Relativistic hydrodynamic approach to the classical ideal gas

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The necessary and sufficient condition for a conservative perfect fluid energy tensor to be the energetic evolution of a classical ideal gas is obtained. This condition forces the square of the speed of sound to have the form $c_s^2 = \frac{\gamma p}{\rho + p}$ in terms of the hydrodynamic quantities, energy density ρ and pressure p, γ being the (constant) adiabatic index. The *inverse problem* for this case is also solved, that is, the determination of all the fluids whose evolutions are represented by a conservative energy tensor endowed with the above expression of c_s^2, and it shows that these fluids are, and only are, those fulfilling a Poisson law. The relativistic compressibility conditions for the classical ideal gases and the Poisson gases are analyzed in depth and the values for the adiabatic index γ for which the compressibility conditions hold in physically relevant ranges of the hydrodynamic quantities ρ, p are obtained. Some scenarios that model isothermal or isentropic evolutions of a classical ideal gas are revisited, and preliminary results are presented in applying our hydrodynamic approach to looking for perfect fluid solutions that model the evolution of a classical ideal gas or of a Poisson gas.

I. INTRODUCTION

In the relativistic framework, a perfect fluid is usually assumed to be a *perfect energy tensor*, $T \equiv (\rho + p)u \otimes u + pg$, solution to the conservation equations $\nabla \cdot T = 0$. It can model a test fluid in any given space-time, or the source of a solution of the Einstein field equations $G = kT$. Nevertheless, complementary physical requirements on the *hydrodynamic quantities* (the *unit velocity* u, the *energy density* ρ, and the *pressure* p) must be imposed for T to represent the energetic evolution of a realistic thermodynamic fluid in local thermal equilibrium.

As it is well known, a fluid whose all possible energetic evolutions in local thermal equilibrium are described by perfect energy tensors is necessarily Pascalian and with vanishing heat conductivity. Its equations are those of the Eckart's thermodynamic scheme [1] for this case, and lead to the introduction of the *thermodynamic quantities*: *matter density n*, *specific internal energy ε*, *temperature* Θ, and *specific entropy s*. On the other hand, we will restrict the fluids obeying these equations to verify general constraints of physical reality, namely the positivity conditions of some of their quantities, the energy conditions [2] and the relativistic compressibility conditions [3,4].

Elsewhere [5] we have shown that, given a perfect energy tensor in a space-time domain, the question of whether or not it admits the above thermodynamic scheme can be detected by conditions just involving the hydrodynamic quantities (u, ρ, p), namely that the *indicatrix function* $\chi = \dot{p}/\dot{\rho}$ be an equation of state, $\chi = \chi(\rho, p)$, and then it is the square of the speed of sound c_s^2. This result offers a purely hydrodynamic characterization of the local thermal equilibrium and solves the *generic direct problem* [6], i.e., the determination of the set $\mathbf{T_F}$ of all perfect energy tensors corresponding to all possible evolutions of the set $\mathbf{F}$ of all perfect fluids. In Ref. [6] we have also solved the *generic inverse problem* for a perfect energy tensor T, i.e., the determination of the set $\mathbf{F}_T$ of all perfect fluids for which T is the energetic description of a particular evolution.

Nevertheless, because of its practical applications, solving a *specific direct problem* for a family of fluids may be more interesting than to solve the generic one. We have studied in [6] the specific direct problem for the family of the *generic ideal gases*, i.e., those defined by the equation of state $p = kn\Theta$. These results have allowed us the study of the Stephani universes that can be interpreted as an ideal gas in local thermal equilibrium [7], and the determination of the associated thermodynamics.

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Furthermore, elsewhere [8] we have shown that two of the relativistic compressibility conditions can be formulated in terms of the hydrodynamic quantity $\chi(\rho, p)$, the square of the speed of sound, so that we have a hydrodynamic characterization for these two constraints.

Here we apply all these hydrodynamic approaches to the *classical ideal gas*, defined by the equations of state, $p = kn\Theta$ and $\epsilon = c_v\Theta$. The interest of this study is twofold. On one hand, the classical ideal gas is defined by simple equations of state and it allows us to go further away in obtaining and interpreting the results, so that the present study is useful to gain a better understanding of the concepts and conclusions obtained in Refs. [6,8]. On the other hand, the classical ideal gas is a good model for realistic fluids in the low temperature range, so that our results can be applied in looking for new physically interesting solutions, or in interpreting already known ones in this range.

In Sec. II we present the basic elements concerning the local thermal equilibrium assumption, and we outline some results on the above-mentioned hydrodynamical approach.

Section III is devoted to studying the specific direct and inverse problems for the classical ideal gas (CIG): we show that the square of the speed of sound $\chi(\rho, p) = \frac{\gamma p}{\rho + p}$, where γ is the *adiabatic index*, characterizes a CIG evolution (direct problem), and we obtain all the thermodynamic quantities of the CIG in terms of the hydrodynamic quantities ρ and p (specific inverse problem). Moreover, for each of the values of γ, we obtain the domain of the variable $\pi = p/\rho$ where the thermodynamic quantities are positive and the relativistic compressibility conditions hold. We show that this domain is relevant for $\gamma > 1$.

Section IV is devoted to solving the generic inverse problem for a CIG indicatrix function, and we show that the fluids that have the same speed of sound that a CIG are, and only are, the *Poisson gases*, that is, the gases fulfilling the *Poisson law* $p = \beta(s)n^\gamma$. Then, we study the compressibility conditions for the Poisson gases and we analyze the stronger restrictions on the adiabatic index obtained by Taub [9] from a kinetic theory approach.

We have pointed out in [6] that energy tensor that fulfill a barotropic relation $p = p(\rho)$ can model particular evolutions of a nonbarotropic perfect fluid. In Sec. V we obtain the barotropic relation of a CIG in isothermal evolution and we model an isothermal atmosphere and a self-gravitating isothermal sphere.

The above models have been established without reference to any heat equation. Otherwise, the current relativistic Fourier equations impose a strong constraint on the temperature when the conductivity coefficient does not vanish. In Sec. VI we take into account this fact in dealing with a model for a self-gravitating CIG sphere in thermal equilibrium.

In Sec. VII we obtain the barotropic relation of a CIG in isentropic evolution and we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) models that fulfill this constraint. The study of the field equations shows that these models are defined by a pressure that is a power of the expansion factor.

In Sec. VIII we present preliminary results on perfect fluid solutions of the field equations that can be interpreted as a CIG in local thermal equilibrium. For the case of fluids in irrotational motion we show that our hydrodynamic characterization implies that, in comoving coordinates, the pressure is, up to an arbitrary function of the spatial coordinates, a power of the determinant of the spatial metric. Then, for the spherically symmetric metrics with geodesic motion and 2-sphere curvature changing with the comoving radial coordinate, we show that the only CIG solutions are the FLRW models considered in Sec. VII.

Interestingly, work in progress seems to show that the known perfect fluid exact solutions to Einstein equations do not verify the CIG constraint exactly. Thus, it can be suitable to look for exact solutions that approximate a CIG at low temperatures, which is the range where the CIG model is realistic. In Sec. VIII we also propose a way to approximate a CIG from a generic ideal gas by using the indicatrix function $\chi(\rho, p)$.

Finally, in Sec. IX we present several remarks and report some work in progress.

II. LOCAL THERMAL EQUILIBRIUM: BASIC CONCEPTS AND HYDRODYNAMIC APPROACH

The energetic description of the evolution of a perfect fluid is given by its energy tensor T:

$$T = (\rho + p)u \otimes u + pg. \tag{1}$$

where ρ, p and u are, respectively, the *energy density*, *pressure* and *unit velocity* of the fluid. A divergencefree T, $\nabla \cdot T = 0$, of this form is called a *perfect energy tensor*. These conservation equations take the expression:

$$dp + \dot{p}u + (\rho + p)a = 0, \tag{2}$$

$$\dot{\rho} + (\rho + p)\theta = 0, \tag{3}$$

where a and θ are, respectively, the acceleration and the expansion of u, a dot denotes the directional derivative with respect to u of a scalar quantity q, $\dot{q} = u(q) = u^\alpha \partial_\alpha q$, and dq denotes the space-time exterior derivative, $dq = \partial_\alpha q dx^\alpha$.

A *barotropic evolution* is an evolution along which the *barotropic relation* $d\rho \wedge dp = 0$ is fulfilled. It is to be noted that barotropic evolutions may be followed by any fluid, whatever its equations of state, barotropic or not. A perfect energy tensor describing energetically a barotropic evolution is called a *barotropic perfect energy tensor*.

The thermodynamic scheme for a relativistic perfect fluid is obtained as the Pascalian and of vanishing heat conductivity restriction of the general thermodynamic

scheme by Eckart [1]. The energy density ρ is decomposed in terms of the *matter density n* and the *specific internal energy ϵ*:

$$\rho = n(1 + \epsilon), \qquad (4)$$

requiring the conservation of matter:

$$\nabla \cdot (nu) = \dot{n} + n\theta = 0, \qquad (5)$$

where $\nabla \cdot X = \nabla_\alpha X^\alpha$ is the divergence of a vector field X. When $n = n(\rho, p) > 0$, and according to a classical argument, it is always possible to identify an integral divisor of the one-form $\Lambda \equiv (1/n)\mathrm{d}\rho + (\rho + p)\mathrm{d}(1/n)$ with the (absolute) *temperature* Θ of the fluid, allowing us to define the *specific entropy s* by the *local thermal equilibrium equation*:

$$\Theta \mathrm{d}s = \mathrm{d}\epsilon + p\mathrm{d}(1/n). \qquad (6)$$

When, in addition to (2) and (3), a perfect energy tensor T also satisfies (4), (5) and (6), we will say that T evolves in *local thermal equilibrium* (l.t.e.).

We have already shown [5,6] that the notion of l.t.e. admits a purely hydrodynamic formulation: a perfect energy tensor T describes a thermodynamic perfect fluid in l.t.e. if, and only if, its hydrodynamic quantities $\{u, \rho, p\}$ fulfill the hydrodynamic sonic condition:

$$(\dot{\rho}\mathrm{d}\dot{p} - \dot{p}\mathrm{d}\dot{\rho}) \wedge \mathrm{d}\rho \wedge \mathrm{d}p = 0. \qquad (7)$$

When the perfect energy tensor is nonisoenergetic, $\dot{\rho} \neq 0$, condition (7) states that the space-time function $\chi \equiv \dot{p}/\dot{\rho}$, called *indicatrix of local thermal equilibrium*, depends only on the quantities p and ρ, $\chi = \chi(p, \rho)$. Moreover this function of state represents physically the square of the *speed of sound* in the fluid, $\chi(\rho, p) \equiv c_s^2$.

Note that the above quoted result solves the *generic direct problem*, i.e., the determination of the set $\mathbf{T_F}$ of all the perfect energy tensors corresponding to all possible energetic evolutions of all perfect fluids $\mathbf{F}$. In Ref. [6] we have also solved the *inverse problem* for a perfect energy tensor T, and we have shown that the set $\mathbf{F}_T$ of all perfect fluids for which T is the energetic description of a particular evolution is determined up to two arbitrary functions of the entropy.

In practice, solving a *restricted direct problem* may be more interesting than solving the generic one. In this way we have solved in [6] the direct problem for the family of ideal gases $\mathbf{G}$, which is defined by the equation of state:

$$p = kn\Theta, \qquad k \equiv \frac{k_B}{m}. \qquad (8)$$

Now the hydrodynamic sonic condition states [6]: the necessary and sufficient condition for a nonbarotropic

$(d\rho \wedge dp \neq 0)$ and nonisoenergetic $(\dot{\rho} \neq 0)$ perfect energy tensor $T = (u, \rho, p)$ to represent the l.t.e. evolution of an ideal gas is that the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ be a function of the quantity $\pi \equiv p/\rho$, $\chi = \chi(\pi) \neq \pi$.

This statement characterizes the set $\mathbf{T_G}$ of all the perfect energy tensors which are the energetic evolution of a nonbarotropic ideal gas $f \in \mathbf{G}$. In Ref. [6] we have solved the *specific inverse problem* by obtaining, in terms of ρ and p, all the thermodynamic quantities that define the thermodynamic scheme of an ideal gas.

III. CLASSICAL IDEAL GAS: HYDRODYNAMIC CHARACTERIZATION

Let us study now the evolution (direct problem) of the *classical ideal gas* (CIG), that is to say, we consider any ideal gas verifying (8) and

$$\epsilon = c_v\Theta, \qquad (9)$$

$c_v > 0$ being the *heat capacity at constant volume*. From (8) and (9) it follows that a CIG satisfies the *classical γ-law*:

$$p = (\gamma - 1)n\epsilon, \qquad \gamma \equiv 1 + \frac{k}{c_v}, \qquad (10)$$

γ being the *adiabatic index*. On the other hand the l.t.e. Eq. (6) implies that a CIG has the characteristic equation

$$\epsilon = \epsilon(n, s) = \bar{\beta}(s)n^{\gamma-1}, \qquad \bar{\beta}(s) \equiv \exp\frac{s - \bar{s}_0}{c_v}. \qquad (11)$$

And from (10) and (11) it follows that any CIG fulfills a *Poisson law*[1]:

$$p = \beta(s)n^\gamma, \qquad \beta(s) = (\gamma - 1)\bar{\beta}(s). \qquad (12)$$

We know [6] that the only intrinsically barotropic ideal gases are those satisfying $\epsilon(\Theta) = c_v\Theta - 1$. Thus a CIG is, necessarily, nonbarotropic $d\rho \wedge dp \neq 0$, and we can take the hydrodynamic quantities (ρ, p) as coordinates in the thermodynamic plane. Then, from the above expressions (4), (8), (9), (10) and (11), we obtain:

$$n(\rho, p) = \rho - \frac{p}{\gamma - 1}, \qquad \epsilon(\rho, p) = \frac{p}{(\gamma - 1)\rho - p}, \qquad (13)$$

$$s(\rho, p) = s_0 + c_v \ln \frac{p}{[\rho(\gamma - 1) - p]^\gamma}. \qquad (14)$$

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[1]The classical γ-law (10) and the Poisson law (12) are equations of state. A fluid that fulfills (10) or (12) is called a *γ-gas* or a *Poisson gas*, respectively. A Poisson gas in isentropic evolution fulfills a *Poisson adiabatic law* $p = \beta_0 n^\gamma$. In Sec. IV Table I gives the equations of state that define the diverse families of perfect fluids concerned in this paper and summarizes some of our results.

Moreover, the square of the speed of sound can be calculated as $\chi(\rho, p) = -s'_\rho/s'_p$, and we obtain:

$$c_s^2 = \chi(\rho, p) \equiv \frac{\gamma p}{\rho + p}. \tag{15}$$

We summarize these results in the following.

Lemma 1: In terms of the hydrodynamic quantities (ρ, p), the matter density n, the specific internal energy ϵ, the specific entropy s and the speed of the sound c_s of a classical ideal gas are given by (13), (14) and (15).

As shown in Ref. [6,8], a nonbarotropic and isoenergetic ($\dot{\rho} = 0$) perfect energy tensor evolves in l.t.e. if, and only if, it is isobaric, $\dot{p} = 0$, and then it admits any thermodynamic scheme. In particular, it will admit the thermodynamics of any CIG. Thus, we have:

Proposition 1: The necessary and sufficient condition for a nonbarotropic and isoenergetic ($\dot{\rho} = 0$) perfect energy tensor $T = (u, \rho, p)$ to represent the l.t.e. evolution of a CIG is that it be isobaroenergetic: $\dot{\rho} = 0$, $\dot{p} = 0$. Then T represents the evolution in l.t.e. of any CIG, and the specific internal energy ϵ, the matter density n, the specific entropy s and the speed of sound c_s are given by (13), (14) and (15).

Note that the richness of CIG associated with a nonbarotropic and isobaroenergetic perfect energy tensor depends on two parameters, the adiabatic index γ and the heat capacity at constant volume c_v. This last one determines through (8) and (10) the particle mass m.

On the other hand, if the perfect energy tensor T is nonisoenergetic, $\dot{\rho} \neq 0$, then the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ equals the square of the speed of sound [6] and it is given by (15) for a CIG. Conversely, when the perfect fluid has this indicatrix, the thermodynamic quantities n, ϵ and s given in (13) and (14) fulfill the CIG characteristic equation (11). Thus we can state:

Theorem 1: The necessary and sufficient condition for a nonbarotropic and nonisoenergetic perfect energy tensor $T = (u, \rho, p)$ to represent the l.t.e. evolution of a classical ideal gas with adiabatic index γ is that the indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ be of the form:

$$\chi = \frac{\gamma p}{\rho + p}. \tag{16}$$

Then, the matter density n and the specific entropy s are given by (13) and (14), and the constants k and c_v are related by

$$\gamma = 1 + \frac{k}{c_v}. \tag{17}$$

After this theorem it is worth remarking the following points:
(i) Theorem 1 offers a purely hydrodynamic characterization of the set **C** of the CIG by solving the associated direct problem: the determination of the

set $\mathbf{T}_f$ of all the perfect energy tensors that are the energetic evolution of any CIG $f \in \mathbf{C}$. It also solves the *specific inverse problem* for **C**: the determination of the set $\mathbf{C}_T$ of the CIG whose evolution is described by a given $T \in \mathbf{T_C}$.

(ii) In practice, it can be more convenient to verify directly the condition that characterizes the full set **C** without specifying the adiabatic index γ:

$$\mathrm{d}\frac{(\rho + p)\dot{p}}{p\dot{\rho}} = 0. \tag{18}$$

If (18) holds, a constant γ exists that satisfies (16), and then the perfect energy tensor $T = (u, \rho, p)$ represents the l.t.e. evolution of a CIG with adiabatic index γ.

(iii) Note that in the nonisoenergetic case the adiabatic index γ, that is, the quotient k/c_v, is fixed by the hydrodynamic quantities. Remember that the values $\gamma = 5/3$ and $\gamma = 7/5$ correspond, respectively, to *monoatomic* and *diatomic* ideal gases. Again, the freedom to choose c_v is related with the particle mass. Some conditions for physical reality imposing constraints on the adiabatic index γ will be analyzed below.

A. Constraints for physical reality: Compressibility conditions

The Plebański [2] energy conditions are basic constraints for physical reality of formal arbitrary media. They impose algebraic restrictions on the hydrodynamic quantities, and for a perfect energy tensor take the expression $-\rho < p \leq \rho$. For a medium that fulfills the equation of state of a generic ideal gas (8), it is reasonable to consider a positive pressure, $p > 0$. Thus, the energy conditions E for these media state:

$$\text{E:} \qquad 0 < \pi \leq 1, \qquad \pi = \frac{p}{\rho}. \tag{19}$$

When the perfect fluid evolves in l.t.e. some physical requirements must be imposed on the thermodynamic quantities. First, the positivity conditions P for the matter density n and the specific internal energy ϵ:

$$\text{P:} \qquad n > 0, \qquad \epsilon > 0, \tag{20}$$

and second, the relativistic compressibility conditions H must be fulfilled [3,4]. We have shown in Ref. [8] that, for a generic ideal gas, the conditions H can be stated in terms of the indicatrix function, $\chi = \chi(\pi) \neq \pi$:

$$\text{H:} \qquad \begin{array}{c} \frac{\pi}{2\pi+1} < \chi < 1, \\ (1 + \pi)(\chi - \pi)\chi' + 2\chi(1 - \chi) > 0. \end{array} \tag{21}$$

In this paper, where a macroscopic approach is carried out, the conditions E, P and H are called the *constraints for physical reality*.

For a CIG, the positivity constraint P can be stated in terms of hydrodynamic quantities as a consequence of (13):

Lemma 2: For a CIG, the positivity constraint P, $n > 0$, $\epsilon > 0$, hold if, and only if,

$$\gamma > 1, \qquad \pi < \pi_m \equiv \gamma - 1. \tag{22}$$

Now we study the compressibility constraints H for a CIG, that is, after (16), for

$$\chi(\pi) = \frac{\gamma \pi}{\pi + 1}, \qquad \chi'(\pi) = \frac{\gamma}{(\pi + 1)^2}. \tag{23}$$

If $b(\pi) = \frac{\pi}{2\pi + 1}$, we have that both derivatives $\chi'(\pi)$ and $b'(\pi)$ are positive and decreasing in the domain [0, 1]. Moreover $b(0) = \chi(0) = 0$; $b(1) = 1/3$, $\chi(1) = \gamma/2$; $b'(0) = 1$, $\chi'(0) = \gamma$. Consequently, when $\gamma > 1$ we have $b(\pi) < \chi(\pi)$ in the domain]0, 1]. On the other hand, $\chi < 1$ if $\pi \in [0, \frac{1}{\gamma-1}[$. Thus, we can state:

Lemma 3: If $\gamma > 1$, the first of the compressibility conditions H in (21) holds in the interval:

$$0 < \pi < \tilde{\pi}_m \equiv \frac{1}{\gamma - 1}. \tag{24}$$

The second compressibility condition in (21) can be analyzed taking into account (23) and we obtain:

Lemma 4: If $\gamma > 1$, the second of the compressibility conditions H in (21) holds in the interval:

$$0 < \pi < \hat{\pi}_m \equiv \frac{\gamma + 1}{2\gamma - 1}. \tag{25}$$

Finally, note that if $\gamma = 2$, $\pi_m = \hat{\pi}_m = \tilde{\pi}$; if $\gamma \leq 2$, $\pi_m < \hat{\pi}_m < \tilde{\pi}$; and if $\gamma \geq 2$, $\tilde{\pi}_m < \hat{\pi}_m < \pi_m$. Thus, we can collect the results of the three lemmas 2, 3 and 4 in the following.

Proposition 2: For a CIG with adiabatic index γ, the constraints for physical reality E, P and H are satisfied for values of π in a nonempty subinterval of [0, 1] if, and only if, $\gamma > 1$. In addition, they hold in the interval:

$$\begin{cases} 0 < \pi < \pi_m \equiv \gamma - 1, & \text{if } 1 < \gamma \leq 2, \\ 0 < \pi < \tilde{\pi}_m \equiv \frac{1}{\gamma-1}, & \text{if } \gamma \geq 2. \end{cases} \tag{26}$$

Usually (see, e.g., [10]) the adiabatic index γ is supposed to be constrained by $1 < \gamma \leq 2$. Proposition above shows that, under reasonable physical requirements, the upper limit for γ can be relaxed. In subsection IV A we will analyze this apparent contradiction and we will also comment about the stronger restrictions imposed by a model based on the relativistic kinetic theory.

IV. THE GENERIC INVERSE PROBLEM FOR AN INDICATRIX OF A CLASSICAL IDEAL GAS

In Sec. III we have solved the specific inverse problem for the CIG, that is, we have determined the set $\mathbf{C}_T$ of CIG having any given perfect energy tensor $T \in \mathbf{T_C}$ as energetic description. Now we analyze the generic inverse problem for the CIG: to determine the set $\mathbf{F}_T$ of all perfect fluids having a given perfect energy tensor $T \in \mathbf{T_C}$ as energetic description. The answer to this problem is tantamount to obtaining the thermodynamics compatible with the expression (15) of the square of the speed of sound. We can make use of the following known result [6]:

Lemma 5: Let T be a nonbarotropic and nonisoenergetic perfect energy tensor that evolves in l.t.e. The admissible thermodynamics are defined by a matter density $n = n(\rho, p)$ and a specific entropy $s(\rho, p)$ of the form $n = \bar{n}R(\bar{s})$ and $s = s(\bar{s})$, where $R(\bar{s})$ and $s(\bar{s})$ are arbitrary real functions, and $\bar{n}(\rho, p)$ and $\bar{s}(\rho, p)$ are, respectively, particular solutions of the equations:

$$n'_\rho + \chi(\rho, p)n'_p = \frac{n}{\rho + p}, \qquad s'_\rho + \chi(\rho, p)s'_p = 0, \tag{27}$$

$\chi(\rho, p)$ being the indicatrix function, $\chi \equiv \dot{p}/\dot{\rho}$.

For a CIG the indicatrix function takes the form (15), with $\gamma > 1$, and then the functions n and s given in (13) and (14) are particular solutions of (27). Then, from lemma 5 a straightforward calculation leads to:

Lemma 6: If a perfect fluid evolves in l.t.e. and the speed of sound is the function of state $c_s^2 = \chi(\rho, p) = \frac{\gamma p}{\rho + p}$, $\gamma > 1$, then it is a Poisson gas, that is, it fulfills the Poisson law:

$$p = \beta(s)n^\gamma, \qquad \gamma > 1. \tag{28}$$

In order to prove the converse statement we need to know the characteristic equation for a Poisson gas, which easily follows from (28) and the local thermal equilibrium relation (6):

Lemma 7: The thermodynamic characteristic equation of a Poisson gas is given by:

$$\epsilon(n, s) = \tilde{\beta}(s)n^{\gamma-1} + \alpha(s), \qquad \tilde{\beta}(s) = \frac{\beta(s)}{\gamma - 1}. \tag{29}$$

In particular, when $\alpha(s) = 0$ this equation characterizes the γ-*gases*, that is, the gases fulfilling a classical γ-law:

$$p = (\gamma - 1)n\epsilon, \qquad \gamma > 1. \tag{30}$$

And if, in addition, $\beta(s) = \beta_0 e^s$ is imposed, the CIG is obtained.

Note that the characteristic equation (29) of a Poisson gas depends on two arbitrary functions of the specific entropy, $\tilde{\beta}(s)$ and $\alpha(s)$. They correspond to the two

arbitrary functions that give the richness of admissible thermodynamic schemes of the generic inverse problem presented in Lemma 5. Thus the generic specific entropy is a function of the specific entropy of a CIG given in (14):

$$s(\rho, p) = f(x), \qquad x \equiv [\rho(\gamma - 1) - p]p^{-1/\gamma}. \qquad (31)$$

From this expression, the square of the speed of sound can be calculated as $c_s^2 = \chi(\rho, p) = -s'_\rho/s'_p$, and we obtain (15). Thus, the indicatrix function of a Poisson gas is that of a classical ideal gas. This result and Lemma 6 allow us to state:

Proposition 3: The Poisson gases defined by (28) are, and only are, the media that evolve in l.t.e. with an indicatrix function of the form $\chi(\rho, p) = \frac{\gamma p}{\rho + p}$, $\gamma > 1$.

From Lemma 7 and Proposition 3 it results that the indicatrix function of a material medium satisfying a classical γ-law is (16). Now $\alpha(s) = 0$ corresponds to taking in Lemma 5 $R(\bar{s}) = 1$ and $\bar{n}(\rho, p)$ equals the matter density of a CIG. Therefore:

Proposition 4: The γ-gases, fulfilling a γ-law, are, and only are, the media that evolve in l.t.e. with an indicatrix function of the form (16) and that have a CIG matter density.

A. Constraints for physical reality

From the above results, the study of the constraints for physical reality presented in subsection III A for a CIG can be extended to the γ-gases and the Poisson gases. The γ-gases are submitted to identical constraints as the CIG as a consequence of Proposition 4. For the Poisson gases the expressions of the matter density n and the specific internal energy ϵ are more generic than in the case of a CIG. Thus, the second of the positivity constraint P in (22) does not hold necessarily, and then the second compressibility constraint H given by (25) applies for $1 < \gamma \leq 2$. Therefore:

Proposition 5: For a γ-gas, defined by the condition (30), the constraints for physical reality are satisfied in the interval given by (26):

$$\begin{cases} 0 < \pi < \pi_m \equiv \gamma - 1, & \text{if } 1 < \gamma \leq 2, \\ 0 < \pi < \tilde{\pi}_m \equiv \frac{1}{\gamma - 1}, & \text{if } \gamma \geq 2. \end{cases}$$

For a Poisson gas, defined by the condition (28), the constraints for physical reality are satisfied in the interval given by

$$\begin{cases} 0 < \pi < \hat{\pi}_m \equiv \frac{\gamma + 1}{2\gamma - 1}, & \text{if } 1 < \gamma \leq 2, \\ 0 < \pi < \tilde{\pi}_m \equiv \frac{1}{\gamma - 1}, & \text{if } \gamma \geq 2. \end{cases} \qquad (32)$$

Most of the thermodynamic expressions presented in previous sections on CIG and Poisson gases can be found in the literature or can be deduced from known

thermodynamic relations. In particular, from (13) and (15) we obtain $c_s^2 = \frac{\gamma \epsilon (\gamma - 1)}{1 + \gamma \epsilon}$, an expression for the speed of sound that can be found, for example, in Ref. [11]. It is also known [10] that the speed of sound in a Poisson gas can be written as (15). Nevertheless, our approach brings new insights on the subject:

(i) This approach offers the sufficient condition of this last statement, that is, it shows that the equation of state (15) characterizes the Poisson gases.

(ii) It also emphasizes the purely hydrodynamic nature of this characterization, and it presents the answer to the inverse problem for the indicatrix function (15) for the set **C** of CIG, the set Γ of the γ-gases, and the set **P** of the Poisson gases. Moreover, it easily clarifies the inclusion relationship between these three sets of fluids, $\mathbf{C} \subset \Gamma \subset \mathbf{P}$, an issue often unclear in the literature, and furthermore it shows that the set $\mathbf{G} \cap \mathbf{P}$ of the ideal gases that are Poisson gases are those with $\epsilon = c_v \Theta + \epsilon_0$; the case $\epsilon_0 = 0$ (CIG) arises for a γ-gas, that is, $\mathbf{G} \cap \Gamma = \mathbf{C}$. On the other hand, this approach allows to solve the inverse problem for the energy tensor, leading, in particular, to the result that the set **P** of the Poisson gases are the answer to the inverse problem for both the CIG and the γ-gases, that is, $\mathbf{P} = \mathbf{F_{T_C}} = \mathbf{F_{T_\Gamma}}$, where $\mathbf{F_{T_C}}$ and $\mathbf{F_{T_\Gamma}}$ are, respectively, the set of all perfect fluids $f \in \mathbf{F}$ whose all possible evolution energy tensors $\mathbf{T_C}$ are those corresponding to any of the CIG, and the set of all perfect fluids $f \in \mathbf{F}$ whose all possible evolution energy tensors $\mathbf{T_\Gamma}$ are those corresponding to any of the γ-gases. Table I summarizes all these results.

(iii) Compressibility conditions for a Poisson gas were analyzed in Ref. [10], and sufficient conditions in the quantities n, p were also presented. Here our approach offers in Propositions 2 and 5 necessary and sufficient conditions for the selected constraints for physical reality, and allow to state them in terms of the purely hydrodynamic equation of state $c_s^2 = \chi(\rho, p)$.

Usually, in the literature on thermodynamic perfect fluids [10,11], the adiabatic index is considered constrained by $1 < \gamma \leq 2$. Nevertheless, our results in Propositions 2 and 5 show that values of the adiabatic index greater than 2 could model physically reasonable media (CIG, γ-gases, or Poisson gases) in physically relevant ranges of the hydrodynamic quantity π.

The constraint $\gamma \leq 2$ was deduced by Taub [9] by imposing the limit $c_s < 1$ for the speed of the sound, a requirement that is included in the compressibility conditions (21). The contradiction between our statements and the Taub's result is only apparent. Indeed, he deduced the constraint for γ by analyzing the behavior of c_s at high temperature, a fact that is consistent with our upper-limit $\pi < \tilde{\pi}$ for $\gamma \geq 2$.

TABLE I.   Different families of fluids that are considered in this paper (first column) and the functions of state that define them (second column). The third column shows their characteristic equation $\epsilon = \epsilon(s, n)$, and the fourth one shows, for some cases, equations of state that also characterize them. Finally, the fifth column shows the square of the speed of sound χ in terms of the hydrodynamic quantity $\pi = p/\rho$. Note that each classical ideal gas is a γ-gas, and each γ-gas is a Poisson gas, and the three families have the same indicatrix function $\chi(\pi)$. Moreover and ideal gas is a Poisson gas if, and only if, the specific internal energy ϵ is a linear function of the temperature Θ. And the classical ideal gases are the ideal gases that are γ-gases.

	Definition	Characteristic equation	Equivalent conditions	$\chi = c_s^2 \equiv -\frac{s_\rho'}{s_p'}$
Ideal gas	$p = kn\Theta$	$\epsilon = \epsilon(s + k\ln n)$		$\chi = \chi(\pi) \neq \pi$
Poisson gas	$p = \beta(s)n^\gamma$	$\epsilon = \tilde{\beta}(s)n^{\gamma-1} + \alpha(s)$	$\chi = \frac{\gamma\pi}{1+\pi}$	$\chi = \frac{\gamma\pi}{1+\pi}$
γ − Gas	$p = (\gamma - 1)n\epsilon$	$\epsilon = \tilde{\beta}(s)n^{\gamma-1}$		$\chi = \frac{\gamma\pi}{1+\pi}$
Poisson ideal gas	$p = kn\Theta$, $p = \beta(s)n^\gamma$	$\epsilon = e^{\frac{s-s_0}{c_v}}n^{\gamma-1} + \epsilon_0$	$p = kn\Theta$, $\epsilon(\Theta) = c_v\Theta + \epsilon_0$	$\chi = \frac{\gamma\pi}{1+\pi}$
Classical ideal gas	$p = kn\Theta$, $\epsilon(\Theta) = c_v\Theta$	$\epsilon = e^{\frac{s-s_0}{c_v}}n^{\gamma-1}$	$p = kn\Theta$, $p = (\gamma - 1)n\epsilon$	$\chi = \frac{\gamma\pi}{1+\pi}$

The difficulties in finding perfect fluid solutions of the Einstein equation modeling realistic media are well known (see, e.g., Sec. VIII). Thus, in the framework of the general relativity theory it may be of interest to deal with equations of state that, not being valid in the whole range, are meaningful in a relevant range of the hydrodynamic quantities ρ and p.

In this paper our point of view on the physical constraints is directly macroscopic, not derived from the kinetic theory of gases. Nevertheless, we believe that it is worthwhile to comment about the Taub inequality obtained from a kinetic approach [9]:

$$n^2 \leq \rho(\rho - 3p). \qquad (33)$$

For a CIG and a γ-gas the mater density takes the expression $n = n(\rho, p)$ given in (13). Then, for $\gamma \geq 5/3$ there is no value of $\pi \in]0, 1[$ fulfilling the Taub inequality (33). Thus, the kinetic theory implies $\gamma < 5/3$ for these media. Nevertheless, the matter density of a Poisson gas is of the form $R(s)n(\rho, p)$, where $R(s)$ is an arbitrary function of the specific entropy. Consequently, for any $\gamma > 1$, we can select in the inverse problem a matter density $\bar{n} = R_0 n(\rho, p)$, with R_0 such that (33) is fulfilled. Moreover, the temperature Θ can be taken such that the gas ideal equation of state (8) is verified. Therefore, Poisson ideal gases with any $\gamma > 1$ are compatible with the Taub inequality (33).

V. ISOTHERMAL EVOLUTION OF A CLASSICAL IDEAL GAS

We know [6] that a general (nonbarotropic) perfect fluid admits barotropic evolutions. For example, this occurs when the fluid evolves by keeping constant a determined function of state. When this function of state is not the specific entropy s, this evolution is, necessarily, isobaroenergetic, $\dot{\rho} = \dot{p} = 0$ [6]. Here we study the isothermal evolution of a CIG, and we offer two elementary examples.

A generic ideal gas in isothermal evolution fulfills a barotropic evolution relation[2] of the form $p = \pi_0 \rho$ [6]. Now we determine what the constant π_0 means in the case of a CIG.

If a CIG evolves at constant temperature Θ_0, from (4), (8) and (9) we obtain:

$$\frac{p}{\rho} = \pi_0 \equiv \frac{k\Theta_0}{1 + c_v\Theta_0} = \frac{(\gamma - 1)\epsilon}{1 + \epsilon} < 1, \qquad (34)$$

$$k\Theta_0 = \frac{(\gamma - 1) - \pi_0}{(\gamma - 1)\pi_0}. \qquad (35)$$

Then, we have:

Proposition 6: A perfect energy tensor $T = (u, \rho, p)$ represents the isothermal evolution of a CIG if, and only if, it is isobaroenergetic, $\dot{\rho} = \dot{p} = 0$, and the following barotropic relation holds:

$$p = \pi_0 \rho, \qquad 0 < \pi_0 < 1. \qquad (36)$$

Conversely, an isobaroenergetic and barotropic energy tensor with barotropic relation (36) represents the isothermal evolution of any CIG. For a given specific adiabatic index γ, the product $k\Theta_0$ is constrained by the condition (35) and the specific internal energy ϵ, the matter density n, and the specific entropy s are given by (13) and (14), the constants k and c_v being related by (17).

A. Relativistic model of isothermal atmosphere

The external gravitational field to a spherically symmetric object is given by the Schwarszchild metric:

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2}dr^2 + r^2 d\Omega^2, \quad \alpha \equiv \sqrt{1 - \frac{2\mu}{r}}. \qquad (37)$$

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[2]Not to be confused with an equation of state.

Let us consider a test classical ideal gas at rest around this object in a spherical configuration. We have then $\dot{\rho} = \dot{p} = 0$. Moreover, the hydrodynamic equations (2)–(3) for the unit velocity $u = \alpha^{-1}\partial_t$, the energy density $\rho(r)$ and the pressure $p(r)$ become:

$$p'(r) = -(\rho + p)\frac{\alpha'}{\alpha}. \tag{38}$$

Then, as a consequence of Proposition 6, if the CIG evolves at constant temperature Θ_0, then $p = \pi_0\rho$, where π_0 is given in (34). Thus, (38) becomes:

$$p'(r) = -2\nu p\frac{\alpha'}{\alpha}, \qquad \nu \equiv \frac{1 + \pi_0}{2\pi_0}, \tag{39}$$

and integrating this equation we have:

$$p(r) = \frac{C}{\alpha^{2\nu}} = \frac{C}{(1 - \frac{2\mu}{r})^{\nu}} = p_0\left[\frac{1 - \frac{2\mu}{r_0}}{1 - \frac{2\mu}{r}}\right]^{\nu}. \tag{40}$$

It is worth remarking that the Newtonian model of isothermal atmosphere of a classical ideal gas leads to:

$$p(r) = Ce^{\frac{\lambda}{r}} = p_0 e^{\lambda(\frac{1}{r} - \frac{1}{r_0})}, \qquad \lambda \equiv \frac{\mu}{k\Theta_0}. \tag{41}$$

For low temperatures, $k\Theta_0 \ll 1$, ν approaches λ, and for weak gravitational field, $\mu \ll r$, (40) and (41) have the same behavior.

B. A model of self-gravitating isothermal sphere

The gravitational field generated by a static spherically symmetric distribution of matter is modeled by the metric:

$$ds^2 = -\alpha^2 dt^2 + \beta^{-2}dr^2 + r^2 d\Omega^2, \tag{42}$$

$$\alpha = \alpha(r), \qquad \beta = \beta(r) \equiv \sqrt{1 - \frac{2m(r)}{r}}, \tag{43}$$

where the mass function $m(r)$ and the pressure $p(r)$ are submitted to the differential system:

$$m'(r) = 4\pi r^2 \rho, \tag{44}$$

$$p'(r) = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}, \tag{45}$$

and they fulfill the initial conditions $m(0) = 0$ and $p(0) = p_c$. Moreover, the gravitational potential $\alpha(r)$ can be obtained from Eq. (38). A way to close the Oppenheimer-Volkoff equations (44)–(45) is to impose a barotropic constraint $\rho = \rho(p)$.

The solution of the system (44)–(45) under the barotropic relation $p = \pi_0\rho$ was studied by Chandrasekhar [12],

and he showed that it becomes an Emden-like equation, as in the classical problem of an isothermal gas. In this paper by Chandrasekhar and in a more recent one by Chavanis [13] this barotropic model is called a relativistic "isothermal" model due to its similarity to the isothermal Newtonian case.

It is worth remarking that this model is indeed an exact isothermal relativistic solution as a consequence of Proposition 6: it performs the evolution at constant temperature Θ_0 of any ideal gas (we can take any adiabatic index γ), with Θ_0 constrained by (35). But it could also model any self-gravitating isothermal generic ideal gas [6] and, particularly, a Synge relativistic gas [14] evolving at constant temperature as considered in Ref. [15].

Of course, this assertion has been established without reference to any heat equation and it remains valid under the hypothesis of vanishing conductivity. Otherwise, the local thermal equilibrium implies necessarily a gradient of temperature attached to the gradient of the gravitational potential as a consequence of a result by Tolman [16,17]. This fact was previously pointed out in Ref. [15] (see also Ref. [13] and next section).

VI. CLASSICAL IDEAL GAS SPHERES IN THERMAL EQUILIBRIUM

If a fluid has a nonvanishing heat conductivity coefficient κ, the energy flux q, the temperature Θ and the fluid acceleration a are constrained by the relativistic Fourier equation [1]:

$$q = -\kappa(\perp d \ln \Theta + a), \tag{46}$$

where $\perp$ denotes the orthogonal projector to the fluid velocity; for a function f, $\perp df = df + \dot{f}u$.

When the energy flux vanishes we obtain the constraint $a = -\perp d \ln \Theta^3$ and, if the fluid is at rest with respect to the static gravitational field (42), this condition leads to:

$$\alpha\Theta = C = \text{constant}, \tag{47}$$

and we recover the above-cited result by Tolman [17].

If we consider a spherical distribution of a CIG in thermal equilibrium, from (4), (8), (10) and (47) we obtain:

$$\rho = \left(\frac{1}{kC}\alpha + \frac{1}{\gamma - 1}\right)p, \tag{48}$$

and the hydrostatic equation (38) becomes:

---

[3]It is known that Eq. (46) leads to a noncausal thermodynamics. The proposed alternatives in causal extended thermodynamics [11] also lead to $a = -\perp d \ln \Theta$ when the energy flux vanishes.

$$p'(\alpha) = -p\left(\frac{1}{kC} + \frac{\gamma}{\gamma-1}\frac{1}{\alpha}\right), \qquad (49)$$

which leads to:

$$p(\alpha) = K\alpha^{-\frac{\gamma}{\gamma-1}}e^{-\frac{\alpha}{kC}}. \qquad (50)$$

Note that (48) and (50) give a barotropic relation, $d\rho \wedge dp = 0$, in a parametric form: $\rho = \rho(\alpha)$, $p = p(\alpha)$. And from here, we can obtain an implicit barotropic relation $\psi(\rho, p) = 0$. On the other hand, from (8) and (47) we obtain:

$$n(\Theta) = \bar{K}\Theta^{\frac{1}{\gamma-1}}e^{-\frac{1}{k\Theta}}. \qquad (51)$$

If we consider a test classical ideal gas in the Schwarzschild gravitational field (37) we have $\alpha^2 = 1 - \frac{2\mu}{r}$. Then (47), (50) and (51) offer a model of atmosphere in thermal equilibrium.

Instead, for a self-gravitating distribution, the stellar structure equations can be written for the functions $m(r)$ and $\alpha(r)$:

$$m'(r) = 4\pi r^2 \rho(\alpha), \qquad (52)$$

$$\alpha'(r) = \alpha\frac{m + 4\pi r^3 p(\alpha)}{r(r-2m)}, \qquad (53)$$

where $\rho(\alpha)$ and $p(\alpha)$ are given in (48) and (50), respectively. A similar reasoning for the case of a classical monoatomic gas ($\gamma = 5/3$), and in particular the expression (51) for this specific case, can be found in the above-cited paper by Tolman [16].

VII. ISENTROPIC EVOLUTION OF A CLASSICAL IDEAL GAS

When a nonbarotropic perfect fluid has an isentropic evolution, this evolution is performed by a barotropic energy tensor, the barotropic relation $p = \phi(\rho)$ depending on the perfect fluid characteristic equation [6]. For a CIG, and also for a γ-gas or a Poisson gas, an isentropic evolution implies that the process is polytropic, that is, an adiabatic Poisson law holds: $p = \beta_0 n^\gamma$, $\beta_0 = $ constant.

From our purely hydrodynamic approach an isentropic evolution means that the function of state $x(\rho, p)$ given in (31) takes a constant value, and we obtain a specific barotropic relation. More precisely we have:

Proposition 7: A perfect energy tensor $T = (u, \rho, p)$ represents the isentropic evolution of a CIG if, and only if, the following barotropic relation holds:

$$(\gamma - 1)\rho = p + Bp^{1/\gamma}, \qquad B = \text{constant}. \qquad (54)$$

Conversely, the barotropic evolution (54) represents the isentropic evolution of a CIG with adiabatic index γ. Moreover, the matter density n is given by

$$n = \frac{B}{\gamma - 1}p^{1/\gamma}. \qquad (55)$$

The adiabatic Poisson law (55) follows from (13) and (54). The first statement in Proposition above is also valid for both a γ-gas and a Poisson gas. And the expression for the matter density in (55) is valid for a γ-gas as a consequence of Proposition 4. On the other hand, for a Poisson gas we evidently have that the matter density fulfills an adiabatic Poisson law $p = An^\gamma$, where A is now a constant that can be taken independent of B.

The analysis of fluids with polytropic evolution has been widely considered in literature. For example, the study of polytropic self-gravitating spheres is a basic topic in both Newtonian and relativistic astrophysics (see, e.g., Refs. [18,19]). A purely hydrodynamic approach to this problem would imply the study of the relativistic structure equations (44)–(45) under the barotropic constraint (54). But we do not consider this question here, focusing instead on the analysis of the FLRW universes that model a CIG in isentropic evolution.

A. Classical ideal gas FLRW models

The Friedmann-Lemaître-Robertson-Walker universes are perfect fluid space-times with line element:

$$ds^2 = -dt^2 + \frac{R^2(t)}{[1 + \frac{1}{4}\epsilon r^2]^2}(dr^2 + r^2 d\Omega^2), \qquad (56)$$

with $\epsilon = 0, 1, -1$, and homogeneous energy density and pressure given by:

$$\rho = \frac{3\dot{R}^2}{R^2} + \frac{3\epsilon}{R^2} \equiv \rho(R), \qquad (57)$$

$$p = -\rho - \frac{R}{3}\partial_R\rho \equiv p(R). \qquad (58)$$

Evidently, we have a barotropic evolution, $d\rho \wedge dp = 0$. Nevertheless, in looking for physically relevant models we must impose a physically realistic barotropic relation $p = \phi(\rho)$. Then, (58) enables us to determine $\rho(R)$, and (57) becomes a *Friedmann equation* for $R(t)$. The significant cosmological models for radiation and matter dominant eras are obtained by taking $\rho = 3p$ and $p = 0$, respectively.

What are the *generalized Friedmann equations* when the energy content is a classical ideal gas with adiabatic index γ? The homogeneity of the hydrodynamic quantities ρ and p implies that, necessarily, the evolution is at constant entropy. Then, as a consequence of Proposition 7, the

barotropic relation is of the form (54). A straightforward calculation allows us to integrate equation (58) and to determine $p(R)$, and then $\rho(R)$, and one obtains:

Proposition 8: The classical ideal gas FLRW models are defined by the generalized Friedmann equation (57) with the energy density $\rho(R)$ given by:

$$\rho(R) = n_0 \left(\frac{R_0}{R}\right)^3 + \frac{p_0}{\gamma - 1}\left(\frac{R_0}{R}\right)^{3\gamma}. \quad (59)$$

In terms of the expansion factor R, the pressure, the matter density and the temperature are given by:

$$p(R) = p_0 \left(\frac{R_0}{R}\right)^{3\gamma}, \quad n(R) = n_0 \left(\frac{R_0}{R}\right)^3, \quad (60)$$

$$\Theta(R) = \frac{p_0}{kn_0}\left(\frac{R_0}{R}\right)^{3(\gamma-1)}. \quad (61)$$

For $\gamma = 5/3$ we obtain a model of monoatomic gas, and for $\gamma = 7/5$ a model of diatomic gas. The cosmological model with decoupled matter and radiation follows by taking $\gamma = 4/3$. And with a limiting procedure we can recover the pressureless solution ($p_0 = 0$) and any γ-model, $p = (\gamma - 1)\rho$, if $n_0 = 0$, and in particular the radiation-dominant solution ($\gamma = 4/3$).

It is worth remarking that the results in Proposition 8 also apply to model a γ-gas in expansion. And for a Poisson gas the model depends on one more parameter allowing two different values for the constants n_0 in (59) and in (60) and (61).

The above classical ideal gas FLRW models have been achieved by imposing the barotropic relation (54). Nevertheless, these models can also be obtained by requiring $p \propto R^{-3\gamma}$. Indeed, Eq. (58) can be solved under this assumption and we obtain the expression (59) for $\rho(R)$. Thus, we have:

Corollary 1: The classical ideal gas FLRW models in Proposition 8 are, and only are, the FLRW models with a pressure of the form (60).

It is worth remarking that all the FLRW models with pressure depending on the metric function R as (60) can be interpreted as a CIG in isentropic evolution.

VIII. ON THE CLASSICAL IDEAL GAS SOLUTIONS OF EINSTEIN EQUATIONS

After theorem 1, the general form for the CIG field equations follows by adding constraint (16), $\dot{p}/\dot{\rho} = \frac{\gamma p}{\rho + p}$, to the usual perfect fluid field equations. And, according to the energy conservation condition (3), this constraint is equivalent to:

$$\frac{\dot{p}}{p} = -\gamma\theta. \quad (62)$$

In some specific kinematic or thermodynamic situations this condition is simpler. Thus, in previous sections we have considered the field equations for a CIG under isothermal or isentropic evolutions, both cases leading to a barotropic evolution relation.

Let us consider now a CIG solution with irrotational motion. Then, in comoving coordinates the metric tensor takes the form:

$$ds^2 = -e^{2\nu}dt^2 + g_{ij}dx^i dx^j. \quad (63)$$

The fluid expansion is $\theta = (\ln\Delta)^{\cdot}$, where Δ^2 is the determinant of the spatial metric g_{ij}. Then, from (62) we obtain:

$$p = P(x^i)\Delta^{-\gamma}, \qquad \Delta \equiv \sqrt{|g_{ij}|}. \quad (64)$$

Thus, the field equations for an irrotational CIG follow by replacing the pressure p for the expression (64) in the field equations in comoving coordinates.

The above result applies to the spherically symmetric metrics and to their hyperbolic and parabolic counterparts. Now the metric tensor takes the form:

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)d\Omega^2, \quad (65)$$

where $d\Omega^2$ is a metric of constant curvature $k = 0, 1, -1$. Then (64) can be written as:

$$p^{-1/\gamma} = 3h(r)Y^2 e^\lambda, \quad (66)$$

a condition that can be added to the usual set of field equations if we look for a classical ideal gas model.

A. Solutions in geodesic motion and admitting a G_3 on S_2 with $Y'(r) \neq 0$

Now we focus on the case $Y' \neq 0$ and a geodesic motion, that is, $\nu = 0$. Then, the field equations can be partially integrated and one has [20]:

$$p(t)Y^2 = -2Y\ddot{Y} - \dot{Y}^2 - \varepsilon f^2(r), \qquad \varepsilon = 0, 1, -1, \quad (67)$$

$$e^\lambda = Y'F(r), \quad F^2[k - \varepsilon f^2] = 1, \quad k = 0, 1, -1. \quad (68)$$

Then, the CIG condition (66) leads to $p^{-1/\gamma} = 3h(r)F(r)Y^2 Y' = h(r)F(r)(Y^3)'$, and we obtain:

$$Z^2 \equiv Y^3 = \alpha(t)a(r) + \beta(t), \quad (69)$$

$$\alpha(t) \equiv p^{-1/\gamma}, \qquad a(r) \equiv \int \frac{dr}{h(r)F(r)}. \quad (70)$$

Note that $Y' \neq 0$ implies $a'(r) \neq 0$ and $\alpha(t) \neq 0$. Moreover, when Y factorizes we have a vanishing shear and an homogeneous expansion [20]. Consequently, the FLRW limit occurs when $\beta = c\alpha$, $c =$ constant.

On the other hand, the change $Z^2 = Y^3$ allows us to write the field equation (67) as [20]:

$$\ddot{Z} + \frac{3}{4} p(t) Z + \frac{3}{4} \epsilon f^2(r) Z^{-\frac{1}{3}} = 0. \qquad (71)$$

When $\epsilon = 0$, Eq. (71) can be solved by quadratures for each election of the function $p(t)$ [21]. But we are interested here in CIG solutions. Then, substituting the expressions (69) and (70) for Z and p in (71) we have:

$$Aa^2 + Ba + C + 3\varepsilon f^2 (\alpha a + \beta)^{\frac{4}{3}} = 0, \qquad (72)$$

$$A = A(t) \equiv 2\alpha\ddot{\alpha} - \dot{\alpha}^2 + 3\alpha^{2-\gamma}, \qquad (73)$$

$$B = B(t) \equiv 2[\alpha\ddot{\beta} + \beta\ddot{\alpha} - \dot{\alpha}\dot{\beta} + 3\beta\alpha^{1-\gamma}], \qquad (74)$$

$$C = C(t) \equiv 2\beta\ddot{\beta} - \dot{\beta}^2 + 3\beta^2\alpha^{-\gamma}. \qquad (75)$$

Thus, we have proved:

Lemma 8: The metrics (65) with $Y' \neq 0$ that model a classical ideal gas with geodesic motion are defined by four functions $\alpha(t) \neq 0$, $\beta(t)$, $f(r)$ and $a(r)$, $a'(r) \neq 0$, submitted to the differential equation (72). The metric functions $Y(t, r)$ and $\lambda(t, r)$ are given in (69)–(70) and (68), respectively, and $\nu = 0$.

Equation (72) can be written as:

$$M^3 = \varphi N^4, \qquad \varphi = \varphi(a) \equiv -27\varepsilon f^6, \qquad (76)$$

$$M = Aa^2 + Ba + C, \qquad N = \alpha a + \beta. \qquad (77)$$

If we isolate φ in (76) and differentiate with respect t we obtain:

$$3\dot{M}N - 4M\dot{N} = 0, \qquad (78)$$

and, taking into account (77), we arrive to:

$$Pa^3 + Qa^2 + Ra + S + 3 = 0, \qquad (79)$$

$$P = P(t) \equiv 3\alpha\dot{A} - 4\dot{\alpha}A, \qquad (80)$$

$$Q = Q(t) \equiv 3\beta\dot{A} + 3\alpha\dot{B} - 4\dot{\alpha}B - 4\dot{\beta}A, \qquad (81)$$

$$R = R(t) \equiv 3\beta\dot{B} + 3\alpha\dot{C} - 4\dot{\beta}B - 4\dot{\alpha}C, \qquad (82)$$

$$S = S(t) \equiv 3\beta\dot{C} - 4\dot{\beta}C. \qquad (83)$$

If $a'(r) \neq 0$, Eq. (79) implies the four equations $P(t) = Q(t) = R(t) = S(t) = 0$ for the two functions $\alpha(t)$ and $\beta(t)$. From these equations we obtain:

$$0 = \frac{1}{4}[\alpha^3 S - \beta\alpha^2 R + \beta^2\alpha Q - \beta^3 P] = \alpha^6 \left[\left(\frac{\beta}{\alpha}\right)^{\cdot}\right]^3, \qquad (84)$$

and, consequently, $\beta = c\alpha$. Then, the four equations reduce to:

$$A = K\alpha^{\frac{4}{3}}, \qquad \varphi(a) = K^3(a + c)^2, \qquad K = \text{constant}. \qquad (85)$$

Thus, the metric becomes a FLRW model (56) with expansion factor $R(t) = \alpha^{\frac{1}{3}}$. Moreover, the pressure is $p \propto R^{-3\gamma}$, and taking into account corollary 1, we have proved:

Proposition 9: The only perfect fluid solutions with geodesic motion and admitting a G_3 on S_2 with $Y'(r) \neq 0$, which can be interpreted as classical ideal gases, are the FLRW models labeled in Proposition 8.

B. On the exact solutions that approach a classical ideal gas behavior

The preliminary result given in the previous subsection is an example of the difficulties in looking for exact solutions to field equations that model a classical ideal gas in local thermal equilibrium. Work in progress reveals this fact. Indeed, elsewhere [22] we have studied class II Szekeres-Szafron models in local thermal equilibrium and, although we have found physically realistic solutions, none of them represents a classical ideal gas. A similar situation occurs in analyzing the thermodynamic meaning of the Stephani universes, a task we undertook years ago [7]: there are no exact classical ideal gas models. Nevertheless, Stephani models representing a generic ideal gas, and approximating a classical one at low temperatures can be found (see also the more recent paper [8]).

The classical ideal gas is a physically realistic model only at low temperatures. Consequently, it may be interesting to obtain solutions with other thermodynamic schemes but with analogous behavior at low temperatures. The most natural way to achieve this is to do what we did in Ref. [7] (see also Ref. [8]) for the Stephani universes: i.e., to look for a generic ideal gas that approximates a classical one.

The generic ideal gases are characterized by an indicatrix function $\chi \equiv \dot{p}/\dot{\rho}$ depending only on the hydrodynamic quantity $\pi \equiv p/\rho$, $\chi = \chi(\pi) \neq \pi$ [6]. In the particular case of a classical ideal gas this function and its first derivative are given in (23). The kth derivative is

$$\chi^{(k)}(\pi) = \gamma \frac{(-1)^{k+1} k!}{(1 + \pi)^{k+1}}, \qquad \chi^{(k)}(0) = \gamma(-1)^{k+1} k!. \qquad (86)$$

Consequently,

$$\chi(\pi) = \gamma \sum_{k=1}^{\infty} (-1)^{k+1} \pi^k. \qquad (87)$$

Then, we say that a generic ideal gas with indicatrix function $\chi = \chi(\pi)$ approximate a classical one at mth order if:

$$\chi(\pi) = \gamma \sum_{k=1}^{m} (-1)^{k+1} \pi^k + O(\pi^{m+1}). \qquad (88)$$

The Stephani ideal gas models obtained in [7] admit a subfamily that approximate at first order classical ideal gas models. Moreover these models fulfill the compressibility conditions in a wide range of the interval [0, 1]. Nevertheless, the class II Szekeres-Szafron metrics that are generic ideal gas models only approximate a classical ideal gas at zero order [22].

IX. DISCUSSION AND WORK IN PROGRESS

In this paper the hydrodynamic approach to the local thermal equilibrium developed in [5,6,8] have been applied to the classical ideal gas. Thus, the specific direct and inverse problems for the CIG case have been solved for both isoenergetic (Proposition 1) and non isoenergetic evolutions (theorem 1). The compressibility conditions for the CIG have been revisited from this hydrodynamic perspective, and we have studied, for any given adiabatic index $\gamma > 1$, the domain of the hydrodynamic quantity $\pi = p/\rho$ where they hold (Proposition 2). The analysis of the extended inverse problem for a CIG shows that the Poisson gases are characterized by having the indicatrix function of a CIG (Proposition 3), and the γ-gases are the Poisson gases with a CIG matter density (Proposition 4). The compressibility conditions have also been analyzed for the γ-gases and the Poisson gases (Proposition 4).

A barotropic perfect energy tensor can model the evolution of a nonbarotropic fluid [6]. Here we obtain the barotropic relation that a CIG in isothermal evolution fulfills (Proposition 6), and we apply this result to model an isothermal atmosphere (Sec. V A), an isothermal self-gravitating sphere (Sec. V B), and a CIG sphere with nonvanishing conductivity coefficient in thermal equilibrium (Sec. VI). The

barotropic relation of a CIG in isentropic evolution (Proposition 7) allows us to present a FLRW solution that models a CIG in l.t.e. (Proposition 8).

Further work will be devoted to obtaining perfect fluid solutions that model CIG in nonbarotropic evolution. A first result in this direction presented in Sec. VIII shows that the only CIG spherically symmetric solutions in geodesic motion and $Y' \neq 0$ are the above-mentioned FLRW models (Proposition 9). A similar limited result seems to derive from the study of the Szekeres-Szafron solutions that model a CIG [22]. One can overcome this situation in looking for exact solutions that approximate a CIG at low temperatures. A way to control this approximation has been outlined in Sec. VIII B.

In our search for physically realistic solutions of the Einstein equations we can directly add the CIG hydrodynamic condition (62) [or (16)] to the common perfect fluid equations, or we can perform a wider study that includes perfect fluids and thermodynamic schemes differing from that of a CIG. This is the method we have built in studying thermodynamic Szekeres-Szafron solutions [22], and likewise the one we shall use for the analysis of the thermodynamic behavior of other families of perfect fluid solutions hereafter.

For a family of perfect fluid solutions of the Einstein field equations with perfect energy tensor $T \equiv (u, \rho, p)$, our method consist in the following steps. In a first step we impose the generic hydrodynamic constraint (7) and obtain the indicatrix function $\chi = \chi(\rho, p)$ for the subfamily that verifies it. In a second step we detect the subfamily with a generic ideal gas indicatrix by imposing $\chi = \chi(\pi) \neq \pi$. Finally, in a third step, when this function does not coincide with the CIG indicatrix (16) for any value of the involved parameters, we can look for solutions that approximates a CIG as proposed in Sec. VIII B.

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