared to (7.26) and (7.27), we see that

$$G_f = \mathrm{e}^{-\sum_{v\in\partial f} \theta_{e'vef} 2X_{ef} + \mathrm{i} \sum_{v\in\partial f} \phi_{e'vef} 2X_{ef}}. \tag{7.31}$$

Given G_f and $\tilde{G}_f$ from a pair of critical solutions with the opposite $\mathrm{sgn}(V)$ orientation, we find

$$g_{ve} \tilde{G}_f^{-1} G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f} \Delta\theta_{e'vef} X_f + 2\mathrm{i} \sum_{v\in\partial f} \Delta\phi_{e'vef} X_f}. \tag{7.32}$$

3. Phase difference

For a pair of globally orientated [constant $\mathrm{sgn}(V)$] critical solutions with the opposite orientation, from (7.5) we have

$$\Delta S_f = -2\mathrm{i} s_f \left(\sum_{v\in\partial f} \Delta\theta_{e'vef} + \gamma \sum_{v\in\partial f} \Delta\phi_{e'vef} \right) \tag{7.33}$$

where $\Delta\theta$ and $\Delta\phi$ are determined by

$$g_{ve} \tilde{G}_f^{-1} G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f} \Delta\theta_{e'vef} X_f + 2\mathrm{i} \sum_{v\in\partial f} \Delta\phi_{e'vef} X_f}. \tag{7.34}$$

$G_f \equiv G_f(e_1, e_0)$ if f is a boundary face. Since $\gamma s_f = n_f/2 \in \mathbb{Z}/2$, we may restrict

$$\sum_{v\in\partial f} \Delta\phi_{e'vef} \in [-\pi, \pi] \tag{7.35}$$

because ΔS_f is an exponent.

After projecting to $\mathrm{SO}_+(1,3)$,

$$g_{ve} \tilde{G}_f^{-1} G_f g_{ev} \to G_{ve} \tilde{G}_f^{-1} G_f G_{ev}, \qquad \mathrm{i} \to *. \tag{7.36}$$

For the spacelike normal vector $u = (0,0,0,1)$, from which it is easy to see that G and $\tilde{G}$ are related by

$$\tilde{G} = R_{e_0} G R_u I \in SO_+(1,3) \tag{7.37}$$

and

$$\tilde{G}_f = R_{u_e} G_f R_{u_e} \tag{7.38}$$

for both internal and boundary triangles f. The equation then leads to

$$G_{ve} \tilde{G}_f^{-1} G_f G_{ev} = G_{ve} R_u G_f^{-1} R_u G_f G_{ev} = R_{N_e} R_{N_{e'}} \tag{7.39}$$

for both internal and boundary triangles f. N_e and $N_{e'}$ here are given by

$$N_e = G_{ve} u, \qquad N_{e'} = G_{ve}(G_f^{-1} u); \tag{7.40}$$

thus, $N_{e'}$ is the parallel transported vector along the face.

Therefore, in both the internal case and boundary case, we have

$$R_{N_e}R_{N_{e'}} = \mathrm{e}^{-2\sum_{v\in\partial f}\Delta\theta_{e'vef}X_f+2*\sum_{v\in\partial f}\Delta\phi_{e'vef}X_f}. \quad (7.41)$$

On the other hand, from the fact that $R_N = GR_uG$, and the fact that $G^\Delta_{ve} = GI^{s_{ve}}(IR_u)^s_v$, we have

$$R_{N_e}R_{N'_e} = R_{N^\Delta_e}R_{N^\Delta_{e'}}. \quad (7.42)$$

Since R_{N^Δ} is a reflection with respect to the spacelike normal N^Δ, we have (see Appendix F)

$$R_{N^\Delta_e}R_{N^\Delta_{e'}} = \mathrm{e}^{2\theta_f\frac{N^\Delta_e\wedge N^\Delta_{e'}}{|N^\Delta_e\wedge N^\Delta_{e'}|}} \quad (7.43)$$

where f is the triangle dual to the face determined by edges e and e'. $\theta_f\in[0,\pi]$ satisfies $N^\Delta_e\cdot N^\Delta_{e'} = \cos(\theta_f)$. From the geometric reconstruction,

$$B_f = n_fX_f = -\frac{1}{\mathrm{Vol}^\Delta}rW^\Delta_eW^\Delta_{e'}*(N^\Delta_{e'}\wedge N^\Delta_e). \quad (7.44)$$

Since $|B_f|^2 = -n_f^2$, we have

$$\left|\frac{1}{\mathrm{Vol}^\Delta}rW^\Delta_eW^\Delta_{e'}\right||N^\Delta_{e'}\wedge N^\Delta_e| = n_f. \quad (7.45)$$

Thus

$$X_f = \frac{B_f}{n_f} = \sigma_f\frac{*(N^\Delta_{e'}\wedge N^\Delta_e)}{|(N^\Delta_{e'}\wedge N^\Delta_e)|} \quad (7.46)$$

where $\sigma_f = -r\mathrm{sign}(W^\Delta_{e'}W^\Delta_e)$. Since N_e and $N_{e'}$ are both spacelike, we have $\sigma_f = -r$. Keep in mind that r is the orientation and is a constant sign on the (sub-)triangulation. Therefore,

$$\mathrm{e}^{2r\sum_{v\in\partial f}\Delta\theta_{e'vef}\frac{*(N^\Delta_{e'}\wedge N^\Delta_e)}{|N^\Delta_{e'}\wedge N^\Delta_e|}+2r\sum_{v\in\partial f}\Delta\phi_{e'vef}\frac{N^\Delta_{e'}\wedge N^\Delta_e}{|N^\Delta_{e'}\wedge N^\Delta_e|}} = \mathrm{e}^{2\theta_f\frac{N^\Delta_e\wedge N^\Delta_{e'}}{|N^\Delta_e\wedge N^\Delta_{e'}|}} \quad (7.47)$$

which implies

$$\sum_{v\in\partial f}\Delta\theta_{e'vef} = 0,$$
$$-r\sum_{v\in\partial f}\Delta\phi_{e'vef} = \theta_f \mod \pi. \quad (7.48)$$

The phase difference is then

$$\Delta S_f = 2\mathrm{i}rA_f\theta_f \mod \mathrm{i}\pi \quad (7.49)$$

where $A_f = \gamma s_f = n_f/2\in\mathbb{Z}/2$ is the area spectrum of the timelike triangle.

The $i\pi$ ambiguity relates to the lift ambiguity from $G_f\in\mathrm{SO}^+(1,3)$ to $\mathrm{SL}(2,\mathbb{C})$. Some ambiguities may be absorbed into gauge transformations $g_{ve}\to -g_{ve}$. First, we consider a single 4-simplex, (7.48) reduces to $\Delta\theta_{e'vef} = 0$, and $\Delta\phi_{e'vef} = -\theta_f \mod \pi$. [Here we use the notation that we move the orientation r from $\Delta\phi$ in (7.48) to the definition of ΔS. Keep in mind ΔS always depends on the orientation r.] However, it is shown in Appendix G that this ambiguity can indeed be absorbed into the gauge transformation of g_{ve}, i.e., if we fix the gauge,

$$\Delta\phi_{e'vef} = -\theta_f(v) \mod 2\pi, \quad (7.50)$$

where $\theta_f(v)$ is the angle between the tetrahedron normals in the 4-simplex at v. Although this fixing of the lift ambiguity only applies to a single 4-simplex, it is sufficient for us to obtain ΔS^Δ_f unambiguously. Applying (7.50) to the case with many 4-simplices

$$\sum_{v\in\partial f}\Delta\phi_{e'vef} = -\sum_{v\in\partial f}\theta_f(v) \mod 2\pi. \quad (7.51)$$

Since $\theta_f(v)$ relates to the dihedral angle $\Theta_f(v)$ by $\theta_f(v) = \pi-\Theta_f(v)$, for an internal f, $\sum_{v\in\partial f}\Delta\phi_{e'vef}$ relates to the deficit angle $\varepsilon_f = 2\pi-\sum_{v\in\partial f}\Theta_f(v)$ by

$$\sum_{v\in\partial f}\Delta\phi_{e'vef} = (2-m_f)\pi-\varepsilon_f \mod 2\pi \quad (7.52)$$

where m_f is the number of $v\in\partial f$. Similarly, for a boundary f, $\sum_{v\in\partial f}\Delta\phi_{e'vef}$ relates to the deficit angle $\theta_f = \pi-\sum_{v\in\partial f}\Theta_f(v)$ by

$$\sum_{v\in\partial f}\Delta\phi_{e'vef} = (1-m_f)\pi-\theta_f \mod 2\pi. \quad (7.53)$$

As a result, the total phase difference is

$$\exp(\Delta S_f) = \exp\left\{2\mathrm{i}r\sum_f{}_{\mathrm{bulk}}A_f[(2-m_f)\pi-\varepsilon_f] + 2\mathrm{i}r\sum_f{}_{\mathrm{boundary}}A_f[(1-m_f)\pi-\theta_f]\right\}. \quad (7.54)$$

The exponent is a Regge action when all the bulk m_f are even, i.e., every internal f has an even number of vertices. Obtaining the Regge calculus only requires all bulk m_f's to be even, while boundary m_f's can be arbitrary, since the boundary terms $A_f(1-m_f)\pi$ do not affect the Regge equation of motion.

The above phase difference is for a general simplicial complex; the result for a single 4-simplex is simply given by removing the bulk terms and letting the boundary $m_f = 1$.

4. Determine the phase for bulk triangles

For the internal faces in the bulk, we can determine the phase at the critical point uniquely.

Recall (7.31), the holonomy $G_f(v) = g_{ve}G_f(e)g_{ev}$ at vertex v reads

$$G_f(v) = e^{-\sum_{v\in\partial f}\theta_{e'vef}2X_f(v)+i\sum_{v\in\partial f}\phi_{e'vef}2X_f(v)}. \tag{7.55}$$

Recall (E73) as we showed in Appendix E, for edges $E_{l1}(v)$ and $E_{l1}(v)$ of the triangle f in the frame of vertex v,

$$\begin{aligned} G_f(v)E_{l1}(v) &= \mu E_{l1}(v),\\ G_f(v)E_{l2}(v) &= \mu E_{l2}(v) \end{aligned} \tag{7.56}$$

where $\mu = (-1)^{\sum_{e\subset\partial f}s_e} = \pm 1$. Here s_e is defined as $s_e = s_{ve} + s_{v'e} + 1$ for edge $e = (v, v')$ with $s_{ve} \in \{0, 1\}$. With edges $E_{l1}(v)$ and $E_{l1}(v)$, the bivector $X_f(v)$ at vertex v can be expressed as

$$X_f(v) = \frac{*(N_{e'}(v)\wedge N_e(v))}{|N_{e'}(v)\wedge N_e(v)|} = \frac{E_{l1}(v)\wedge E_{l2}(v)}{|E_{l1}(v)\wedge E_{l2}(v)|}. \tag{7.57}$$

From (7.56) and (7.57), with the fact that $e^{X_f(v)}$ is a boost, one immediately sees $\mu_e = 1$ and

$$G_f(v) = e^{i\sum_{v\in\partial f}\phi_{e'vef}2X_f(v)} = e^{2r\sum_{v\in\partial f}\phi_{e'vef}\frac{N^\Delta_{e'}\wedge N^\Delta_e}{|N^\Delta_{e'}\wedge N^\Delta_e|}} \tag{7.58}$$

where we use (7.46). As we proved in Appendix F, there exists the spacelike normalized vector $\tilde{N}$ in the plane spanned by N_e and $N_{e'}$ such that

$$G_f(v) = R_N R_{\tilde{N}}. \tag{7.59}$$

From (7.38),

$$\begin{aligned} G_{ve}\tilde{G}_f(e)G_f(e)G_{ev} &= G_{ve}R_u G_f(e)R_u G_f(e)G_{ev}\\ &= R_N G_f(v)R_N G_f(v). \end{aligned} \tag{7.60}$$

Then, it is straightforward to show

$$\begin{aligned} G_{ve}\tilde{G}_f(e)G_f(e)G_{ev} &= R_N G_f(v)R_N G_f(v)\\ &= R_N R_N R_{\tilde{N}} R_N R_N R_{\tilde{N}} = R_{\tilde{N}} R_{\tilde{N}} = 1. \end{aligned} \tag{7.61}$$

Thus,

$$e^{2\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef})*X_f} = 1 \tag{7.62}$$

which leads to

$$\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef}) = 0 \mod \pi. \tag{7.63}$$

The π ambiguity here relates to the lift ambiguity again. Note that, the fixing of the lift ambiguity to these 4-simplices sharing the triangle f as in Appendix G leads to $g_{ve}\tilde{G}_f(e)G_f(e)g_{ev} = 1$. Then we have

$$\sum_{v\in\partial f}(\tilde{\phi}_{e'vef}+\phi_{e'vef}) = 0 \mod 2\pi \tag{7.64}$$

where the π ambiguity is fixed. Combined with (7.52), we have

$$\begin{aligned} \sum_{v\in\partial f}\phi_{e'vef} &= -\sum_{v\in\partial f}\tilde{\phi}_{e'vef}\\ &= \frac{(2-m_f)\pi-\varepsilon_f}{2} \mod \pi. \end{aligned} \tag{7.65}$$

As a result, the total phase for bulk triangles is

$$\exp(S_f) = \exp\left\{ir\sum_{f\text{ bulk}}A_f[(2-m_f)\pi-\varepsilon_f]\right\}. \tag{7.66}$$

Again, the exponent is a Regge action when all bulk m_f are even; i.e., every internal f has an even number of vertices.

Note that the above derivation assumes a uniform orientation $\mathrm{sgn}(V)$, but the asymptotic formula of the spin foam amplitude is given by summing over all possible configurations of orientations. As suggested by [14], at a critical solution, one can make a partition of $\mathcal{K}$ into subregions such that each region has a uniform orientation, so that the above derivation can be applied.

B. Split signature solutions

In this subsection, we focus on a single 4-simplex. We consider a pair of the degenerate solutions $g^\pm_{ve}$ which can be reformulated as nondegenerate solutions in the flipped signature space $(-++-)$ here. When degenerate solutions are gauge equivalent, there exists only a single critical point; then there is a single phase depending on boundary coherent states.

Since (7.25) and (7.32) hold for all $\mathrm{SL}(2,\mathbb{C})$ elements which solve critical equations, they also hold for degenerate solutions $g^\pm_{ve}$. Thus, from (7.22), we have

$$\begin{aligned} g^\pm_{ev}g^\mp_{ev}g^\mp_{ve'}g^\pm_{e'v} &= e^{\mp 2\Delta\theta_{e'vef}X^\pm_f \pm 2i\Delta\phi_{e'vef}X^\pm_f}\\ &= e^{\mp 2\Delta\theta_{e'vef}X^\pm_f}. \end{aligned} \tag{7.67}$$

Notice that since all $g^\pm_{ve} \in \mathrm{SU}(1,1) \subset \mathrm{SL}(2,\mathbb{C})$, we have $2\Delta\phi_{e'vef} = 0 \mod 2\pi$ ($*X^\pm_f$ generates rotations in the $v^g - u$ plane).

From (E87), we have

$$\Phi^{\pm}(g_{ev}\tilde{g}_{ev}\tilde{g}_{ve'}g_{e'v}) = \Phi^{\pm}(g_{ev})\Phi^{\pm}(\tilde{g}_{ev})\Phi^{\pm}(\tilde{g}_{ve'})\Phi^{\pm}(g_{e'v}) = g^{\pm}_{ev}g^{\mp}_{ev}g^{\mp}_{ve'}g^{\pm}_{e'v}. \quad (7.68)$$

Since $\tilde{G}_{ve} = R_u G_{ve} R_u$, we have

$$\Phi^{\pm}(R_{N_e}R_{N_{e'}}) = G^{\pm}_{ev}G^{\mp}_{ev}G^{\mp}_{ve'}G^{\pm}_{e'v}. \quad (7.69)$$

For X_f in flipped signature space M', from the definition of $\Phi^{\pm}$ in (5.10), we have

$$\Phi^{\pm}(*'X_f) = \pm\Phi^{\pm}(X_f) = \pm v^{g\pm}_{ef} = \pm\Phi^{\pm}(X^{\pm}_f) \quad (7.70)$$

where we know $X^{\pm}_f = v^{g\pm}_{ef} \wedge u$ in the degenerate case, and $X^{\pm}_f$ can be regarded as bivectors in $\mathrm{so}(V) \sim \wedge^2 V$. Then we have

$$\Phi^{\pm}(\mathrm{e}^{2\Delta\theta_{e'vef}*'X_f}) = \mathrm{e}^{\mp 2\Delta\theta_{e'vef}X^{\pm}_f} \quad (7.71)$$

where we identify the SO(1,2) acting on V to the one acting on M'.

Therefore, the $\Delta\theta$ contribution to the phase difference in degenerate solutions $\{g^{\pm}\}$ is identified to the $\Delta\theta$ written in flipped signature solutions $\{g\}$ satisfying $\Phi^{\pm}(g) = g^{\pm}$. $\Delta\theta$ is given by

$$R_{N_e}R_{N_{e'}} = \mathrm{e}^{2\Delta\theta_{e'vef}*'X_f} \quad (7.72)$$

where X_f is the bivector from flipped signature solutions

$$X_f = \frac{B_f}{n_f} = -r\frac{*'(N^{\Delta}_{e'} \wedge N^{\Delta}_e)}{|*'(N^{\Delta}_{e'} \wedge N^{\Delta}_e)|}. \quad (7.73)$$

From the fact that geometrically,

$$R_{N_e}R_{N_{e'}} = R_{N^{\Delta}_e}R_{N^{\Delta}_{e'}} = \mathrm{e}^{2\theta_f\frac{N^{\Delta}_e \wedge N^{\Delta}_{e'}}{|N^{\Delta}_e \wedge N^{\Delta}_{e'}|}}, \quad (7.74)$$

where $\theta_f \in \mathbb{R}$ is a boost dihedral angle. We have

$$-r\Delta\theta_{e'vef} = \theta_f, \qquad 2\Delta\phi_{e'vef} = 0 \mod 2\pi \quad (7.75)$$

and the phase difference is

$$\Delta S^{\Delta}_f = 2\mathrm{i}rs_f\theta_f = 2\mathrm{i}r\frac{1}{\gamma}A_f\theta_f \mod \pi\mathrm{i}. \quad (7.76)$$

We can again fix the πi ambiguity by using the method in Appendix G. There is no ambiguity in θ_f since it is a boost angle. As a result,

$$\exp(\Delta S_f) = \exp\left(2\mathrm{i}r\frac{1}{\gamma}A_f\theta_f\right). \quad (7.77)$$

The generalization to the simplicial complex is similar to the nondegenerate case, by substituting every g and $\tilde{g}$ there with $g^{\pm}$.

VIII. CONCLUSION AND DISCUSSION

The present work studies the large-j asymptotics limit of the spin foam amplitude with timelike triangles in a most general configuration on a 4D simplicial manifold with many 4-simplices. It turns out the asymptotics of the spin foam amplitude is determined by the critical configurations of the corresponding spin foam action on the simplicial manifold. The critical configurations have geometrical interpretations as different types of geometries in separated subregions: Lorentzian $(-+++)$ 4-simplices, split $(--++)$ 4-simplices, or degenerate vector geometries. The configurations come in pairs which correspond to opposite global orientations in each subregion. In each subcomplex with globally oriented 4-simplices coming with the same signature, the asymptotic contribution to the spin foam amplitude is an exponential of the Regge action, up to a boundary term which does not affect the Regge equation of motion.

An important remark is that, for a vertex amplitude containing at least one timelike and one spacelike tetrahedron, critical configurations only give Lorentzian 4-simplices, while Euclidean and degenerate vector geometries do not appear. In all known examples of Lorentzian Regge calculus, the geometries are corresponding to such configuration, e.g., the Sorkin triangulation [30] where each 4-simplex contains 4 timelike tetrahedra and 1 spacelike tetrahedron. Since such a configuration only gives Regge-like critical configurations, which is supposed to be the result of the simplicity constraint in spin foam models [5], the result could open a new and promising way towards a better understanding of the imposition of the simplicity constraint. Furthermore, such a configuration also naturally inherits the causal structure to spin foam models, which may open the possibility to build the connection between spin foam models and causal sets theory [31] or causal dynamical triangulation theories [32,33].

With this work, the asymptotics of the Conrady-Hnybida spin foam model, with arbitrary timelike or spacelike nondegenerate boundaries, is now complete. In the present work, we mainly concentrate on the case where each tetrahedron contains both timelike and spacelike triangles, which is the case in all Regge calculus geometry examples. The geometrical interpretation of the case where the tetrahedron contains only timelike triangles is much more complicated and we only identify its critical configurations on special cases where the boundary data satisfies the length matching condition and orientation matching condition. Further investigation is needed for all possible critical configurations in such cases.

Moreover, in the present analysis we do not give the explicit form of measure factors of the asymptotics formula, which is important for the evaluation of the spin foam propagator and amplitude. The measure factor in the EPRL model is related to the Hessian matrix at the critical configuration [34,35]. However, the measure factor for the triangulation with timelike triangles is a much more complicated function of second derivatives of the action, due to the appearance of singularities. A further study of such a kind of multidimensional stationary phase approximation, in particular, the derivation of the measure factor, would be interesting.

The present work opens the possibility to have Regge geometries in Lorentzian Regge calculus emerge as critical configurations from the spin foam model, which may leads to a semiclassical effective description of the spin foam model. Especially, this may lead to an effective equation of motion for symmetry reduced models, e.g., Friedmann-Lemaître-Robertson-Walker cosmology or black holes, from the semiclassical limit of spin foam models.

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APPENDIX A: DERIVATION OF THE REPRESENTATION MATRIX

This Appendix shows the Wigner matrix of the continuous series in unitary irreps of the SU(1,1) group in the large s approximation. We begin with the introduction of the Wigner matrix of the continuous series given in [36]. Then by transformations of hypergeometric functions and the saddle point approximation, we obtain the representation matrix in the large s limit.

1. Wigner matrix

First, let us introduce the parametrization of the SU(1,1) group element v:

$$v(z) = \mathrm{e}^{\mathrm{i}\phi J^3}\mathrm{e}^{\mathrm{i}tK^2}\mathrm{e}^{\mathrm{i}uK^1} = \begin{pmatrix} v_1 & v_2 \\ \bar{v}_2 & \bar{v}_1 \end{pmatrix} \tag{A1}$$

where

$$v_1 = \mathrm{e}^{\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right)\cosh\left(\frac{u}{2}\right) - \mathrm{i}\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{u}{2}\right)\right) \tag{A2}$$

$$v_2 = e^{\frac{\mathrm{i}\phi}{2}}\left(\mathrm{i}\cosh\left(\frac{u}{2}\right)\sinh\left(\frac{t}{2}\right) - \cosh\left(\frac{t}{2}\right)\sinh\left(\frac{u}{2}\right)\right). \tag{A3}$$

Note that the generators defined here are a complex version of what we used in the main part. In this parametrization, the Wigner matrix, which is defined as

$$D^j_{m\lambda\sigma}(v) = \langle j, m|v|j\lambda\sigma\rangle, \tag{A4}$$

can be expressed by [36]

$$D^j_{m\lambda\sigma} = \mathrm{e}^{\mathrm{i}m\phi}d^j_{m\lambda\sigma}\mathrm{e}^{\mathrm{i}\lambda u} = \mathrm{e}^{\mathrm{i}m\phi}S^j_{m\lambda\sigma}(T^j_{m\lambda}F^j_{m,\mathrm{i}\lambda}(\beta) - (-1)^\sigma T^j_{-m\lambda}F^j_{-m,\mathrm{i}\lambda}(\bar{\beta}))\mathrm{e}^{\mathrm{i}\lambda u} \tag{A5}$$

where

$$F^j_{m,\mathrm{i}\lambda}(\beta) = (1-\beta)^{(m-\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}{}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta) \tag{A6}$$

$$T^j_{m\lambda} = \frac{1}{\Gamma(-m-j)\Gamma(m+1+\mathrm{i}\lambda)}. \tag{A7}$$

Here ${}_2F_1(a,b,c,z)$ refers to Gaussian hypergeometric function, and $\Gamma(z)$ is the Gamma function. The normalization factor $S^j_{m\lambda\sigma}$ reads

$$S^j_{m\lambda\sigma} = \sqrt{\frac{\Gamma(m-j)}{\Gamma(m+j+1)}}\frac{2^{j-1}\Gamma(-j+\mathrm{i}\lambda)}{\mathrm{i}^\sigma \sin(\pi/2(-j-\mathrm{i}\lambda+\sigma))} \tag{A8}$$

with $\beta = (1-\mathrm{i}\sinh(t))/2$.

Above, Eq. (A5) can be written in terms of normalized spinors $v=(v_1,v_2)$ in the SU(1,1) inner product $\langle v,v\rangle = 1$. According to the parametrization, we have

$$v_1+v_2 = \mathrm{e}^{-\frac{u}{2}+\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right)+\mathrm{i}\sinh\left(\frac{t}{2}\right)\right), \qquad v_1-v_2 = \mathrm{e}^{\frac{u}{2}+\frac{\mathrm{i}\phi}{2}}\left(\cosh\left(\frac{t}{2}\right)-\mathrm{i}\sinh\left(\frac{t}{2}\right)\right). \tag{A9}$$

The Wigner matrix D can be written in terms of v and $\bar{v}$

$$D^j_{m\lambda\sigma} = S^j_{m\lambda\sigma}(T^j_{m\lambda}F^j_{m,\mathrm{i}\lambda}(v) - (-1)^\sigma T^j_{-m\lambda}F^j_{-m,\mathrm{i}\lambda}(\bar{v})) \tag{A10}$$

with

$$\begin{aligned} F^j_{m,\mathrm{i}\lambda}(v) &= 2^{-m}(v_1+v_2)^{(m-\mathrm{i}\lambda)}(v_1-v_2)^{(m+\mathrm{i}\lambda)} \\ &\quad\times {}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; (\bar{v}_1+\bar{v}_2)(v_1-v_2)/2). \end{aligned} \tag{A11}$$

2. Asymptotics of Gauss hypergeometric function

According to (A5), we need to evaluate the hypergeometric function

$${}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta), \qquad {}_2F_1(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta). \tag{A12}$$

The function itself is complicated. However, we only need the asymptotics behavior with $j\sim m\sim\lambda\gg 1$ in our case. According to (2.29), m is chosen to be $n/2$ which is related to $j=-1/2+\mathrm{i}s$ by the simplicity constraint (2.9). Correspondingly, λ is also chosen to be related to s.

a. Transformation of original function

First, we would like to transform the original function to a more convenient form. According to the transformation properties of hypergeometric function, we have

$${}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta) = (1-\beta)^{-m+\mathrm{i}\lambda}{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta) \tag{A13}$$

$${}_2F_1(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta) = (\beta)^{m+\mathrm{i}\lambda}{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta) \tag{A14}$$

$$\begin{aligned} &\frac{\sin(\pi(-m+\mathrm{i}\lambda))}{\pi\Gamma(m+\mathrm{i}\lambda+1)}{}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta) \\ &\quad= \beta^{-m-\mathrm{i}\lambda}\frac{{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta)}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \\ &\quad - (1-\beta)^{-m+\mathrm{i}\lambda}\frac{{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta)}{\Gamma(-m+\mathrm{i}\lambda+1)\Gamma(-j+m)\Gamma(j+m+1)} \end{aligned} \tag{A15}$$

$$\begin{aligned} &\times\frac{\sin(\pi(m+\mathrm{i}\lambda))}{\pi\Gamma(-m+\mathrm{i}\lambda+1)}{}_2F_1(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta) \\ &= (1-\beta)^{m-\mathrm{i}\lambda}\frac{{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; -m-\mathrm{i}\lambda+1; \beta)}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \\ &\quad - (\beta)^{m+\mathrm{i}\lambda}\frac{{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta)}{\Gamma(m+\mathrm{i}\lambda+1)\Gamma(-j-m)\Gamma(j-m+1)}. \end{aligned} \tag{A16}$$

From (A14) and (A15), we have

$$
\begin{aligned}
{}_2F_1(-j-m, j-m+1; -m+\mathrm{i}\lambda+1; 1-\beta) &= \Gamma(-m+\mathrm{i}\lambda+1)\Gamma(-j+m)\Gamma(j+m+1) \\
&\times\Bigg(-\frac{(\beta)^{m+\mathrm{i}\lambda}\sin(\pi(-m+\mathrm{i}\lambda))}{\pi\Gamma(m+\mathrm{i}\lambda+1)}\,{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta) \\
&+\frac{(1-\beta)^{m-\mathrm{i}\lambda}}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\,{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta)\Bigg). \qquad \text{(A17)}
\end{aligned}
$$

Similarly, from (A13) and (A16), we have

$$
\begin{aligned}
{}_2F_1(-j+m, j+m+1; m+\mathrm{i}\lambda+1; \beta) &= \Gamma(m+\mathrm{i}\lambda+1)\Gamma(-j-m)\Gamma(j-m+1) \\
&\times\Bigg(-\frac{(1-\beta)^{-m+\mathrm{i}\lambda}\sin(\pi(m+\mathrm{i}\lambda))}{\pi\Gamma(-m+\mathrm{i}\lambda+1)}\,{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta) \\
&+\frac{\beta^{-m-\mathrm{i}\lambda}}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\,{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; -m-\mathrm{i}\lambda+1; \beta)\Bigg). \qquad \text{(A18)}
\end{aligned}
$$

Then, in terms of (A13) and (A17), the function $d^j_{m\lambda\sigma}$ can be written as

$$
\begin{aligned}
d^j_{m\lambda\sigma}(\beta) = S^j_{m\lambda\sigma}\Bigg[&(1+(-1)^\sigma\tan(\pi(-m+\mathrm{i}\lambda))) \\
&\times\frac{(1-\beta)^{(-m+\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}\,{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta)}{\Gamma(-m-j)\Gamma(m+\mathrm{i}\lambda+1)} \\
&-(-1)^\sigma\frac{\beta^{(-m-\mathrm{i}\lambda)/2}(1-\beta)^{(m-\mathrm{i}\lambda)/2}\,{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta)}{\Gamma(m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j+m+1)}\Bigg]. \qquad \text{(A19)}
\end{aligned}
$$

Now we only need to evaluate the hypergeometric function ${}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta)$, since ${}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; m-\mathrm{i}\lambda+1; 1-\beta)$ is nothing else but the complex conjugation of the previous one. Similarly, starting from (A14) and (A18), we have

$$
\begin{aligned}
d^j_{m\lambda\sigma}(\beta) = S^j_{m\lambda\sigma}\Bigg[&(-\tan(\pi(m+\mathrm{i}\lambda))-(-1)^\sigma) \\
&\times\frac{(1-\beta)^{(-m+\mathrm{i}\lambda)/2}\beta^{(m+\mathrm{i}\lambda)/2}\,{}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; -m+\mathrm{i}\lambda+1; 1-\beta)}{\Gamma(m-j)\Gamma(-m+\mathrm{i}\lambda+1)} \\
&+\frac{\beta^{(-m-\mathrm{i}\lambda)/2}(1-\beta)^{(m-\mathrm{i}\lambda)/2}\,{}_2F_1(j-\mathrm{i}\lambda+1, -j-\mathrm{i}\lambda; -m-\mathrm{i}\lambda+1; \beta)}{\Gamma(-m-\mathrm{i}\lambda+1)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j-m+1)}\Bigg]. \qquad \text{(A20)}
\end{aligned}
$$

Clearly the two expression obey the relation $d^j_{m\lambda\sigma}(\beta) = -(-1)^\sigma d^j_{-m\lambda\sigma}(\bar{\beta})$.

b. Saddle point approximation

From (A19), we need the large s approximation of the hypergeometric function ${}_2F_1(j+\mathrm{i}\lambda+1, -j+\mathrm{i}\lambda; m+\mathrm{i}\lambda+1; \beta)$. Here we will only concentrate on the parameters such that $m = n/2 = \gamma s$ and $\lambda \sim s$ are satisfied. In this choice, all the parameters will scale together with s. A choice of λ is $\lambda = -s$. The generalization to parameters where m and λ scale with Λ but takes a different value is straightforward. Note that the smearing of λ requires us to calculate $\lambda = -s_0 + \epsilon$ where $\epsilon \ll \lambda$.

For simplicity, we will transform the original function as

$$\begin{aligned}&{}_2F_1(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta)\\&=(1-\beta)^{-1/2}{}_2F_1\left(j+\mathrm{i}\lambda+1,j+m+1;m+\mathrm{i}\lambda+1;\frac{\beta}{\beta-1}\right)\quad\text{with}\quad\lambda=-s,m=\gamma s,\gamma>0\\&=(1-\beta)^{-1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2}+(\gamma+\mathrm{i})s;(\gamma-\mathrm{i})s+1;\frac{\beta}{\beta-1}\right).\end{aligned}\tag{A21}$$

We will use the integral representation for hypergeometric functions [37]:

$${}_2F_1(a,b;c;z)=\frac{\Gamma(1+b-c)\Gamma(c)}{2\pi\mathrm{i}\Gamma(b)}\int_0^{1+}\frac{t^{b-1}(t-1)^{c-b-1}}{(1-zt)^a}dt,\quad\text{if } c-b\notin N\ \&\ \Re(b)>0.\tag{A22}$$

The validity region for these equations is $|\arg(1-z)|<\pi$. In (A22), the integration path is the anticlockwise loop that starts and ends at $t=0$, encircles the point $t=1$, and excludes the point $t=1/z$. In our case, we have $\Re(c-b)=1/2$ and $\Re(b)=1/2+m=1/2+\gamma s$ which satisfy the requirement. Thus, with (A22) we rewrite the original hypergeometric function as

$${}_2F_1\left(\frac{1}{2},\frac{1}{2}+(\gamma+\mathrm{i})s;(\gamma-\mathrm{i})s+1;\frac{\beta}{\beta-1}\right)=\frac{G(s)}{2\pi\mathrm{i}}\int_0^{1+}dtf(t,\beta)\mathrm{e}^{s\Psi(t)}\tag{A23}$$

where $\Psi(t)$ and $f(t,\beta)$ are

$$\Psi(t)=(\gamma+\mathrm{i})\ln t-2\mathrm{i}\ln(t-1),\qquad f(t,\beta)=\left(t(t-1)\left(1-\frac{\beta t}{\beta-1}\right)\right)^{-\frac{1}{2}}\tag{A24}$$

and $G(s)$ is

$$G(s)=\frac{\Gamma(\frac{1}{2}+2\mathrm{i}s)\Gamma((\gamma-\mathrm{i})s+1)}{\Gamma(\frac{1}{2}+(\gamma+\mathrm{i})s)}\sim\frac{\sqrt{2\pi(\gamma-\mathrm{i})s}((\gamma-\mathrm{i}))^{(\gamma-\mathrm{i})s}(2\mathrm{i})^{2\mathrm{i}s}}{((\gamma+\mathrm{i}))^{(\gamma+\mathrm{i})s}}.\tag{A25}$$

Here we use the asymptotic formula of Γ functions

$$\Gamma(z)\sim\sqrt{2\pi}z^{z-1/2}\mathrm{e}^{-z}.\tag{A26}$$

Note that $|G(s)|\sim\sqrt{s}\exp(-\pi s)$. We will see later the contribution form $\exp(-\pi s)$ will cancel the contribution form $|\exp(s\Psi(t))|$ at the saddle point t_0.

Clearly when $\beta/(\beta-1)\neq 1$, we have three branch points $t=0$, $t=1$, and $t=(\beta-1)/\beta$ for $f(t,z)$ and two branch points $t=0$ and $t=1$ for $\Psi(t)$. The branch cuts for $\Psi(t)$ on the real axis are given by $(-\infty,0]$ and $(0,1]$, which can be seen in Fig. 1. We need to exclude the point $t_\beta=(\beta-1)/\beta$ from the path.

There is one saddle point t_0 given by the solution of the equation $\Psi'(t)=0$

$$t_0=\frac{\gamma+\mathrm{i}}{\gamma-\mathrm{i}}.\tag{A27}$$

Consequently, at the saddle point $\Re(\Psi(t_0))=\pi$. The steepest decent and ascent curves are shown in Fig. 1. The original integration path then can be deformed as the steepest decent curve and two equal real part curves of $\Psi(t)$.

The corresponding value at the saddle point t_0 reads

$$\mathrm{e}^{s\Psi(t_0)}=\left(\frac{\gamma+\mathrm{i}}{\gamma-\mathrm{i}}\right)^{(\gamma+\mathrm{i})s}\left(\frac{2\mathrm{i}}{\mathrm{i}+\gamma}\right)^{-2\mathrm{i}s},\qquad f(t_0,\beta)=\frac{(2\mathrm{i})^{\mathrm{i}\epsilon}}{\sqrt{2\mathrm{i}}}\left(\frac{\gamma-\mathrm{i}(1-2\beta)}{1-\beta}\right)^{-\frac{1}{2}-\mathrm{i}\epsilon}\left(\frac{\gamma+\mathrm{i}}{(\gamma-\mathrm{i})^3}\right)^{-\frac{1}{2}}\tag{A28}$$

and

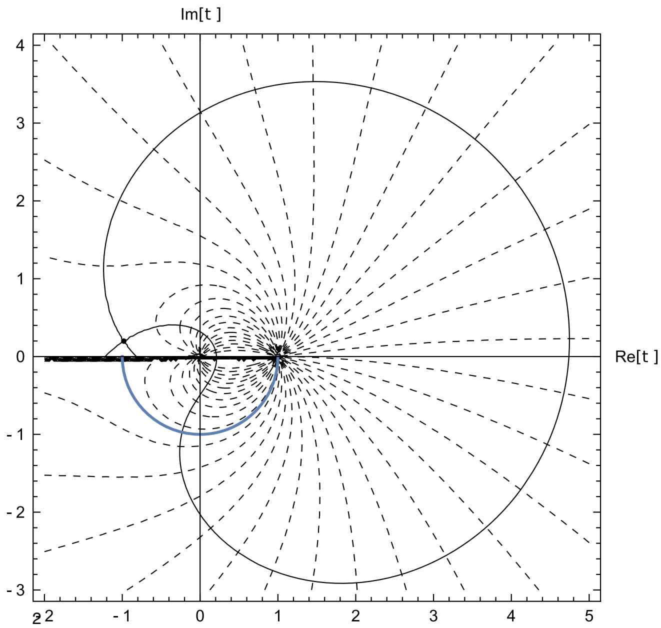

FIG. 1. The value of $\Re(\Phi(t))$ (dash line) and the steepest decent and ascent path (black line) over the t-complex plane for $\gamma = 0.1$. The blue line shows the position of possible poles t_β of f.

$$\Phi''(t_0) = \frac{-\mathrm{i}(\gamma-\mathrm{i})^3}{2(\gamma+\mathrm{i})},$$
$$\alpha = \arg(n\Psi''(t_0)) = \frac{\pi}{2} - \arg\left(\frac{\mathrm{sgn}(\gamma+\mathrm{i})}{\mathrm{sgn}(\gamma-\mathrm{i})^3}\right),$$
$$\theta = \frac{\pi-\alpha}{2}. \tag{A29}$$

Then by the saddle point approximation we have

$$I = \frac{G(n)}{2\pi\mathrm{i}}\int_C dt f(t)\mathrm{e}^{s\Psi(t)}$$
$$\sim \frac{\mathrm{e}^{s\Psi(t_0)+\mathrm{i}\theta}}{\sqrt{n}}\left(f(t_0)\sqrt{\frac{2\pi}{|\Phi''(t_0)|}} + \mathcal{O}(s^{-1})\right), \quad \text{as } s\to\infty$$
$$\sim \sqrt{\gamma-\mathrm{i}}\left(\frac{\gamma-\mathrm{i}(1-2\beta)}{1-\beta}\right)^{-1/2} + \mathcal{O}(s^{-1/2}). \tag{A30}$$

Note that the generalization to $\lambda = -s_0+\delta$ or $s = s_0+\delta$ leads to a modification with $\left(\frac{\gamma-\mathrm{i}(1-2\beta)}{1-\beta}\right)^{-\mathrm{i}\delta}$.

We also need to consider the branch point $t_\beta = (\beta-1)/\beta$. When it lives outside the contour C, the integration over contour C is exactly the path required by (A22). Thus, in this case we get the asymptotics of the hypergeometric function with the usual saddle point method as (A30). However, when $(\beta-1)/\beta$ inside the contour, we need to deform the contour to exclude the branch point and the branch cut due to $(\beta-1)/\beta$. A possible way is we choose the branch cut along one of the steepest decent paths starting at $(1-\beta)/\beta$, and deform the contour C excluding the branch point and branch cut, which may give a nontrivial contribution to the asymptotic expansion. Since $t_\beta = (\beta-1)/\beta$ is a 1/2 order branch point, according to [38], in this case, the contribution coming from the branch point is given by

$$I_1 \sim 2\sqrt{\pi}\frac{G(n)}{2\pi\mathrm{i}}\mathrm{e}^{s\Psi(t_\beta)}f(t_\beta,\beta)\left(t_\beta - \frac{\beta-1}{\beta}\right)^{\frac{1}{2}}\left(\frac{1}{s|\Psi'(t_\beta)|}\right)^{\frac{1}{2}} + \mathcal{O}(s^{-1/2})$$
$$\sim (1-\beta)^{(\gamma+\mathrm{i})s}\beta^{(-\gamma+\mathrm{i})s}\frac{\sqrt{2(\gamma-\mathrm{i})}(-1)^{\gamma s}2^{2\mathrm{i}s}((\gamma-\mathrm{i}))^{(\gamma-\mathrm{i})s}}{((\gamma+\mathrm{i}))^{(\gamma+\mathrm{i})s}}\sqrt{\frac{1-\beta}{|-\mathrm{i}(1-2\beta)+\gamma|}} + \mathcal{O}(s^{-1/2}). \tag{A31}$$

Since the asymptotics contribution contains the power of s in terms of $\mathrm{e}^{s\Psi(t)}$, the full asymptotics of the function will come from the largest $\mathrm{Re}(\Psi(t))$ of t_0 and t_β. In our case, t_β is in the negative imaginary half plane

$$t_\beta = \frac{\beta-1}{\beta} = \frac{\bar{\beta}}{\beta}. \tag{A32}$$

And it is easy to show

$$\Re(\Psi(t_\beta)) = \begin{cases} -\pi, & t<0 \\ \pi & t>0 \end{cases}. \tag{A33}$$

When $t>0$, the contribution from t_β is lower than t_0 in arbitrary order after multiplying by the power of s, and the final result is given by (A30). The contribution from the branch point only exists when $\sinh(t)+\gamma<\epsilon_0<0$ and the contribution reads

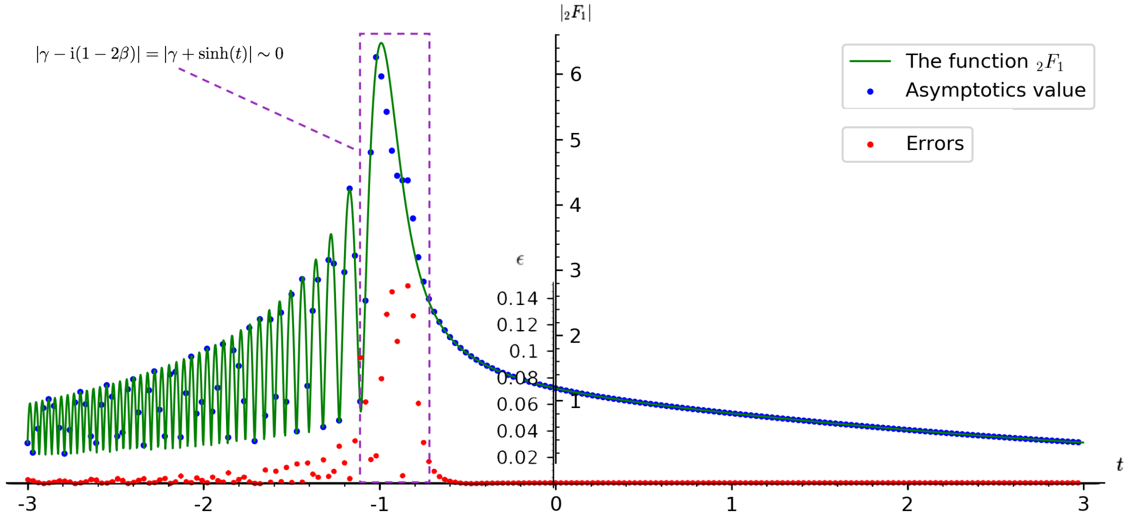

FIG. 2. The function ${}_2F_1(j+\mathrm{i}\lambda+1,-j+\mathrm{i}\lambda;m+\mathrm{i}\lambda+1;\beta)$ as shown in (A21) and its asymptotics result I given as (A30), (A34), and (A35), respectively, with $t\in[-3,3]$, $s=100$, $\gamma=1$. The absolute error is defined as $\epsilon=|(|I|-|{}_2F_1|)|/|{}_2F_1|$.

$$I=I_0-I_1. \tag{A34}$$

And in this case, the final asymptotics is given by the sum of (A30) and (A31). A special case is when the branch point is located near the critical point $|t_0-t_\beta|\le\epsilon_0$, where the result is

$$\begin{aligned} I&\sim\frac{G(n)}{2\pi\mathrm{i}}\left(\frac{\pi\mathrm{e}^{\mathrm{i}\pi(-1/4+\theta/2)}}{\Gamma(1/4)}f(t_0)\left(t_0-\frac{\beta-1}{\beta}\right)^{\frac{1}{2}}\left(\frac{2}{s|\Psi''(t_0)|}\right)^{-\frac{1}{4}}\mathrm{e}^{s\Psi(t_0)}+\mathcal{O}(s^{-3/4})\right)\\ &\sim\frac{2\sqrt{\pi}s^{1/4}}{\Gamma(1/4)}(-\mathrm{i}(\gamma-\mathrm{i})(\gamma+\mathrm{i}))^{1/4}+\mathcal{O}(s^{-1/4}). \end{aligned} \tag{A35}$$

Note that, for the continuous approximation on β, we have $\epsilon_0\sim s^{-1/2}$. Figure 2 shows the error level of the above asymptotics result when $s=100$.

c. Result

Now we can write out the final result. According to (A21), we have

$${}_2F_1(j-\mathrm{i}s+1,-j-\mathrm{i}s;n/2-\mathrm{i}s+1;\beta(t))\sim\frac{\sqrt{\gamma-\mathrm{i}}(1+\mathrm{i})}{\sqrt{2(\mathrm{i}\gamma+(1-2\beta))}}+\mathcal{O}(s^{-1/2}). \tag{A36}$$

From (A19), for $\sinh(t)>-\gamma$ we have

$$\begin{aligned} d_{0_{n/2,-\mathrm{i}\lambda,\sigma}}^{j}\sim S_{m\lambda\sigma}^{j}\left(\frac{1}{\sqrt{s(\gamma-\mathrm{i}(1-2\beta))}}+\mathcal{O}(s^{-1})\right)&\left(\frac{(1-(-1)^\sigma\mathrm{i})(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right.\\ &\left.-(-1)^\sigma\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right) \end{aligned} \tag{A37}$$

where we use the approximation

$$\Gamma\left(-\frac{n}{2}-j\right)\Gamma\left(\frac{n}{2}+\mathrm{i}\lambda+1\right)\sim2\pi\sqrt{(\gamma-\mathrm{i})s}(-(\gamma+\mathrm{i})s)^{-(\gamma+\mathrm{i})s}((\gamma-\mathrm{i})s)^{(\gamma-\mathrm{i})s}\mathrm{e}^{2\mathrm{i}s} \tag{A38}$$

$$\Gamma\left(\frac{n}{2}-\mathrm{i}\lambda+1\right)\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)\Gamma^{-1}(j+m+1)\sim\sqrt{2}\pi\sqrt{(\gamma+\mathrm{i})s}(-2\mathrm{i}s)^{-2\mathrm{i}s}\mathrm{e}^{2\mathrm{i}s} \tag{A39}$$

for $\sinh(t) < -\gamma$, the contribution from the extra branch point reads

$$d_{1}{}^{j}_{n/2,-\mathrm{i}s,\sigma} \sim S^j_{m\lambda\sigma}\left(\frac{\sqrt{2}}{\sqrt{s|\gamma-\mathrm{i}(1-2\beta)|}}+\mathcal{O}(s^{-1})\right)\left(\frac{(1-(-1)^\sigma\mathrm{i})(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}}{\sqrt{2}\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}-(-1)^\sigma\frac{\sqrt{2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right). \tag{A40}$$

One checks that the final result is approximately

$$\begin{aligned} d^j_{n/2,-\mathrm{i}s,\sigma} = d_{0}{}^{j}_{n/2,-\mathrm{i}s,\sigma} - d_{1}{}^{j}_{n/2,-\mathrm{i}s,\sigma} &\sim S^j_{m\lambda\sigma}\left(\frac{1}{\sqrt{s|\gamma-\mathrm{i}(1-2\beta)|}}++\mathcal{O}(s^{-1})\right) \\ &\times\left(\frac{(1-(-1)^\sigma\mathrm{i})(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}-(-1)^\sigma\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right). \end{aligned} \tag{A41}$$

When $|\gamma-\mathrm{i}(1-2\beta)|<\epsilon$, which means the branch point near the saddle point, we have

$$\begin{aligned} d^j_{n/2,-\mathrm{i}s,\sigma} \sim S^j_{m\lambda\sigma}&\left(\frac{2\sqrt{\pi}(-\mathrm{i}(1+\gamma^2))^{1/4}s^{1/4}}{\Gamma(1/4)\sqrt{s}}+\mathcal{O}(s^{-3/4})\right)\left(\frac{(1-(-1)^\sigma\mathrm{i})(1-\beta)^{(-\frac{n}{2}+\mathrm{i}\lambda)/2}\beta^{(\frac{n}{2}+\mathrm{i}\lambda)/2}}{\Gamma(-\frac{n}{2}-j)\Gamma(\frac{n}{2}+\mathrm{i}\lambda+1/2)}\right. \\ &\left.-(-1)^\sigma\frac{\beta^{(-\frac{n}{2}-\mathrm{i}\lambda)/2}(1-\beta)^{(\frac{n}{2}-\mathrm{i}\lambda)/2}}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)}\right). \end{aligned} \tag{A42}$$

3. Full representation matrix

According to (A10), now we can write out the D matrix in terms of the group elements v:

$$\begin{aligned} D_{m,\lambda}(z) = \frac{S^j_{m,\lambda,\sigma}}{\sqrt{s_0}}&\left(\frac{H(|\gamma+\Im(\bar{v}_1v_2)|-\epsilon)}{\sqrt{|\gamma+\Im(\bar{v}_1v_2)|}}+H(\epsilon-|\gamma+\Im(\bar{v}_1v_2)|)\frac{2\sqrt{\pi}(1+\gamma^2)^{1/4}s_0^{1/4}}{\sqrt{\pi}\Gamma(1/4)}\right) \\ &\times\left(T^j_{+\sigma}\left(\frac{v_1-v_2}{\sqrt{2}}\right)^{m+\mathrm{i}\lambda}\left(\overline{\frac{v_1-v_2}{\sqrt{2}}}\right)^{-m+\mathrm{i}\lambda}-T^j_{-\sigma}\left(\frac{v_1+v_2}{\sqrt{2}}\right)^{m-\mathrm{i}\lambda}\left(\overline{\frac{v_1+v_2}{\sqrt{2}}}\right)^{-m-\mathrm{i}\lambda}\right)+\mathcal{O}(s^{-3/4}) \end{aligned} \tag{A43}$$

where H is the Heaviside step function

$$H(x)\begin{cases}0 & x\leq 0\\ 1 & x>0\end{cases} \tag{A44}$$

and ϵ is defined as

$$\epsilon=\frac{\Gamma(1/4)^2}{4\pi\sqrt{(1+\gamma^2)s}} \tag{A45}$$

such that D is continuous for v. Note that the contribution from $|\gamma+\Im(\bar{v}_1v_2))|<\epsilon$ is actually a regulator of the $1/2$ order singular points because of $|\gamma+\Im(\bar{v}_1v_2))|$. In the inner product this regulator naturally arises as the asymptotics with the $1/2$ order singular points. In this sense, we can ignore the regulator since we are only interested in the inner product in the amplitude. The constant is given by

$$T^j_{+\sigma}=\frac{1-(-1)^\sigma\mathrm{i}}{\Gamma(-m-j)\Gamma(m-j)} \tag{A46}$$

$$T^j_{-\sigma}=\frac{(-1)^\sigma}{\Gamma(j+\mathrm{i}\lambda+1)\Gamma(\mathrm{i}\lambda-j)} \tag{A47}$$

with S given in (A8). In the asymptotics limit, we have

$$S^j_{m\lambda\sigma}\bar{S}^j_{m\lambda\sigma}\sim\frac{\pi}{2\cosh(2\pi s)}, \tag{A48}$$

$$\begin{aligned} T^j_1\bar{T}^j_1 &\sim \frac{2\cos(\pi(-m-\mathrm{i}s))\cos(\pi(m-\mathrm{i}s))}{\pi^2} \\ &\sim\frac{\cosh(2\pi s)}{\pi^2},\quad \text{when } s\gg 1 \end{aligned} \tag{A49}$$

$$T^j_2\bar{T}^j_2\sim\frac{\cosh(2\pi s)}{\pi^2}, \tag{A50}$$

where we use the asymptotic approximation of the Gamma function

$$\lim_{z\to\infty}\frac{\Gamma(z+\epsilon)}{\Gamma(z)z^{\epsilon}}=1. \tag{A51}$$

From the parity property of the representation matrix, we have

$$D^{\sigma j}_{-m,\lambda}(v)=-(-1)^{\sigma}\mathrm{e}^{-\mathrm{i}\pi m}D^{\sigma j}_{m,\lambda}(\bar{v}). \tag{A52}$$

APPENDIX B: ANALYSIS OF SINGULARITIES AND CORRESPONDING STATIONARY PHASE APPROXIMATION

In this Appendix we concentrate on the analysis of singularities appearing in the denominator of the integrand of the vertex amplitude.

1. Analysis of singularities

For simplicity, we consider one vertex case for some v mainly. As we show, the amplitude enrolls the integration in the form

$$I=\int\prod_e dg_{ve}\int\prod_f \Omega_{vf}\prod_f\frac{1}{h_{vef}h_{ve'f}}e^{S_{vf}} \tag{B1}$$

where h is a real valued function

$$h_{vef}=|\langle Z_{vef},Z_{vef}\rangle|\sqrt{\left|\gamma-\mathrm{i}\left(1-\frac{2\langle l^-_{ef},Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle}{\langle Z_{vef},Z_{vef}\rangle}\right)\right|}. \tag{B2}$$

Here, each dual face is determined by two edges $f=(e,e')$. Note that the square root part inside h_{vef} is the spinor representation for the square root term inside the Wigner d matrix:

$$\left|\gamma-\mathrm{i}\left(1-\frac{2\langle l_{ef},Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle}{\langle Z_{vef},Z_{vef}\rangle}\right)\right|=|\gamma+\Im(v_1\bar{v}_2)|. \tag{B3}$$

The zero sets of h are given by $\langle Z_{vef},Z_{vef}\rangle=0$ or $|\gamma+\Im(v_1\bar{v}_2)|=0$.

We can rewrite the original $\langle Z_{vef},Z_{vef}\rangle$ as

$$\langle Z_{vef},Z_{vef}\rangle=2\Re(\langle l^-_{ef},Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle)=\Re(f) \tag{B4}$$

where we define f as

$$f\coloneqq 2\langle l^-_{ef},Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle. \tag{B5}$$

In this notation h_{vef} becomes

$$h_{vef}=|\Re(f)|\sqrt{\left|\gamma+\frac{\Im(f)}{\Re(f)}\right|}=|f||\cos(\phi_f)|\sqrt{|\gamma+\tan(\phi_f)|}. \tag{B6}$$

Suppose the functions f are linearly independent to each other. This requirement is the same as requirement that the boundary tetrahedron $l^{\pm}_{ef}$ be nondegenerate. In this case, we can define a coordinate transformation among the set of the original coordinates $(z,g)\to(\Re(f),\Im(f),z',g')$. The coordinate transformation only transfers among the number of f variables and leaves the left invariant; e.g., we only transfer 40 variables in one vertex case and leave the other four invariant. The elements of the Jacobian matrix of the transformation $J(f)$ is given by

$$\frac{\partial(\Re(f_{vef}))}{\partial z}=\overline{\frac{\partial(\Re(f_{vef}))}{\partial\bar{z}}}=\delta_z\langle Z_{vef},Z_{vef}\rangle=(g_{ve}\eta Z_{vef})^T \tag{B7}$$

$$\frac{\partial(\Im(f_{vef}))}{\partial z}=\mathrm{i}(\delta_z\langle Z_{vef},Z_{vef}\rangle-2\delta_z\langle l^-_{ef},Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle=\mathrm{i}((g_{ve}\eta Z_{vef}-2g\eta l^+_{ef}\langle l^-_{ef},Z_{vef}\rangle)^T \tag{B8}$$

$$\frac{\partial(\Re(f_{vef}))}{\partial g}=\delta_g\langle Z_{vef},Z_{vef}\rangle=\langle L^\dagger Z_{vef},Z_{vef}\rangle+\langle Z_{vef},L^\dagger Z_{vef}\rangle \tag{B9}$$

$$\frac{\partial(\Im(f_{vef}))}{\partial g}=\mathrm{i}(\langle L^\dagger Z_{vef},Z_{vef}\rangle+\langle Z_{vef},L^\dagger Z_{vef}\rangle-2\langle l^-_{ef},Z_{vef}\rangle\langle L^\dagger Z_{vef},l^+_{ef}\rangle-2\langle l^-_{ef},L^\dagger Z_{vef}\rangle\langle Z_{vef},l^+_{ef}\rangle) \tag{B10}$$

where L represents the generators of $\mathrm{SL}(2,\mathbb{C})$. Note that $\delta_g\langle Z_{vef},Z_{vef}\rangle$ is zero when L are SU(1,1) generators. However, the Jacobian is nonzero in general; e.g., in one vertex case of vertex v, we have the nontrivial contribution from terms like

$$\partial_{g_1}(13,14,15),\qquad\partial_{g_2}(21,24,25),\qquad\partial_{g_3}(31,32,35),\qquad\partial_{g_4}(42,43,45),\qquad\partial_z(12,23,34,41,51,52,53,54) \tag{B11}$$

where 12 is the representation of the ef label in terms of numbers labeling edges and corresponding faces (e_1, e_2). Apart from those 0 in (B9), other zeros of the matrix elements are only possible when $Z = \zeta l^{\pm}$. The Jacobian matrix in this case is given by ($Z = \zeta l^{+}$ as an example),

$$\frac{\partial(\Re(f_{vef}))}{\partial z} = \overline{\frac{\partial(\Re(f_{vef}))}{\partial \bar{z}}} = (g_{ve}\eta Z_{vef})^T \tag{B12}$$

$$\frac{\partial(\Im(f_{vef}))}{\partial z} = \overline{\frac{\partial(\Im(f_{vef}))}{\partial \bar{z}}} = -\mathrm{i}(g_{ve}\eta Z_{vef})^T \tag{B13}$$

$$\frac{\partial(\Re(f_{vef}))}{\partial g} = \langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle = \begin{cases} 0, & L = F \\ 2\langle Z_{vef}, L^\dagger Z_{vef}\rangle, & L = \mathrm{i}F \end{cases} \tag{B14}$$

$$\frac{\partial(\Im(f_{vef}))}{\partial g} = \mathrm{i}(\langle Z_{vef}, L^\dagger Z_{vef}\rangle - \langle L^\dagger Z_{vef}, Z_{vef}\rangle) = \begin{cases} 2\mathrm{i}\langle Z_{vef}, L^\dagger Z_{vef}\rangle, & L = F \\ 0, & L = \mathrm{i}F \end{cases}. \tag{B15}$$

Clearly the Jacobian matrix is still well defined and leads to a nonzero Jacobian.

After this coordinate transformation, the original integration becomes

$$I = \prod_v \int \frac{\Omega'}{J(f)} \prod_{e,f} \mathrm{d}\Re(f_{vef})\mathrm{d}\Im(f_{vef}) \prod_f \frac{e^{S_{vf}}}{|\Re(f_{vef})||\Re(f_{ve'f})|\sqrt{\left|\gamma + \frac{\Im(f_{vef})}{\Re(f_{vef})}\right|}\sqrt{\left|\gamma + \frac{\Im(f_{ve'f})}{\Re(f_{ve'f})}\right|}}. \tag{B16}$$

With a further polar coordinate transformation

$$\rho_{vef} = \sqrt{\Re(f_{vef})^2 + \Im(f_{vef})^2}, \qquad \phi_{vef} = \arg(f_{vef}) \in [0, \pi/2) \tag{B17}$$

whose Jacobian is given by

$$J^1_{vef} = \frac{1}{\rho_{vef}}. \tag{B18}$$

The Jacobian is well defined except on the points where $|f| = 0$. After the coordinates transformation, we have

$$I = \int \Omega' \prod_{e,f} \int \mathrm{d}\rho_{vef} \int_0^{\pi/2} \mathrm{d}\phi_{vef} \frac{1}{J(\rho,\phi)} \prod_f \frac{e^{S_{vf}}}{|\cos(\phi_{vef})||\cos(\phi_{ve'f})|\sqrt{|\gamma + \tan(\phi_{vef})||\gamma + \tan(\phi_{ve'f})|}}. \tag{B19}$$

Clearly all possible singular points are 1/2 order. The singular points due to $|\gamma + \tan(\phi_{ve'f})|$ and due to $|\cos(\phi_{ve'f})|$ are separated. The integration with respect to ρ does not have singularities.

2. Multidimensional stationary phase approximation

In Appendix A, we already use the saddle point approximation when there is a branch point appearing in the nonscaled function $g(x)$. When adapting to the stationary phase approximation, for the 1/2 order singular point located exactly at the critical point, the result is the following:

$$I = \int \frac{g(x)}{\sqrt{x}} \mathrm{e}^{\Lambda S(x)} \sim g(x_c) \frac{\pi \mathrm{e}^{\mathrm{i}\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda|S''(x_c)|}\right)^{1/4} \mathrm{e}^{\Lambda S(x_c)} \tag{B20}$$

where $\Lambda \sim \infty$ and S is purely imaginary. Note that the dominant part here is given by the $-1/4$ order of Λ instead of $-1/2$ as in the asymptotic formula without singularities. The regulator appearing in (A43) is exactly this 1/4 order difference.

However, this asymptotic formula only holds for the single variable integral. We will generalize this single variable approximation to the multivariables case. Recall Fubini's theorem:

Theorem B.1. Let $w = f(x_1, x_2, \ldots, x_n)$ be an n variable valued complex function. If the integral of f on the domain $B = \prod_i^n I_n$ where I_n are intervals in $\mathbb{R}$ is absolutely convergent:

$$\int_B |f(x_1, x_2, \ldots, x_n)| d(x_1, x_2, \ldots, x_2) < \infty, \tag{B21}$$

then the multiple integral will give the same result as the iterated integral,

$$\begin{aligned}\int_{A\times B} |f(x,y)| d(x,y) &= \int_A \left(\int_B f(x,y) dy \right) dx \\ &= \int_B \left(\int_A f(x,y) dx \right) dy.\end{aligned} \tag{B22}$$

The result is independent of the iterate order.

Here from (B19) we have the integral in the form

$$I = \int d^n x \prod_{i=1}^{j} (x_i)^{-1/2} g(x) \mathrm{e}^{\mathrm{i}S(x)} \tag{B23}$$

where $S(x) \in \mathbb{R}$, $x \in \mathbb{R}^n$, $j < n$ and $g(x)$ is analytic. $j < n$ illustrates the fact that only in a subspace of the total variables will space have singularities. Then in a closed region M where the stationary phase points (solutions of $\delta S = 0$) exists, we have

$$\begin{aligned}&\int_M d^n x \Big| \prod_i (x_i)^{-1/2} g(x) \mathrm{e}^{\mathrm{i}S(x)} \Big| \\ &\sim \int_M d^n x \Big| \prod_i (x_i)^{-1/2} \tilde{g}(x) \Big| < \infty.\end{aligned} \tag{B24}$$

From Fubini's theorem, we then can write the multidimensional integral as an iterated integral. For the original variables, since the singularities exist only in a subspace of the total variable space, we can always perform a coordinate transformation, such that variables with singularities are separated from those do not have them, as we show in (B19). Then, the final result is given by performing the stationary phase approximation iteratively. In each step one may use the usual stationary phase approximation or the one with singularities. The lowest order of the total integration is given by picking the lowest order approximation of each single integration.

However, due to technical reason, we would like to derive the saddle point equations directly from $S(x)$ instead of evaluating it iteratively. According to the approximation, each single valued integral is dominated by the phase $S(x_0)$ where x_0 is the solution of the saddle point equation $\delta_x S(x) = 0$. Then iteratively, the saddle points are given by

$$\begin{aligned}&\delta_{x_1} S(x_1, x_2, \ldots, x_n) = 0, \\ &\delta_{x_2} S(x_1^0, x_2, \ldots, x_n) = \left(\delta_{x_1} S(x) \frac{\partial x_1^0}{\partial x_2} + \delta_{x_2} S(x) \right) \Big|_{x_1 = x_1^0} = \delta_{x_2} S(x)|_{x_1 = x_1^0} = 0, \\ &\quad\vdots \\ &\delta_{x_n} S(x_1^0, x_2^0, \ldots, x_n) = \delta_{x_n} S(x)|_{x_1 = x_1^0, x_2 = x_2^0, \ldots, x_{n-1} = x_{n-1}^0} = 0\end{aligned} \tag{B25}$$

where $x_i^0(x_{i+1}, \ldots, x_n)$ is the solution of the corresponding equation of motion $\delta_{x_i}(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}, \ldots, x_n)$ with respect to x_i. As one can see from (B25), the above equation of motion is nothing else, but we solve the original equation of motion $\{E_n = \delta S(x)\}$ iteratively. Thus, they have the same solutions. The saddle points given by the two methods will coincide to each other. Note that, for variables whose saddle points are near the singularities, the induced measure which contains second derivatives of the action will be given in the order $1/4$ in contrast to $1/2$ for those do not have singularities. As a result, there is no general Hessian term in contrast to the previous EPRL approximation, and the measure is more involved as some special functions of second derivatives of the action. As a result, finally we have order $I \sim g(\Lambda)\Lambda^{-a/2-b/4}$ for b variables that have singular points.

APPENDIX C: ANALYSIS OF CRITICAL POINTS IN BIVECTOR REPRESENTATION

In this Appendix we will analyze and reformulate the critical point equations we get in Sec. III in the bivector representation. The analysis is done for all possible actions appearing in the amplitude (2.42).

1. S_{vf+} case

From (3.8) and (3.13) in the S_{vf+} case,

$$\begin{aligned}g_{ve} \eta l_{ef}^+ &= \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'} \eta l_{e'f}^+ \\ g_{ve} J \tilde{Z}_{vef} &= \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'} J \tilde{Z}_{vef}\end{aligned} \tag{C1}$$

we have

$$g_{ve}\eta l_{ef}^{+}\otimes(l_{ef}^{-}+\alpha_{vef}l_{ef}^{+})^{\dagger}g_{ev}=g_{ve'}\eta l_{ef}^{+}\otimes(l_{e'f}^{-}+\alpha_{ve'f}l_{e'f}^{+})^{\dagger}g_{e'v} \quad (C2)$$

with the fact that $\langle l^{+},l^{+}\rangle=0$ and $\langle l^{-},l^{+}\rangle=1$. With (3.54), the above equation can be written as

$$g_{ve}(V_{ef}+\mathrm{i}\bar{\alpha}_{vef}W_{ef}^{+})g_{ev}=g_{ve'}(V_{e'f}+\mathrm{i}\bar{\alpha}_{ve'f}W_{e'f}^{+})g_{e'v}. \quad (C3)$$

In the spin-1 representation, this equation reads

$$g_{ve}(V_{ef}+(\Im(\alpha_{vef})+\Re(\alpha_{vef})*)W_{ef}^{+})g_{ev}=g_{ve'}(V_{e'f}+(\Im(\alpha_{ve'f})+\Re(\alpha_{ve'f})*)W_{e'f}^{+})g_{e'v}. \quad (C4)$$

We can define a bivector X_{vef}

$$X_{vef}=V_{ef}+(\Im(\alpha_{vef})+\Re(\alpha_{vef})*)W_{ef}^{+}. \quad (C5)$$

Easy to check, X is a simple bivector which can be expressed as

$$X=*(v+\Im(\alpha)w^{+})\wedge(u-\Re(\alpha)w^{+})=*(\tilde{v}\wedge\tilde{u}). \quad (C6)$$

Here, by the definition of v and w, we have

$$\tilde{v}^{I}=(\tilde{v}^{0},-\tilde{v}^{2},\tilde{v}^{1},0),\qquad \tilde{w}^{I}=(w^{+0},-w^{+2},w^{+1},0), \quad (C7)$$

where

$$\tilde{v}^{i}=-2\langle l^{-}+\mathrm{i}\Im(\alpha)l^{+},F^{i}l^{+}\rangle,\qquad w^{+i}=2\mathrm{i}\langle l^{+},F^{i}l^{+}\rangle. \quad (C8)$$

One can check $\tilde{v}^{I}\tilde{v}_{I}=\tilde{u}^{I}\tilde{u}_{I}=1$; thus, X is timelike. (C4) implies

$$(G_{ve}\tilde{v}_{vef})\wedge(G_{ve}\tilde{u}_{vef})=(G_{ve'}\tilde{v}_{ve'f})\wedge(G_{ve'}\tilde{u}_{ve'f}), \quad (C9)$$

which reminds us to define

$$X_{f}(v)\coloneqq G_{ve}X_{vef}G_{ev}=G_{ve'}X_{ve'f}G_{e'v} \quad (C10)$$

Note that, from this equation, we have

$$(G_{ve}u)_{I}X_{f}^{IJ}(v)=-\Re(\alpha_{vef})(G_{ve}w_{ef}^{+})^{J} \quad (C11)$$

which is 0 only when $\Re(\alpha_{vef})=0$.

Going back to the equations we get from the variation with respect to g, clearly (3.40) and (3.41) can be written as

$$\sum_{f}\epsilon_{ef}(v)\langle l^{-}+\mathrm{i}\Im(\alpha)l^{+},F^{i}l^{+}\rangle=0 \quad (C12)$$

$$\sum_{f}\epsilon_{ef}(v)\Re(\alpha)\langle l^{+},F^{i}l^{+}\rangle=0. \quad (C13)$$

In terms of 4-vectors $\tilde{v}$ and w, these equations read

$$\sum_{f}\epsilon_{ef}(v)G_{ve}\tilde{v}_{vef}=0$$
$$\sum_{f}\epsilon_{ef}(v)\Re(\alpha_{vef})G_{ve}w_{ef}^{+}=0 \quad (C14)$$

where $\tilde{v}$ is defined by (C7). Then we can write (C14) as

$$\sum_{f}\epsilon_{ef}(v)X_{f}(v)=0 \quad (C15)$$

which is a closure condition to the bivectors.

2. S_{vf-} case

In this case, from (3.8) and (3.14) we have

$$g_{ve}\eta n_{vef}=\frac{\bar{\zeta}_{vef}\Re(\alpha_{vef})}{\bar{\zeta}_{ve'f}\Re(\alpha_{ve'f})}g_{ve'}\eta n_{ve'f} \quad (C16)$$

$$g_{ve}J\tilde{Z}_{vef}=\frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}}g_{ve'}J\tilde{Z}_{vef} \quad (C17)$$

where $n_{ef}\coloneqq l_{ef}^{+}+\mathrm{i}(\gamma\Re(\alpha_{vef})+\Im(\alpha_{vef}))l_{ef}^{-}$. Note with Eq. (3.32), we see n does not change for a different vertex v: $n_{ef}(v)=n_{ef}(v')$. n defined here satisfies the relation in Lemma III.1; thus, according to Lemma III.2, $\{n,l^{-}\}$ forms a null basis. With n and l^{-}, $\tilde{Z}$ can be rewritten as

$$Z=l^{+}+\alpha l^{-}=n+(1-\mathrm{i}\gamma)\Re(\alpha)l^{-}. \quad (C18)$$

This leads to the tensor product equation

$$g_{ve}\frac{\eta n_{ef}}{\Re(\alpha_{ef})}\otimes(n_{ef}+(1-\mathrm{i}\gamma)\Re(\alpha_{vef})l_{ef}^{-})^{\dagger}g_{ev}=(e\to e'). \quad (C19)$$

The right part of the above equation means we exchange all the e in the left part to e'.

In terms of the bivector variables, according to (3.54), we have

$$g_{ve}\left(V_{ef}+\frac{(\mathrm{i}-\gamma)W_{ef}^{+}}{(1+\gamma^{2})\Re(\alpha_{vef})}\right)g_{ev}=(e\to e'). \quad (C20)$$

Note now that V is the spacelike bivector generated by n with l^{-} and W^{+} is null bivector generated by n with itself. Again the bivector $X_{vef}\coloneqq V_{ef}-(\gamma-*)W_{ef}/((1+\gamma^{2})\Re(\alpha))$ is a simple bivector. X_{vef} can be written as

$$X_{vef} = *\left(\left(v_{ef} - \frac{\gamma}{(1+\gamma^2)\Re(\alpha_{vef})} w_{ef}^+\right) \wedge \left(u - \frac{\gamma}{(1+\gamma^2)\Re(\alpha_{vef})} w_{ef}^+\right)\right) = *(\tilde{v}_{vef} \wedge \tilde{u}_{vef}) \tag{C21}$$

where

$$\tilde{v}^I = (\tilde{v}^0, -\tilde{v}^2, \tilde{v}^1, 0), \qquad w^{+i} = 2\mathrm{i}\langle n, F^i n\rangle. \tag{C22}$$

Here

$$\tilde{v}^i = 2\left\langle n, F^i\left(l^- - \frac{\mathrm{i}\gamma n}{(1+\gamma^2)\Re(\alpha)}\right)\right\rangle, \quad w^{+i} = 2\mathrm{i}\langle n, F^i n\rangle \tag{C23}$$

and $\tilde{v}^I\tilde{v}_I = \tilde{u}^I\tilde{u}_I = 1$ implies X is timelike.

Then (C19) leads to

$$X_f(v) \coloneqq G_{ve}X_{vef}G_{ev} = G_{ve'}X_{ve'f}G_{e'v} \tag{C24}$$

which is the parallel transport of X between edges e and e'. With (C21), we can write $X_f(v)$ as

$$X_f(v) = G_{ve}\tilde{v}_{vef} \wedge G_{ve}\tilde{u}_{vef}. \tag{C25}$$

Note here again we have

$$(G_{ve}u)_I X_{vf}^{IJ} = -\frac{1}{(1+\gamma^2)\Re(\alpha)}(G_{ve}w_{ef}^+)^J \tag{C26}$$

which is some null vector and cannot be 0.

From (3.40) and (3.42), we have the following equations of motion from the variation with respect to g:

$$\sum_f \epsilon_{ef}(v)\left\langle n, F^\dagger\left(l^- - \frac{\mathrm{i}\gamma n}{(1+\gamma^2)\Re(\alpha)}\right)\right\rangle = 0$$
$$\sum_f \epsilon_{ef}(v)\frac{\langle n, F^\dagger n\rangle}{\Re(\alpha)} = 0. \tag{C27}$$

In terms of 4-vectors,

$$\sum_f \epsilon_{ef}(v)G_{ve}v_{ef} = 0 \qquad \sum_f \epsilon_{ef}(v)\frac{G_{ve}w_{ef}^+}{\Re(\alpha)} = 0 \tag{C28}$$

which leads to

$$\sum_f \epsilon_{ef}(v)X_f(v) = 0. \tag{C29}$$

3. S_{vfx} case

We will use S_{vfx-} as an example, and the S_{vfx+} will be exactly the same except for switching e and e' here. From the critical point equations (3.8) and (3.15),

$$(\gamma - \mathrm{i})s_f\frac{g_{ve}\eta l_{vef}^+}{\bar{\zeta}_{vef}} = -\mathrm{i}s_f\frac{g_{ve'}\eta n_{ve'f}}{\bar{\zeta}_{ve'f}\Re(\alpha_{ve'f})},$$
$$g_{ve}\bar{\zeta}_{vef}J(l_{ef}^- + \alpha_{vef}l_{ef}^+) = g_{ve'}\bar{\zeta}_{ve'f}J(l_{e'f}^+ + \alpha_{ve'f}l_{e'f}^-). \tag{C30}$$

With Eq. (3.40) from the variation with respect to SU(1,1) group elements v_{ef}, in this case $n = l^+$, and $\tilde{Z}_{ve'f}$ can be written as $\tilde{Z}_{ve'f} = l_{e'f}^+ + (1-\mathrm{i}\gamma)\Re(\alpha_{ve'f})l_{e'f}^-$.

The tensor product between the two equations leads to

$$(\mathrm{i}\gamma + 1)g_{ve}(\eta l_{ef}^+ \otimes (l_{ef}^-)^\dagger + \bar{\alpha}_{vef}\eta l_{ef}^+ \otimes (l_{ef}^+)^\dagger)g_{ev}$$
$$= g_{ve'}\eta n_{ve'f} \otimes \left(\frac{n_{ve'f}}{\Re(\alpha_{ve'f})} + (1-\mathrm{i}\gamma)l_{e'f}^-\right)^\dagger g_{e'v}$$
$$= g_{ve'}\left(\frac{\eta n_{ve'f} \otimes n_{ve'f}}{\Re(\alpha_{ve'f})} + (1+\mathrm{i}\gamma)\eta n_{ve'f} \otimes (l_{e'f}^-)^\dagger\right)g_{e'v}. \tag{C31}$$

In bivector representation

$$g_{ve}(V_{ef} + \mathrm{i}\bar{\alpha}_{vef}W_{ef}^+)g_{ev} = g_{ve'}\left(V_{e'f} + \frac{(\mathrm{i}-\gamma)W_{ve'f}^+}{\Re(\alpha_{e'f})(1+\gamma^2)}\right) \times g_{e'v}. \tag{C32}$$

It is easy to see that one recovers the corresponding bivectors in the $S_{vf\pm}$ case, respectively. Thus, the equation implies

$$X_f(v) \coloneqq g_{ve}X_{vef}g_{ev} = g_{ve'}X_{ve'f}g_{e'v} \tag{C33}$$

with X_{vef} defined by (C6) and $X_{ve'f}$ defined by (C21). The closure constraint, in these cases, are the combinations of the corresponding equations in (C14) or (C28) according to their representations in S_+ or S_-. Then we still have

$$\sum_f \epsilon_{ef}(v)X_f(v) = 0. \tag{C34}$$

APPENDIX D: BRIEF REVIEW OF CRITICAL POINT EQUATIONS WITH SPACELIKE TRIANGLES IN TIMELIKE TETRAHEDRA

In this Appendix, we briefly summarize the critical point equations for spacelike triangles in a timelike tetrahedron. The result was derived in [21]. As we described before, spacelike faces correspond to the discrete series representation of the SU(1,1) group. In this case, the simplicity constraint implies

$$\rho_f = \gamma j_f, \qquad n_f/2 = j_f \tag{D1}$$

with the areas spectrum asymptotically given by $A_f = \gamma\sqrt{j_f(j_f+1)} \sim \rho_f = \gamma j_f$.

The embedded coherent state reads

$$f_\xi^{j\alpha} = (\alpha\langle z, z\rangle)^{\mathrm{i}\rho/2-1-j}(\alpha\langle \xi^\alpha, \bar{z}\rangle)^{-2j} \tag{D2}$$

where $\alpha = \pm = \langle z, z\rangle$ for spinors z. ξ are spinors defined as

$$\xi^\alpha = v^{-1\dagger}\xi_0^\alpha, \quad \text{with} \quad \begin{cases} \xi_0^+ = (1,0)^T \\ \xi_0^- = (0,1)^T \end{cases}, \qquad v \in \mathrm{SU}(1,1). \tag{D3}$$

With these coherent states, we immediately see the action read

$$S_{vf}^\pm = \mathrm{i}\gamma j_f \ln\frac{\langle Z_{vef}, Z_{vef}\rangle}{\langle Z_{ve'f}, Z_{ve'f}\rangle} - j_f \ln\frac{\langle \xi_{e'f}^\pm, Z_{ve'f}\rangle^2 \langle Z_{vef}, \xi_{ef}^\pm\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle\langle Z_{ve'f}, Z_{ve'f}\rangle}. \tag{D4}$$

Here we use the simplicity constraint $\rho_f = 2\gamma j_f$. Z_{vef} is again defined by $Z_{vef} = g_{ve}^\dagger \bar{z}_{vf}$. The real parts of the action read

$$\Re S = -j_f \Re \ln\frac{\langle \xi_{e'f}, Z_{ve'f}\rangle^2\langle Z_{vef}, \xi_{ef}\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle\langle Z_{ve'f}, Z_{ve'f}\rangle} \leq 0. \tag{D5}$$

From $\Re S_0 = 0$, we have

$$Z_{vef} = \zeta_{vef}\xi_{ef}^\pm. \tag{D6}$$

Because of $Z_{vef} = g_{ve}^\dagger \bar{z}_{vf}$, this equation leads to

$$g_{ve}J\xi_{ef}^\pm = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}J\xi_{e'f}^\pm. \tag{D7}$$

The variation of the action reads

$$\delta S_{vf} = j_f(1+\mathrm{i}\gamma)\frac{\delta\langle Z_{vef}, Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} + j_f(1-\mathrm{i}\gamma)\frac{\delta\langle Z_{ve'f}, Z_{ve'f}\rangle}{\langle Z_{ve'f}, Z_{ve'f}\rangle} - 2j_f\left(\frac{\delta\langle \xi_{ef}^\pm, Z_{vef}\rangle}{\langle \xi_{ef}^\pm, Z_{vef}\rangle} + \frac{\delta\langle Z_{ve'f}, \xi_{ef}^\pm\rangle}{\langle Z_{ve'f}, \xi_{e'f}^\pm\rangle}\right). \tag{D8}$$

1. Critical point equation

Note that the variation takes the same properties as in timelike triangle case, where the variation with respect to z leads to

$$\delta_z S = j_f(1+\mathrm{i}\gamma)\frac{(g_{ve}\eta Z_{vef})^T}{\langle Z_{vef}, Z_{vef}\rangle} + j_f(1-\mathrm{i}\gamma)\frac{(g_{ve'}\eta Z_{ve'f})^T}{\langle Z_{ve'f}, Z_{ve'f}\rangle} - 2j_f\frac{(g_{ve'}\eta\xi_{e'f}^\pm)}{\langle Z_{ve'f}, \xi_{e'f}^\pm\rangle}. \tag{D9}$$

After inserting (D6), we have

$$g_{ve}\eta\xi_{ef}^\pm = \frac{\bar{\zeta}_{vef}}{\bar{\zeta}_{ve'f}} g_{ve'}\eta\xi_{e'f}^\pm. \tag{D10}$$

One can check that the variation with respect to the SU(1,1) group elements v_{ef} is trivial. The variation with respect to the $\mathrm{SL}(2,\mathbb{C})$ group elements g_{ve} leads to

$$\begin{aligned}\delta S = &\sum_{f+} j_f(1+\mathrm{i}\gamma)\frac{\langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} - 2j_f\frac{\langle \xi_{ef}^\pm, L^\dagger Z_{vef}\rangle}{\langle \xi_{ef}^\pm, Z_{vef}\rangle} \\ &\times \sum_{f-} j_f(1-\mathrm{i}\gamma)\frac{\langle L^\dagger Z_{vef}, Z_{vef}\rangle + \langle Z_{vef}, L^\dagger Z_{vef}\rangle}{\langle Z_{vef}, Z_{vef}\rangle} - 2j_f\frac{\langle L^\dagger Z_{vef}, \xi_{ef}^\pm\rangle}{\langle Z_{vef}, \xi_{ef}^\pm\rangle}.\end{aligned} \tag{D11}$$

Applying (D6), we have

$$\begin{aligned}\delta S = &\sum_{f+} j_f(1+\mathrm{i}\gamma)(\langle L^\dagger\xi_{vef}^\pm, \xi_{vef}^\pm\rangle + \langle \xi_{vef}^\pm, L^\dagger\xi_{vef}^\pm\rangle) - 2j_f\langle \xi_{ef}^\pm, L^\dagger\xi_{ef}^\pm\rangle \\ &+ \sum_{f-} j_f(1-\mathrm{i}\gamma)(\langle L^\dagger\xi_{vef}^\pm, \xi_{vef}^\pm\rangle + \langle \xi_{vef}^\pm, L^\dagger\xi_{vef}^\pm\rangle) - 2j_f\langle L^\dagger\xi_{ef}^\pm, \xi_{ef}^\pm\rangle\end{aligned} \tag{D12}$$

where $f\pm$ means face f is either an incoming or outgoing edge e correspondingly. This leads to six equations with the generators of the $\mathrm{SL}(2,\mathbb{C})$ group, which reads

$$\delta S = -2\sum_f \epsilon_{ef}(v) j_f \langle \xi^{\pm}_{ef}, F^{\dagger}\xi^{\pm}_{ef}\rangle \tag{D13}$$

$$\delta S = 2\mathrm{i}\gamma\sum_f \epsilon_{ef}(v) j_f \langle \xi^{\pm}_{ef}, \tilde{F}^{\dagger}\xi^{\pm}_{ef}\rangle. \tag{D14}$$

Again $\epsilon_{ef}(v)$ here is the signature determined up to a global sign by

$$\epsilon_{ef}(v) = -\epsilon_{e'f}(v), \qquad \epsilon_{ef}(v) = -\epsilon_{ef}(v') \tag{D15}$$

for the triangle f shared by the tetrahedra t_e and $t_{e'}$.

2. Geometrical interpretation

We can define a vector from ξ_{ef}

$$n^i_{ef} = -2\mathrm{i}\langle \xi^{\pm}_{ef}, F^i \xi^{\pm}_{ef}\rangle \tag{D16}$$

which is the SU(1,1) action on the unit timelike vector $n_0 = -2\mathrm{i}\langle \xi^{\pm}_0, F^i\xi^{\pm}_0\rangle = \{\pm1,0,0\}$. The encoding of this vector in four-dimensional Minkowski space is given by

$$n^I_{ef} = \{n^3_{ef}, -n^2_{ef}, n^1_{ef}, 0\} = \langle \xi^{\pm}_{ef}|\sigma^I|\xi^{\pm}_{ef}\rangle - \langle \xi^{\pm}_{ef}||\xi^{\pm}_{ef}\rangle. \tag{D17}$$

Clearly n^I_{ef} is the timelike vector and future directed with ζ^+_{ef} while the past is directed with ζ^-_{ef}.

Then there is a nature $\mathrm{SL}(2,\mathbb{C})$ bivector defined by

$$X_{ef} = -2\mathrm{i}\langle \xi^{\pm}_{ef}, F^i\xi^{\pm}_{ef}\rangle E_i = -\mathrm{i}\left(\eta\xi^{\pm}_{ef}\otimes(\xi^{\pm}_{ef})^{\dagger} - \frac{1}{2}I_2\right) \tag{D18}$$

which, in spin-1 representation, reads

$$X^{IJ}_{ef} = \begin{pmatrix} 0 & n^1 & n^2 & 0 \\ -n^1 & 0 & n^3 & 0 \\ -n^2 & -n^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = *(n^I_{ef}\wedge u^I). \tag{D19}$$

Clearly from (D7) and (D10), X_{ef} satisfy the parallel transport equation

$$X_f(v) = g_{ve}X_{ef}g_{ev} = g_{ve'}X_{e'f}g_{e'v} \tag{D20}$$

and satisfies

$$(G_{ve}u).X_f(v) = 0. \tag{D21}$$

The bivector is then again scaled as $B_f(v) = 2A_f X_f(v) = 2\gamma j_f X_f(v)$, where $|B_f| = 2A_f$. Equations (D13) and (D14) then can be written as equations of B_f:

$$\delta_g S = \frac{\mathrm{i}}{2\gamma}\sum_f \epsilon_{ef}(v)B_f(v) = 0 \tag{D22}$$

$$\delta_{\tilde{g}} S = \frac{1}{2}\mathrm{i}\sum_f \epsilon_{ef}(v)B_f(v) = 0. \tag{D23}$$

APPENDIX E: GEOMETRIC INTERPRETATION AND RECONSTRUCTION

In this Appendix we summarize the geometric reconstruction theorems for the tetrahedron with spacelike triangles only in [12–15,21], and extending them to general tetrahedron may contain also timelike triangles. We start with a single simplex σ_v corresponding to a vertex v, and then generalize the result to a general simplicial manifold with many simplices. For simplicity, we introduce a shorthand notation for a single simplex σ_v:

$$N_i \coloneqq N_{e_i}(v) \quad B^G_{ij} = -B^G_{ji} = \epsilon_{e_ie_j}(v)B_{e_ie_j}(v)$$
$$B^G_{ij} = *(v^G_{ij}\wedge N_i) \tag{E1}$$

where e_ie_j represents the face determined by the dual edges e_i and e_j, and $i = 0,1,\ldots,4$, and v_{ij} here is the triangles that are normal scaled with the area: $v^2_{ij} = \pm 4A^2_{ij}$.

Note that here we will assume our boundary data to be a geometric boundary data, which means they satisfy the length matching condition and orientation matching condition. The detailed meaning of these conditions will become clear later. The geometric boundary data is necessary to get a Regge-like geometric solution. For nongeometric boundary data, there will be at most one solution up to gauge equivalence, which is an analogy to the result in the EPRL model [12,13].

1. Nondegenerate condition and classification of the solution

To begin with, we would like to introduce the nondegenerate condition. We will first consider nondegenerate simplices and then move to the degenerate case. For the boundary data, nondegenerate means that for a boundary tetrahedron any three out of four face normal vectors n_{ef} span a three-dimensional space. With nondegenerate boundary data, for any three different edges i, j, k in a 4-simplex one of the following holds:

(i) $N_{ei} = \pm N_{ej}$ and $N_{ej} = \pm N_{ek}$,
(ii) $N_{ei} \neq N_{ej}$

The first case can be further proved that leads to all N_i are parallel by using the closure constraint of B_{ij}. This result was first proved in [12] and later by [21].

The only nondegenerate case is then specified by the following nondegeneracy condition:

$$\prod_{e1,e2,e3,e4=0}^{5} \det(N_{e1}, N_{e2}, N_{e3}, N_{e4}) \neq 0 \qquad \text{(E2)}$$

which means any four out of five normals are linear independent and span a four-dimensional Minkowski space. Since $N_e(v) = g_{ve}N^0$, it is easy to see the non-degenerate condition is actually a constraint on $\{g_{ve}\}$.

2. Nondegenerate geometry on a 4-simplex

For simplicity, we start with one 4-simplex σ_v in four-dimensional Minkowski space $M = R^4$ here. For each 4-simplex σ_v dual to the vertex v, we associate it with a reference frame. In this reference frame, the five vertices of the 4-simplex $[p_0, p_1, p_2, p_3, p_4]$ have the coordinates $p_i: (x_i^I) = (x_i^0, x_i^1, x_i^2, x_i^3)$. Based on these coordinates, we introduce vectors y_i, a as well as covector A in an auxiliary space R^5,

$$y_i = (x_i^I, 1)^T, \quad \text{and} \quad a = (0, \ldots, 0, 1)^T, \qquad A = a^T. \qquad \text{(E3)}$$

We define the $k+1$-vector in R^5

$$\tilde{V}_{\alpha_0,\ldots,\alpha_k} = y_{\alpha_0} \wedge \ldots \wedge y_{\alpha_k} \qquad \text{(E4)}$$

where $\alpha_i \in \{0, \ldots, 5\}$. With covector A, for k-vectors Ω in R^5 satisfying $A \llcorner \Omega = 0$, we can identify it with a k-vector in M. For example, since $A \llcorner A \llcorner \tilde{V}_{\alpha_0,\ldots,\alpha_5} = 0$, we then induce a 4-vector in M from $\tilde{V}_{\alpha_0,\ldots,\alpha_5}$,

$$V_{\alpha_0,\ldots,\alpha_5} = A \llcorner \tilde{V}_{\alpha_0,\ldots,\alpha_k} = (y_{\alpha_1} - y_{\alpha_0}) \wedge \ldots \wedge (y_{\alpha_5} - y_{\alpha_0}). \qquad \text{(E5)}$$

This vector is actually 4! times the volume 4-vector of the 4-simplex:

$$\begin{aligned} V_{\alpha_0,\ldots,\alpha_4} &= (x_{\alpha_1} - x_{\alpha_0}) \wedge \ldots \wedge (x_{\alpha_4} - x_{\alpha_0}) \\ &= E_{\alpha_1\alpha_0} \wedge \ldots \wedge E_{\alpha_5\alpha_0}. \end{aligned} \qquad \text{(E6)}$$

$E^I_{\alpha_i\alpha_0} = x^I_{\alpha_i} - x^I_{\alpha_0}$ is the edge vector related to the oriented edge $l_{\alpha_i\alpha_0} = [p_{\alpha_i}, p_{\alpha_0}]$. Notice that the volume 4-vector comes with a sign with respect to the order of points.

We further define the 3-vector and bivector by skipping some points

$$V_i = (-1)^i V_{0\ldots\hat{i}\ldots4} \qquad \text{(E7)}$$

$$B_{ij} = A \llcorner \tilde{V}_{0\ldots\hat{i}\ldots n} = \begin{cases} (-1)^{i+j+1} V_{0\ldots\hat{i}\ldots\hat{j}\ldots4} & i < j \\ (-1)^{i+j} V_{0\ldots\hat{j}\ldots\hat{i}\ldots4} & i > j \end{cases} \qquad \text{(E8)}$$

where $\hat{i}$ means omitting the ith elements. We have the following properties for V_i and B_{ij}:

$$\sum_i V_i = 0, \qquad \text{(E9)}$$

$$B_{ij} = -B_{ij}m \quad \forall_i \sum_{j\neq i} B_{ij} = 0. \qquad \text{(E10)}$$

One can further check that B_{ij} can be written as

$$B_{ij} = \frac{1}{2}(-1)^{\mathrm{sgn}(\sigma)} \epsilon^{\mathrm{ijkmn}} E_{mk} \wedge E_{nk}. \qquad \text{(E11)}$$

And one has $B_{ij}^2 = \pm 4A_{ij}^2$ where A_{ij} is the area of the corresponding spacelike or timelike triangles in the non-degenerate case.

Suppose the volume 4-vector of the 4-simplex $V_{0,\ldots,4}$ is nondegenerate. In this case any of the four out of the five y_i are linearly independent. One can introduce the dual bases $\hat{y}_i$ and $\tilde{y}_i$ defined by

$$\hat{y}_i \lrcorner y_j = \delta_{ij}, \qquad \hat{y}_i = \tilde{y}_i + \mu_i A, \qquad \tilde{y}_i \lrcorner a = 0 \qquad \text{(E12)}$$

with properties

$$\sum_i \hat{y}_i = A, \qquad \sum_i \tilde{y}_i = 0, \qquad \text{(E13)}$$

$\tilde{y}_i$ here can be regarded as covectors belonging to M. With $\tilde{y}_i$, we have

$$V_i = -\tilde{y}_i \lrcorner V_{0\ldots4}, \qquad B_{ij} = \tilde{y}_j \lrcorner \tilde{y}_i \lrcorner V_{0\ldots4}. \qquad \text{(E14)}$$

Thus, the covectors $\tilde{y}_i$ are conormal to subsimplices V_i. And by using the Hodge star, we have

$$V_i = -\mathrm{Vol} * \tilde{y}_i, \qquad B_{ij} = -\mathrm{Vol} * (\tilde{y}_j \wedge \tilde{y}_i) \qquad \text{(E15)}$$

where the volume $\mathrm{Vol} > 0$ is the absolute value of the oriented 4-volume

$$V_4 \coloneqq \det(V_{0,\ldots,4}) = \mathrm{sgn}(V_4)\mathrm{Vol}. \qquad \text{(E16)}$$

It can be shown that

$$\frac{1}{V_4} = \epsilon^{ijkl} \det(\tilde{y}_i, \tilde{y}_j, \tilde{y}_k, \tilde{y}_l) \qquad \text{(E17)}$$

and the coframe vector E_{ij} is given by

$$E_{ij} = V_4 \epsilon_{ijklm}(v) * (\tilde{y}^k \wedge \tilde{y}^l \wedge \tilde{y}^m). \qquad \text{(E18)}$$

If the subsimplices V_i are nondegenerate, by introducing normalized vectors N_i, we can write $\tilde{y}_i$ as

$$\tilde{y}_i = \frac{1}{\mathrm{Vol}} W_i N_i, \qquad N_i \cdot N_i = t_i, \qquad W_i > 0 \quad \text{(E19)}$$

where $t_i = \pm 1$ distinguish spacelike or timelike normals, respectively. This leads to

$$B_{ij} = -\frac{1}{\mathrm{Vol}} W_i W_j * (N_j \wedge N_i), \qquad \sum_i W_i N_i = 0. \quad \text{(E20)}$$

In order to make the normal out-pointing, we redefine the normalized normal vectors N_i by

$$N_i^\Delta = -t_i N_i, \quad W_i^\Delta = -t_i W_i \quad \sum_i W_i^\Delta N_i^\Delta = 0 \quad \text{(E21)}$$

such that N_i^Δ are out-pointing.

3. Reconstructing geometry from nondegenerate critical points

We begin with the reconstruction of the normals. Recall in the critical point equation (3.91), the normals N_e satisfy

$$\forall_{f \in t_e} \eta_{IJ} N_e{}^I B_f(v)^{JK} = 0. \quad \text{(E22)}$$

If there is another normal vector N satisfying the same condition for some edge e, it is easy to see that we have

$$\forall_{f \in t_e} B_f(v) \sim *(N \wedge N_e) \quad \text{(E23)}$$

which means for an edge e, B_{ef} are proportional to each other. This is clearly contrary to the fact that we have a nondegenerate solution. Thus, for the given bivectors which are the solution of the critical point equation, if we require a vector, N satisfies

$$\forall_{f \in t_e} \eta_{IJ} N^I B_f(v)^{JK} = 0 \quad \text{(E24)}$$

for an edge tetrahedron t_e, and we then have $N = \pm N_e$ after normalization. The condition (E24) is sufficient and necessary.

Considering a 4-simplex σ_v at some vertex v, the critical point equation (3.91) can be written in the shorthand notation we introduce in (E1) as

$$B_f(v) = B_{ij}^{\{G\}} = -B_{ji}^G, \quad N_i \llcorner B_{ij}^{\{G\}} = 0, \quad \sum_j B_{ij}^{\{G\}} = 0. \quad \text{(E25)}$$

Now we give normalized vectors N_i satisfying the nondegenerate condition. If we require that the bivectors satisfy (E25), they are uniquely determined up to a constant $\lambda \in R$

$$B'_{ij} = \lambda W_i W_j * (N_j \wedge N_i). \quad \text{(E26)}$$

Here $W_i \in R$ are nonzero and determined by

$$\sum_i W_i N_i = 0. \quad \text{(E27)}$$

The proof is stated first in [14] and later in [21]. Note that the bivector B_{ij} is independent of the choice of the signature of normal vectors N since the signs of W and N will change simultaneously. λ can be fixed up to a sign by the normalization of B'_{ij}

$$|B_f|^2 = -4\gamma^2 s_f^2 = -4A_f^2. \quad \text{(E28)}$$

Then it can be proved that the nondegenerate geometric solution determines the 4-simplex specified by the bivectors B^Δ uniquely up to shift and inversion such that

$$B_{ij}^\Delta = r B_{ij}^{\{G\}} \quad \text{(E29)}$$

where $r = \pm 1$ is the geometric Plebanski orientation. The construction can be done as follows. With five given normals N_i, we take any five planes orthogonal to N_i. With the nondegeneracy condition, they cut out a 4-simplex Δ' which is uniquely determined up to shifts and scaling. According to (E20) and (E26), the bivectors of the reconstructed 4-simplex $B_{ij}^{\Delta'}$ relate to B_{ij} as

$$B_{ij}^{\Delta'} = \lambda B_{ij}^{\{G\}}. \quad \text{(E30)}$$

Then the identity of the normalization will determine the scaling up to a sign

$$B_{ij}^{\{G\}} = r B_{ij}^{\Delta'} = -\frac{1}{\mathrm{Vol}} r W_i^\Delta W_j^\Delta * (N_j^\Delta \wedge N_i^\Delta) \quad \text{(E31)}$$

where Vol is the 4!-volume of the 4-simplex.

Let us move to the boundary tetrahedron. Since G_e is a SO(1,3) rotation, its action then keeps the shape of tetrahedrons. Thus, the tetrahedron with bivectors $B_{ij} = *(v_{ij} \wedge u_i)$ has the same shape with the tetrahedron with the face bivectors $B_{ij}^{\{G\}} = G_i * (v_{ij} \wedge u_i)$. For given v_{ij}, when the boundary data is nondegenerate, we can cut out a tetrahedron with planes perpendicular to v_{ij} in the three-dimensional Minkowski space orthogonal to u. Clearly, the face bivectors of this tetrahedron satisfy

$$B_{ij} = \lambda'_{ij} * (v_{ij} \wedge u) \quad \text{(E32)}$$

with the λ'_{ij} arbitrary real number. However, from the closure constraint, we have

$$\sum_{j:j\neq i} B'_{ij} = *\left(\sum_{j:j\neq i} \lambda'_{ij} v_{ij}\right) \wedge u = 0. \quad \text{(E33)}$$

Since $\forall_j v_{ij}.u = 0$, the above closure equation implies

$$\sum_{j:j\neq i} \lambda'_{ij} v_{ij} = 0 \tag{E34}$$

which, according to closure with v_{ij}, leads to

$$\exists_\lambda : \lambda'_{ij} = \lambda. \tag{E35}$$

Thus, for every edge e_i, there exists a tetrahedron determined uniquely up to inversion and translation with face bivectors

$$B_{ij} = r_i(v_{ij} \wedge u) \tag{E36}$$

in the subspace perpendicular to N_i with $r_i = \pm 1$.

The edge lengths of the tetrahedron are then determined uniquely by v_{ij}. We denote ${l^i_{jk}}^2$ the signed square lengths of the edge between faces ij and ik. The length matching condition can be expressed as

$$l^2_{(ijk)} \coloneqq {l^i_{jk}}^2 = {l^j_{ik}}^2 = {l^k_{ij}}^2. \tag{E37}$$

The nondegenerate solution exists if and only if the lengths satisfy the length matching condition. In case the length matching condition is satisfied, we can write $l^2_{(ijk)}$ using the missing indices which are different from i, j, k as $l^2_{(ml)}$. With this notation, one introduces the lengths Gram matrix of the 4-simplex

$$G^l = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & l^2_{01} & \cdots & l^2_{04} \\ 1 & l^2_{10} & 0 & \cdots & l^2_{24} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & l^2_{40} & l^2_{41} & \cdots & 0 \end{pmatrix}. \tag{E38}$$

The signature of G^l corresponds to the signature of the reconstructed 4-simplex. We denote the signature as (p, q). Based on if G^l is degenerate or not, we have the following:

(i) If G^l is nondegenerate, then there exist a unique up to rotation, shift, and reflection nondegenerate 4-simplex with signature (p, q). There are two nonequivalent 4-simplices up to rotations and shift. The normals of two reconstructed 4-simplices $\{N_i\}$ and $\{N'_i\}$ are related by

$$N'_i = (-1)^{s_i} G N_i = G I^{s_i} N_i. \tag{E39}$$

(ii) If G^l is degenerate, then there exist a unique up to rotation and shift degenerate 4-simplex with signature (p, q). The 4-volume in this case is 0.

The signature here is related to the signature of the boundary tetrahedron. For all boundary tetrahedra being timelike, the possible signatures are Lorentzian $(- + ++)$, split $(- + +-)$, or degenerate $(- + +0)$. For all boundary tetrahedra being spacelike, the possible signatures are Lorentzian $(- + ++)$, Euclidean $(+ + ++)$, or degenerate $(0 + ++)$. For boundary data containing both spacelike and timelike tetrahedra, the only possible reconstructed 4-simplex is in Lorentzian signature $(- + ++)$.

4. Gauge equivalent class of solutions

Suppose we have a nondegenerate geometric boundary data and the 4-volume is nondegenerate, then we can reconstruct the geometric nondegenerate 4-simplex up to the orthogonal transformations. Suppose we have this reconstructed 4-simplex with the geometric bivectors B^Δ_{ij} with normals N^Δ_i. From these normals, we can introduce

$$v^\Delta_{ij} = -\frac{1}{\mathrm{Vol}}\left(W^\Delta_i W^\Delta_j N^\Delta_j - \frac{W^\Delta_i W^\Delta_j N^\Delta_i \cdot N^\Delta_j}{(N^\Delta_i)^2} N^\Delta_i \right). \tag{E40}$$

It is easy to check that $v^\Delta_{ij} \cdot N^\Delta_i = 0$ and $B^\Delta_{ij} = *(v^\Delta_{ij} \wedge N^\Delta_i)$. Thus, these are nothing else but normals of faces of the ith tetrahedron recovered from the bivectors B^Δ_{ij}. It is easy to check that we have

$$v^\Delta_{ij} \cdot v^\Delta_{ik} = v_{ij} \cdot v_{ik} \tag{E41}$$

by the fact that $B^\Delta_{ij} \cdot B^\Delta_{ik} = B_{ij} \cdot B_{ik}$. We can introduce group elements $G^\Delta_i \in O$ for each i satisfying

$$G^\Delta_i u = N^\Delta_i, \qquad \forall_{j:j\neq i} G^\Delta_i v_{ij} = v^\Delta_{ij}. \tag{E42}$$

Note that there are only four independent conditions out of five.

We would like compare these group elements G^Δ_i obtained from B^Δ_{ij} with G_i from the critical point solution. From the reconstruction of bivectors and normals, we know that

$$B^\Delta_{ij} = (-1)^s B^{\{G\}}_{ij}, \qquad N_i = (-1)^{s_i} N^\Delta_i \tag{E43}$$

where $(-1)^s$ with $s \in \{0, 1\}$ and $s_i \in \{0, 1\}$. The condition leads to

$$\begin{aligned} *\,(G_i v_{ij} \wedge N_i) &= B^{\{G\}}_{ij} = (-1)^s B^\Delta_{ij} \\ &= (-1)^s *(v^\Delta_{ij} \wedge N^\Delta_i) = *((-1)^{s+s_i} v^\Delta_{ij} \wedge N_i). \end{aligned} \tag{E44}$$

Since $N_i \cdot v^\Delta_{ij} = N_i \cdot G_i v_{ij} = 0$, we have

$$G_i v_{ij} = (-1)^{s+s_i} v^\Delta_{ij}, \qquad G_i N = (-1)^{s_i} N^\Delta_i \tag{E45}$$

which implies

$$G_i = G_i^\Delta I^{s_i}(IR_N)^s. \tag{E46}$$

For $G_i \in \mathrm{SO}$, we have $\det G_i = 1$, then from (E46)

$$\det G_i^\Delta = (-1)^s. \tag{E47}$$

Since there is only one reconstructed 4-simplex up to rotations from O, thus two G^Δ solutions are related by

$$G_i^{\Delta'} = GG_i^\Delta, \qquad G \in O \tag{E48}$$

which means

$$\forall_i \frac{\det G_i^{\Delta'}}{\det G_i^\Delta} = \det G. \tag{E49}$$

This condition reminds us to introduce an orientation matching condition for the boundary data where the reconstructed 4-simplex has

$$\forall_i \det G_i^\Delta = r \quad r \in \{-1, 1\}. \tag{E50}$$

We call the boundary data as the geometric boundary data if they satisfy the length matching condition and orientation matching condition.

After we choose the reconstructed 4-simplex, we have fixed the value of s by

$$r = (-1)^s \tag{E51}$$

and it is the Plebanski orientation. However, s_i is still arbitrary.

With (E46) and (E47), we can identify the geometric solution and reconstructed 4-simplicies. Up to SO rotations, there are two reconstructed 4-simplices. The two classes of simplices solutions are related by the reflection with respect to any normalization 4-vector e_α

$$B_{ij}^{\tilde{G}} = R_{e_\alpha}(B_{ij}^{\{G\}}), \qquad s' = s + 1 \tag{E52}$$

which means

$$\tilde{G}_i = R_{e_\alpha} G_i(IR_u) \in \mathrm{SO}(1,3). \tag{E53}$$

With the gauge choice that $G_i \in \mathrm{SO}_+(1,3)$, we can rewrite (E53) as

$$\tilde{G}_i = R_{e_0} I^{r_i} G_i R_u \tag{E54}$$

such that $\tilde{G}_i \in \mathrm{SO}_+(1,3)$. It is direct to see $r_i = 0$ for u timelike and $r_i = 1$ for u spacelike.

5. Simplicial manifold with many simplices

The above interpretation and reconstruction are within the single 4-simplex case. Now we will generalize the result to simplicial manifold with many simplices. We will consider two neighboring 4-simplices where the corresponding center v and v' are connected by a dual edge $e = (v, v')$. For a shorthand notation, we will use prime to represent the parallel transported bivector and normals from the simplex with center v' to v, e.g., $N_i' = G_{vv'} N_i(v')$. We denote the edge $e = (v, v')$ as e_0.

Since $N_e(v) = G_{ve} u$ and $N_e(v') = G_{v'e} u$, we have $N_e(v) = G_{vv'} N_e(v')$ for $G = (v, v')$. From the reconstruction theorem, with (E43), we have

$$N_0^\Delta = (-1)^{s_0 + s'_0} N'^\Delta_0. \tag{E55}$$

From the parallel transport equation $X_f(v) = g_{vv'} X_f(v') g_{v'v}$, with the fact that $\epsilon_{ef}(v) = -\epsilon_{ef}(v')$, we have

$$B_{0i}^{\{G\}} = -r(v) \frac{1}{\mathrm{Vol}} W_i^\Delta W_0^\Delta * (N_i^\Delta \wedge N_0^\Delta) = r(v') \frac{1}{\mathrm{Vol}'} W'^\Delta_i W'^\Delta_0 * (N'^\Delta_i \wedge N'^\Delta_0) \tag{E56}$$

where B_{0i}^Δ is the geometric bivector corresponding to the triangle f dual to the face determined by e, e_i, e_i'. Now, similar to (E40), we can define

$$v_{0i}^\Delta(v) = -\frac{1}{\mathrm{Vol}} \left(W_0^\Delta(v) W_i^\Delta(v) N_i^\Delta(v) - \frac{W_0^\Delta(v) W_i^\Delta(v) N_0^\Delta(v) \cdot N_i^\Delta(v)}{(N_0^\Delta(v))^2} N_0^\Delta(v) \right). \tag{E57}$$

which satisfies $v_{0i}^\Delta(v) \cdot N_0^\Delta(v) = 0$. The geometrical group elements $\Omega_{vv'}^\Delta \in O(1,3)$ is defined from

$$v_{0i}^\Delta(v) = \Omega_{vv'}^\Delta v_{0i}^\Delta(v'), \qquad N_0^\Delta(v) = \Omega_{vv'}^\Delta N_0^\Delta(v'). \tag{E58}$$

(E56) now reads

$$B_{0i}^{\{G\}} = r(v) * (v_{0i}^\Delta(v) \wedge N_0^\Delta(v)) = -r(v') * (G_{vv'} v_{0i}^\Delta(v') \wedge G_{vv'} N_0^\Delta(v')). \tag{E59}$$

From (E55) and (E59), with the fact that $v_{0i}^\Delta(v) \cdot N_0^\Delta(v) = G_{vv'} v_{0i}^\Delta(v') \cdot G_{vv'} N_0^\Delta(v') = 0$, we have

$$v_{0i}^{\Delta}(v) = -(-1)^{s_0+s_0'} r(v) r(v') G_{vv'} v_{0i}^{\Delta}(v'), \qquad N_0^{\Delta}(v) = (-1)^{s_0+s_0'} G_{vv'} N_0^{\Delta}(v'). \tag{E60}$$

Compared with (E58),

$$\Omega_{vv'}^{\Delta} = G_{vv'} I I^{s_0+s_0'} (I R_{N_0(v')})^{s+s'}, \qquad \det \Omega_{vv'}^{\Delta} = (-1)^{s+s'} \tag{E61}$$

where s and s' are determined by $(-1)^s = r(v)$ and $(-1)^{s'} = r(v')$. Note that from the fact $N_0(v') = G_0(v')u = I^{s_0'} N_0^{\Delta}(v')$, and $R_N = G R_u G^{-1}$, we have $R_{N_0^{\Delta}} = R_{N_0}$. One can check that (E61) can be written as

$$\Omega_{vv'}^{\Delta} = I I^{s_0+s_0'} I^{s+s'} G_{ve} R_u^{s+s'} G_{ev'} = I G_{ve}^{\Delta} G_{ev'}^{\Delta} \tag{E62}$$

which coincide with the geometric solution for the single simplex. Note that, after fixing a pair of compatible values of s and s', another pair of compatible values are given by $s+1$ and $s'+1$ due to the common tetrahedron t_e shared by two 4-simplices. This is nothing but reflecting that every 4-simplex simultaneously connects with each other. Then, according to (E53), these two possible non-gauge-equivalent solutions are related by

$$\tilde{G}_f = \begin{cases} R_{u_e} G_f(e) R_{u_e} & \text{internal faces} \\ I^{r_{e1}+r_{e0}} R_{u_{e1}} G_f(e_1, e_0) R_{u_{e0}} & \text{boundary faces} \end{cases} \tag{E63}$$

where $G_f = \prod_{v\subset\partial f} G_{e'v} G_{ve}$ is the face holonomy.

For a simplicial manifold, we will introduce the consistent orientation. For two 4-simplices σ_v and $\sigma_{v'}$ share the same tetrahedron t_e, and we say they are consistently oriented if their orientation satisfies $[p_0, p_1, p_2, p_3, p_4]$ and $-[p_0, p_1, p_2, p_3, p_4]$. Therefore, we have $\epsilon^{01234}(v) = -\epsilon^{01234}(v')$ for the orientation in (E11). The orientated volume then contains a minus sign in V'.

From (E55) and (E56), we have

$$N'^{\Delta}_i = -(-1)^{s_0+s'_0} r(v) r(v') \frac{W_i^{\Delta} W_0^{\Delta} \mathrm{Vol}'}{W'^{\Delta}_i W'^{\Delta}_0 \mathrm{Vol}} N_i^{\Delta} + a_i N_0^{\Delta} \tag{E64}$$

where a_i are some coefficients s.t. $\sum_i W'^{\Delta}_i N'^{\Delta}_i = -W'^{\Delta}_0 N'^{\Delta}_0$. We introduce $\tilde{y}$ where $\tilde{y}_i = \frac{1}{\mathrm{Vol}} W_i^{\Delta} N_i^{\Delta}$, then

$$B_{0i}^G = -r(v) \mathrm{Vol} * (\tilde{y}_i \wedge \tilde{y}_0), \qquad \tilde{y}'_i = -(-1)^{s_0+s'_0} r(v) r(v') \frac{W_0^{\Delta}}{W'^{\Delta}_0} \tilde{y}_i + \tilde{a}_i \tilde{y}_0 \tag{E65}$$

where $\tilde{a}_i$ are coefficients s.t. $\sum_i \tilde{y}_i = -\tilde{y}_0$. We then have

$$-\frac{1}{V'} = \det(\tilde{y'}_0, \tilde{y'}_1, \tilde{y'}_2, \tilde{y'}_3) = (-r(v) r(v'))^3 \left(\frac{W_0^{\Delta}}{W'^{\Delta}_0}\right)^2 \frac{\mathrm{Vol}}{\mathrm{Vol}'} \det(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = -\tilde{r}(v) \tilde{r}(v') \left(\frac{W_0^{\Delta}}{W'^{\Delta}_0}\right)^2 \frac{1}{V'} \tag{E66}$$

where we define $\tilde{r}(v) = r(v) \mathrm{sgn}(V(v))$. The equation results in $\tilde{r}(v) = \tilde{r}(v') = \tilde{r}$. Therefore, $\tilde{r} = \mathrm{sgn}(V(v)) r(v)$ is a global sign on the entire triangulation after we choose compatible orientation. The equation also implies $|W_0^{\Delta}| = |W'^{\Delta}_0|$. With the fact that the normal vectors N_0^{Δ} and N'^{Δ}_0 are of the same type (spacelike or timelike), we have $W_0^{\Delta} = W'^{\Delta}_0$. Thus (E64) leads to

$$N'^{\Delta}_i = -(-1)^{s_0+s'_0} \mathrm{sgn}(VV') \frac{W_i^{\Delta} W_0^{\Delta} \mathrm{Vol}'}{W'^{\Delta}_i W'^{\Delta}_0 \mathrm{Vol}} N_i^{\Delta} + a_i N_0^{\Delta} = \mu_e N_i^{\Delta} + a_i N_0^{\Delta} \tag{E67}$$

where we define a sign factor $\mu_e \coloneqq -(-1)^{s_0+s'_0} \mathrm{sgn}(VV')$. One can see that, for an edge E_{lm} in the tetrahedron t_e shared by σ_v and $\sigma_{v'}$, we have

$$E'_{lm} = V' \epsilon_{lmjk}(v') * (\tilde{y'}^j \wedge \tilde{y'}^k \wedge \tilde{y'}^0) = \mu_e V \epsilon_{lmjk}(v) * (\tilde{y}^j \wedge \tilde{y}^k \wedge \tilde{y}^0) = \mu_e E_{lm}. \tag{E68}$$

The equation thus implies the coframe vectors on all edges of the tetrahedron t_e at neighboring vertices v and v' are related by

$$E_l(v) = \mu_e G_{vv'} E_l(v'). \tag{E69}$$

Since $E_l(v') \perp N_0(v')$, the relation is a direct consequence of (E61) with the fact that $\tilde{r}(v) = \tilde{r}(v') = \tilde{r}$. This relation shows that the vectors E in a tetrahedron shared by two 4-simplices σ_v and $\sigma_{v'}$ satisfy

$$g_{l_1 l_2} \coloneqq \eta_{IJ} E^I_{l_1}(v) E^J_{l_2}(v) = \eta_{IJ} E^I_{l_1}(v') E^J_{l_2}(v') \tag{E70}$$

where $g_{l_1 l_2}$ is the induced metric on the tetrahedron and it is independent of v. If the oriented volume of these two neighboring 4-simplices come with the same signature, i.e., $\mathrm{sgn}(V(v)) = \mathrm{sgn}(V(v'))$, we can associated a reference frame in each 4-simplex σ_v and the frame transformation is given by $\Omega_{vv'} = \mu_e G_{vv'} \in \mathrm{SO}(1,3)$. The matrix $\Omega_{e=(v,v')}$ is a discrete spin connection compatible with the coframe then. Note that, since $\tilde{r}(v) = r(v)\mathrm{sgn}(V(v))$ is a global sign, globally orienting $\mathrm{sgn}(V(v))$ will make $r = r(v)$ a global orientation on the dual face.

Let us go back to the original geometric rotation $\Omega^\Delta_{vv'}$. Suppose we orient consistently all pairs of 4-simplices on the simplicial complex $\mathcal{K}$. We then choose a subcomplex with the boundary such that, within it the oriented volume $\mathrm{sgn}(V)$ is a constant. Then for the holonomy along the edges of an internal face, we have

$$\Omega^\Delta_f(v) = \Omega^\Delta_{v_0 v_n} \Omega^\Delta_{v_n v_{n-1}} \cdots \Omega^\Delta_{v_1 v_0} = I^n I^{s_{0n} + s_{n,n-1} + \cdots + s_{10}} G_{v_0 v_n} G_{v_n v_{n-1}} \cdots G_{v_1 v_0} = \mu_e G_f(v) \tag{E71}$$

while for a boundary face,

$$\Omega^\Delta_f(v_n, v_0) = \Omega_{v_n v_{n-1}} \cdots \Omega_{v_1 v_0} = I^n I^{s_{n,n-1} + \cdots + s_{10}} G_{v_0 v_n} G_{v_n v_{n-1}} \cdots G_{v_1 v_0} = \mu_e G_f(v_n, v_0) \tag{E72}$$

where n is the number of internal edges belonging to the face f. Here $\mu_e = I^n \prod_{e\in f} I^{s_e} = \pm 1$, and $s_{e=(v,v')} = s_{ve} + s_{ve'}$ is independent from orientation.

Suppose the edges of the triangle due to face f res given by $E_{l1}(v)$ and $E_{l2}(v)$. Then from (E69) and (E71)–(E72), we have

$$G_f(v) E_l(v) = \mu_e E_l(v), \quad \text{or} \quad G_f(v_n, v_0) E_l(v_0) = \mu_e E_l(v_n). \tag{E73}$$

For the normals $N_0(v)$ and $N_1(v)$ which are orthogonal to the triangle due to f, from (E67) and (E71)–(E72), we have

$$G_f(v) N_1(v)^\Delta = a N_0(v)^\Delta + b N_1(v)^\Delta, \qquad G_f N_1(v) \cdot E_{l1}(v) = G_f N_1(v) \cdot E_{l2}(v) = 0. \tag{E74}$$

For boundary faces with the boundary tetrahedron t_{e_n} and t_{e_0}, similarly, we have

$$G_f(v_n, v_0) N_{e0}(v_0) \cdot E_{l1}(v_n) = G_f(v_n, v_0) N_{e0}(v_0) \cdot E_{l2}(v_n) = 0. \tag{E75}$$

6. Flipped signature solution and vector geometry

Now let us consider the degenerate case, where the 4-volume is 0 and G_i can be a gauge fixed to its subgroup $G_i \in SO(1,2)$ for the timelike tetrahedron. In this case, the 4-normals of the boundary tetrahedra are then gauge fixed to be $\forall_i N_i = u$. We can introduce an auxiliary space $M^{4\prime}$ with metric $g'_{\mu\nu}$ from M^4 by flipping the norm of u

$$g'_{\mu\nu} = g_{\mu\nu} - 2u_\mu u_\nu \tag{E76}$$

where $g_{\mu\nu}$ is the metric in M^4. We will use prime to all the operations in $M^{4\prime}$. For the norm of u, we have

$$t = u \cdot u, \qquad t' = -t = u \cdot' u. \tag{E77}$$

Notice that for the subspace V orthogonal to u, the restriction of both scalar products coincides. Thus, for the vectors in V we can use both scalar products. The Hodge dual operation satisfies $*'^2 = -*^2 = t = -t'$.

For the subspace V, we can introduce maps $\Phi^\pm$

$$\Phi^\pm : \Lambda^2 M^{4\prime} \to V,$$
$$\Phi^\pm(B) = t'(\pm B - t' *' B) \cdot' u = (\mp B + *' B) \cdot' u \tag{E78}$$

where B is a bivector in $M^{4\prime}$. Clearly for a vector $v \in V$, we have

$$\Phi^\pm(*'(v \wedge u)) = v. \tag{E79}$$

The map $\Phi^\pm$ naturally induces a map from $G \in \mathrm{SO}(2,2)$ to the subgroup $h \in \mathrm{SO}(1,2)$, which is defined by

$$\Phi^\pm(GBG^{-1}) = \Phi^\pm(G)\Phi^\pm(B) \tag{E80}$$

where

$$\Phi^{\pm}(G) \in O(V). \tag{E81}$$

It is easy to see when $G = h \in \mathrm{SO}(1,2)$, we have $\Phi^{\pm}(h) = h$. And one can further prove that the condition is sufficient and necessary as shown in [21].

Clearly, for given bivectors $B_{ij}^{\{G\}} = G_i * (v_{ij} \wedge u)$ in M', if $B_{ij}^{\{G\}} = -B_{ji}^{\{G\}}$, we have

$$v_{ij}^{\{G\}\pm} = -v_{ji}^{\{G\}\pm}, \qquad v_{ij}^{\{G\}\pm} = \Phi^{\pm}(G)v_{ij} = \Phi^{\pm}(B_{ij}^{\{G\}}) \tag{E82}$$

and the closure $\sum_i B_{ij}^g = 0$ leads to

$$\sum_i v_{ij}^{\{G\}\pm} = 0. \tag{E83}$$

One can prove the condition is necessary. In other words, if we have $g_i^{\pm}$ such that $v_{ij}^{\{G\}\pm} = -v_{ji}^{\{G\}\pm}$, we can always build unique $G_i \in SO(M')$ (up to I^{s_i}) which constitutes a $SO(M')$ solution.

In summary we see that there is an 1-1 correspondence between

(i) the pair of two nongauge equivalent vector geometries,

(ii) the geometric $SO(M')$ nondegenerate solution.

The two vector geometries are obtained from the $SO(M')$ solutions $\{g_{ve}\}$ as $g_{ve}^{\pm} = \Phi^{\pm}(g_{ve})$. This is the flipped signature case for a Gram matrix with given geometric boundary data. For example, with all boundary tetrahedra timelike, the signature of the reconstructed nondegenerate 4-simplex is split $(- + + -)$.

From the reconstruction for nondegenerate solutions, we have the orientation matching condition for the geometric group elements $G^{\Delta\pm} \in O(V)$ where

$$G_i^{\Delta\pm} v_{ij} = v_{ij}^{\Delta\pm}, \qquad v_{ij}^{\Delta\pm} = \Phi^{\pm}(B_{ij}^{\Delta}). \tag{E84}$$

One can show that, in the flipped signature case, this condition becomes

$$\det G_{ve}^{\Delta} = \det G_{ve}^{\Delta\pm}. \tag{E85}$$

The critical point solutions are in 1-1 correspondence with reconstructed 4-simplices up to reflection and shift. As a direct result from (E53), for nondegenerate boundary data satisfying the length matching condition and orientation matching condition, there are two gauge inequivalent solutions corresponding to reflected 4-simplices which are related by

$$\tilde{G} = R_u G R_u \tag{E86}$$

where $\tilde{G}$ and G represent two gauge equivalent series. Two nonequivalent geometric $SO(M')$ nondegenerate solutions then satisfy

$$\Phi^{\pm}(\tilde{G}) = \Phi^{\pm}(R_u G R_u) = \Phi^{\mp}(g). \tag{E87}$$

Finally, when the $SO(M')$ solution is degenerate, we can assume $N_i = u$ by gauge transformations. In this case, we see $\Phi^{+}(G) = \Phi^{-}(G) = h$. Thus, the vector geometries are gauge equivalent. The inverse is also true. When the vector geometries are gauge equivalent, we have $\Phi^{+}(G) = \Phi^{-}(G)$, which means there exists G_i (uniquely up to gauge transformations) such that after gauge transformations $N_i = G_i u = u$. This corresponds to the degenerate reconstructed 4-simplex with zero 4-volume.

APPENDIX F: DERIVATION OF ROTATION WITH DIHEDRAL ANGLES

In this Appendix, we prove the following equation:

$$R_{N_i} R_{N_j} = \Omega_{ij} = \mathrm{e}^{2\theta_{ij}\frac{N_i \wedge N_j}{|N_i \wedge N_j|}} \tag{F1}$$

which is used in Sec. VII. For two normalized spacelike vectors N_i, N_j, $N_i^I N_{iI} = N_j^J N_{jJ} = 1$, compatible with (7.1) and (7.2), we have

$$N_i^I N_{jI} = \cos\theta_{ij}, \tag{F2}$$

$$|N_j \wedge N_i|^2 = -|* N_j \wedge N_i|^2 = \sin^2(\theta_{ij}). \tag{F3}$$

For the N_i, N_j that are timelike and the signature of the plane spanned by $N_i \wedge N_j$ that is mixed in the flipped signature case, we have

$$N_i^I N_{jI} = \cosh\theta_{ij}, \tag{F4}$$

$$|N_j \wedge N_i|^2 = |*' N_j \wedge N_i|^2 = -\sinh^2(\theta_{ij}). \tag{F5}$$

Now from

$$(R_N)^I_J = I - \frac{2N^I N_J}{N \cdot N} = I - 2tN^I N_J \tag{F6}$$

where we define $t \coloneqq N^I N_I$. It is easy to see for a vector v in the $N_i \wedge N_j$ plane,

$$\begin{aligned} R_{N_i} R_{N_j} v &= (I - 2tN_i^K N_{iI})(I - 2tN_j^I N_{jJ})v^J \\ &= v - 2t(N_i \cdot v)N_i - 2t(N_j \cdot v)N_j \\ &\quad + 4(N_i \cdot N_j)(N_j \cdot v)N_i \end{aligned} \tag{F7}$$

which leads to

$$R_{N_i} R_{N_j} - R_{N_j} R_{N_i} = 4(N_i \cdot N_j) N_i \wedge N_j \tag{F8}$$

$$\mathrm{Tr}(R_{N_i}R_{N_j}) = 4(N_i \cdot N_j)^2 - 2. \tag{F9}$$

Let us introduce spacetime rotations $\Omega \in SO_{\pm}(1,3)$. For connected components in the Lorentzian group, two group elements Ω and Ω' are equal if they satisfy

$$\Omega - \Omega^{-1} = \Omega' - \Omega'^{-1}, \qquad \mathrm{Tr}(\Omega) = \mathrm{Tr}(\Omega'). \tag{F10}$$

The space rotation can be written using bivectors as

$$\Omega_{ij} = \mathrm{e}^{2\theta_{ij}\frac{N_i \wedge N_j}{|N_i \wedge N_j|}} = \cos(2\theta_{ij}) + \sin(2\theta_{ij})\frac{N_i \wedge N_j}{|N_i \wedge N_j|} \tag{F11}$$

and for spacelike normal vectors we have

$$\Omega_{ij} - \Omega_{ji} = 2\sin(2\theta_{ij})\frac{N_i \wedge N_j}{|N_i \wedge N_j|} = 4(N_i \cdot N_j)(N_i \wedge N_j) \tag{F12}$$

$$\begin{aligned}\mathrm{Tr}(\Omega_{ij}) &= 2\cos(2\theta_{ij}) = 2(2\cos^2(\theta_{ij}) - 1)\\ &= 4(N_i \cdot N_j)^2 - 2\end{aligned} \tag{F13}$$

while for timelike normal vectors that span a mixed signature plane, Ω is a boost,

$$\Omega_{ij} = \mathrm{e}^{2\theta_{ij}\frac{N_i \wedge N_j}{|N_i \wedge N_j|}} = \cosh(2\theta_{ij}) + \sinh(2\theta_{ij})\frac{N_i \wedge N_j}{|N_i \wedge N_j|} \tag{F14}$$

with

$$\Omega_{ij} - \Omega_{ji} = 2\sinh(2\theta_{ij})\frac{N_i \wedge N_j}{|N_i \wedge N_j|} = 4(N_i \cdot N_j)(N_i \wedge N_j) \tag{F15}$$

$$\begin{aligned}\mathrm{Tr}(\Omega_{ij}) &= 2\cosh(2\theta_{ij}) = 2(2\cosh^2(\theta_{ij}) - 1)\\ &= 4(N_i \cdot N_j)^2 - 2.\end{aligned} \tag{F16}$$

Notice that here $|N_i \wedge N_j|$ is defined as

$$|N_i \wedge N_j| = \sqrt{||N_i \wedge N_j|^2|}. \tag{F17}$$

Thus, in both cases we have

$$R_{N_i}R_{N_j} = \Omega_{ij} = \mathrm{e}^{2\theta_{ij}\frac{N_i \wedge N_j}{|N_i \wedge N_j|}} \tag{F18}$$

where θ_{ij} is the angle between normals and related to the dihedral angle by (7.1) and (7.2).

APPENDIX G: FIX THE AMBIGUITY IN THE ACTION

In this Appendix we show how to choose the $\mathrm{SL}(2,\mathbb{C})$ lift to fix the ambiguity in the action. Note that here we only fix the ambiguity for a single 4-simplex σ_v with the boundary data, where the deficit angle $\Theta_f = \theta_f$ is the angle between normals. The ambiguity (in one 4-simplex σ_v with boundary) which due to odd n_f can be expressed as

$$\Delta S - \Delta S^{\Delta} = \mathrm{i}r\sum_{f:n_f\mathrm{odd}} \begin{cases}\Delta\phi - \Theta_f & \text{non degenerate case}\\ \Delta\phi & \text{split signature case.}\end{cases} \tag{G1}$$

The procedure we use here is an extension of the one used for spacelike triangles in [21].

1. Nondegenerate case

Suppose we have nondegenerate solutions $\{G_{ve}^0 \in SO(1,3)\}$ with normals v_{ef}^0 of triangles of nondegenerate boundary tetrahedra. The area of these triangles is given by spins $\gamma s_f^0 = \frac{n_f^0}{2}$. Define the following continuous path:

$$G_{ve}(t), \qquad v_{ef}(t), \qquad u(t) = u = (0,0,0,1)^T, \tag{G2}$$

where $\forall\, e G_{ve}^0 = G_{ve}(0)$, $v_{ef}^0 = v_{ef}(0)$ such that

(i) $\forall\, t \in [0,1]$, $\{G_{ve}(t)\}$ is a solution of the critical point equations with boundary data where the normals of the triangles of the boundary tetrahedra are $v_{ef}(t)$,

(ii) $\forall t \neq 1$ boundary data is nondegenerate, and $v_{ef}(1) \neq 0$,

(iii) $\forall t \neq 1$ solution $\{G_{ve}(t)\}$ is nondegenerate,

(iv) for $t = 1$, the pair of solutions $\{G_{ve}(t)\}$ and $\{\tilde{g}_{ve}(t) = R_{e_\alpha} g_{ve}(t) R_u\}$ are gauge equivalent.

In this path, the function

$$f(t) = \sum_{f:n_f\mathrm{odd}} \Delta\phi_{eve'f}(t) - r\Theta_f(t) \mod 2\pi \tag{G3}$$

takes values in $\{0,\pi\}$ and changes continuously with the phase and the difference from the stationary points determined by $\{G_{ve}(t)\}$ and $\{\tilde{G}_{ve}(t) = R_{e_\alpha} G_{ve}(t) R_u\}$. Thus, $f(t)$ is a constant. Since at $t = 1$, we have two geometric solutions that are gauge equivalent to each other, which means the lifts $g_{ve}, \tilde{g}_{ve}$ of the solutions satisfy

$$\forall_e \tilde{g}_{ve} = (-1)^{r_{ve}} g g_{ve}, \qquad r_{ve} = \{0,1\}. \tag{G4}$$

From (7.22),

$$(-1)^{r_{ve}+r_{ve'}} = g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1}g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef}X_f + 2\mathrm{i}\Delta\phi_{e'vef}X_f} \tag{G5}$$

which leads to $\Delta\phi_{eve'f}(1) = (r_{ve} + r_{ve'})\pi$ mod 2π since we have $(2X_f)^2 = 1$. We shall consider a subgraph of the spin network which contains those odd n links. The subgraph has even valence nodes. Thus, we can decompose into Euler cycles. In those cycles every link of odd n will appears exactly once. For a Euler cycle consisting of edges with odd n, every edge will be counted twice. Thus we have

$$\sum_{e\in cycle} \Delta\phi_{eve'f}(1) = \sum_{e\in cycle} 2r_{ve}\pi = 0 \mod 2\pi. \quad \text{(G6)}$$

Also, from the fact that two geometrical solutions are gauge equivalent, $\forall_e \tilde{G}_{ve} = GG_{ve}$, we have $R_{N_e}R_{N_{e'}} = G_{ve}(\tilde{G}_{e'v}\tilde{G}_{ve})^{-1}G_{e'v} = 1$, thus

$$\Theta_f(1) = \tilde{r}_f\pi \mod 2\pi, \qquad \tilde{r}_f = \tilde{r}_{ve} + \tilde{r}_{ve'} \in \{0,1\}, \quad \text{(G7)}$$

which can be fixed again using Euler cycles for $\Delta\phi$.

The path can be achieved by deforming solutions in the following way: First choose a timelike plane with the simple normalized bivector V at some vertex v that satisfies

$$\forall_f V \wedge *B_f \neq 0. \quad \text{(G8)}$$

The path is made by contracting the two directions in $*V$, and we donate the $t = 1$ as the limit for contracting directions to 0. From the above condition we have that $\lim_{t\to 1} B_f$ exist and keep nonzero. The dual action of the shrinking on the geometric normal vectors N^Δ also has a limit which is their normalized components lying in the $*V$ plane (after normalization). By a suitable definition of boundary data, we can assume $G_{ve}(1) = \lim_{\to 1} G_{ve}(t)$ exist. Now we end up with a highly degenerate 4-simplex which is contained in a 2D plane and all bivectors are proportional to V.

2. Split signature case

The treatment concerns degenerate solutions following the similar method. We start from the nondegenerate boundary data, where normals of the triangles of boundary tetrahedra are given by v_{ef}^0 and an area of these triangles are related to spins $n_f/2$. Suppose from these boundary data, we can reconstruct a nondegenerate 4-simplex in flipped signature space M'. In this case, we have two non-gauge-equivalent solutions $\{g_{ve}^\pm\}$. We define the following path:

$$g_{ve}^\pm(t), \qquad v_{ef}(t), \qquad u(t) = u = (0,0,0,1)^T, \quad \text{(G9)}$$

where $\forall\, e g_{ve}^{0\pm} = g_{ve}^\pm(0)$, $v_{ef}^0 = v_{ef}(0)$. The path satisfies

(i) $\forall t \in [0,1]$, $\{g_{ve}^\pm(t)\}$ are solutions of the critical point equation with boundary data given by $v_{ef}(t)$,

(ii) $\forall t \in [0,1]$ boundary data is nondegenerate, e.g., the boundary tetrahedron is nondegenerate,

(iii) $\forall t \neq 1$ solutions $\{g_{ve}^\pm\}$ are non-gauge-equivalent, thus we have a nondegenerate reconstructed 4-simplex in M'

(iv) for $t = 1$, the reconstructed 4-simplex is degenerate in M'.

Now the constant function $f(t) \in \{0, \pi\}$ reads

$$f(t) = \sum_{f:n_f \mathrm{odd}} \Delta\phi_{eve'f}(t) \mod 2\pi. \quad \text{(G10)}$$

Following the same argument in the nondegenerate case, we have for the lifts

$$g_{ve}^+(1) = (-1)^{r_{ve}} g_{ve}^-(1). \quad \text{(G11)}$$

Based on the same consideration using Euler cycles, we have

$$f(1) = \sum_{f:n_f \mathrm{odd}} \Delta\phi_{eve'f}(t) = 0 \mod 2\pi. \quad \text{(G12)}$$

Thus we have

$$\Delta S^0 - \Delta S^{\Delta 0} = 0 \mod 2\pi. \quad \text{(G13)}$$

The path is built by the following way: We choose a spacelike normal such that, in flipped signature space

$$\forall_f N \wedge B_f \neq 0. \quad \text{(G14)}$$

The path is then made by contracting in the direction of N in the flipped space M'. The contraction leads to a continuous path of nondegenerate solutions in M' until $t = 1$ where the 4-simplex is degenerate.

[1] T. Thiemann, Modern canonical quantum general relativity, arXiv:gr-qc/0110034.

[2] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2014).

[3] *Loop Quantum Gravity*, edited by A. Ashtekar and J. Pullin, 100 Years of General Relativity (World Scientific, Singapore, 2017), Vol. 4.

[4] M. Han, W. Huang, and Y. Ma, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2007).

[5] A. Perez, The spin foam approach to quantum gravity, Living Rev. Relativity **16**, 3 (2013).
[6] J. Engle, E. Livine, R. Pereira, and C. Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. **B799**, 136 (2008).
[7] F. Conrady and J. Hnybida, A spin foam model for general Lorentzian 4-geometries, Classical Quantum Gravity **27**, 185011 (2010).
[8] F. Conrady, Spin foams with timelike surfaces, Classical Quantum Gravity **27**, 155014 (2010).
[9] J. Rennert, Timelike twisted geometries, Phys. Rev. D **95**, 026002 (2017).
[10] F. Conrady and L. Freidel, On the semiclassical limit of 4d spin foam models, Phys. Rev. D **78**, 104023 (2008).
[11] J. D. Christensen, I. Khavkine, E. R. Livine, and S. Speziale, Sub-leading asymptotic behaviour of area correlations in the Barrett-Crane model, Classical Quantum Gravity **27**, 035012 (2010).
[12] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes, and F. Hellmann, Asymptotic analysis of the EPRL four-simplex amplitude, J. Math. Phys. (N.Y.) **50**, 112504 (2009).
[13] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, F. Hellmann, and R. Pereira, Lorentzian spin foam amplitudes: Graphical calculus and asymptotics, Classical Quantum Gravity **27**, 165009 (2010).
[14] M. Han and M. Zhang, Asymptotics of spinfoam amplitude on simplicial manifold: Lorentzian theory, Classical Quantum Gravity **30**, 165012 (2013).
[15] M.-X. Han and M. Zhang, Asymptotics of spinfoam amplitude on simplicial manifold: Euclidean theory, Classical Quantum Gravity **29**, 165004 (2012).
[16] M. Han, On spinfoam models in large spin regime, Classical Quantum Gravity **31**, 015004 (2014).
[17] M. Han, Semiclassical analysis of spinfoam model with a small Barbero-Immirzi parameter, Phys. Rev. D **88**, 044051 (2013).
[18] M. Han, Einstein equation from covariant loop quantum gravity in semiclassical continuum limit, Phys. Rev. D **96**, 024047 (2017).
[19] H. M. Haggard, M. Han, W. Kamiski, and A. Riello, SL(2,C) ChernSimons theory, a non-planar graph operator, and 4D quantum gravity with a cosmological constant: Semiclassical geometry, Nucl. Phys. **B900**, 1 (2015).
[20] H. M. Haggard, M. Han, W. Kaminski, and A. Riello, SL(2,C) Chern-Simons theory, flat connections, and four-dimensional quantum geometry, arXiv:1512.07690.
[21] W. Kaminski, M. Kisielowski, and H. Sahlmann, Asymptotic analysis of the EPRL model with timelike tetrahedra, Classical Quantum Gravity **35**, 135012 (2018).
[22] J. W. Barrett, M. Galassi, W. A. Miller, R. D. Sorkin, P. A. Tuckey, and R. M. Williams, A Paralellizable implicit evolution scheme for Regge calculus, Int. J. Theor. Phys. **36**, 815 (1997).
[23] A. P. Gentle and W. A. Miller, A fully $(3+1)$-D Regge calculus model of the Kasner cosmology, Classical Quantum Gravity **15**, 389 (1998).
[24] A. P. Gentle, A cosmological solution of Regge calculus, Classical Quantum Gravity **30**, 085004 (2013).
[25] L. Freidel and K. Krasnov, A new spin foam model for 4d gravity, Classical Quantum Gravity **25**, 125018 (2008).
[26] H. Liu and K. Noui, Gravity as an $\mathfrak{su}(1,1)$ gauge theory in four dimensions, Classical Quantum Gravity **34**, 135008 (2017).
[27] G. Lindblad and B. Nagel, Continuous bases for unitary irreducible representations of $su(1,1)$, in *Annales de l'IHP Physique Théorique* (Gauthier-Villars, 1970), Vol. 13, pp. 27–56.
[28] F. Conrady and J. Hnybida, Unitary irreducible representations of SL(2,C) in discrete and continuous SU(1,1) bases, J. Math. Phys. (N.Y.) **52**, 012501 (2011).
[29] N. Bleistein, Uniform asymptotic expansions of integrals with stationary point near algebraic singularity, Commun. Pure Appl. Math. **19**, 353 (1966).
[30] R. Sorkin, Time evolution problem in Regge calculus, Phys. Rev. D **12**, 385 (1975); Erratum, Phys. Rev. D **23**, 565(E) (1981).
[31] R. D. Sorkin, Causal sets: Discrete gravity, in *Lectures on Quantum Gravity. Proceedings, School of Quantum Gravity, Valdivia, Chile, 2002* (Springer, Boston, 2003), pp. 305–327.
[32] J. Ambjorn, J. Jurkiewicz, and R. Loll, Dynamically triangulating Lorentzian quantum gravity, Nucl. Phys. **B610**, 347 (2001).
[33] J. Ambjørn, A. Görlich, J. Jurkiewicz, and R. Loll, Quantum gravity via causal dynamical triangulations, in *Springer Handbook of Spacetime*, edited by A. Ashtekar and V. Petkov (Springer-Verlag, Berlin, Heidelberg, 2014), pp. 723–741.
[34] E. Alesci and C. Rovelli, The complete LQG propagator. II. Asymptotic behavior of the vertex, Phys. Rev. D **77**, 044024 (2008).
[35] E. Bianchi and Y. Ding, Lorentzian spinfoam propagator, Phys. Rev. D **86**, 104040 (2012).
[36] G. Lindblad, Eigenfunction expansions associated with unitary irreducible representations of su(1,1), Phys. Scr. **1**, 201 (1970).
[37] F. W. Olver, *NIST Handbook of Mathematical Functions Hardback and CD-ROM* (Cambridge University Press, Cambridge, 2010).
[38] P. D. Miller, *Applied Asymptotic Analysis* (American Mathematical Society, Rhode Island, 2006), Vol. 75.