Covariant equations of motion of extended bodies with arbitrary mass and spin multipoles

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Gravitational wave detectors allow us to test general relativity and to study the internal structure and orbital dynamics of neutron stars and black holes in inspiralling binary systems with a potentially unlimited rigor. Currently, analytic calculations of a gravitational wave signal emitted by inspiralling compact binaries are based on the numerical integration of the asymptotic post-Newtonian expansions of the equations of motion in a pole-dipole approximation that includes masses and spins of the bodies composing the binary. Further progress in the accurate construction of gravitational wave templates of the compact binaries strictly depends on our ability to significantly improve the theoretical description of gravitational dynamics of extended bodies by taking into account the higher-order (quadrupole, octupole, etc.) multipoles in equations of motion of the bodies both in the radiative and conservative approximations of general relativity and other viable alternative theories of gravity. This paper employs the post-Newtonian approximations of a scalar-tensor theory of gravity along with the mathematical apparatus of the Cartesian symmetric trace-free tensors and the Blanchet-Damour multipole formalism to derive translational and rotational equations of motion of $\mathbb{N}$-extended bodies having arbitrary distribution of mass and velocity of matter. We assume that a spacetime manifold can be covered globally by a single coordinate chart which asymptotically goes over to the Minkowskian coordinate chart at spatial infinity. We also introduce $\mathbb{N}$ local coordinate charts adapted to each body and covering a finite domain of space around the body. The gravitational field in the neighborhood of each body is parametrized by an infinite set of mass and spin multipoles of the body as well as by the set of tidal gravitoelectric and gravitomagnetic multipoles of external $\mathbb{N}-1$ bodies. The origin of the local coordinates is set moving along the accelerated worldline of the center of mass of the corresponding body by an appropriate choice of the internal and external dipole moments of the gravitational field. Translational equations of motion of the body's center of mass and rotational equations of motion for its spin are derived by integrating microscopic equations of motion of the body's matter and applying the method of the asymptotic matching technique to splice together the post-Newtonian solutions of the field equations of the scalar-tensor theory of gravity for the metric tensor and scalar field obtained in the global and local coordinate charts. The asymptotic matching is also used for separating the post-Newtonian self-field effects from the external gravitational environment and constructing the effective background spacetime manifold. It allows us to present the equations of translational and rotational motion of each body in covariant form by making use of the Einstein principle of equivalence. This relaxes the slow-motion approximation and makes the covariant post-Newtonian equations of motion of extended bodies with weak self-gravity applicable for the case of relativistic speeds. Though the covariant equations of the first post-Newtonian order are still missing terms from the second post-Newtonian approximation, they may be instrumental in getting a glimpse of the last several orbital revolutions of stars in an ultracompact binary system just before merging. Our approach significantly generalizes the Mathisson-Papapetrou-Dixon covariant equations of motion with regard to the number of the body's multipoles and the post-Newtonian terms having been taken into account. The equations of translational and rotational motion derived in the present paper include the entire infinite set of covariantly defined mass and spin multipoles of the bodies. Thus, they can be used for a much more accurate prediction of orbital dynamics of tidally deformed stars in inspiralling binary systems and construction of templates of gravitational waves at the merger stage of a coalescing binary when the strong tidal distortions and gravitational coupling of higher-order mass and spin multipoles of the stars play a dominant role in the last few seconds of the binary life.

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I. INTRODUCTION

The mathematical problem of derivation of relativistic equations of motion of extended bodies has been attracting theorists since the discovery of general relativity. An enormous progress in solving this problem has been reached for the case of an isolated gravitating system consisting of spinning massive bodies in the so-called pole-dipole particle approximation [1–3] that was originally discussed by Mathisson [4,5], Papapetrou [6], and Dixon [7–11] (see also papers of the other researchers [12–15] and references therein). These types of equations of motion are used for a comprehensive study of the nature of gravity through the monitoring orbital and rotational motion of bodies in the Solar System [16,17], binary pulsars [18–21], and inspiralling compact binary systems made of neutron stars and/or black holes [22]. A new branch of relativistic astrophysics, gravitational wave astronomy can test general relativity in a strong field, fast-motion regime of coalescing binaries to unprecedented accuracy and probe the internal structure of neutron stars by measuring their Love numbers [23–27] through the gravitational response of their internal multipoles subject to the immense strength of the tidal gravitational field of an inspiralling binary just before the merger [28]. Therefore, a more advanced study of the dynamics of a relativistic $\mathbb{N}$-body system is required to take into account gravitational perturbations generated by higher-order multipoles of extended bodies (quadrupole, octupole, etc.) that can significantly affect the orbital motion of the pole-dipole massive particles [29–35]. The study of these perturbations is also important for improving the Solar System experimental tests of various gravity theories [36,37] and for building more precise relativistic models of astronomical data processing [38–41].

Over the last three decades most theoretical efforts in derivation of equations of motion were focused on solving the two-body problem in general relativity in order to work out an exact analytic description of the higher-order post-Newtonian corrections beyond the quadrupole radiative approximation of Landau and Lifshitz [42] that would allow one to construct sufficiently accurate waveforms of a gravitational signal emitted by inspiralling the binary systems. One of the main obstacles in solving this problem is the self-interaction of a gravitational field that strongly affects the orbital dynamics of inspiralling binaries through nonlinearity of Einstein's field equations [29,43,44]. The nonlinearity of a gravitational field severely complicates derivation of equations of motion and computation of the waveform templates that are used for detecting a gravitational wave signal by a matched filtering technique and for estimating physical parameters of the binary system [45]. The nonlinearity of the field equations leads to the appearance of formally divergent integrals in the post-Newtonian approximations [46] that have to be regularized to prescribe them a unique and unambiguous finite value making physical sense. Major computational difficulty arises from using the Dirac delta function as a source of gravitational field of point particles in curved spacetime [47]. Dirac's delta function works well in a linear field theory like electrodynamics but it is not directly applicable for solving nonlinear field equations in general relativity to account for the self-field effects of massive stars. This difficulty had been recognized by Infled and Plebanski [48] who pioneered the use of distributions in general relativity to replace the field singularities used in the original derivation of the Einstein-Infeld-Hoffmann (EIH) equations of motion [49]. In order to circumvent the mathematical difficulty arising from the usage of the delta functions in the nonlinear approximations of general relativity, the Lorentz-invariant Hadamard "partie finie" method has been developed by French theorists [50–53]. It has been successfully used to regularize the divergent integrals up to the 3D post-Newtonian approximation but faces certain limitations beyond it due to the presence of a specific pole in the quadrupole of the point-particle binary being intimately associated with the dimension of space and leading to ambiguities [44]. Therefore, the Hadamard partie finie method was replaced with a more powerful method of dimensional regularization [52] to calculate equations of motion of pointlike massive bodies in higher-order post-Newtonian approximations [3,44,54]. There are other methods to calculate equations of motion of pointlike particles in general relativity based on the matched asymptotic expansions [55–57] such as the application of surface integral techniques like in the EIH approach [49,58] and the strong-field point-particle limit approach [30,43,59].

It is well understood that the pointlike particle approximation is not enough for a sufficiently accurate calculation of gravitational waveforms emitted by inspiralling compact binaries so that various types of mutual gravitational coupling of higher multipoles of moving bodies (spin, quadrupole, etc.) should be taken into account. Spin effects have been consistently tackled in a large number of papers [1–3,60–68] while only a few papers, e.g., [62,64], attempted to compute the orbital post-Newtonian effects due to a body's mass quadrupole demonstrating a substantial complexity of calculations. A new generation of gravitational wave detectors will allow us to measure much more subtle effects of the multipolar coupling present in gravitational waveforms emitted by inspiralling compact binaries. Among them, especially promising from the fundamental point of view, are the effects associated with the elastic properties of tidally induced multipoles of neutron stars and black holes as they provide us with direct experimental access to nuclear physics of condensed matter at ultrahigh density of the neutron star's core and exploration of the true nature of astrophysical black holes. Therefore, one of the challenging tasks for theorists working in gravitational wave astronomy is to derive equations of motion in the relativistic ℕ-body problem while accounting for all effects of multipolar harmonics of extended bodies. This task is daunting and the progress in finding its solution is slow. The theoretical approach to resolving the primary difficulties in derivation of the equations of motion in isolated astronomical systems consisting of ℕ-extended bodies with arbitrary multipoles has been introduced in a series of papers by Brumberg and Kopeikin (BK) [69–73] and further advanced by Damour, Soffel and Xu (DSX) [74–77]. The two approaches are essentially similar but the advantage of the DSX formalism is the employment of the Blanchet-Damour (BD) multipoles of extended bodies which take into account the post-Newtonian corrections in the definition of the body's multipoles. The BD mass multipoles were introduced by Blanchet and Damour [78] and the spin multipoles were devised by Damour and Iyer [79]; see also [80,81]. The BD formulation of a multipolar structure of a gravitational field significantly improves the mathematical treatment of relativistic multipoles by Thorne [82] which suffers from the appearance of divergent integrals from the Landau-Lifshitz pseudotensor of a gravitational field [42] entering the integral kernels. The BK-DSX formalism was adopted by the International Astronomical Union as a primary framework for dealing with the problems of relativistic celestial mechanics of the Solar System [17,83]. Racine and Flanagan [84] and Racine *et al.* [85] implemented it for a comprehensive study of the post-Newtonian dynamics of ℕ-extended, arbitrarily structured bodies and for derivation of their translational equations of motion while accounting for all mass and spin BD multipoles. However, Racine and Flanagan [84] neither derived the post-Newtonian rotational equations of motion of the bodies nor did they provided a covariant generalization of the equations of motion.

In this paper we also use the BK-DSX formalism to derive translational and rotational equations of motion of ℕ-extended bodies in the post-Newtonian (PN) approximation of a scalar-tensor theory of gravity with a full account of an arbitrary internal structure of the bodies which is mapped to the infinite set of the BD multipoles extended to the case of the scalar-tensor theory. Our mathematical approach deals explicitly with all integrals depending on the internal structure of the extended bodies and in this respect is different from the formalism applied by Racine-Vines-Flanagan (RVF) [84,85]. Besides the metric tensor, a scalar field is also a carrier of the long-range gravitational interaction in the scalar-tensor theory of gravity that brings about complications in computing the equations of motion. In particular, instead of two sets of general-relativistic BD multipoles we have to deal with an additional set of multipoles associated with the presence of the scalar field [17,86,87]. We assume that the background value of the scalar field changes slowly which allows us to parametrize the scalar-tensor theory of gravity with two covariantly defined parameters, β and γ, which correspond to the parameters of the parametrized post-Newtonian (PPN) formalism [88]. The β-γ parametrization of the equations of motion in the ℕ-body problem is a powerful tool to test general relativity against the scalar-tensor theory of gravity in the Solar System [36,88,89], in binary pulsars [18,20,90], as well as with gravitational wave detectors [91–93] and pulsar-timing arrays [93–95]. The present paper significantly extends the result of papers [84,85] to the scalar-metric sector of gravitational physics and checks its consistency in Appendix B. Moreover, the present paper derives the post-Newtonian rotational equations for spins of massive bodies of the ℕ-body system including all their multipoles.

Post-Newtonian dynamics of extended bodies on curved spacetime manifold M is known in literature as relativistic celestial mechanics—the term coined by Brumberg [96,97]. Mathematical properties of the manifold M are fully determined in general relativity by the metric tensor $g_{\alpha\beta}$ which is found by solving Einstein's field equations. General-relativistic celestial mechanics admits a minimal number of fundamental constants characterizing geometry of curved spacetime—the universal gravitational constant G and the fundamental speed of gravity c which is assumed to be equal the speed of light in vacuum [98,99]. For experimental purposes Will [88] denotes the fundamental speed in a gravity sector as c_g to distinguish it from the fundamental speed c in a matter sector of theory, but he understands it in a rather narrow sense as the speed of weak gravitational waves propagating in a radiative zone of an isolated gravitating system. On the other hand, Kopeikin [100] defines c_g more generally as the fundamental speed that determines the rate of change of a gravity field in both near and radiative zones. In the near zone c_g defines the strength of a gravitomagnetic field caused by rotational and/or translational motion of matter [17,100,101]. Einstein postulated that in general relativity $c_g = c$ but this postulate along with general relativity itself is a matter of experimental testing by radio interferometry [102,103] or with gravitational wave detectors [104]. The presence of additional (hypothetical) long-range fields coupled to gravity brings about other fundamental parameters of the scalar-tensor theory like β and γ which are well known in PPN formalism [88]. The basic principles of the parametrized relativistic celestial mechanics of extended bodies in scalar-tensor theory of gravity remain basically the same as in general relativity [17,99].

Post-Newtonian celestial mechanics deals with an isolated gravitating ℕ-body system whose theoretical concept cannot be fully understood without careful study of three aspects: asymptotic structure of spacetime, approximation methods, and equations of motion [105,106].[1] In what follows, we adopt that spacetime is asymptotically flat at

[1]The initial value problem is tightly related to the questions about origin and existence (stability) of an isolated gravitating system as well [106–110] but we do not elaborate on it in the present paper.

infinity [106,111,112] and the post-Newtonian approximations (PNA) can be applied for solving the field equations. Strictly speaking, this assumption is not valid as our physical Universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric which is conformally flat at infinity. Relativistic dynamics of extended bodies in a FLRW universe requires development of the post-Friedmannian approximations for solving the field equations in case of an isolated gravitating system placed on the FLRW spacetime manifold.[2] The post-Friedmannian approximation method is more general than the post-Newtonian approximations and includes an additional small parameter that is the ratio of the characteristic length of the isolated gravitating system to the Hubble radius of the Universe. A rigorous mathematical approach for doing the post-Friedmannian approximations is based on the field theory of the Lagrangian perturbations of pseudo-Riemannian manifolds [119], and it has been worked out in a series of our papers [120–122]. Relativistic celestial mechanics of an isolated gravitating system in cosmology leads to a number of interesting predictions [123,124]. More comprehensive studies are required to fully incorporate various cosmological effects to the Bondi-Sachs formalism [125] that deals entirely with gravitational waves in asymptotically flat space time.

Equations of motion of an $\mathbb{N}$-body system describe the time evolution of a set of independent variables in the configuration space of the system. These variables are integral characteristics of the continuous distribution of mass and current density of matter inside the bodies, and they are known as mass and spin (or current) multipoles of a gravitational field [78,80,82]. Among them, mass monopole, mass dipole, and spin dipole of each body play a primary role in the description of translational and rotational degrees of freedom (d.o.f.). Higher-order multipoles of each body couples with the external gravitational field of other bodies of the isolated system and perturbs the evolution of the lower-order multipoles of the body in the configuration space. Equations of motion are subdivided into three main categories corresponding to various d.o.f. of the system configuration variables [126]. They are the following:

(I) translational equations of motion of the linear momentum and the center of mass of each body,
(II) rotational equations of motion of the intrinsic angular momentum (spin) of each body,
(III) evolutionary equations of the higher-order (quadrupole, etc.) multipoles of each body.

Translational and rotational equations of motion are sufficient for describing the dynamics of pole-dipole massive particles which are physically equivalent to spherically symmetric and rigidly rotating bodies. A deeper understanding of celestial dynamics of arbitrarily structured extended bodies requires derivation of the evolutionary equations of the higher-order multipoles. Usually, a simplifying assumption of the rigid intrinsic rotation about the center of mass of each body is used for this purpose [97,126–129]. However, this assumption works only until one can neglect the tidal deformation of the body caused by the presence of other bodies in the system and, certainly, cannot be applied at the latest stages of a compact binary's inspiral before merger. It is worth noticing that some authors refer to the translational and rotational equations of the linear momentum and spin of the bodies as to the laws of motion and precession [58,105,130,131] relegating the term *equations of motion* to the center of mass and angular velocity of rotation of the bodies. We do not follow this terminology in the present paper.

The most works on the equations of motion of massive bodies have been done in some particular coordinate charts from which the most popular are the ADM and harmonic coordinates [31,66,132].[3] However, the coordinate description of relativistic dynamics of an $\mathbb{N}$-body system must have a universal physical meaning and predict the same dynamical effects irrespective of the choice of coordinates on spacetime manifold M. The best way to eliminate the appearance of possible spurious coordinate-dependent effects would be derivation of covariant equations of motion based entirely on the covariant definition of the configuration variables. To this end Mathisson [4,5], Papapetrou [6,134] and, especially, Dixon [7–11,135,136] had published a series of programmatic papers suggesting constructive steps toward the development of such fully covariant algorithm for derivation of the set of equations of motion[4] known as Mathisson's variational dynamics or the Mathisson-Papapetrou-Dixon (MPD) formalism [135,136]. However, the ambitious goal to make the MPD formalism independent of a specific theory of gravity and applicable to an arbitrary pseudo-Riemannian manifold created a number of hurdles that slowed down the advancement in developing the covariant dynamics of extended bodies. Nonetheless, continuing efforts to elaborate on the MPD theory had never stopped [12,13,135,139–145].

In order to make the covariant MPD formalism connected to the more common coordinate-based derivations of the equations of motion of extended bodies the metric tensor $\bar{g}_{\alpha\beta}$ of the effective background spacetime manifold $\bar{M}$ must be specified and Dixon's multipoles of the stress-energy skeleton [9,11] have to be linked to the covariant definition of the BD multipoles of extended bodies. To find out this connection we tackle the problem of the covariant

[2]Notice that the term "post-Friedmannian" is used differently by various authors in cosmology [113–116]. We use this term in the sense used by Milillo *et al.* [117] and Rampf *et al.* [118].

[3]The ADM and harmonic coordinate charts are in general different structures but they can coincide under certain circumstances [133].

[4]See also [137,138].

formulation of the equations of motion in a particular gauge associated with the class of conformal harmonic coordinates introduced by Nutku [146,147]. Covariant formulation of the equations of motion is achieved at the final stage of our calculations by building the effective background manifold $\bar{M}$ and applying the Einstein equivalence principle for mapping the locally defined BD multipoles to the arbitrary coordinates. This procedure has been proposed by Landau and Lifshitz [42] and consistently developed and justified by Thorne and Hartle [58]. It works perfectly on torsionless manifolds with the affine connection being fully determined by the metric tensor. Its extension to the pseudo-Riemannian manifolds with torsion and/or nonminimal coupling of matter to gravity requires further theoretical study which is not pursued in the present paper. Some steps forward in this direction have been made, for example, by Yasskin and Stoeger [148], Mao *et al.* [149], March *et al.* [150], Flanagan and Rosenthal [151], Hehl *et al.* [152], and Puetzfeld and Obukhov [143,153].

Dynamics of matter in an isolated gravitating system consisting of $\mathbb{N}$-extended bodies is naturally split in two components: the orbital motion of the center of mass of each body and the internal motion of matter with respect to the body's center of mass. Therefore, the coordinate-based derivation of equations of motion of $\mathbb{N}$-extended bodies in the isolated gravitating system suggests a separation of the problem of motion in two parts: external and internal [126,154]. The external problem deals with the derivation of translational equations of bodies relative to each other while the internal problem provides the definition of physical multipoles of each body and translational equations of motion of the center of mass of the body with respect to the origin of the body-adapted local coordinates. The internal problem also gives us the evolutionary equations of the body's physical multipoles including the rotational equations of motion of the body's spin. A solution of the external problem is rendered in a single global coordinate chart covering the entire manifold M. A solution of the internal problem is executed separately for each body in the body-adapted local coordinates. There are $\mathbb{N}$-local coordinate charts for $\mathbb{N}$ bodies making the atlas of the spacetime manifold. Mathematical construction of the global and local coordinates relies upon and is determined by the solutions of the field equations of the scalar-tensor theory of gravity. The coordinate-based approach to solving the problem of motion provides the most effective way for the unambiguous separation of the internal and external d.o.f. of matter and for the definition of the internal multipoles of each body. Matching of the asymptotic expansions of the solutions of the field equations in the local and global coordinates allows us to find out the structure of the coordinate transition functions on the manifold and to build the effective background metric $\bar{g}_{\alpha\beta}$ on spacetime manifold $\bar{M}$ that is used for transforming the coordinate-dependent form of the equations of motion to the covariant one which can be compared with the MPD covariant equations of motion.

The global coordinate chart is introduced for describing the orbital dynamics of the body's center of mass. It is not unique and is defined up to the group of diffeomorphisms which are consistent with the assumption that spacetime is asymptotically flat at null infinity. This group is called the Bondi-Metzner-Sachs (BMS) group [125,155] and it includes the Poincare transformations as a subgroup. It means that in case of an isolated astronomical system embedded to the asymptotically flat spacetime we can always introduce a nonrotating global coordinate chart with the origin located at the center of mass of the system such that at infinity (1) the metric tensor approaches the Minkowski metric, $\eta_{\alpha\beta}$, and (2) the global coordinates smoothly match the inertial coordinates of the Minkowski spacetime. The global coordinate chart is not sufficient for solving the problem of motion of extended bodies as it is not adequately adapted for the description of the internal structure and motion of matter inside each body in the isolated $\mathbb{N}$-body system. This description is done more naturally in a local coordinate chart attached to each gravitating body as it allows us to exclude various spurious effects appearing in the global coordinates (like Lorentz contraction, geodetic precession, etc.) which have no relation to the motion of matter inside the body [69,156]. The body-adapted local coordinates replicate the inertial Lorentzian coordinates only in a limited domain of spacetime manifold M inside a world tube around the body under consideration. Thus, a complete coordinate-based solution of the external and internal problems of celestial mechanics requires introduction of $\mathbb{N}+1$ coordinate charts—one global and $\mathbb{N}$ local ones [83,99]. It agrees with the topological structure of manifold defined by a set of overlapping coordinate charts making the atlas of spacetime manifold [157,158]. The equations of motion of the bodies are intimately connected to the differential structure of the manifold characterized by the metric tensor and its derivatives. It means that the mathematical presentations of the metric tensor in the local and global coordinates must be diffeomorphically equivalent; that is, the transition functions defining spacetime transformation from the local to global coordinates must map the components of the metric tensor of the internal problem of motion to those of the external problem and vice verse. The principle of covariance is naturally satisfied when the law of transformation from the global to local coordinates is derived by matching the global and local asymptotic solutions of the field equations for the metric tensor. The coordinate transformation establishes a mutual functional relation between various geometric objects that appear in the solutions of the field equations, and determines the equation of motion of the origin of the local coordinates adapted to each body. The coordinate transformations are also employed to map the equations of motion of the center of mass of each body to the coordinate-free, covariant form.

The brief content of our study is as follows. Next, Sec. II summarizes the main concepts and notations used in the present paper. In Sec. III we discuss a scalar-tensor theory of gravity in application to the post-Newtonian celestial mechanics of an $\mathbb{N}$-body system including the β-γ parametrization of the field equations, the small parameters, the post-Newtonian approximations, and gauges. Parametrized post-Newtonian coordinate charts covering the entire spacetime manifold M globally and in a local neighborhood of each body are set up in Sec. IV. They make up an atlas of spacetime manifold. Geometrical properties of coordinates in relativity are characterized by the functional form of the metric tensor and its corresponding parameters—the internal and external multipoles of a gravitational field—which are also introduced and explained in Sec. IV along with the multipolar structure of the scalar field. The local differential structure of spacetime manifold M presumes that the functional forms of the metric tensor and scalar field given in different coordinates must smoothly match each other in the buffer regions where the coordinate charts overlap. The procedure of matching of the asymptotic expansions of the metric tensor and scalar field in the global and local coordinates is described in Sec. V that establishes (1) the functional structure of the body-frame external multipoles of a gravitational field in terms of the volume integrals taken from the distribution of mass density, matter current, pressure, etc., (2) defines the worldline $\mathcal{W}$ of the origin of the body-adapted local coordinates and yields the equation of its translational motion with respect to the global coordinate chart, and (3) defines the effective background metric, $\bar{g}_{\alpha\beta}$, for each extended body that is used later on for derivation of the covariant equations of motion of the bodies.

Section VI provides details of how the local coordinate chart adapted to each extended body is used for a detailed description of the body's own gravitational field inside and outside of the body and for definition of its mass, center of mass, linear and angular momentum (spin). This section also derives the equations of motion of a body's center of mass moving along worldline $\mathcal{Z}$, and its spin in the body-adapted local coordinates. Translational equations of motion of a body's center of mass in the global coordinates follow immediately after substituting the local equations of motion to the parametric description of the worldline $\mathcal{W}$ of the origin of the local coordinates with respect to the global coordinates. The parametric description of worldline $\mathcal{W}$ follows through the multipolar expansion of the external gravitational potentials in Sec. VII and that of the external multipoles in Sec. VIII. Section IX derives the equations of translational motion of the worldline $\mathcal{Z}$ of the center of mass of each body in terms of the complete set of the Blanchet-Damour internal multipoles of the bodies comprising the $\mathbb{N}$-body system. Rotational equations of motion for spin of each body with the torque expressed in terms of the Blanchet-Damour multipoles are derived in Sec. X. Finally, Sec. XI introduces the reader to the basic concepts of the Mathisson-Papapetrou-Dixon variational dynamics and establishes a covariant form of the post-Newtonian translational and rotational equations of motion of extended bodies derived previously in the conformal harmonic coordinates in Secs. IX and X.

The paper has four Appendices. Appendix A sets out auxiliary mathematical relationships for symmetric trace-free (STF) tensors. Appendix B compares our equations of translational motion from Sec. VII with similar equations derived by Racine and Flanagan [84] and Racine *et al.* [85] and analyzes the reason for the seemingly different appearance of the equations. Appendix C explains the concept of Dixon's multipole moments of extended bodies and discusses their mathematical correspondence with the Blanchet-Damour multipole moments. Finally, Appendix D compares Dixon's covariant equations of translational and rotational motion of extended bodies with our covariant equations of motion from Sec. XI.

II. PRIMARY CONCEPTS AND MATHEMATICAL NOTATIONS

We consider an isolated gravitating system consisting of $\mathbb{N}$-extended bodies in the framework of a generic scalar-tensor theory of gravity. The bodies are indexed by either of three capital letters B, C, D from the Roman alphabet. Each of these indices takes values from 1 to $\mathbb{N}$. The bodies have arbitrary but physically admissible distributions of mass, internal energy, pressure, and velocity of matter which can depend on time. We exclude processes of the matter exchange between the bodies so that they interact between themselves only through the coupling to the gravity and/or scalar field force. We also exclude processes of nuclear transmutation of matter particles.

It is now well understood [17,84,97,99,159] that the solution of the problem of motion of $\mathbb{N}$-body system requires introduction of one global coordinate chart, x^α, covering the entire spacetime manifold and $\mathbb{N}$-local coordinate charts, $w^\alpha_{\rm B}$, adapted to each body B of the system. If there is no confusion with other bodies the subindex B in the notation of the local coordinate chart of the body B is omitted.

Equations of scalar-tensor theory of gravity admit a class of conformal transformations of the metric tensor which allows us to put the gravity field equations in two different forms which are referred to as the Einstein and Jordan frames respectively. The field equations in the Einstein frame makes the field equations look exactly as Einstein's equations of general relativity with the scalar field entering solely the right-hand side of the field equations in the form of the stress-energy tensor. The metric tensor in the Einstein frame is coupled with the scalar field explicitly while the

Ricci tensor is uncoupled from the scalar field. In the Jordan frame the situation is opposite—the Ricci tensor couples with the scalar field explicitly while the metric tensor is uncoupled from the scalar field. It was debated for a while which frame—Einstein or Jordan—is physical [160,161]. The answer is that all classical physical predictions are to be conformal-frame invariant [162]. Therefore, the choice of the frame is a matter of mathematical convenience. In the present paper we shall primarily work in the Jordan frame in which matter is minimally coupled to the gravitational field like in general relativity.

Let us single out a body B in the $\mathbb{N}$-body system and consider the metric tensor in the local, body-adapted coordinates. The metric outside the body is parametrized by two infinite sets of configuration parameters which are called the *internal* and *external* multipoles. The multipoles are purely spatial, 3-dimensional, symmetric trace-free Cartesian tensors [50,82,163] residing on the hypersurface $\mathcal{H}_{u_\mathrm{B}}$ of constant coordinate time u_B passing through the origin of the local coordinate chart, w^α_B. The internal multipoles characterize the gravitational field and internal structure of the body B itself and they are of two types—the mass multipoles $\mathcal{M}^L_\mathrm{B}$, and the spin multipoles $\mathcal{S}^L_\mathrm{B}$ where the multi-index $L = i_1 i_2 \ldots i_l$ consists of a set of spatial indices with l denoting the rank of the STF tensor $(l \geq 0)$. If there is no confusion, the index B of the internal multipoles is dropped off. There are also two types of external multipoles—the gravitoelectric multipoles $\mathcal{Q}_L$, and the gravitomagnetic multipoles $\mathcal{C}_L$. The external multipoles with rank $l \geq 2$ characterize tidal gravitational field in the neighborhood of body B produced by other (external) bodies residing outside body B. Gravitoelectric dipole $\mathcal{Q}_i$ describes local acceleration of the origin of the local coordinates adapted to body B. Gravitomagnetic dipole $\mathcal{C}_i$ is the angular velocity of rotation of the spatial axes of the local coordinates. In what follows we set $\mathcal{C}_i = 0$. The scalar field of the scalar-tensor theory of gravity has its own multipolar decomposition with the internal and external multipoles. The external multipoles of the scalar field are denoted as $\mathcal{P}_L$. The above-mentioned multipoles are called *canonical* as they are directly related to 2 d.o.f. of vacuum gravitational field and one d.o.f. of the scalar field. The overall theory also admits the appearance of *noncanonical* STF multipoles in the process of derivation of the equations of motion. These multipoles are related to the gauge d.o.f. and can be eliminated from the equations of motion by the appropriate choice in the definition of the canonical multipoles and the center of mass of body B.

Definitions of the *canonical* STF multipoles must be consistent with the differential structure of spacetime manifold M determined by the solutions of the gravity field equations in the global and local coordinate charts. The consistency is achieved by applying the method of asymptotic matching of the external and internal solutions of the field equations that allows us to express the external multipoles, $\mathcal{Q}_L$ and $\mathcal{C}_L$, in terms of the internal multipoles, $\mathcal{M}^L_\mathrm{B}$ and $\mathcal{S}^L_\mathrm{B}$. The internal multipoles of an extended body B are defined by the integrals taken over the body's volume from the correspondingly chosen internal distribution of mass energy inside the body. This distribution includes not only the internal characteristics of the body B (mass density, pressure, compression energy, etc.) but also the energy density of the tidal gravitational field produced by the external bodies.

There are two important reference worldlines associated with the translational motion of each body B: a worldline $\mathcal{W}$ of the origin of the body adapted, local coordinates, w^α_B, and a worldline $\mathcal{Z}$ of the center of mass of the body. Equations of motion of the origin of the local coordinates are obtained by performing the asymptotic matching of the internal and external solutions of the field equations for the metric tensor. Equations of motion of the center of mass of the body are derived by integrating the macroscopic post-Newtonian equations of motion of matter which are the consequence of the local law of conservation of the stress-energy tensor. The center of mass of each body is defined by the condition of vanishing of the internal mass dipole of the body in the multipolar expansion of the metric tensor in the Einstein frame, $\mathcal{I}^i_\mathrm{B} = 0$. This definition imposes a constraint on the local acceleration $\mathcal{Q}_i$ that makes worldline $\mathcal{W}$ coincide with $\mathcal{Z}$. It also eliminates the other extraneous (*noncanonical*) types of STF multipoles of the gravitational field from the translational and rotational equations of motion.

We use G to denote the observed value of the universal gravitational constant and c as a fundamental speed both in gravity and matter sectors of the theory. Every time, when there is no confusion about the system of units, we choose a geometrical system of units such that $G = c = 1$ so that G and c do not appear in equations explicitly. We put a hat above any function that describes a contribution from the internal distribution of mass, velocity, etc., of body B in the local coordinates adapted to the body. A bar over any function denotes functions produced by the distributions of mass, velocity, etc., from the bodies being external with respect to body B. The bar also denotes the gravitational potentials entering the external multipoles as well as the metric tensor, $\bar{g}_{\alpha\beta}$, of the effective background manifold, $\bar{M}$, that is used to construct covariant equations of motion of the bodies in Sec. XI.

Primary mathematical symbols and notations used in the present paper are as follows:

(i) The capital Roman indices B,C,D label the extended bodies of the $\mathbb{N}$-body system. Each of them takes values from the set $\{1, 2, \ldots, N\}$.

(ii) The small Greek letters $\alpha, \beta, \gamma, \ldots$ denote spacetime indices of tensors and run through values 0,1,2,3.

(iii) The small Roman indices $i, j, k, \ldots$ denote spatial tensor indices and take values 1,2,3.

(iv) The capital Roman letters L, K, N, S denote spatial tensor multi-indices, for example, $L \equiv \{i_1 i_2 \ldots i_l\}$, $N \equiv \{i_1 i_2 \ldots i_n\}$, $K-1 \equiv \{i_1 i_2 \ldots i_{k-1}\}$, etc.

(v) The Einstein summation rule is applied for repeated (dummy) indices and multi-indices, for example, $\mathcal{P}^\alpha \mathcal{Q}_\alpha \equiv \mathcal{P}^0 \mathcal{Q}_0 + \mathcal{P}^1 \mathcal{Q}_1 + \mathcal{P}^2 \mathcal{Q}_2 + \mathcal{P}^3 \mathcal{Q}_3$, $\mathcal{P}^i \mathcal{Q}_i \equiv \mathcal{P}^1 \mathcal{Q}_1 + \mathcal{P}^2 \mathcal{Q}_2 + \mathcal{P}^3 \mathcal{Q}_3$, $\mathcal{P}^L \mathcal{Q}_L = \mathcal{P}^{i_1 i_2 \ldots i_l} \mathcal{Q}_{i_1 i_2 \ldots i_l}$, $\mathcal{P}^{K-1} \mathcal{Q}_{K-1} = \mathcal{P}^{i_1 i_2 \ldots i_{k-1}} \mathcal{Q}_{i_1 i_2 \ldots i_{k-1}}$, etc.

(vi) The Kronecker symbol $\delta_{ij} = \delta^{ij} = \delta^i_j = \delta^j_i$ in 3-dimensional space is a unit matrix

$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(vii) The Levi-Civita fully antisymmetric symbol, $\varepsilon_{ijk} = \varepsilon^{ijk}$, in 3-dimensional space is defined as $\varepsilon_{123} = +1$, and

$$\varepsilon_{ijk} \equiv \begin{cases} +1 & \text{if the set } \{i,j,k\} \text{ forms an even permutation}, \\ -1 & \text{if the set } \{i,j,k\} \text{ forms an odd permutation}, \\ 0 & \text{if, at least, two indices from the set } \{i,j,k\} \text{ coincide}. \end{cases}$$

(viii) $E_{\alpha\beta\gamma\delta}$ is a 4-dimensional generalization of the fully antisymmetric, 3-dimensional Levi-Civita symbol.

(ix) $g_{\alpha\beta}$ is a full metric of spacetime manifold M.

(x) $\bar{g}_{\alpha\beta}$ is the effective metric of the background spacetime manifold $\bar{M}$.

(xi) $\eta_{\alpha\beta} = \mathrm{diag}\{-1,+1,+1,+1\}$ is the Minkowski metric.

(xii) $h_{\alpha\beta}$ is the metric perturbation of the Minkowski spacetime in the global coordinate chart.

(xiii) $\hat{h}_{\alpha\beta}$ is the metric perturbation of the Minkowski spacetime in the local coordinate chart of body B.

(xiv) $w^\alpha_{\mathrm{B}} = (w^0_{\mathrm{B}}, w^i_{\mathrm{B}}) = (u_{\mathrm{B}}, w^i_{\mathrm{B}})$ are the local coordinates adapted to a body B with u_{B} being the local coordinate time. Every time, when there is no confusion, we drop the sub-index B from the notations of the local coordinates. Thus, by default $w^\alpha = (w^0, w^i) = (u, w^i)$ are the local coordinates adapted to body B with u being the local coordinate time.

(xv) $x^\alpha = \{x^0, x^i\} = \{t, x^i\}$ are the global coordinates covering the entire spacetime manifold M or $\bar{M}$. Notation for the manifold should not be confused with the mass internal monopole of body B which is denoted with $\mathcal{M}_{\mathrm{B}}$.

(xvi) $\partial_\alpha = \partial/\partial x^\alpha$ is a partial derivative with respect to coordinate x^α.

(xvii) $\hat{\partial}_\alpha = \partial/\partial w^\alpha$ is a partial derivative with respect to the local coordinate w^α.

(xviii) Shorthand notations for the multi-index partial derivatives with respect to coordinates x^α are $\partial_L \equiv \partial_{i_1 \ldots i_l} = \partial_{i_1} \partial_{i_2} \ldots \partial_{i_l}$, $\partial_{L-1} \equiv \partial_{i_1 \ldots i_{l-1}}$, $\partial_{pL-1} \equiv \partial_{p i_1 \ldots i_{l-1}}$, etc.

(xix) Shorthand notations for the multi-index partial derivatives with respect to coordinates w^α are $\hat{\partial}_L \equiv \hat{\partial}_{i_1 \ldots i_l} = \hat{\partial}_{i_1} \hat{\partial}_{i_2} \ldots \hat{\partial}_{i_l}$, $\hat{\partial}_{L-1} \equiv \hat{\partial}_{i_1 \ldots i_{l-1}}$, $\hat{\partial}_{pL-1} \equiv \hat{\partial}_{p i_1 \ldots i_{l-1}}$, etc.

(xx) $\bar{\nabla}$ standing in front of a group of p tensor indices denotes an operator of the covariant derivative of the pth order with respect to the background metric $\bar{g}_{\alpha\beta}$, for example, $\bar{\nabla}_{\alpha_1 \alpha_2 \ldots \alpha_p} = \bar{\nabla}_{\alpha_1} \bar{\nabla}_{\alpha_2} \ldots \bar{\nabla}_{\alpha_p}$.

(xxi) ∇ standing in front of a group of p tensor indices denotes a covariant derivative of the pth order with respect to the full metric $g_{\alpha\beta}$, that is $\nabla_{\alpha_1 \alpha_2 \ldots \alpha_p} = \nabla_{\alpha_1} \nabla_{\alpha_2} \ldots \nabla_{\alpha_p}$.

(xxii) $\frac{\mathcal{D}}{\mathcal{D}\tau} = \bar{u}^\alpha \bar{\nabla}_\alpha$ denotes a covariant derivative along vector $\bar{u}^\alpha$.

(xxiii) $\frac{\mathcal{D}_{\mathrm{F}}}{\mathcal{D}\tau}$ denotes a Fermi-Walker covariant derivative along vector $\bar{u}^\alpha$ [[164], Chapter 1, Sec. 4],

(xxiv) Tensor (Greek) indices of geometric objects on spacetime manifold M are raised and lowered with the full metric $g_{\alpha\beta}$.

(xxv) Tensor (Greek) indices of geometric objects on the effective background manifold $\bar{M}$ are raised and lowered with the background metric $\bar{g}_{\alpha\beta}$.

(xxvi) Tensor (Greek) indices of the metric tensor perturbation $h_{\alpha\beta}$ are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$.

(xxvii) The spatial (Roman) indices of geometric objects are raised and lowered with the Kronecker symbol δ^{ij}. Effectively, it means that the position of the spatial indices—either superscript or subscript—does not matter.

(xxviii) A symbol of summation over *all* $\mathbb{N}$ bodies of an $\mathbb{N}$-body system is denoted as $\sum_{\mathrm{B}} \equiv \sum_{\mathrm{B}=1}^{N}$, or $\sum_{\mathrm{C}} \equiv \sum_{\mathrm{C}=1}^{N}$, etc.

(xxix) The symbol of summation over $\mathbb{N}-1$ bodies of an $\mathbb{N}$-body system excluding, let us say body C, is $\sum_{\mathrm{B}\neq\mathrm{C}} \equiv \sum_{\substack{\mathrm{B}=1 \\ \mathrm{B}\neq\mathrm{C}}}^{N}$.

(xxx) The ordinary factorial is $l! = l(l-1)(l-2)\ldots 2\cdot 1$.

(xxxi) The double factorial means

$$l!! \equiv \begin{cases} l(l-2)(l-4)\ldots 4\cdot 2 & \text{if } l \text{ is even,} \\ l(l-2)(l-4)\ldots 3\cdot 1 & \text{if } l \text{ is odd.} \end{cases}$$

(xxxii) The round parentheses around a group of tensor indices denote full symmetrization,

$$T_{(\alpha_1\alpha_2\ldots\alpha_l)} = \frac{1}{l!}\sum_{\sigma\in S} T_{\sigma(\alpha_1)\sigma(\alpha_2)\ldots\sigma(\alpha_l)},$$

where σ is a permutation of the set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$

$$\sigma = \begin{Bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_l \\ \sigma(\alpha_1) & \sigma(\alpha_2) & \sigma(\alpha_3) & \ldots & \sigma(\alpha_l) \end{Bmatrix},$$

for example, $T_{(\alpha\beta\gamma)} = \frac{1}{3!}(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} + T_{\beta\alpha\gamma} + T_{\alpha\gamma\beta} + T_{\gamma\beta\alpha})$, etc.

(xxxiii) The curled parentheses around a group of tensor indices denote *un-normalized* symmetrization over the smallest set of the index permutations, for example, $T_{\{\alpha}\delta_{\beta\gamma\}} \equiv T_\alpha\delta_{\beta\gamma} + T_\beta\delta_{\alpha\gamma} + T_\gamma\delta_{\alpha\beta}$, etc.

(xxxiv) The square parentheses around a pair of tensor indices denote antisymmetrization, for example, $T^{[\alpha\beta]\gamma} = \frac{1}{2}(T^{\alpha\beta\gamma} - T^{\beta\alpha\gamma})$, etc.

(xxxv) The angular brackets around tensor indices denote a symmetric trace-free projection of tensor $T_L = T_{i_1 i_2\ldots i_l}$. The STF projection $T_{\langle L\rangle}$ of tensor T_L is constructed from its symmetric part,

$$S_L \equiv T_{(L)} = T_{(i_1 i_2\ldots i_l)}, \tag{1}$$

by subtracting all the permissible traces. This makes $T_{\langle L\rangle}$ fully symmetric and trace free on all pairs of indices. The general formula for the STF projection is [50,82]

$$T_{\langle L\rangle} \equiv \sum_{n=0}^{[l/2]} \frac{(-1)^n}{2^n n!}\frac{l!}{(l-2n)!}\frac{(2l-2n-1)!!}{(2l-1)!!} \times \delta_{(i_1 i_2}\ldots\delta_{i_{2n-1} i_{2n}} S_{i_{2n+1}\ldots i_l) j_1 j_1\ldots j_n j_n}, \tag{2}$$

where $[l/2]$ is the largest integer less than or equal to $l/2$.

(xxxvi) The STF spatial derivative is denoted by the angular parentheses embracing the STF indices, for example, $\partial_{\langle L\rangle} \equiv \partial_{\langle i_1 i_2\ldots i_l\rangle}$ or $\partial_{\langle K\rangle} \equiv \partial_{\langle i_1 i_2\ldots i_k\rangle}$.

(xxxvii) The Christoffel symbols on a spacetime manifold M are $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}(\partial_\beta g_{\gamma\sigma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma})$.

(xxxviii) The Christoffel symbols of the effective background manifold $\bar{M}$ are $\bar{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}\bar{g}^{\alpha\sigma}(\partial_\beta\bar{g}_{\gamma\sigma} + \partial_\gamma\bar{g}_{\beta\sigma} - \partial_\sigma\bar{g}_{\beta\gamma})$.

(xxxix) The sign of the Riemann tensor on spacetime manifold M is defined by convention (it is the same as in [165])

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_{\alpha\nu}g_{\beta\mu} + \partial_{\beta\mu}g_{\alpha\nu} - \partial_{\beta\nu}g_{\alpha\mu} - \partial_{\alpha\mu}g_{\beta\nu}) + g_{\rho\sigma}(\Gamma^\rho_{\alpha\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\rho_{\alpha\mu}\Gamma^\sigma_{\beta\nu}). \tag{3}$$

(xxxx) The Riemann tensor of the effective background manifold $\bar{M}$ is

$$\bar{R}_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_{\alpha\nu}\bar{g}_{\beta\mu} + \partial_{\beta\mu}\bar{g}_{\alpha\nu} - \partial_{\beta\nu}\bar{g}_{\alpha\mu} - \partial_{\alpha\mu}\bar{g}_{\beta\nu}) + \bar{g}_{\rho\sigma}(\bar{\Gamma}^\rho_{\alpha\nu}\bar{\Gamma}^\sigma_{\beta\mu} - \bar{\Gamma}^\rho_{\alpha\mu}\bar{\Gamma}^\sigma_{\beta\nu}). \tag{4}$$

The sign conventions (3) and (4) for the Riemann tensor are opposite to that from the Weinberg textbook [[166], Eq. 6.6.2].

Other notations will be introduced and explained in the main text of the paper as they appear. Useful algebraic and differential identities of STF tensors are given in Appendix A of the present paper.

III. SCALAR-TENSOR THEORY AND POST-NEWTONIAN APPROXIMATIONS

We consider an isolated $\mathbb{N}$-body system comprised of $\mathbb{N}$-extended bodies with a nonsingular interior described by the stress-energy tensor $T^{\alpha\beta}$ of baryonic matter. The bodies have a localized matter support and are supposed to be isolated one from another in space in the sense that accretion, transfer, and other fluxes of baryonic matter outside of the bodies are excluded.

Post-Newtonian celestial mechanics describes orbital and rotational motions of the bodies on a curved spacetime manifold M defined by the metric tensor $g_{\alpha\beta}$ obtained as a solution of the field equations of a metric-based theory of gravitation in the slow-motion and weak-field approximation. The class of viable metric theories of gravity, which can be employed for developing relativistic celestial mechanics, ranges from general theory of relativity [42,97] to a scalar-vector-tensor theory of gravity proposed by Bekenstein [167] for describing orbital motion of galaxies in clusters at cosmological scale. It is not the goal of the present paper to review all these theories and we refer the reader to reviews by Will [36] and Turyshev [37] for further details.

We shall build the parametrized post-Newtonian celestial mechanics in the framework of a scalar-tensor theory of gravity introduced by Jordan [168,169] and Fierz [170], and independently rediscovered later by Brans and Dicke [171] and Dicke [172,173]. The Jordan-Fierz-Brans-Dicke (JFBD) theory extends the Lagrangian of general relativity by introducing a long-range, nonlinear scalar field (or fields [86]) being minimally coupled to gravity. The presence of the scalar field causes deviation of the metric-based gravity theory from a pure geometric phenomenon. The scalar field effects are superimposed on gravitational effects of general relativity, thus, highlighting the geometric role of the metric tensor and making physical content of the theory richer. Recent discovery of the scalar Higgs boson at LHC [174] and its possible connection to the effects of a JFBD scalar field in gravitation and cosmology [175] reinforce the significance of application of the scalar-tensor theory in relativistic astrophysics and celestial mechanics of isolated gravitating systems.

A. Lagrangian and field equations

A gravitational field in the scalar-tensor theory of gravity is described by the metric tensor $g_{\alpha\beta}$ and a long-range scalar field Φ with nonlinear self-interaction described by means of a coupling function $\omega(\Phi)$. Field equations in the Jordan frame of scalar-tensor theory are derived from the action [88]

$$S = -\frac{1}{16\pi}\int \Phi R\sqrt{-g}d^4x + \frac{1}{8\pi}\int L^{\Phi}\sqrt{-g}d^4x + \int L^{\mathrm{M}}\sqrt{-g}d^4x, \tag{5}$$

where $g = \det[g_{\alpha\beta}] < 0$ is the determinant of the metric tensor $g_{\alpha\beta}$, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar, $R_{\alpha\beta}$ is the Ricci tensor,

$$L^{\Phi} = \frac{\omega(\Phi)}{2\Phi}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - V(\Phi) \tag{6}$$

is the Lagrangian of the scalar field with $V(\Phi)$ being the potential of the scalar field, and $L^{\mathrm{M}} \equiv L(g_{\alpha\beta}, \psi)$ is the Lagrangian of matter of the $\mathbb{N}$-body system with ψ denoting the dynamic variables characterizing the matter of the extended bodies comprising the system. We keep the self-coupling function $\omega(\Phi)$ of the scalar field unspecified for making covariant parametrization of possible deviations of the scalar-tensor theory from general relativity. Moreover, we assume the minimal coupling of the metric tensor $g_{\alpha\beta}$ with matter variables ψ without coupling to the scalar field Φ. It explains why the Lagrangian L^{M} does not depend on the scalar field Φ.

The action (5) is written in the Jordan frame in which the metric tensor $g_{\alpha\beta}$ has a standard physical meaning of observable quantity used in the definitions of the proper time, the proper length, and in the geodesic equation of motion of test particles [88]. Taking variational derivatives from the action (5) with respect to the metric tensor, we obtain gravitational field equations for the metric tensor,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{\Phi}(\nabla_{\mu\nu}\Phi - g_{\mu\nu}\Box_g\Phi + T^{\Phi}_{\mu\nu}) + \frac{8\pi}{\Phi}T^{\mathrm{M}}_{\mu\nu}, \tag{7}$$

where, here and everywhere else, the operator ∇_μ denotes a covariant derivative on the spacetime manifold with the metric $g_{\alpha\beta}$, the g-box symbol

$$\Box_g \equiv g^{\mu\nu}\nabla_{\mu\nu} = g^{\mu\nu}\partial_{\mu\nu} - g^{\mu\nu}\Gamma^\alpha_{\mu\nu}\partial_\alpha \tag{8}$$

denotes the differential Laplace-Beltrami operator [165,176] on manifold with metric $g_{\alpha\beta}$, and $T^{\Phi}_{\mu\nu}$ and $T^{M}_{\mu\nu}$ are stress-energy tensors of the scalar field and matter of the $\mathbb{N}$-body system respectively. In particular,

$$T^{\Phi}_{\mu\nu} = \frac{\omega(\Phi)}{\Phi}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\Phi\partial_\alpha\Phi\right) + g_{\mu\nu}V(\Phi), \tag{9}$$

and

$$T^{\mathrm{M}}_{\mu\nu} = \rho(1+\Pi)u_\mu u_\nu + \mathfrak{s}_{\mu\nu}, \tag{10}$$

where ρ and Π are the density and the specific internal energy of the baryonic matter, $u^\alpha = dx^\alpha/cd\tau$ is the 4-velocity of the matter with τ being the proper time along the worldline of the matter's volume element, and $\mathfrak{s}^{\alpha\beta}$ is an arbitrary (but physically admissible) symmetric tensor of spatial stresses being orthogonal to the 4-velocity of matter

$$u^\alpha\mathfrak{s}_{\alpha\beta} = 0. \tag{11}$$

Equation (11) means that the stress tensor has only spatial components in the frame comoving with matter.

Equation for the scalar field Φ is obtained by variation of action (5) with respect to Φ. After making use of a contracted form of (7) it yields [88]

$$\Box_g\Phi = \frac{1}{3+2\omega(\Phi)} \times \left[8\pi T^{\mathrm{M}} - \frac{d\omega}{d\Phi}\partial^\alpha\Phi\partial_\alpha\Phi - 2\Phi\frac{dV}{d\Phi} + 4V(\Phi)\right], \tag{12}$$

where $T^{\mathrm{M}} = g^{\alpha\beta}T^{\mathrm{M}}_{\alpha\beta}$ is the trace of the stress-energy tensor of matter which serves as a source of the scalar field along with its own kinetic (due to the self-coupling) and potential energies.

A gravitational field and matter are tightly connected via the Bianchi identities of the field equations for the metric tensor [42,165] which read

$$\nabla_\nu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) \equiv 0. \tag{13}$$

The Bianchi identities make four out of ten components of the metric tensor fully independent. This freedom is usually fixed by picking up a specific gauge condition, which imposes four constraints on the components of the metric tensor and/or its first derivatives. At the same time the Bianchi identity (13) imposes four differential constraints on the stress-energy tensor of matter and scalar field which constitute microscopic equations of motion of matter [42]. Due to the Bianchi identities (13) the source of the gravitational field standing in the right-hand side of (7) is also conserved. The law of conservation of this tensor is convenient to write down in the following form:

$$8\pi\nabla_\nu T^{\mu\nu}_{\rm M} = -\nabla_\nu T^{\mu\nu}_{\Phi} + \frac{\nabla^\mu\Phi}{2\Phi}(8\pi T_{\rm M} + T_\Phi - 3\Box_g\Phi). \quad (14)$$

After taking the covariant derivative from the stress-energy tensor of the scalar field (9), and making use of the scalar field equation (12) we can check by direct calculation that the right-hand side of (14) vanishes. It yields the laws of conservation of the stress-energy tensor of baryonic matter of an $\mathbb{N}$-body system,

$$\nabla_\nu T^{\mu\nu}_{\rm M} = 0. \quad (15)$$

The conservation of the stress-energy leads to the (exact) equation of continuity

$$\nabla_\alpha(\rho u^\alpha) = \frac{1}{\sqrt{-g}}\partial_\alpha(\rho\sqrt{-g}u^\alpha) = 0, \quad (16)$$

and to the thermodynamic law of conservation of energy that is expressed as a differential relation between the specific internal energy Π and the stress tensor of matter

$$\rho u^\alpha\partial_\alpha\Pi + \mathfrak{s}^{\alpha\beta}\nabla_\alpha u_\beta = 0. \quad (17)$$

These equations will be employed later on for solving the field equations and for derivation of equations of motion of the extended bodies.

B. Post-Newtonian approximations

We shall assume that the potential $V(\Phi)$ of the scalar field can be neglected in the following calculations. Discarding the potential $V(\Phi)$ is justified from an observational point of view in a weak gravitational field (like in the Solar System) as it does not reveal any measurable effect in orbital and rotational motion of celestial bodies on sufficiently long intervals of time [36,37]. On the other hand, if the potential of the scalar field is not identically nil, it may become important in astrophysical systems having a strong gravitational field like compact binary neutron stars or black holes, and its inclusion to the theory leads to important physical consequences [86,177].

Neglecting the scalar field potential simplifies the field equations (7) and (12) and reduces them to the following form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{\Phi}\left[8\pi T_{\mu\nu} + \frac{\omega(\Phi)}{\Phi}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\Phi\partial_\alpha\Phi\right) + \nabla_{\mu\nu}\Phi - g_{\mu\nu}\Box_g\Phi\right], \quad (18)$$

$$\Box_g\Phi = \frac{1}{3+2\omega(\Phi)}\left(8\pi T - \frac{d\omega}{d\Phi}\partial^\alpha\Phi\partial_\alpha\Phi\right), \quad (19)$$

where we suppressed index M at the stress-energy tensor of the baryonic matter for simplicity: $T^{\mu\nu} \equiv T^{\mu\nu}_M$ and $T \equiv T^\alpha{}_\alpha$.

Field equations (18) and (19) of the scalar-tensor theory of gravity represent a system of eleventh nonlinear differential equations in partial derivatives. It is challenging to find their solution in the case of an $\mathbb{N}$-body system made of extended bodies with a sufficiently strong gravitational field whose backreaction on the geometry of a spacetime manifold cannot be neglected. Like in general relativity, an exact solution of this problem is not known and may not be available in analytic form. Hence, one has to resort to approximations to apply the analytic methods. Two basic methods have been worked out in asymptotically flat spacetime: the post-Minkowskian (PMA) and the post-Newtonian (PNA) approximations [17,30,154]. Post-Newtonian approximations are applicable in cases when matter moves slowly and the gravitational field is weak everywhere—the conditions, which are satisfied, e.g., within the Solar System. Post-Minkowskian approximations relax the requirement of the slow motion but the weak-field limitation remains. A strong field regime requires more involved techniques [43]. We use the method of the post-Newtonian approximations in this paper which is remarkably effective and consistent in describing the gravitational field of isolated gravitating systems including binary pulsars containing dense neutron stars and a binary black hole inspiralling toward a final merger [178].

The post-Newtonian approximation scheme suggests that the metric tensor can be expanded in the near zone of an $\mathbb{N}$-body system in powers with respect to the inverse powers of the fundamental speed c.[5] This expansion may be not analytic at higher post-Newtonian approximations in a certain class of coordinate charts including the harmonic coordinates [50,180]. The exact mathematical formulation of the basic axioms underlying the post-Newtonian expansion was given by Rendall [181]. Practically, it requires one to have several small parameters characterizing the $\mathbb{N}$-body system and the interior structure of the bodies. They are $\epsilon_{\rm i} \sim v_{\rm i}/c$, $\epsilon_{\rm e} \sim v_{\rm e}/c$, and $\eta_{\rm i} \sim U_{\rm i}/c^2$, $\eta_{\rm e} \sim U_{\rm e}/c^2$, where $v_{\rm i}$ is a characteristic internal velocity of motion of matter inside an extended body, $v_{\rm e}$ is a characteristic velocity of the relative motion of the bodies with respect to each other,

[5]For historical reasons the speed c in all sectors of fundamental interactions is called "the speed of light" [179]. It is clear that in the gravity sector its physical meaning is the speed of gravity [98,100].

$U_{\rm i}$ is the internal Newtonian gravitational potential inside each body, and $U_{\rm e}$ is the external Newtonian gravitational potential in the regions of space between the bodies. If we denote a characteristic radius of an extended body as L and a characteristic distance between the bodies as R, the internal and external gravitational potentials will have the following estimates: $U_{\rm i} \simeq GM/L$ and $U_{\rm e} \simeq GM/R$, where M is a characteristic mass of the body. Due to the virial theorem of the Newtonian gravity [42] the small parameters are not fully independent. Specifically, one has $\epsilon_e^2 \sim \eta_e$ and $\epsilon_{\rm i}^2 \sim \eta_{\rm i}$ if the internal motions of matter inside the body are governed by the gravitational field of the body through macroscopic equations of motion. The slow-motion parameter $\epsilon_{\rm i}$ is not related to the weak-field parameter $\eta_{\rm i}$ in all other cases like rotational motion of the body, convection of matter, sound waves, etc. Parameters $\epsilon_{\rm i}$ and $\epsilon_{\rm e}$ are the primary parameters in calculating the post-Newtonian expansions of the solutions of the field equations for the metric tensor and scalar field. In what follows, we use a single notation ϵ to quantify the order of the parametric expansion in the post-Newtonian series.

Besides the small parameters ϵ and η, the post-Newtonian approximation utilizes two more small parameters: $\delta \sim L/R$ characterizing the dependence of the body's gravitational field on its finite size L, and the asphericity parameter $\lambda \simeq \Delta L/L$ estimating how much the shape of the body under consideration deviates from the sphere. These parameters appear in vacuum multipolar expansion of the metric tensor and scalar field. As the metric tensor has ten algebraically independent components, we might expect appearance of ten different types of tensor multipoles but only two types of them (mass and spin multipoles) are physically significant because eight types of the tensor multipoles are gauge dependent and can be eliminated from the multipolar expansion of the metric tensor by using the gauge freedom of the theory [50,78,82]. Multipolar expansion of the scalar field has naturally one type of the (scalar) multipoles which is fully independent of the choice of the metric gauge. The property of disappearance of eight types of the tensor multipoles in the multipolar expansion of the metric tensor is known as the *effacing* principle [154] which tells us that the only information about the internal structure of the body obtained from the measurement of its vacuum gravitational field, can be extracted from the *canonical* mass and spin multipoles of the body. It imposes certain limitations on our ability to get unambiguous information about the distribution of mass, velocity, pressure, and other internal characteristics of the body, for example, the gravitational field of an extended body having spherically symmetric distribution of mass cannot be distinguished from that of a massive pointlike particle having the same mass due to the Birkhoff theorem that is valid in the scalar-tensor theory of gravity as well as in general relativity [182].

In principle, translational and/or rotational equations of motion of extended bodies might depend on more than the two (canonical) types of the multipoles of the bodies. This is because derivation of the equations of motion is based on integration of macroscopic equations of motion of matter over finite volumes of the bodies and it is not evident that the result of a such integration will not produce additional *noncanonical* types of the multipoles entering the gravitational force and/or torque exerted on each body. Had this happened the parameter $\delta = L/R$ would appear in the post-Newtonian expansions even if the bodies comprising the $\mathbb{N}$-body system were spherically symmetric. Scrutiny into the theoretical study of the problem of motion in general relativity has shown that such noncanonical multipoles do not appear in the equations of motion of an $\mathbb{N}$-body system and the internal structure of extended bodies is completely effaced up to 2.5 PN approximation for spherically symmetric bodies [154,183–186] and up to 1 PN approximation for arbitrarily structured bodies [75,84,87]. We demonstrate in the present paper that the effacing principle is also valid in the scalar-tensor theory of gravity in 1PN approximation. The effacing of the internal structure and disappearance of the noncanonical multipoles of the bodies from equations of motion indicates that the equations can be extrapolated to the case of structureless bodies like black holes in compact binaries.

The multipoles of extended bodies have some *bare* values in cases when the body is nonrotating and fully isolated from an external gravitational environment. The numerical value of the multipoles will deviate from the bare value if the body rotates and interacts gravitationally with other members of the $\mathbb{N}$-body system as it brings about intrinsic deformations in the distribution of matter inside the body. The measured value of each multipole is a sum of its bare value and the induced deformations. The magnitude of the induced deformations depends on the parameters of elasticity of each body which are intrinsically related to the equation of state of the body's matter. These parameters are known as Love's numbers $\kappa_{{\rm n}l}$ where subindex ${\rm n} = 1$, 2, 3 indicates the physical type of the Love number and l is the multipole number [187–190]. Measurement of the Love numbers of neutron stars and black holes in compact inspiralling binaries is one of the main goals of gravitational wave astronomy [23–26]. Generally speaking, the Love numbers $\kappa_{{\rm n}l}$ depend on the frequency of orbital harmonics and are different for each multipole [27]. Therefore, a complete study of the internal structure of neutron stars by means of the gravitational wave astronomy requires including all multipoles of the bodies to the translational and rotational equations of motion in order to get an exhaustive amount of information about their internal physical characteristics—equation of state, radius, distribution of mass density, etc. The present paper accounts for all internal multipoles of the bodies.

C. Post-Newtonian expansions

Post-Newtonian series are expansions of the metric tensor, scalar field, and matter variables around their background values with respect to the small parameters introduced above. We denote Φ_0 the background value of the scalar field Φ and assume that the dimensionless perturbation of the field, ϕ, is small compared with Φ_0. In the cosmological case, Φ_0 is not constant and changes subject to the Hubble expansion of the Universe [191]. The inverse value of the background scalar field is proportional to the universal gravitational constant $G \sim 1/\Phi_0$ as shown below in (47). Therefore, the time variation of Φ_0 causes a secular evolution of the universal gravitational constant $G = G_0 + \dot{G}(t - t_0)$ as well as other PPN parameters of the scalar-tensor theory [124]. The rate of the hypothetical secular variation of the universal gravitational constant has been measured by lunar laser ranging and is negligibly small—$\dot{G}/G_0 = (7.1 \pm 7.6) \times 10^{-14}\ \mathrm{yr}^{-1}$ [192]. Other techniques yield similar constraints [90,193,194]. In this paper we consider the case of asymptotically flat space time and treat Φ_0 as constant. We write *exact* decomposition

$$\frac{\Phi}{\Phi_0} = 1 + \phi, \tag{20}$$

where ϕ is the dimensionless value of the scalar field Φ normalized to Φ_0.

According to theoretical expectations [191] and experimental limitation on PPN parameters [36,37,192], the post-Newtonian perturbation ϕ of the scalar field has a very small magnitude, so that we can expand all quantities depending on the scalar field in a Maclaurin series with respect to ϕ using it as a small parameter in the expansion. In particular, the post-Newtonian decomposition of the coupling function $\omega(\Phi)$ can be written as

$$\omega(\Phi) = \omega_0 + \omega_0' \phi + \mathcal{O}(\phi^2), \tag{21}$$

where $\omega_0 \equiv \omega(\Phi_0)$, $\omega_0' \equiv (d\omega/d\phi)_{\Phi=\Phi_0}$, and we impose the boundary condition on the scalar field such that ϕ approaches zero as the distance from the $\mathbb{N}$-body system approaches infinity; see Eqs. (60) and (61). The post-Newtonian expansion of the perturbation ϕ is given in the form

$$\phi = \epsilon^2 \phi^{(2)} + \mathcal{O}(\epsilon^3), \tag{22}$$

where the post-Newtonian correction $\phi^{(2)}$ will be defined below, and the symbol $\mathcal{O}(\epsilon^3)$ indicates the expected magnitude of the residual terms. Notice that the linear term being proportional to ϵ does not appear in (22) as it is incompatible with the field equations (19).

The unperturbed value of the metric tensor $g_{\alpha\beta}$ in asymptotically flat spacetime is the Minkowski metric, $\eta_{\alpha\beta}$. The metric tensor is expanded in the post-Newtonian series with respect to parameter ϵ around the Minkowski metric as follows:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \epsilon^3 h^{(3)}_{\alpha\beta} + \epsilon^4 h^{(4)}_{\alpha\beta} + \mathcal{O}(\epsilon^5). \tag{23}$$

The generic post-Newtonian expansion of the metric tensor is not analytic with respect to parameter ϵ [50,154,180]. However, the nonanalytic (logarithmic) terms emerge only in higher post-Newtonian approximations and do not affect the results of the present paper since we restrict ourselves with the first post-Newtonian approximation. Notice also that the linear, with respect to ϵ, terms in the metric tensor expansion (23) do not originate from the field equations (18) and are a pure coordinate-dependent effect. Hence, they can be eliminated by making an appropriate adjustment of the coordinate chart [58,87,195]. If we kept them, they would make the coordinate grid nonorthogonal and rotating at classic (Newtonian) level. Reference frames with such properties are rarely used in astronomy and astrophysics. Therefore, we shall postulate that the linear term in expansion (23) is absent.

After eliminating the linear terms in the post-Newtonian expansion of the metric tensor and substituting the expansion to the field equations (18) we can check by inspection that various components of the metric tensor and the scalar field have in the first post-Newtonian approximation the following form [195]:

$$g_{00} = -1 + \epsilon^2 h^{(2)}_{00} + \epsilon^4 h^{(4)}_{00} + \mathcal{O}(\epsilon^6), \tag{24}$$

$$g_{0i} = \epsilon^3 h^{(3)}_{0i} + \mathcal{O}(\epsilon^5), \tag{25}$$

$$g_{ij} = \delta_{ij} + \epsilon^2 h^{(2)}_{ij} + \mathcal{O}(\epsilon^4), \tag{26}$$

where each term of the expansions will be defined and explained below. In order to simplify notations, we shall use the following abbreviations for the metric tensor perturbations:

$$h_{00} \equiv h^{(2)}_{00}, \qquad l_{00} \equiv h^{(4)}_{00}, \qquad h_{0i} \equiv h^{(3)}_{0i}, \qquad h_{ij} \equiv h^{(2)}_{ij}, \qquad h \equiv h^{(2)}_{kk}. \tag{27}$$

Post-Newtonian expansion of the metric tensor (24)–(26) introduces a corresponding expansion of the stress-energy tensor of matter (10),

$$T_{00} = T^{(0)}_{00} + \epsilon^2 T^{(2)}_{00} + \mathcal{O}(\epsilon^4), \tag{28}$$

$$T_{0i} = \epsilon T^{(1)}_{0i} + \mathcal{O}(\epsilon^3), \tag{29}$$

$$T_{ij} = \epsilon^2 T^{(2)}_{ij} + \mathcal{O}(\epsilon^4), \tag{30}$$

where

$$T^{(0)}_{00} = \rho^*, \tag{31}$$

$$T^{(1)}_{0i} = -\rho^* v^i, \tag{32}$$

$$T^{(2)}_{ij} = \rho^* v^i v^j + \mathfrak{s}^{ij}, \tag{33}$$

$$T^{(2)}_{00} = \rho^* \left(\frac{v^2}{2} + \Pi - h_{00} - \frac{h}{2}\right), \tag{34}$$

$v^i = cu^i/u^0 = dx^i/dt$ is 3-dimensional velocity of matter, and

$$\rho^* \equiv \sqrt{-g}u^0\rho = \rho + \frac{\epsilon^2}{2}\rho(v^2 + h) + \mathcal{O}(\epsilon^4) \tag{35}$$

is the invariant density of matter that is a useful mathematical variable in relativistic hydrodynamics [126] due to the exact law (16) of conservation of rest mass. This conservation law can be recast, following (16), to the equation of continuity [126]

$$\partial_t \rho^* + \partial_i(\rho^* v^i) = 0, \tag{36}$$

which has the *exact* Newtonian form in arbitrary coordinates. Since Eq. (36) is exact it makes calculation of the total time derivative from a volume integral of arbitrary differentiable function $f(t, \boldsymbol{x})$ simple,

$$\frac{d}{dt}\int_{\mathcal{V}_{\rm B}} \rho^*(t, \boldsymbol{x}) f(t, \boldsymbol{x}) d^3x = \int_{\mathcal{V}_{\rm B}} \rho^*(t, \boldsymbol{x}) \frac{df(t, \boldsymbol{x})}{dt} d^3x, \tag{37}$$

where $\mathcal{V}_{\rm B}$ denotes a volume of body B, and the operator of the total time derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}. \tag{38}$$

In derivation of (37) we have taken into account that the boundary of the volume of body B can change as time progresses [17] but there is no flux of baryonic matter through the boundary of the body. We also notice that Eq. (37) is exact.

In what follows, we shall give up on the post-Newtonian expansion parameter ϵ in all subsequent equations because we work only in the first post-Newtonian approximation, and leaving out ϵ should not cause confusion. We also use the geometric system of units, $G = c = 1$. Physical units like SI or CGS can be easily put back to our equations by making use of dimensional analysis [196].

D. Conformal harmonic gauge

The post-Newtonian field equations for the post-Newtonian components of the metric tensor and scalar field variables can be derived after substituting the post-Newtonian series of the previous section to the covariant equations (18) and (19), and arranging the terms in the expansion in the order of smallness with respect to parameter ϵ. The post-Newtonian equations are covariant like the original field equations that is their form is independent of the choice of spacetime coordinates. Hence, their solutions are determined up to four arbitrary functions reflecting a freedom of coordinate transformations called the gauge freedom of the metric tensor. It is a common practice to limit the coordinate arbitrariness by imposing a gauge condition which limits the choice of coordinates on spacetime manifold. The gauge condition does not fix the freedom in choosing coordinates completely—a restricted class of coordinate transformations within the imposed gauge still remains. This class of transformations is called a *residual* gauge freedom which plays an important role in theoretical formulation of relativistic dynamics of an $\mathbb{N}$-body system.

One of the most convenient gauge conditions in a scalar-tensor theory of gravity was proposed by Nutku [146,147] as a generalization of the harmonic gauge of general relativity

$$\partial_\nu(\Phi\sqrt{-g}g^{\mu\nu}) = 0. \tag{39}$$

The Nutku gauge condition (39) is equivalent to the following condition imposed on the Christoffel symbols:

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = g^{\alpha\beta}\partial_\beta \ln\Phi. \tag{40}$$

Let us consider now the Laplace-Beltrami operator introduced above in (8) and write it down in the Nutku gauge in the case of an arbitrary scalar function $F \equiv F(x^\alpha)$. It yields

$$\Box_g F \equiv g^{\alpha\beta}(\partial_{\alpha\beta}F - \partial_\alpha F \partial_\beta \ln\Phi). \tag{41}$$

Any function F that is subject to the homogeneous Laplace-Beltrami equation, $\Box_g F = 0$, is called harmonic. The Laplace-Beltrami operator (41) applied to each particular coordinate being considered as a scalar function $F = x^\alpha$, gives us

$$\Box_g x^\alpha = -g^{\alpha\beta}\partial_\beta \ln\Phi \neq 0, \tag{42}$$

which means that the coordinates x^α are not harmonic functions on the spacetime manifold in the Jordan frame and in the Nutku gauge. Nonetheless, such nonharmonic coordinates are more convenient in the scalar-tensor theory of gravity because they allow us to eliminate more coordinate-dependent terms from the field equations as compared with the harmonic gauge condition $\Box_g x^\alpha = 0$ which is not equivalent to the Nutku gauge (39). We call the class of the coordinates satisfying the Nutku gauge (39) the conformal harmonic coordinates [87]. As we have learned above, these coordinates are not harmonic in the Jordan frame but it can be shown that they are harmonic functions of spacetime manifold in the conformal Einstein frame with the metric $\tilde{g}_{\alpha\beta} \equiv \Phi g_{\alpha\beta}$. Indeed, in the Einstein frame, the Nutku gauge condition (40) reads $\partial_\beta(\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}) = 0$, which is exactly the harmonic gauge condition.

The conformal harmonic coordinates have many properties similar to the harmonic coordinates in general relativity. Our preferences in choosing the conformal harmonic coordinates for constructing a theory of motion of extended celestial bodies are justified by three factors:

(1) the conformal harmonic coordinates become harmonic coordinates in general relativity when the scalar field is switched off, $\Phi \to 0$,
(2) the conformal harmonic coordinates represent a natural generalization of the IAU 2000 resolutions [83] on relativistic reference frames from general relativity to scalar-tensor theory of gravity, and
(3) the Nutku gauge condition (39) significantly simplifies the field equations and facilitates finding their solutions like in the case of the harmonic gauge in general relativity.

Harmonic coordinates in the Jordan frame have been used by Klioner and Soffel [197] for constructing post-Newtonian reference frames in PPN formalism. The conformal harmonic coordinates were employed in our publications [17,87] for discussing relativistic celestial mechanics of the Solar System. We shall also use the conformal harmonic coordinates in the present paper.

The gauge condition (40) does not fix the conformal harmonic coordinates uniquely. Let us change the coordinates

$$x^\alpha \mapsto w^\alpha = w^\alpha(x^\alpha) \tag{43}$$

but keep the Nutku gauge condition (40) intact in the new coordinates. After applying the coordinate transformation (43) to (40) it is straightforward to show that the new conformal harmonic coordinates w^α must satisfy a homogeneous wave equation

$$g^{\mu\nu}(x^\beta)\frac{\partial^2 w^\alpha}{\partial x^\mu \partial x^\nu} = 0, \tag{44}$$

which describes the residual gauge freedom in choosing the conformal harmonic coordinates that remain after imposing the Nutku gauge condition on the metric tensor. Equation (44) has the infinite number of nontrivial solutions defining the entire set of the conformal harmonic coordinates on a spacetime manifold. The residual gauge freedom in the scalar-tensor theory of gravity is similar to that existing in the harmonic gauge of general relativity. We shall specify the set of the conformal harmonic coordinates used for derivation of equations of motion of celestial bodies in $\mathbb{N}$-body system in Sec. IV.

E. Post-Newtonian field equations

Before writing down the field equations, it is worth noticing that the post-Newtonian approximation of the scalar-tensor theory of gravity with a variable coupling function $\omega(\Phi)$ has two parameters, ω_0 and ω'_0, characterizing deviation from general relativity. It is more convenient to bring these parameters to the standard form of PPN parameters, γ and β [88]

$$\gamma = \frac{\omega_0 + 1}{\omega_0 + 2}, \tag{45}$$

$$\beta = 1 + \frac{\omega'_0}{(2\omega_0 + 3)(2\omega_0 + 4)^2}. \tag{46}$$

General relativity is obtained as a limiting case of the scalar-tensor theory when parameters $\gamma = \beta = 1$ or $\omega_0 \to \infty$. Notice that in order to get this limit convergent, the derivative of the coupling function, ω'_0, must grow slower than ω_0^3 as ω_0 approaches infinity. Currently, there are no experimental data restricting the asymptotic behavior of $\omega'_0 \sim \omega_0^3\beta$ which could help us to understand better the nature of the coupling function $\omega(\Phi)$. This makes the parameter β a primary target for experimental study in the near-future gravitational experiments [198–200] including the advanced lunar laser ranging [201–203] and gravitational wave detectors [91]. The background scalar field Φ_0 and the parameter of coupling ω_0 determine the observed numerical value of the universal gravitational constant

$$G = \frac{2\omega_0 + 4}{2\omega_0 + 3}\Phi_0^{-1}. \tag{47}$$

Had the background value Φ_0 of the scalar field been driven by cosmological evolution, the measured values of the universal gravitational constant G and parameters β and γ would depend on time [124]. Notice also that in the geometric system of units $G = 1$, and Eq. (47) reads

$$\Phi_0 = \frac{2\omega_0 + 4}{2\omega_0 + 3} = \frac{2}{\gamma + 1}, \tag{48}$$

which allows us to express the background value Φ_0 of the scalar field in terms of the PPN parameter γ.

Let us now substitute the post-Newtonian expansions given by Eqs. (24)–(30) to the field equations (18) and (19) and make use of the conformal harmonic gauge condition (39) in the first post-Newtonian approximation. It reads

$$\partial_0(h_{kk} + h_{00}) + 2(1-\gamma)\partial_0\varphi = 2\partial_j h_{0j}, \tag{49}$$

$$\partial_i(h_{kk} - h_{00}) + 2(1-\gamma)\partial_i\varphi = 2\partial_j h_{ij}, \tag{50}$$

where, for the sake of simplifying the field equations, we have introduced a new notation of the post-Newtonian perturbation, $\phi^{(2)}$, of the scalar field, namely,

$$\phi^{(2)} \equiv (1-\gamma)\varphi. \tag{51}$$

It is worth noting that in the first post-Newtonian approximation the metric tensor component $h^{(4)}_{00} \equiv l_{00}$ does not enter (49) and (50) and should be taken into account only at the second post-Newtonian approximation which we do not consider in the present paper.

After making use of the stress-energy tensor (31)–(34), definitions of the PPN parameters (45)–(46) and (48), one obtains the final form of the post-Newtonian field equations:

$$\Box_\eta \varphi = -4\pi\rho^*, \tag{52}$$

$$\Box_\eta h_{00} = -8\pi\gamma\rho^*, \tag{53}$$

$$\Box_\eta h_{ij} = -8\pi\gamma\rho^*\delta_{ij}, \tag{54}$$

$$\Box_\eta h_{0i} = 8\pi(1+\gamma)\rho^* v^i, \tag{55}$$

$$\Box_\eta l_{00} = -8\pi\rho^*\left[\left(\gamma+\frac{1}{2}\right)v^2 + \Pi + \gamma\frac{\mathfrak{s}^{kk}}{\rho^*} - \frac{h_{kk}}{6} - (2\beta-\gamma-1)\varphi\right] - \frac{1}{2}\Box_\eta[h^2_{00} + 4(\beta-1)\varphi^2], \tag{56}$$

where the η-box symbol, $\Box_\eta \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, is the D'Alembert (wave) operator of the Minkowski spacetime. Equations (52)–(56) are valid in the conformal harmonic coordinate charts defined by the gauge condition (39) imposed on the components of the metric tensor. Their solution depends on the boundary conditions imposed on the metric tensor and the scalar field perturbations. In their own turn, the boundary conditions singled out a certain type of coordinate chart. We discuss the coordinate charts in next section.

IV. PARAMETRIZED POST-NEWTONIAN COORDINATES

Standard textbooks on the post-Newtonian celestial mechanics [16,17,48,96,101,126,159] derives post-Newtonian equations of motion in a particular gauge to suppress the gauge-dependent effects and to bring the equations to a form which is suitable for finding analytic solutions and for computational applications like numerical orbital simulations, data processing, etc. The coordinate-based approach is also used for solving the field equations and deriving relativistic equations of motion of compact inspiralling binaries for the purposes of gravitational wave astronomy [29–31,204]. The post-Newtonian equations admit a large freedom in making the gauge (coordinate) transformations on spacetime manifold as well as in the configuration space of the orbital parameters characterizing motion of bodies [205,206]. Therefore, each single term taken in such post-Newtonian equations separately from the others makes no physical sense—it can be always changed or even eliminated by making the post-Newtonian coordinate transformations. Only after the equations are solved and their solutions are substituted to observables can we unambiguously discuss gravitational physics because the observables are invariantly defined. Therefore, a primary goal of the present paper is to derive the post-Newtonian equations of translational and rotational motion of arbitrarily structured bodies in the $\mathbb{N}$-body problem in a fully covariant form. Nonetheless, the coordinate-dependent form of equations of motion is more convenient for practical use in various applications. This is why we, first, derive the equations of motion in the conformal harmonic coordinates and, then, establish their correspondence to the covariant form of the equations of motion.

Derivation of the covariant equations of motion of bodies from the field equations can be achieved directly by the methods of differential geometry like in the Mathisson-Papapetrou-Dixon formalism. They can be compared with the coordinate-dependent form of the equations of motion by projecting the corresponding covariant quantities onto the coordinate basis but we use an alternative approach in the present paper. More specifically, we build a set of $\mathbb{N}$ local coordinate charts adapted to each body, derive equations of motion of each body in the local chart, and then, prolongate the coordinate-dependent description to the covariant form by making use of the Einstein principle of equivalence (EEP) applied on the effective background spacetime manifold $\bar{M}$ to the multipoles propagated along the accelerated worldline of the origin of the local coordinates. This procedure is equivalent to "comma-goes-to-semicolon" rule [165][Chapter 16] applied on the worldline of the origin of the local coordinates. EEP effectively allows us to replace each spatial partial derivative $\hat{\partial}_i$ in the local coordinates with a covariant derivative $\bar{\nabla}_\alpha$ projected on the hypersurface being orthogonal to the 4-velocity $\bar{u}^\alpha$ of the origin of the local coordinates. It also replaces each time derivative in the local coordinates with the Fermi-Walker covariant derivative of the Fermi-Walker transport; see Sec. XI E for more details.

Nonetheless, it is not guaranteed that taking the first post-Newtonian equations of motion and "covariantizing" them by making use of the comma-goes-to-semicolon rule will automatically lead to results which are even formally valid in the fast motion and thus, for binaries, strong-field regime. Each term in the "generalized" covariant equations of motion results from a corresponding term in the post-Newtonian equations of motion, which have themselves relied on the post-Newtonian field equations for their derivation. It is certainly conceivable and perhaps even likely, especially at sufficiently high orders in the multipole expansions, that there could exist higher-order nonlinearities and higher-order time-derivative terms in some appropriate formally valid covariant equations of motion which would leave no imprint on the appropriately expanded post-Newtonian equations of motion. Such terms would then not be produced by the covariantization procedure as implemented in the present paper. The limits of application of the EEP to the derivation of the covariant equations of motion beyond the first post-Newtonian approximation requires additional study.

Direct derivation of the covariant equations of motion of extended bodies having an arbitrary set of multipoles has been proposed in general relativity by Mathisson [4,5], further developed by Tulczyjew [207], Tulczyjew and Tulczyjew [208], Papapetrou [6,134,209], Taub [137], Madore [138] and, especially, by Dixon [7–11] with some improvements made by Ehlers and Rudolph [139], Schattner [140] and Dixon [136]. Subsequent development of the MPD covariant approach [143,144,210,211] brought more progress to our understanding of the covariant nature of motion but it has not yet been elaborated to the extent that allows us to apply the formalism in astrophysical work.

The MPD approach operates on worldlines of the center-of-mass of the extended bodies which are considered as pointlike particles endowed with an infinite set of Dixon's multipoles [9]. Such treatment of the extended bodies requires one to replace the continuous stress-energy tensor of matter with a, so-called, stress-energy skeleton defined in terms of distributions [212]. The skeleton must lead to the same solution of the field equations and to the same equations of motion as the continuous stress-energy tensor. This identity has been checked in the linearized approximation of general relativity but it is not yet clear how to build the skeleton in the nonlinear gravity regime that hampers extension of the MPD approach to astrophysical objects with strong gravity like neutron stars and black holes whose equations of motion are currently derived by the matched asymptotic expansions technique [55–57,213,214].

The MPD covariant approach to the problem of motion of an $\mathbb{N}$-body system of extended bodies has an ambiguity concerning the most optimal definition of the center of mass of an extended body. There are four competing mathematical definitions based on the, so-called, spin supplementary condition demanding the intrinsic angular momentum (spin) of the body to be orthogonal to either 4-velocity of the center of mass (Mathisson-Pirani condition) or to the body's linear momentum (Tulczyjew-Dixon condition) or to some timelike vector (Newton-Wigner condition) or to the unit vector being tangent to the coordinate time axis (Corinaldesi-Papapetrou condition). Depending on the choice of the spin supplementary condition, the MPD equations of motion take different forms leading to different solutions of the equations of motion which are intensively discussed in literature—see, e.g., [15,211,215–217]—but there is no general agreement which solution corresponds to a real physical motion of the body.

The above-mentioned problems with the MPD formalism convinced us to use a more practical, coordinate-based route to the derivation of covariant equations of motion used along with the method of asymptotic matching of the solutions of the internal and external problems in the $\mathbb{N}$-body problem and the Blanchet-Damour (BD) multipole formalism. The employment of a set of global and local coordinates is a necessary intermediate step in building the covariant theory of motion of extended bodies. Coordinates are necessary to give a physically meaningful definition of the BD multipoles of the bodies in the nonlinear gravity regime, to unambiguously single out the center of mass of each body and its worldline, and to separate the self-action force of each body from the external gravitational force of the other bodies of an $\mathbb{N}$-body system. The coordinate description is practically useful in astrophysics for computation of orbital motion of inspiralling binaries and in the relativistic celestial mechanics of the Solar System [17]. On the other hand, the coordinate description of the equations of motion can be easily converted to the covariant form as soon as the theory is completed. As we have learned above, discussion of the dynamics of the $\mathbb{N}$-body problem requires introduction of one global and $\mathbb{N}$ local coordinate charts adapted to each body. Geometric properties of the coordinate charts as well as their kinematic and dynamic characteristics are defined by the boundary conditions imposed on the metric tensor and scalar field.

A. Global coordinate chart

1. Boundary conditions

We consider an isolated system consisting of $\mathbb{N}$-extended bodies which are gravitationally bound, occupy a finite volume of space, and there is no other matter outside it. Since there is no matter outside the system, the spacetime manifold with the metric tensor $g_{\alpha\beta}$ can be considered at infinity as asymptotically approaching to flat spacetime with the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)$. We further assume, in accordance with the post-Newtonian approximations, that there are no physical singularities on the manifold like black holes, wormholes, etc., among the bodies of the system, and that the bodies move slowly and the gravitational field is weak everywhere.

These founding assumptions allow us to cover the whole spacetime manifold with a global coordinate chart denoted as $x^\alpha = (x^0, x^i)$, where $x^0 = t$ is the coordinate time and $x^i \equiv \boldsymbol{x}$ are the spatial coordinates. The global coordinates are used for describing orbital dynamics of the bodies, for calculating generation and propagation of gravitational waves emitted by the isolated system, and for formulating the global laws of conservation and conserved quantities [119]. The coordinate time, t, and spatial coordinates, x^i, have no immediate physical meaning in the regions of space where the gravitational field is not negligible. However, when one approaches to infinity the global coordinates approximate the Lorentz coordinates of the inertial observer in the Minkowski space. For this reason, one can interpret the coordinate time t and the spatial coordinates x^i respectively as the proper time and the proper distance measured by a set of the inertial observers located at rest at spatial infinity [126]. The global coordinates are not defined uniquely but up to a group of transformation preserving the asymptotic flatness of spacetime. Contrary to the original expectations this group of transformation is not a 10-parametric Poincaré group but the infinite-dimensional BMS group which is isomorphic to the semidirect product of the homogeneous Lorentz group with the Abelian group of supertranslations [106]. The Poincaré group is a subgroup of the BMS group.

A precise mathematical description of properties of the global post-Newtonian coordinates can be given in terms of the metric tensor that is the solution of the field equations (53)–(56) with the boundary conditions imposed at infinity. To formulate the boundary conditions, we introduce the metric perturbation

$$h_{\alpha\beta}(t,\boldsymbol{x}) \equiv g_{\alpha\beta}(t,\boldsymbol{x}) - \eta_{\alpha\beta}, \tag{57}$$

where $h_{\alpha\beta}$ is the full post-Newtonian series defined in (23). The global coordinates must match asymptotically with the inertial coordinates of the Minkowski spacetime which presumes that the products $rh_{\alpha\beta}$ and $r^2 h_{\alpha\beta,\gamma}$ where $r = |\boldsymbol{x}|$, are bounded at spatial infinity [126,184], while at the future null infinity

$$\lim_{\substack{r\to\infty \\ t+r=\text{const.}}} h_{\alpha\beta}(t,\boldsymbol{x}) = 0. \tag{58}$$

An additional boundary condition must be imposed on the first derivatives of the metric tensor to exclude nonphysical (advanced) radiative solutions associated with gravitational waves incoming to the $\mathbb{N}$-body system from infinity. This condition is imposed because we have assumed that there are no sources of gravitational waves outside of the isolated $\mathbb{N}$-body system. It is formulated as follows [126,184]:

$$\lim_{\substack{r\to\infty \\ t+r=\text{const.}}} [\partial_r(rh_{\alpha\beta}) + \partial_t(rh_{\alpha\beta})] = 0, \tag{59}$$

where ∂_r and ∂_t denote the partial derivatives with respect to radial coordinate r and time t, respectively. Though, the first post-Newtonian approximation does not include gravitational waves, the boundary condition (59) tells us to choose the retarded solution of the field equations (53)–(56).

Similarly, we impose the "no-incoming-radiation" conditions on the perturbation φ of the scalar field defined in (51),

$$\lim_{\substack{r\to\infty \\ t+r=\text{const.}}} \varphi(t,\boldsymbol{x}) = 0, \tag{60}$$

$$\lim_{\substack{r\to\infty \\ t+r=\text{const.}}} [\partial_r(r\varphi) + \partial_t(r\varphi)] = 0. \tag{61}$$

These conditions eliminates the advanced radiative solution for the scalar field.

2. Scalar field

The scalar field in the global coordinates is obtained as a solution of the field equation (52) with the no-incoming (scalar) radiation boundary conditions (60), (61). This solution is a retarded potential

$$\varphi(t,\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\rho^*(t - |\boldsymbol{x}-\boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{62}$$

where the integration is performed over the entire space $\mathbb{R}^3$. The post-Newtonian expansion of the retarded potential is obtained by expanding the integrand in (62) around the instant of time t, and integrating each term of the expansion. In what follows, we need merely the first term of the expansion. Moreover, since the density of matter ρ^* vanishes outside the bodies of the $\mathbb{N}$-body system, the integration is carried out over only the volumes of the bodies, which yield

$$\varphi(t,\boldsymbol{x}) = U(t,\boldsymbol{x}). \tag{63}$$

Here,

$$U(t,\boldsymbol{x}) = \sum_{\mathrm{C}} U_{\mathrm{C}}(t,\boldsymbol{x}) \tag{64}$$

is a linear superposition of the Newtonian gravitational potentials $U_{\mathrm{C}}(t,\boldsymbol{x})$ of the bodies $(C = 1, 2, \ldots, N)$, and

$$U_{\mathrm{C}}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{65}$$

where $\mathcal{V}_{\mathrm{C}}$ denotes the spatial volume occupied by the body C.

Subsequent derivation requires one to single out one of the bodies, let say a body B, and split the scalar field in two parts—internal and external,

$$U(t,\boldsymbol{x}) = U_{\mathrm{B}}(t,\boldsymbol{x}) + \bar{U}(t,\boldsymbol{x}), \tag{66}$$

where U_{B} denotes the internal gravitational potential produced by the body B alone,

$$U_{\mathrm{B}}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{67}$$

and

$$\bar{U}(t,\boldsymbol{x}) = \sum_{\mathrm{C}\neq\mathrm{B}} U_{\mathrm{C}}(t,\boldsymbol{x}), \tag{68}$$

denotes the external gravitational potential of all other bodies of the $\mathbb{N}$-body system but the body B.

3. Metric tensor

The metric tensor $g_{\alpha\beta}(t,\boldsymbol{x})$ in the global coordinates is obtained by solving the field equations (53)–(56) with the boundary conditions (58)–(59). It yields [17,87]

$$h_{00}(t,\boldsymbol{x}) = 2U(t,\boldsymbol{x}), \tag{69}$$

$$h_{ij}(t,\boldsymbol{x}) = 2\gamma\delta_{ij}U(t,\boldsymbol{x}), \tag{70}$$

$$h_{0i}(t,\boldsymbol{x}) = -2(1+\gamma)U^i(t,\boldsymbol{x}), \tag{71}$$

$$l_{00}(t,\boldsymbol{x}) = 2\Psi(t,\boldsymbol{x}) - 2\beta U^2(t,\boldsymbol{x}) - \partial_{tt}\chi(t,\boldsymbol{x}), \tag{72}$$

where the operator $\partial_{tt} \equiv \partial^2/\partial t^2$, the post-Newtonian potential

$$\Psi(t,\boldsymbol{x}) \equiv \left(\gamma+\frac{1}{2}\right)\Psi_1(t,\boldsymbol{x}) + (1-2\beta)\Psi_2(t,\boldsymbol{x}) + \Psi_3(t,\boldsymbol{x}) + \gamma\Psi_4(t,\boldsymbol{x}), \tag{73}$$

and parameters γ and β have been defined in (45) and (46) respectively.

Newtonian gravitational potential U has been defined above in (64). Post-Newtonian potentials U^i, χ, Ψ_n ($n = 1$, 2, 3, 4) are linear combinations of the gravitational potentials produced by the bodies of the $\mathbb{N}$-body system,

$$U^i(t,\boldsymbol{x}) = \sum_{\mathrm{C}} U^i_{\mathrm{C}}(t,\boldsymbol{x}),$$

$$\Psi_n(t,\boldsymbol{x}) = \sum_{\mathrm{C}} \Psi_{\mathrm{C}n}(t,\boldsymbol{x}), \qquad \chi(t,\boldsymbol{x}) = \sum_{\mathrm{C}} \chi_{\mathrm{C}}(t,\boldsymbol{x}). \tag{74}$$

Here, the summation index $C = 1, 2, \ldots, N$ numerates the bodies of the $\mathbb{N}$-body system, and the gravitational potentials of body C are defined as integrals performed over a spatial volume $\mathcal{V}_{\mathrm{C}}$ occupied by the body's matter,

$$U^i_{\mathrm{C}}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\rho^*(t,\boldsymbol{x}')v^i(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{75}$$

$$\Psi_{\mathrm{C}1}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\rho^*(t,\boldsymbol{x}')v^2(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{76}$$

$$\Psi_{\mathrm{C}2}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\rho^*(t,\boldsymbol{x}')U(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{77}$$

$$\Psi_{\mathrm{C}3}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\rho^*(t,\boldsymbol{x}')\Pi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{78}$$

$$\Psi_{\mathrm{C}4}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{C}}} \frac{\mathfrak{s}^{kk}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{79}$$

where $v^i = v^i(t,\boldsymbol{x})$ is velocity of the element of matter located at time t at a spatial point $x^i = \boldsymbol{x}$ in the global coordinates, and $v^2 = \delta_{ij}v^iv^j$.

Superpotential χ_{C} is determined as a particular solution of the inhomogeneous Poisson equation

$$\triangle\chi_{\mathrm{C}}(t,\boldsymbol{x}) = -2U_{\mathrm{C}}(t,\boldsymbol{x}) \tag{80}$$

where $\triangle \equiv \delta^{ij}\partial_i\partial_j$ is the Laplace operator in the Euclidean space. The source of the superpotential χ_{C} is the Newtonian gravitational potential U_{C} that presents everywhere in a whole space. Nevertheless, because it falls off as $1/r$ at infinity, the solution of the Poisson equation (80) has a compact support, and is given by an integral taken over the finite volume of body C [88,126]

$$\chi_{\mathrm{C}}(t,\boldsymbol{x}) = -\int_{\mathcal{V}_{\mathrm{C}}} \rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'| d^3x'. \tag{81}$$

It is useful to emphasize that all of above-given volume integrals defining the metric tensor in the global coordinates are taken on the spacelike hypersurface $\mathcal{H}_t$ of constant coordinate time t. Changing the time coordinate does not change the functional form of the integrals but transforms the time hypersurface that makes the numerical value of the integrals different. This remark is important for understanding the post-Newtonian transformations and the technique of matched asymptotic expansions of the metric tensor and scalar field which we explain below in Sec. V.

In what follows we single out a body B, and split all post-Newtonian potentials in two parts—internal and external—like we did above in (66) for the Newtonian gravitational potential

$$U^i(t,\boldsymbol{x}) = U^i_{\mathrm{B}}(t,\boldsymbol{x}) + \bar{U}^i(t,\boldsymbol{x}), \tag{82}$$

$$\Psi(t,\boldsymbol{x}) = \Psi_{\mathrm{B}}(t,\boldsymbol{x}) + \bar{\Psi}(t,\boldsymbol{x}), \tag{83}$$

$$\chi(t,\boldsymbol{x}) = \chi_{\mathrm{B}}(t,\boldsymbol{x}) + \bar{\chi}(t,\boldsymbol{x}). \tag{84}$$

Here, functions with subindex B denote the internal potentials produced by the body B alone,

$$U^i_{\mathrm{B}}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\rho^*(t,\boldsymbol{x}')v^i(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{85}$$

$$\Psi_{\mathrm{B}1}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\rho^*(t,\boldsymbol{x}')v^2(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{86}$$

$$\Psi_{\mathrm{B}2}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\rho^*(t,\boldsymbol{x}')U(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{87}$$

$$\Psi_{\mathrm{B}3}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\rho^*(t,\boldsymbol{x}')\Pi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{88}$$

$$\Psi_{\mathrm{B}4}(t,\boldsymbol{x}) = \int_{\mathcal{V}_{\mathrm{B}}} \frac{\mathfrak{s}^{kk}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{89}$$

$$\chi_{\rm B}(t,\boldsymbol{x}) = -\int_{\mathcal{V}_{\rm B}} \rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|d^3x', \tag{90}$$

and functions covered with a bar denote the external potentials,

$$\bar{U}^i(t,\boldsymbol{x}) = \sum_{\rm C\neq B} U^i_{\rm C}(t,\boldsymbol{x}), \qquad \bar{\Psi}(t,\boldsymbol{x}) = \sum_{\rm C\neq B}\Psi_{\rm C}(t,\boldsymbol{x}),$$
$$\bar{\chi}(t,\boldsymbol{x}) = \sum_{\rm C\neq B}\chi_{\rm C}(t,\boldsymbol{x}), \tag{91}$$

where potentials $U^i_{\rm C}, \Psi_{\rm C}, \chi_{\rm C}$ are given by integrals (75)–(79) respectively. It is worth emphasizing [218] that the integrand of integrals (77), (87) depends on the *total* gravitational potential U of all bodies of an $\mathbb{N}$-body system as defined in (64). It is also important to notice that the Newtonian gravitational potential $U(t,\boldsymbol{x})$ has a double camouflage in the scalar-tensor theory of gravity. It appears in the solution (63) of the field equation for scalar field φ, and, also, in (69) and (70) describing perturbations of the metric tensor components h_{00} and h_{ij}. It would be wrong, however, to interpret the metric tensor component $h_{00}=2U$, and the trace $h\equiv\delta^{ij}h_{ij}=h_{kk}=6U$ like scalars; they can be expressed in terms of the scalar field φ alone only in the global coordinates. By definition, the metric tensor perturbations, h_{00} and h_{kk}, are transformed as tensors not as scalars.

A mathematical description of orbital dynamics of extended bodies in an $\mathbb{N}$-body system would be significantly simplified if we could keep the position of the center of mass of an $\mathbb{N}$-body system at the origin of the global coordinates for any instant of time. This condition suggests that the dipole, $\mathbb{D}^i$, of the gravitational field of an $\mathbb{N}$-body system in the multipolar expansion of $h_{00}(t,\boldsymbol{x})$ component of the metric tensor perturbation vanishes along with the dipole (linear momentum), $\mathbb{P}^i$, in the multipolar expansion of the h_{0i} component [165]. This condition cannot be satisfied at higher post-Newtonian approximations due to the gravitational wave recoil which makes the system's center of mass moving with acceleration [219]. Nonetheless, in the first and second post-Newtonian approximations the orbital dynamics of an $\mathbb{N}$-body system is fully determined by the Lagrangian admitting ten conservation laws corresponding to ten infinitesimal generators of the Poincaré group preserving the invariance of the Lagrangian of the $\mathbb{N}$-body problem [48,126,220–222]. The post-Newtonian law of conservation of the total linear momentum, $\mathbb{P}^i$, allows one to hold the center of mass of an $\mathbb{N}$-body system always at the origin of the global coordinate chart [17].

B. Local coordinate chart

1. Boundary conditions

We label the local coordinates adapted to body B by letters $w^\alpha_{\rm B}=(w^0_{\rm B},w^i_{\rm B})=(u_{\rm B},w^i_{\rm B})$ where $u_{\rm B}$ stands for the local coordinate time and $w^i_{\rm B}$ denote the spatial coordinates $({\rm B}=1,2,\ldots,N)$. There are $\mathbb{N}$ local coordinate charts—one for each body. In a case when there is no confusion, we drop off the subindex B in the notation of the local coordinates. Hence, by default the local coordinates adapted to body B will be denoted by $w^\alpha=(u,w^i)\equiv(u_{\rm B},w^i_{\rm B})$. The origin of the local coordinates adapted to body B moves along a reference worldline $\mathcal{W}$ which is chosen to be sufficiently close to the worldline $\mathcal{Z}$ of the center of mass of body B. Initially, the two worldlines are different but can be made identical after careful study of the problem of definition of the center of mass and its equations of motion relative to $\mathcal{W}$. This will be done in Sec. VI.

The local coordinates are used to describe the internal motion of matter inside the body, to define its center of mass, linear momentum, spin and the other, higher-order internal multipoles of body's gravitational field. The importance of the local coordinates for adequate mathematical description of relativistic dynamics of extended, self-gravitating massive bodies in an $\mathbb{N}$-body system was emphasized by Fock [126]. Concrete mathematical construction of the body-adapted, local coordinates was achieved in the post-Newtonian approximation by the technique of asymptotic matching in papers [69,156]—for extended bodies, and in papers [56,223]—for black holes. Later on, a more rigorous mathematical BK-DSX formalism of construction of the local coordinates has been elaborated in a series of publications [72–76] which led to the development and adoption of the IAU 2000 resolutions on general-relativistic reference frames in the Solar System [17,83,159]. Below we extend this formalism to the scalar-tensor theory of gravity.

The scalar field and metric tensor in the local coordinates adapted to body B are solutions of the field equations (52)–(54) inside a bounded spatial domain enclosing worldline $\mathcal{Z}$ of the center of mass of body B and having radius spreading out to another nearest body from the $\mathbb{N}$-body system. Thus, the right side of the inhomogeneous equations (52)–(56) includes only matter of body B. In order to distinguish solutions of the field equations in the local coordinates from the corresponding solutions of the field equations in the global coordinates, we put a hat over functions of the local coordinates. The solution of the field equation for metric tensor or scalar field in the local coordinates is a linear combination of a particular solution of the inhomogeneous equation and a general solution of a homogeneous equation. The particular solution yields the internal gravitational field of body B alone while the general solution of the homogeneous equation pertains to the external field of other bodies ${\rm C}\neq{\rm B}$. The nonlinear nature of the field equation (56) brings in mixed terms l_{00} to the metric tensor perturbation describing a coupling between the first-order perturbations.

The post-Newtonian solution of the scalar field equation (52) in the local coordinates adapted to body B is written as a sum of two terms

$$\hat{\varphi}(u,\boldsymbol{w}) = \hat{\varphi}^{\rm int}(u,\boldsymbol{w}) + \hat{\varphi}^{\rm ext}(u,\boldsymbol{w}), \tag{92}$$

describing contributions of the internal matter of body B and external bodies C ≠ B respectively. If we had no other bodies but the body B, the internal solution had to vanish at infinity. Hence, it obeys the boundary conditions similar to (60) and (61). The external solution must be regular at the origin of the local coordinates and diverges at infinity.

Perturbation of the metric tensor in the local coordinates is denoted

$$\hat{h}_{\mu\nu}(u,\boldsymbol{w}) = \hat{g}_{\mu\nu}(u,\boldsymbol{w}) - \eta_{\mu\nu}, \tag{93}$$

where each component of $\hat{h}_{\mu\nu}$ is expanded in the post-Newtonian series similar to (24)–(26),

$$\hat{h}_{00}(u,\boldsymbol{w}) = \epsilon^2\hat{h}^{(2)}_{00}(u,\boldsymbol{w}) + \epsilon^4\hat{h}^{(4)}_{00}(u,\boldsymbol{w}) + \mathcal{O}(\epsilon^6), \tag{94}$$

$$\hat{h}_{0i}(u,\boldsymbol{w}) = \epsilon^3\hat{h}^{(3)}_{0i}(u,\boldsymbol{w}) + \mathcal{O}(\epsilon^5), \tag{95}$$

$$\hat{h}_{ij}(u,\boldsymbol{w}) = \delta_{ij} + \epsilon^2\hat{h}^{(2)}_{ij}(u,\boldsymbol{w}) + \mathcal{O}(\epsilon^4), \tag{96}$$

and each term of the post-Newtonian series will be denoted

$$\hat{h}_{00} \equiv \hat{h}^{(2)}_{00}, \qquad \hat{l}_{00} \equiv \hat{h}^{(4)}_{00}, \qquad \hat{h}_{0i} \equiv \hat{h}^{(3)}_{0i}, \\ \hat{h}_{ij} \equiv \hat{h}^{(2)}_{ij}, \qquad \hat{h} \equiv \hat{h}^{(2)}_{kk}. \tag{97}$$

The post-Newtonian solution of the field equations (53)–(56) in the local coordinates is given as a sum of three terms [58]

$$\hat{h}_{\mu\nu}(u,\boldsymbol{w}) = \hat{h}^{\rm int}_{\mu\nu}(u,\boldsymbol{w}) + \hat{h}^{\rm ext}_{\mu\nu}(u,\boldsymbol{w}) + \hat{h}^{\rm mix}_{\mu\nu}(u,\boldsymbol{w}), \tag{98}$$

where $\hat{h}^{\rm int}_{\mu\nu}$ describes the gravitational field generated by the internal matter of body B, $\hat{h}^{\rm ext}_{\mu\nu}$ describes the tidal gravitational field produced by external bodies C ≠ B, and the term $\hat{h}^{\rm mix}_{\mu\nu}$ is a contribution due to the nonlinear coupling of the internal and external metric perturbations in the field equation (56). In the first post-Newtonian approximation the coupling term $\hat{h}^{\rm mix}_{\mu\nu}$ appears only in the $\hat{l}_{00}(u,\boldsymbol{w})$ component of the metric tensor perturbation. The body-frame field $\hat{h}^{\rm int}_{\mu\nu}(u,\boldsymbol{w})$ is the same as if the other bodies of the $\mathbb{N}$-body system were absent. Therefore, it is defined by imposing the boundary conditions similar to (58) and (59). Since the external metric perturbation $\hat{h}^{\rm ext}_{\mu\nu}(u,\boldsymbol{w})$ has a physical meaning of the tidal field caused by external bodies, it must be regular on the worldline $\mathcal{W}$ of the origin of the local coordinates. The coupling field $\hat{h}^{\rm mix}_{\mu\nu}(u,\boldsymbol{w})$ is obtained directly by finding a particular solution of the nonlinear part of the field equation (56). Since the internal and external part of the metric tensor perturbation have been already specified, there is no need to impose a separate boundary condition on the coupling component of the metric tensor perturbation.

The origin of the local coordinates moves along some, yet unspecified, worldline, $\mathcal{W}$, which will be determined later on by matching the solutions of the field equations obtained in the local and global coordinates in the buffer domain where the two coordinate charts overlap. Because we are interested in derivation of equations of motion of the center of mass of each body, we wish to make the origin of the local coordinates coinciding with the center of mass of the body under consideration at any instant of time. This requires a precise post-Newtonian definition of the center of mass. Any deficiency in the definition of a body's center of mass introduces to the equations of motion fictitious forces and torques that have no direct physical meaning. We prove in the present paper that the freedom in choosing the position of the center of mass is large enough to completely remove such fictitious forces and torques from the equations of motion of extended bodies in the scalar-tensor theory of gravity.

We should also impose a limitation on the rotation of spatial axes of the local coordinates as they move along worldline $\mathcal{W}$. Spatial axes of the local coordinates are called kinematical nonrotating if their spatial orientation does not change with respect to the spatial axes of the global coordinates at infinity as time goes on [224,225]. Dynamical nonrotating spatial coordinates are defined by demanding that equations of motion of test particles in the local coordinates do not have the Coriolis and centrifugal forces [224]. Because an $\mathbb{N}$-body system is isolated the spatial axes of the global coordinate do not rotate in any sense. On the other hand, the local coordinates are adapted to a single body B that is not fully isolated from external gravitational environment of other bodies of an $\mathbb{N}$-body system. Therefore, we have to postulate whether the spatial axes of the local coordinates are nonrotating in a kinematic or dynamic sense. For the sake of mathematical simplifications in writing solutions of the field equations it is more convenient to postulate that the spatial axes of the local coordinates are not rotating dynamically. Relativistic nature of gravitational interaction suggests that the spatial axes of the dynamically nonrotating local coordinates will be slowly rotating (precessing) in the kinematic sense with respect to the spatial axes of the global coordinates. Relativistic precession of the spatial axes of the local coordinates has a pure geometric origin and includes three physically different terms that are called respectively de-Sitter (geodetic), Lense-Thirring (gravitomagnetic), and Thomas precession [165]. The exact formula for the matrix of the kinematic precession of spatial axes of the local coordinates is given below in Eq. (151).

2. Scalar field: Internal and external solutions

In the local coordinates adapted to body B, the internal, $\hat{\varphi}^{\rm int}(u,\boldsymbol{w})$, and external, $\hat{\varphi}^{\rm ext}(u,\boldsymbol{w})$, parts of scalar field perturbation (92) have the following form:

$$\hat{\varphi}^{\rm int}(u,\boldsymbol{w}) = \hat{U}_{\rm B}(u,\boldsymbol{w}), \tag{99}$$

$$\hat{\varphi}^{\rm ext}(u,\boldsymbol{w}) = \sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{P}_L w^L. \tag{100}$$

Here, the scalar field $\hat{\varphi}^{\rm int}(u,\boldsymbol{w})$ is a particular solution of inhomogeneous equation (52) with the right-hand side depending solely on the matter density ρ^* of body B. It is expressed in terms of the Newtonian gravitational potential of body B, $\hat{U}_{\rm B}(u,\boldsymbol{w})$, that is defined below in Eq. (106). The scalar field, $\hat{\varphi}^{\rm ext}(u,\boldsymbol{w})$, is a general solution of a homogeneous Laplace equation (52) without sources. As $\hat{\varphi}^{\rm ext}(u,\boldsymbol{w})$ must be regular at the origin of the local coordinates, the solution is given in the form of a Maclaurin series with respect to STF harmonic polynomials, $w^L \equiv w^{\langle i_1\ldots i_l\rangle}$, made out of the products of the spatial local coordinates w^i and the Kronecker symbols δ^{ij}; see definition of STF tensor projection in (2). Coefficients of the expansion are scalar external multipoles, $\mathcal{P}_L \equiv \mathcal{P}_{\langle i_1\ldots i_l\rangle}(u)$, which are STF Cartesian tensors in 3-dimensional Euclidean space that is tangent to hypersurface $\mathcal{H}_u$ of constant coordinate time u taken at the origin of the local coordinates adapted to body B.

3. Metric tensor: Internal solution

The boundary conditions imposed on the internal solution $\hat{h}^{\rm int}_{\alpha\beta}$ for the metric tensor perturbation in the local coordinates adapted to body B are identical with those given in Eqs. (58) and (59). For this reason the internal solution has the same form as in the global coordinates but all functions now refer solely to body B. We obtain

$$\hat{h}^{\rm int}_{00}(u,\boldsymbol{w}) = 2\hat{U}_{\rm B}(u,\boldsymbol{w}), \tag{101}$$

$$\hat{h}^{\rm int}_{0i}(u,\boldsymbol{w}) = -2(1+\gamma)\hat{U}^i_{\rm B}(u,\boldsymbol{w}), \tag{102}$$

$$\hat{h}^{\rm int}_{ij}(u,\boldsymbol{w}) = 2\gamma\delta_{ij}\hat{U}_{\rm B}(u,\boldsymbol{w}), \tag{103}$$

$$\hat{l}^{\rm int}_{00}(u,\boldsymbol{w}) = 2\hat{\Psi}_{\rm B}(u,\boldsymbol{w}) - 2\beta\hat{U}^2_{\rm B}(u,\boldsymbol{w}) - \partial_{uu}\hat{\chi}_{\rm B}(u,\boldsymbol{w}), \tag{104}$$

where the partial time derivative $\partial_{uu} \equiv \partial^2/\partial u^2$,

$$\hat{\Psi}_{\rm B}(u,\boldsymbol{w}) = \left(\gamma+\frac{1}{2}\right)\hat{\Psi}_{\rm B1}(u,\boldsymbol{w}) + (1-2\beta)\hat{\Psi}_{\rm B2}(u,\boldsymbol{w}) + \hat{\Psi}_{\rm B3}(u,\boldsymbol{w}) + \gamma\hat{\Psi}_{\rm B4}(u,\boldsymbol{w}), \tag{105}$$

and index B indicates that the potential having this index is generated by matter of body B only. All the potentials are defined as integrals over volume $\mathcal{V}_{\rm B}$ occupied by matter of body B:

$$\hat{U}_{\rm B}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{106}$$

$$\hat{U}^i_{\rm B}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')\nu^i(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{107}$$

$$\hat{\Psi}_{\rm B1}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')\nu^2(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{108}$$

$$\hat{\Psi}_{\rm B2}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')\hat{U}_{\rm B}(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{109}$$

$$\hat{\Psi}_{\rm B3}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')\Pi(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{110}$$

$$\hat{\Psi}_{\rm B4}(u,\boldsymbol{w}) = \int_{\mathcal{V}_{\rm B}}\frac{\mathfrak{s}^{kk}(u,\boldsymbol{w}')}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{111}$$

$$\hat{\chi}_{\rm B}(u,\boldsymbol{w}) = -\int_{\mathcal{V}_{\rm B}}\rho^*(u,\boldsymbol{w}')|\boldsymbol{w}-\boldsymbol{w}'|d^3w'. \tag{112}$$

$\nu^i = dw^i/du$ is the coordinate velocity of body's matter with respect to the origin of the local coordinates. Notice that the integrals (106)–(112) are taken over hypersurface $\mathcal{H}_u$ of coordinate time u that is different from the hypersurface $\mathcal{H}_t$ of constant coordinate time t, which is used for spatial integration in Eqs. (65), (75)–(79) defining gravitational potentials in the global coordinates x^α. This is important for the post-Newtonian transformation of gravitational potentials as it requires one to use a Lie transport of functions from hypersurface $\mathcal{H}_u$ to hypersurface $\mathcal{H}_t$; for more details, see [[17], Sec. 5. 2. 3].

The internal potentials of the metric tensor in the local coordinates given by (101) and (107) are connected through the exact equation

$$\partial_u\hat{U}_{\rm B}(u,\boldsymbol{w}) + \partial_i\hat{U}^i_{\rm B}(u,\boldsymbol{w}) = 0, \tag{113}$$

which is a direct consequence of the equation of continuity (36) applied in the local coordinates.

4. Metric tensor: External solution

The solution of the homogeneous field equations (53)–(55) for the linearized metric tensor perturbation in the local coordinates adapted to body B yields the tidal gravitational field of external bodies of an $\mathbb{N}$-body system in terms of the external STF multipoles [17,87]. The external solution is convergent at the origin of the local coordinates and its most general form is given by Kopeikin *et al.* [17] and Kopeikin and Vlasov [87]

$$\hat{h}^{\rm ext}_{00}(u,\boldsymbol{w}) = 2\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_L w^L, \tag{114}$$

$$\hat{h}^{\rm ext}_{0i}(u,\boldsymbol{w}) = \sum_{l=2}^{\infty}\frac{l}{(l+1)!}\varepsilon_{ipq}\mathcal{C}_{pL-1}w^{qL-1} + \sum_{l=0}^{\infty}\frac{1}{l!}Z_{iL}w^L + \sum_{l=0}^{\infty}\frac{1}{l!}S_L w^{iL}, \tag{115}$$

$$\hat{h}^{\rm ext}_{ij}(u,\boldsymbol{w}) = 2\delta_{ij}\sum_{l=1}^{\infty}\frac{1}{l!}A_L w^L + \sum_{l=0}^{\infty}\frac{1}{l!}B_L w^{ijL} + \sum_{l=1}^{\infty}\frac{1}{l!}[D_{iL-1}w^{jL-1} + \varepsilon_{ipq}E_{pL-1}w^{jqL-1}]_{{\rm sym}(ij)} + \sum_{l=2}^{\infty}\frac{1}{l!}[F_{ijL-2}w^{L-2} + \varepsilon_{pq(i}G_{j)pL-2}w^{qL-2}], \tag{116}$$

where A_L, B_L, etc., are STF Cartesian tensors defined on worldline $\mathcal{W}$ of the origin of the local coordinate, and the symbol ${\rm sym}(ij)$ denotes symmetrization.

Tensors A_L, B_L, etc., are the *external* multipoles which depend on the coordinate time u only, that is $A_L \equiv A_L(u)$, $B_L \equiv B_L(u)$, etc. Four gauge conditions (49) and (50) imposed on the components (114)–(116) of the metric tensor perturbations reveal that only six out of ten external multipoles are algebraically independent. This allows one to eliminate four multipoles, B_L, E_L, S_L, D_L, from the local metric perturbation [17,87]. The remaining six multipoles, $A_L, \mathcal{C}_L, F_L, G_L, \mathcal{Q}_L, Z_L$, can be constrained by making use of the residual gauge freedom allowed by the differential equation (44) that excludes four other multipoles—A_L, F_L, G_L, Z_L [17,87]. Finally, only two families of the external multipoles—gravitoelectric multipoles $\mathcal{Q}_L$ and gravitomagnetic multipoles $\mathcal{C}_L$—have real physical meaning reflecting the existence of 2 d.o.f. (polarization states) for the tidal gravitational field of the metric tensor.

After fixing the gauge freedom as indicated above, the external metric tensor assumes in the local coordinates the following form:

$$\hat{h}^{\rm ext}_{00}(u,\boldsymbol{w}) = 2\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_L w^L, \tag{117}$$

$$\hat{h}^{\rm ext}_{0i}(u,\boldsymbol{w}) = \frac{1-\gamma}{3}\dot{\mathcal{P}}w^i + \sum_{l=1}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\mathcal{C}_{pL}w^{qL} + 2\sum_{l=1}^{\infty}\frac{2l+1}{(2l+3)(l+1)!}[2\dot{\mathcal{Q}}_L + (\gamma-1)\dot{\mathcal{P}}_L]w^{iL}, \tag{118}$$

$$\hat{h}^{\rm ext}_{ij}(u,\boldsymbol{w}) = 2\delta_{ij}\sum_{l=1}^{\infty}\frac{1}{l!}[\mathcal{Q}_L + (\gamma-1)\mathcal{P}_L]w^L, \tag{119}$$

where the scalar external multipoles appear in the metric perturbations through the gauge conditions (49) and (50), and a dot above the external multipoles denotes a total derivative with respect to time u. The external dipole $\mathcal{Q}_i$ is acceleration of worldline $\mathcal{W}$ of the origin of the local frame adapted to body B with respect to a worldline of a freely falling particle, and monopole $\mathcal{P}$ is the value of the scalar field generated by external bodies $C \neq B$, taken at the origin of the local coordinates [17]. It cannot be excluded from the $\hat{h}^{\rm ext}_{0i}$ component by gauge transformation. On the other hand, the monopole $\mathcal{Q}$ in the metric perturbation is gauge dependent and has been eliminated by rescaling of the local coordinate time.

The nonlinear part $\hat{l}_{00}$ of the perturbation of the external metric tensor is determined as a particular solution of the field equation (56) that yields [87]

$$\hat{l}^{\rm ext}_{00}(u,\boldsymbol{w}) = -2\left(\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_L w^L\right)^2 - 2(\beta-1)\left(\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{P}_L w^L\right)^2 + \sum_{l=1}^{\infty}\frac{1}{(2l+3)l!}\ddot{\mathcal{Q}}_L w^L w^2, \tag{120}$$

where, here and everywhere else, a double dot above a function denotes a second derivative with respect to time u. We have excluded the scalar field components $\mathcal{P}^2$ and $\mathcal{P}\mathcal{P}_i$ from the second term in the right-hand side of (120) because $\mathcal{P}^2$ is removed by rescaling of the local coordinate time while $\mathcal{P}\mathcal{P}^i$ is absorbed to, yet unknown, acceleration $\mathcal{Q}_i$, in (117). We might also decompose the product of two sums in (120) in algebraic sum of irreducible components and absorb the STF part of the decomposition to multipoles $\mathcal{Q}_L$ $(l \geq 2)$. However, this way of writing solution (120) complicates calculations and we do not implement it.

5. Metric tensor: The coupling component

The coupling of the internal and external solutions of the linearized metric tensor perturbations is described by the mixed term $\hat{l}^{\rm mix}_{00}$. It is found as a particular solution of the inhomogeneous field equation (56) with the right side taken as a product of the internal and external solutions found on the previous step of the post-Newtonian iterations. Solving (56) yields

$$\hat{l}^{\rm mix}_{00}(u,\boldsymbol{w}) = -2\left\{\eta\mathcal{P} + 2\sum_{l=1}^{\infty}\frac{1}{l!}[\mathcal{Q}_L + (\beta-1)\mathcal{P}_L]w^L\right\}\hat{U}_{\rm B}(u,\boldsymbol{w}) - 2\sum_{l=1}^{\infty}\frac{1}{l!}[\mathcal{Q}_L + 2(\beta-1)\mathcal{P}_L] \times \int_{\mathcal{V}_{\rm B}}\frac{\rho^*(u,\boldsymbol{w}')w'^L}{|\boldsymbol{w}-\boldsymbol{w}'|}d^3w', \tag{121}$$

where $\eta \equiv 4\beta - \gamma - 3$ is called the Nordtvedt parameter [88], and $\mathcal{V}_{\rm B}$ denotes the volume of body B. The best experimental limitation on the numerical value of Nordtvedt's parameter, $|\eta| < 5 \times 10^{-4}$, is known from the lunar laser ranging experiment [226]. Gravitational wave astronomy will improve its measurement by many orders of magnitude. Equation (121) completes derivation of the metric tensor in the local coordinates in the post-Newtonian approximation.

6. Body-frame internal multipoles

Multipolar decomposition of the metric tensor of an isolated gravitating system residing in asymptotically flat spacetime has been thoroughly studied by a number of researchers [82,227–230]. The most useful technique for the case of the post-Newtonian approximations has been worked out by Blanchet and Damour [78] and Damour and Iyer [79,80]. This technique has been extended to the case of a self-gravitating system embedded to a curved, non-asymptotically flat spacetime in general relativity [58,74] and in scalar-tensor theory of gravity [87], and is used in the present paper.

A single body B from an $\mathbb{N}$-body system interacts gravitationally with other bodies of the system and this interaction cannot be ignored in multipolar decomposition of the gravitational field of the body. The presence of the external bodies brings about the interaction field (121) to the metric tensor in the local coordinates whose energy density gives rise to the contribution of the gravitational field of the external fields to the definition of the internal multipoles of body B. It, first, looked like an ambiguity as it was unclear whether the contribution of the external fields has to be included to the definition of the body multipoles or not [58]. This issue was resolved in general relativity by Damour *et al.* [74] and in scalar-tensor theory of gravity by Kopeikin and Vlasov [87] who demonstrated that the contribution of the interaction field is to be included in the definition of the body's internal multipoles in order to eliminate the *noncanonical* multipoles, $\mathcal{N}^L$ and $\mathcal{R}^L$—see (123) and (124)—originating from the nonlinear part of the metric tensor perturbation (121), from the equations of motion of extended bodies. This effectively erases any dependence of the equations of motion on the internal structure of extended bodies and promotes application of the *effacing principle* [154,185] from spherically symmetric bodies to all multipoles.

There are two families of the canonical internal multipoles in general relativity which are called mass and spin multipoles [50,78,83]. In scalar-tensor theory of gravity the mass multipoles are additionally split in two algebraically independent families which are called *active* and *conformal* multipoles [88]. The active mass multipoles of a body B from an $\mathbb{N}$-body system are defined by equation [17,87]

$$\mathcal{M}^L = \int_{\mathcal{V}_B} \sigma(u,\boldsymbol{w})\left\{1-(2\beta-\gamma-1)\mathcal{P} - \sum_{k=1}^{\infty}\frac{1}{k!}[\mathcal{Q}_K + 2(\beta-1)\mathcal{P}_K]w^{\langle K\rangle}\right\}w^{\langle L\rangle}d^3w + \frac{1}{(2l+3)}\left[\frac{1}{2}\ddot{\mathcal{N}}^{\langle L\rangle} - 2(1+\gamma)\frac{2l+1}{l+1}\dot{\mathcal{R}}^{\langle L\rangle}\right] \tag{122}$$

where the angular brackets around spatial indices denote STF Cartesian tensor [50,82], and

$$\mathcal{N}^L = \int_{\mathcal{V}_B} \sigma(u,\boldsymbol{w})w^2 w^{\langle L\rangle}d^3w, \tag{123}$$

$$\mathcal{R}^L = \int_{\mathcal{V}_B} \sigma^i(u,\boldsymbol{w})w^{\langle iL\rangle}d^3w \tag{124}$$

are two additional noncanonical sets of STF multipoles, and $\mathcal{V}_B$ is volume of body B over which the integration is performed. Noncanonical multipoles $\mathcal{N}^L$ generalize the second-order rotational moment of inertia of body B,

$$\mathcal{N} = \int_{\mathcal{V}_B} \rho^* w^2 d^3w, \tag{125}$$

with respect to the origin of the local coordinates, and $\mathcal{R}^L$ are noncanonical multipoles associated with matter currents inside the body. The density σ in (122) is called the *active* mass density [87],

$$\sigma(u,\boldsymbol{w}) = \rho^*(u,\boldsymbol{w})\left[1+\left(\gamma+\frac{1}{2}\right)\nu^2(u,\boldsymbol{w}) + \Pi(u,\boldsymbol{w}) - (2\beta-1)\hat{U}_B(u,\boldsymbol{w})\right] + \gamma\mathfrak{s}^{kk}(u,\boldsymbol{w}), \tag{126}$$

and the vector

$$\sigma^i(u,\boldsymbol{w}) = \rho^*(u,\boldsymbol{w})\nu^i(u,\boldsymbol{w}) \tag{127}$$

is the matter's current density. All integrals in (122)–(125) are performed over hypersurface $\mathcal{H}_u$ of a constant coordinate time u.

The *conformal* mass multipoles of the body B are defined as follows [17,87]:

$$\mathcal{I}^L = \int_{\mathcal{V}_B} \varrho(u,\boldsymbol{w})\left[1-(1-\gamma)\mathcal{P} - \sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{Q}_K w^{\langle K\rangle}\right]w^{\langle L\rangle}d^3w + \frac{1}{(2l+3)}\left[\frac{1}{2}\ddot{\mathcal{N}}^{\langle L\rangle} - 4\frac{2l+1}{l+1}\dot{\mathcal{R}}^{\langle L\rangle}\right], \tag{128}$$

where, again, the integration is performed over a hypersurface $\mathcal{H}_u$ of constant coordinate time u, and

$$\varrho = \rho^*(u,\boldsymbol{w})\left[1+\frac{3}{2}\nu^2(u,\boldsymbol{w}) + \Pi(u,\boldsymbol{w}) - \hat{U}_B(u,\boldsymbol{w})\right] + \mathfrak{s}^{kk}(u,\boldsymbol{w}) \tag{129}$$

is the conformal mass density of matter which does not depend on the PPN parameters β and γ as contrasted to the definition (126) of the *active* mass density.

There is one more type of the multipoles called *scalar* multipoles, $\tilde{I}^L$. However, they are not independent and relate to the *active* and *conformal* multipoles by a simple formula [87]

$$\tilde{I}^L = 2\mathcal{M}^{\langle L\rangle} - (1+\gamma)\mathcal{I}^{\langle L\rangle}. \tag{130}$$

In addition to the gravitational mass multipoles, $\mathcal{M}^L$ and $\mathcal{I}^L$, there is a set of internal spin multipoles. In the Newtonian approximation they are defined by expression [87]

$$\mathcal{S}^L = \int_{\mathcal{V}_B} \varepsilon^{pq\langle i_l} w^{i_{l-1}\dots i_1\rangle p}\sigma^q(u,\boldsymbol{w})d^3w, \tag{131}$$

where the matter's current density σ^q has been defined in (127). All multipoles of body B are functions of time u only. They are the STF Cartesian tensors in the tangent Euclidean space attached to the worldline $\mathcal{W}$ of the origin of local coordinates adapted to body B. Definition (131) is sufficient for deriving the post-Newtonian translational equations of motion of the extended bodies in an $\mathbb{N}$-body system. However, derivation of the post-Newtonian rotational equations of motion requires a post-Newtonian definition of the body's angular momentum (spin). We shall discuss it later in Sec. VI D.

V. MATCHED ASYMPTOTIC EXPANSIONS AND COORDINATE TRANSFORMATIONS

A. Basic principles

Post-Newtonian transformations between the global and local coordinate charts are derived by the method of *matched asymptotic expansions* [231]. It involves finding several different approximate solutions of the field equation, each of which is valid for a specific domain of space, and then combining these different solutions together in a buffer domain where all different solutions overlap, in order to obtain a single approximate solution. The technique of matched asymptotic expansions in general relativity was first implemented by Demiański and Grishchuk [232] for deriving equations of motion of black holes in the Newtonian limit. D'Eath [56,223] significantly extended this technique to the next approximations of general relativity and it is now commonly used for derivation of equations of motion of black holes [57,213,214]. Matching asymptotic expansions are indispensable in case of the singular perturbations of the field equations but the method turned out to be very effective also for derivation of equations of motion of extended bodies [69,73–75,156] and for constructing a post-Newtonian theory of reference frames in the Solar System [17,72,83,159].

In the present paper the independent dynamic field variables are the scalar field and metric tensor which describe the asymptotic solutions of the field equations in the form of the post-Newtonian expansions which are valid in the spatial domains covered by the global or local coordinates. These solutions describe one and the same value of the dynamic variables in any type of coordinates which means that the solutions can be spliced in the spatial region where the coordinate charts overlap. The splicing relies upon the tensor transformation law applied to the post-Newtonian expansions of the metric tensor and scalar field. The post-Newtonian transition functions entering the transformation establish the correspondence between the global and local coordinates. Coordinate distance from the origin of the local coordinates to the first singular points of the Jacobian of the transformation determines the domain of applicability of the local coordinates [17].

The matching procedure is organized as follows. We use conformal harmonic coordinates defined by the Nutku gauge condition (40). Transition functions of the post-Newtonian coordinate transformation are constrained by this condition and must obey differential equation (44) describing the residual gauge freedom. Solutions of this homogeneous equation are to be continuously differentiable functions that are regular at the origin of the local coordinates. These functions can be represented in the form of a Taylor series of the harmonic polynomials of the spatial local coordinates. Coefficients of the Taylor series are the STF Cartesian tensors defined on the worldline $\mathcal{W}$ of the origin of the local coordinates. The transition functions are to be substituted to the matching equations describing the splicing of the internal and external solutions of the field equations in the global and local coordinates. Matching the asymptotic post-Newtonian expansions of the scalar field and the metric tensor allows us to fix all degrees of the residual gauge freedom in the final form of the post-Newtonian coordinate transformation and to determine a functional form of all external multipoles except for the external dipole $\mathcal{Q}_i$ which is not constrained by the matching conditions and must be found separately from the equations of motion of the center of mass of body B in the body-adapted local coordinates.

Physically, the post-Newtonian transformation between coordinate times, t and u, describes the Lorentz (velocity-dependent) and Einstein (gravitational-field-dependent) time dilation associated with the different simultaneity of events in the two coordinate charts [69,156]. It also includes an infinite series of the polynomial terms [72,233]. The post-Newtonian transformation between the spatial coordinates, x^i and w^i, is a quadratic function of spatial coordinates. The linear part of the transformation includes the Lorentz and Einstein contractions of length as well as a matrix of rotation describing the post-Newtonian precession of the spatial axes of the local coordinates with respect to the global coordinates due to the translational and rotational motion of the bodies [154,183]. The Lorentz length contraction takes into account the kinematic aspects of the post-Newtonian transformation and depends on the relative velocity of motion of the local coordinates with respect to the global coordinates. The Einstein (gravitational) length contraction accounts for static effects of the scalar field and the metric tensor [17,87]. The quadratic part of the spatial transformation depends on the orbital acceleration of the local coordinates and accounts for the effects of the affine connection (the Christoffel symbols) of the spacetime manifold.

Let us now discuss the mathematical structure of the post-Newtonian transformation between the local coordinates, $w^\alpha = (w^0, w^i) = (u, \boldsymbol{w})$, and the global coordinates, $x^\alpha = (x^0, x^i) = (t, \boldsymbol{x})$ in more detail. This coordinate transformation must be compatible with the weak-field and slow-motion approximation used in the post-Newtonian expansions. Hence, the coordinate transformation is given as a post-Newtonian expansion:

$$u = t + \xi^0(t, \boldsymbol{x}), \tag{132}$$

$$w^i = R^i_{\rm B} + \xi^i(t, \boldsymbol{x}), \tag{133}$$

where ξ^0 and ξ^i are the post-Newtonian corrections to the Galilean transformation, $u = t$, $R^i_{\rm B} \equiv x^i - x^i_B(t)$, and $x^i_{\rm B}(t)$ is a spatial position of the origin of the local coordinates in the global coordinates. We denote velocity and acceleration of the origin of the local coordinates as $v^i_{\rm B} \equiv \dot{x}^i_{\rm B}$ and $a^i_{\rm B} \equiv \ddot{x}^i_{\rm B}$ respectively, where a dot above a function denotes a derivative with respect to time t. At this step, we do not know yet equations for worldline $\mathcal{W}$ of the origin of the local coordinates adapted to body B nor for worldline $\mathcal{Z}$ of the body's center of mass. Therefore, it is natural to assume that originally the two worldlines, $\mathcal{W}$ and $\mathcal{Z}$, are different. Later on, we shall show that the two worldlines can be made identical by demanding the conservation of the linear momentum of body B. It can be always achieved by choosing the external dipole $\mathcal{Q}_i$ to compensate the noninertial acceleration of the body's center of mass caused by tidal forces [17,69,87]. The presence of nonvanishing dipole $\mathcal{Q}_i$ in the local metric (117) makes the local coordinates adapted to body B to be noninertial.

It is instructive to notice that the local coordinates used by Thorne and Hartle [58] are inertial that is the origin of the Thorne-Hartle local coordinates moves along a geodesic worldline of the *effective* spacetime manifold $\bar{M}$ with metric, $\bar{g}_{\alpha\beta} = \eta + \bar{h}_{\alpha\beta}$, which is obtained from the original spacetime manifold M with metric, $g_{\alpha\beta} = \eta + h_{\alpha\beta}$, by deleting from $h_{\alpha\beta}$ the internal part of the metric $h^{\rm int}_{\alpha\beta}$. In such local inertial coordinates the external dipole $\mathcal{Q}_i \equiv 0$ but the center of mass of body B does not move along the geodesic in the most general case due to the tidal interaction of the internal multipoles $\mathcal{M}_L$ and $\mathcal{S}_L$ of the body with an external gravitational field of other bodies.

The asymptotic matching equations for independent dynamic variables—the scalar field φ and the metric tensor $g_{\mu\nu}$—are given by the laws of coordinate transformations of these geometric objects [157]

$$\varphi(t, \boldsymbol{x}) = \hat{\varphi}(u, \boldsymbol{w}), \tag{134}$$

$$g_{\mu\nu}(t, \boldsymbol{x}) = \hat{g}_{\alpha\beta}(u, \boldsymbol{w}) \frac{\partial w^\alpha}{\partial x^\mu} \frac{\partial w^\beta}{\partial x^\nu}. \tag{135}$$

Equations (134) and (135) are valid in the spacetime region that is covered simultaneously by the local and global coordinates. Functions on the left-hand side of these equations are known and given in Sec. IV A 3 as integrals from the body's matter variables (density, pressure, etc.) performed over volumes of all bodies of the $\mathbb{N}$-body system on hypersurface $\mathcal{H}_t$ of constant time t. The right-hand side of the matching equations contains, besides the known integrals from the matter variables of body B taken on hypersurface $\mathcal{H}_u$ of constant time u, yet unknown external multipoles, P_L, $\mathcal{Q}_L$, $\mathcal{C}_L$ of the external part of the metric tensor in the local coordinates and the transition functions $\xi^\alpha = (\xi^0, \xi^i)$ from the coordinate transformations (132) and (133). We prove below that both the external multipoles and the transition functions can be determined by solving matching Eqs. (134) and (135) that also yield equations of motion of the origin of the local coordinates, $x^i_{\rm B} = x^i_{\rm B}(t)$. Matching the post-Newtonian expansions of the metric tensor and scalar field does not yield equations of motion of the center of mass of body B. An additional procedure of integration of the microscopic equations of motion of matter of body B is required for this purpose to determine the motion of the center of mass of body B with respect to the origin of the local coordinates and to derive rotational equations of motion of the body's spin. It is explained in Sec. VI.

B. Transition functions

A comprehensive description of the matching procedure establishing the correspondence between the global and local coordinates in the $\mathbb{N}$-body problem is given in [17,87,159]. Here, we summarize the main results of the matching.

Solving matching Eqs. (134) and (135) begins from the $\hat{g}_{0i}$ component of the metric tensor perturbation in the local coordinates adapted to body B. This component does not contain 0.5 post-Newtonian term of the order of $\mathcal{O}(\epsilon)$ because we have chosen the spatial axes of the local coordinates dynamically nonrotating and orthogonal to worldline $\mathcal{W}$ of its origin at any instant of time. It eliminates the angular and linear velocity terms of the order of $\mathcal{O}(\epsilon)$ in $\hat{g}_{0i}$ and implies that function $\xi^0(t, \boldsymbol{x})$ in (132) satisfies the following constraint [74,87]:

$$\partial_i \xi^0(t, \boldsymbol{x}) = -v^i_{\rm B} + \partial_i \kappa(t, \boldsymbol{x}), \tag{136}$$

where $\kappa(t, \boldsymbol{x})$ is the post-Newtonian, yet unknown correction of the order of $\mathcal{O}(\epsilon^2)$. Integration of the partial differential equation (136) yields

$$\xi^0(t, \boldsymbol{x}) = \mathcal{A}(t) - v^k_{\rm B} R^k_{\rm B} + \kappa(t, \boldsymbol{x}), \tag{137}$$

where $\mathcal{A}(t)$ is a constant of integration depending on time.

At second step we use differential equation (44) in order to find out the transition functions κ from (137) and ξ^i

from (133). We replace (137) to (132) and substitute it along with w^i from (133) in Eq. (44) which yields two decoupled inhomogeneous Poisson equations for the post-Newtonian components of the transition functions,

$$\triangle\kappa(t,\boldsymbol{x}) = 3v^k_{\rm B}a^k_{\rm B} + \ddot{\mathcal{A}} - \dot{a}^k_{\rm B}R^k_{\rm B}, \tag{138}$$

$$\triangle\xi^i(t,\boldsymbol{x}) = -a^i_{\rm B}, \tag{139}$$

where $\triangle \equiv \delta^{ij}\partial_i\partial_j$ is the Laplace operator in the Euclidean space. A general solution of these elliptic-type equations must be regular at the origin of the local coordinates adapted to body B and consists of two parts—a fundamental solution of the homogeneous Laplace equation and a particular solution of the inhomogeneous Poisson equation [74,87]

$$\kappa = \left(\frac{1}{2}v^k_{\rm B}a^k_{\rm B} - \frac{1}{6}\ddot{\mathcal{A}}\right)R^2_{\rm B} - \frac{1}{10}\dot{a}^k_{\rm B}R^k_{\rm B}R^2_{\rm B} + \Xi(t,\boldsymbol{x}), \tag{140}$$

$$\xi^i = -\frac{1}{6}a^i_{\rm B}R^2_{\rm B} + \Xi^i(t,\boldsymbol{x}). \tag{141}$$

Here, functions Ξ and Ξ^i are the fundamental solutions of the homogeneous Laplace equation—the harmonic polynomials with respect to the local spatial coordinates expressed in terms of the global coordinates, $w^i = R^i_{\rm B} + \mathcal{O}(\epsilon^2)$,

$$\Xi(t,\boldsymbol{x}) = \sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{B}^L R^{\langle L\rangle}_{\rm B}, \tag{142}$$

$$\Xi^i(t,\boldsymbol{x}) = \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{D}^{iL}R^{\langle L\rangle}_{\rm B} + \sum_{l=0}^{\infty}\frac{\varepsilon_{ipq}}{(l+1)!}\mathcal{F}^{pL}R^{\langle qL\rangle}_{\rm B} + \sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{E}^L R^{\langle iL\rangle}_{\rm B}, \tag{143}$$

where the coefficients, $\mathcal{B}^L$, $\mathcal{D}^L$, $\mathcal{F}^L$ and $\mathcal{E}^L$ of the expansions are STF Cartesian tensors which should not be confused with the external multipoles entering the local metric tensor. These coefficients are defined on the worldline $\mathcal{W}$ of the origin of the local coordinates and depend only on time t of the global coordinates. An explicit form of coefficients $\mathcal{B}^L, \mathcal{D}^L, \mathcal{F}^L$ is derived by substituting transitions functions $w^\alpha = (u, w^i)$ in the form of (132), (133), (137), (140)–(143) to matching Eqs. (134)–(135) and solving them. This solution also determines the external multipoles and the equations of motion for the origin of the local coordinates—worldline $\mathcal{W}$. The overall procedure of solving the matching equations is rather long and technical and we do not describe it here. The reader can find its comprehensive description in papers [87,234] and in the book [[17], Chapter 5]. The matching solution is given in Sec. V C below.

C. Matching solution

1. Post-Newtonian coordinate transformation

Parametrized post-Newtonian transformation between the local coordinates w^α adapted to body B and the global coordinates x^α is given by two equations [17,69],

$$u = t + \frac{1}{c^2}(\mathcal{A} - v^k_{\rm B}R^k_{\rm B}) + \frac{1}{c^4}\left[\mathcal{B} + \left(\frac{1}{3}v^k_{\rm B}a^k_{\rm B} - \frac{1}{6}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{10}\dot{a}^k_{\rm B}R^k_{\rm B}\right)R^2_{\rm B} + \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{B}^L R^L_{\rm B}\right] + \mathcal{O}(c^{-6}), \tag{144}$$

$$w^i = R^i_{\rm B} + \frac{1}{c^2}\left[\left(\frac{1}{2}v^i_{\rm B}v^k_{\rm B} + \delta^{ik}\gamma\bar{U}(t,\boldsymbol{x}_{\rm B}) + F^{ik}_{\rm B}\right)R^k_{\rm B} + a^k_{\rm B}R^i_{\rm B}R^k_{\rm B} - \frac{1}{2}a^i_{\rm B}R^2_{\rm B}\right] + \mathcal{O}(c^{-4}), \tag{145}$$

where $R^i_{\rm B} = x^i - x^i_{\rm B}$ is the coordinate distance on the hypersurface $\mathcal{H}_t$ of constant time t between the point of matching, x^i, and the origin of the local coordinates, $x^i_{\rm B} = x^i_{\rm B}(t)$, and we have shown in these equations the fundamental speed c explicitly to attenuate the post-Newtonian order of different terms.

Functions $\mathcal{A}$ and $\mathcal{B}$ depend on the global coordinate time t and define transformation between the local time u and the global coordinate time t at the origin of the local coordinates. They obey the ordinary differential equations,

$$\frac{d\mathcal{A}}{dt} = -\frac{1}{2}v^2_{\rm B} - \bar{U}(t,\boldsymbol{x}_{\rm B}), \tag{146}$$

$$\frac{d\mathcal{B}}{dt} = -\frac{1}{8}v^4_{\rm B} - \left(\gamma + \frac{1}{2}\right)v^2_{\rm B}\bar{U}(t,\boldsymbol{x}_{\rm B}) + \frac{1}{2}\bar{U}^2(t,\boldsymbol{x}_{\rm B}) + 2(1+\gamma)v^k_{\rm B}\bar{U}^k(t,\boldsymbol{x}_{\rm B}) - \bar{\Psi}(t,\boldsymbol{x}_{\rm B}) + \frac{1}{2}\partial_{tt}\bar{\chi}(t,\boldsymbol{x}_{\rm B}) \tag{147}$$

that describe the post-Newtonian transformation between time u of the local coordinates and time t of the global coordinates. The other functions entering (144) and (145) are defined by algebraic relations

$$\mathcal{B}^i = 2(1+\gamma)\bar{U}^i(t,\boldsymbol{x}_{\rm B}) - (1+2\gamma)v^i_{\rm B}\bar{U}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{2}v^i_{\rm B}v^2_{\rm B}, \tag{148}$$

$$\mathcal{B}^{ij} = 2(1+\gamma)\partial^{\langle i}\bar{U}^{j\rangle}(t,\boldsymbol{x}_{\rm B}) - 2(1+\gamma)v^{\langle i}_{\rm B}\partial^{j\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) + 2a^{\langle i}_{\rm B}a^{j\rangle}_{\rm B}, \tag{149}$$

$$\mathcal{B}^{iL} = 2(1+\gamma)\partial^{\langle L}\bar{U}^{i\rangle}(t,\boldsymbol{x}_{\rm B}) - 2(1+\gamma)v^{\langle i}_{\rm B}\partial^{L\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) \quad (l \geq 2), \tag{150}$$

where the angular brackets denote STF projection of indices, and the external (with respect to body B) potentials $\bar{U}$, $\bar{U}^i$, $\bar{\Psi}$, $\bar{\chi}$ are defined in (68) and (91). Notations $\bar{U}(t, \boldsymbol{x}_{\rm B})$, $\bar{U}^i(t, \boldsymbol{x}_{\rm B})$, $\bar{\Psi}(t, \boldsymbol{x}_{\rm B})$, and $\bar{\chi}(t, \boldsymbol{x}_{\rm B})$ mean that the potentials are taken at the origin of the local coordinates adapted to body B at instant of time t.

The skew-symmetric rotational matrix $F^{ij}_{\rm B}$ is a solution of the ordinary differential equation

$$\frac{dF^{ij}_{\rm B}}{dt} = 2(1+\gamma)\partial^{[i}\bar{U}^{j]}(t, \boldsymbol{x}_{\rm B}) + (1+2\gamma)v^{[i}_{\rm B}\partial^{j]}\bar{U}(t, \boldsymbol{x}_{\rm B}) + v^{[i}_{\rm B}\mathcal{Q}^{j]}, \qquad (151)$$

describing the rate of the kinematic rotation of the spatial axes of the local coordinates adapted to body B with respect to the global coordinates [69,87]. Equation (151) has been derived here for arbitrarily structured bodies by the method of matched asymptotic expansions. The same equation was obtained independently for spinning test particle (gyroscope) through the Fermi-Walker transport of spin [[165], § 40.7]. The first term on the right-hand side of (151) describes the Lense-Thirring (gravitomagnetic) precession which is also called the dragging of inertial frames [101,165]. The second term on the right-hand side of (151) describes the de-Sitter (geodetic) precession, and the third term describes the Thomas precession depending on the local (nongeodesic) acceleration $\mathcal{Q}^i = \delta^{ij}\mathcal{Q}_j$ of the origin of the local coordinates with respect to a geodesic worldline of a freely falling test particle. In the scalar-tensor theory both the Lense-Thirring and de-Sitter precession depend on the PPN parameter γ while the Thomas precession does not. The reason is that the Thomas precession is generically a special relativistic effect [235] that cannot depend on a particular version of an alternative theory of gravity.

The Lense-Thirring and geodetic precession have been recently measured in Gravity Probe B gyroscope experiment [236] and by the satellite laser ranging technique [237,238]. Relativistic precession is an attractive mechanism for theoretical explanation of quasiperiodic oscillations (QPO) in the optical power density spectra of accreting black holes [239]. It is also important to include relativistic precession of spins of stars in merging compact binaries for adequate prediction and analysis of gravitational waveforms emitted by the binaries [240–243].

2. Body's self-action force and bootstrap effect

Self-action force is a key concept in gravitational dynamics of extended bodies both in the Newtonian and relativistic gravity theories [141,142,244]. It is defined as the net action of the gravitational field generated by a single body from an ℕ-body system on the body itself. The self-action force includes a conservative part and dissipative terms which are known as the gravitational radiation-reaction force [245–247]. The self-action of the gravitational radiation appears for the first time at 1.5 PN approximation in scalar-tensor theory of gravity due to the emission of dipolar scalar field radiation [88,248] and at 2.5 PN approximation in general relativity [184,220,249–251] due to the emission of quadrupole gravitational waves by the moving bodies [42,165]. Calculation of the radiation-reaction force beyond 2.5 post-Newtonian approximation is a challenging theoretical task [47,245–247] whose solution is of paramount importance for correct prediction of inspiral motion of compact binaries, especially in the extreme mass ratio limit [252,253].

Chicone *et al.* [254] studied the origin of the self-action force by means of the mathematical theory of delay equations which include the field-retardation effects, and predicted that all of them must have runaway modes. It was shown that when retardation effects are small, the physically significant solutions belong to the so-called *slow manifold* of the dynamic system which is identified with the attractor in the state space of the delay equation. It was also demonstrated via an example that when retardation effects are no longer small, the motion of the system exhibits bifurcation phenomena that are not contained in the local equations of motion. The bifurcation behavior of the solutions of the delay equations pointed out by Chicone *et al.* [254] is absent in the conservative post-Newtonian approximations but has to be studied more attentively by analysts computing the gravitational waveforms of inspiral binary systems.

Radiation-reaction force does not prevent a sufficiently compact and nonspinning body from moving on a geodesic in a particularly chosen, regular effective external metric if a singular part of the full metric is properly removed by regularization [255]. Thus, the regular part of radiation-reaction force does not violate the Einstein principle of equivalence [256]. The singular part of the metric corresponds to the conservative part of the self-action force which apparently must obey the third Newton's law to get a vanishing net internal force, thus, preventing self-accelerated runaway motion of the body which we call a *bootstrap* effect. Bootstrapping can happen only in some nonconservative (nonviable) alternative theories of gravity [88]. It does not occur in the first post-Newtonian approximation of scalar-tensor theory for arbitrarily structured bodies as one can see from matching Eqs. (134) and (135) where all the terms depending on a body's internal gravitational potentials mutually cancel out. The bootstrap effect is also absent in the second post-Newtonian approximation both in general relativity [183,186] and in scalar-tensor theory of gravity [248].

3. Worldline of the origin of the local coordinates

The origin of the local coordinates adapted to body B moves in spacetime along worldline $\mathcal{W}$. Matching Eq. (135) for the metric tensors in the local and global coordinates yields equations of translational motion of the origin of the local coordinates, $x^i_{\rm B} = x^i_{\rm B}(t)$, with respect to the global coordinates. It reads [[17], Eq. 5.88]

$$
\begin{aligned}
a^i_{\rm B} = {} & \partial^i \bar{U}(t,\boldsymbol{x}_{\rm B}) - \mathcal{Q}^i + F^{ij}_{\rm B}\mathcal{Q}_j + \partial^i\bar{\Psi}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{2}\partial_{tt}\partial^i\bar{\chi}(t,\boldsymbol{x}_{\rm B}) \\
& + 2(1+\gamma)\dot{\bar{U}}^i(t,\boldsymbol{x}_{\rm B}) - 2(1+\gamma)v^j_{\rm B}\partial^i\bar{U}^j(t,\boldsymbol{x}_{\rm B}) \\
& - (1+2\gamma)v^i_{\rm B}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B}) + (2-2\beta-\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})\partial^i\bar{U}(t,\boldsymbol{x}_{\rm B}) \\
& + (1+\gamma)v^2_{\rm B}\partial^i\bar{U}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{2}v^i_{\rm B}v^j_{\rm B}\partial^j\bar{U}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{2}v^i_{\rm B}v^j_{\rm B}a^j_{\rm B} \\
& - v^2_{\rm B}a^i_{\rm B} - (2+\gamma)a^i_{\rm B}\bar{U}(t,\boldsymbol{x}_{\rm B}), \qquad (152)
\end{aligned}
$$

where a dot above a function denotes a total derivative with respect to time t, $v^i_{\rm B} \equiv \dot{x}^i_{\rm B}$ and $a^i_{\rm B} \equiv \ddot{x}^i_{\rm B}$ are velocity and acceleration of the origin of the local coordinates relative to the global coordinates, and $\mathcal{Q}^i = \delta^{ij}\mathcal{Q}_j$ is a dipole term ($l = 1$) in the external solution for $\hat{h}^{\rm ext}_{00}$ component of the metric tensor perturbation (114) which describes a local acceleration of the worldline $\mathcal{W}$.

The right-hand side of (152) is a gravitational force per unit mass causing the coordinate acceleration $a^i_{\rm B}$ of the origin of the local coordinates of body B with respect to the global coordinates. The force is explicitly expressed in terms of the external gravitational potentials, $\bar{U}$, $\bar{U}^i$, $\bar{\Psi}$, $\bar{\chi}$, and their time and/or spatial derivatives. It also depends on the external dipole, $\mathcal{Q}^i = \delta^{ij}\mathcal{Q}_j$, which represents a local acceleration of worldline $\mathcal{W}$ with respect to a timelike geodesic on the effective spacetime manifold $\bar{M}$ which is explained in more detail in Sec. XI B. Function $\mathcal{Q}^i$ does not depend on the choice of gauge condition and constitutes a part of the definition of the state of motion of the origin of the local coordinates [257]. Only after specification of $\mathcal{Q}^i$ as a function of time, formula (152) becomes an ordinary differential equation whose solution yields worldline $\mathcal{W}$ of the origin of the local coordinates as a known function of time $x^i_{\rm B}(t)$.

A trivial choice of the local acceleration, $\mathcal{Q}^i = 0$, looks attractive as it immediately converts (152) to a fully determined differential equation. It is this choice that has been made, for example, by Dixon [11] and Thorne and Hartle [58] which means that worldline $\mathcal{W}$ of the origin of the local coordinates is a geodesic of the effective background manifold $\bar{M}$. However, this choice does not allow us to keep the origin of the local coordinates always at the center of mass of body B if the body has nonvanishing internal multipoles $\mathcal{M}^L$ and $\mathcal{S}^L$ which interact with the tidal field multipoles $\mathcal{Q}_L$ and $\mathcal{C}_L$ of the external bodies C ≠ B from the $\mathbb{N}$-body system. The interaction exerts a force on the body B and makes its center of mass moving along a nongeodesic worldline having $\mathcal{Q}_i \neq 0$ [58,69]. Thus, worldline $\mathcal{Z}$ of the center of mass of body B is not geodesic in the most general case. If we want to retain the center of mass of body B at the origin of the body-adapted local coordinates at any instant of time, the acceleration $\mathcal{Q}^i$ must obey the equations of motion of the body's center of mass with respect to the local coordinates. Derivation of this equation cannot be achieved by the method of matched asymptotic expansions and requires either integration of microscopic equations of matter over the volume of body B in the local coordinates [74,75,87] or finding asymptotes of the surface integrals in the buffer region of overlapping the local and global coordinates [30,58]. We deal with a regular distribution of matter inside the extended bodies and apply the technique of integration of the microscopic equations of motion to find the local acceleration $\mathcal{Q}_i$ in Sec. VI E.

4. Body-frame external multipoles

Scalar-field multipoles.—Matching determines the external (with respect to body B) tidal multipoles in terms of the partial derivatives from the gravitational potentials of external bodies [17,87]. The external scalar field multipoles are obtained by solving (134) and read

$$\mathcal{P}_L = \partial_L\bar{\varphi}(t,\boldsymbol{x}_{\rm B}), \quad (l \geq 0) \qquad (153)$$

where the external scalar field $\bar{\varphi}$ is expressed in terms of the external Newtonian potential $\bar{U}$

$$\bar{\varphi}(t,\boldsymbol{x}) = \bar{U}(t,\boldsymbol{x}). \qquad (154)$$

We remind the reader that the scalar field perturbation φ is coupled either with the factor $\gamma - 1$ or $\beta - 1$, so that all physical effects of the scalar field are proportional to these factors and can be easily identified in the equations that follow. It should be noticed that the external scalar field monopole $\mathcal{P}$ ($l = 0$) and dipole $\mathcal{P}_i$ ($l = 1$) cannot be removed from observable gravitational effects by rendering a coordinate transformation to a freely falling frame because the scalar field is a true scalar. In other words, the gradient of scalar field is not equivalent to the inertial force caused by acceleration as it cannot be eliminated by changing the state of motion of observer. It was the primary reason why Einstein abandoned a pure scalar field theory of gravity in favor of general relativity where the gravitational field is identified with the components of the metric tensor, and, unlike a scalar field, can be removed by transformation to the local inertial frame.

Rather remarkable, this difference in transformation properties between scalar field and metric tensor has no direct consequence for equivalence between inertial and gravitational masses of test bodies. It was discovered [258] that the inertial and gravitational masses of massive test bodies remain equal in a wide class of scalar-tensor theories of gravity and the freely falling test bodies move in the same way independently of their mass. This observation forces us to carefully discriminate between various formulations of the weak equivalence principle (WEP) in scalar-tensor theories.

Gravitoelectric multipoles.—External gravitoelectric multipoles $\mathcal{Q}_L \equiv \mathcal{Q}_{\langle i_1 i_2 \ldots i_l\rangle}$ ($l \geq 2$) are obtained by solving (135) and given by the following equation [[17], Eq. 5.89][6]:

[6]Be mindful that the spatial indices are raised and lowered with the Kronecker symbol δ^{ij} so that the position of the spatial indices does not matter.

$$\mathcal{Q}^L = \partial^{\langle L\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) + \partial^{\langle L\rangle}\bar{\Psi}(t,\boldsymbol{x}_{\rm B}) - \frac{1}{2}\partial_{tt}\partial^{\langle L\rangle}\bar{\chi}(t,\boldsymbol{x}_{\rm B}) + 2(1+\gamma)\partial^{\langle L-1}\dot{\bar{U}}^{i_l\rangle}(t,\boldsymbol{x}_{\rm B}) - 2(1+\gamma)v^j_{\rm B}\partial^{\langle L\rangle}\bar{U}^j(t,\boldsymbol{x}_{\rm B})$$
$$+ (l-2\gamma-2)v^{\langle i_l}_{\rm B}\partial^{L-1\rangle}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B}) + (1+\gamma)v^2_{\rm B}\partial^{\langle L\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) - \frac{l}{2}v^j_{\rm B}v^{\langle i_l}_{\rm B}\partial^{L-1\rangle j}\bar{U}(t,\boldsymbol{x}_{\rm B}) + (2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})\partial^{\langle L\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})$$
$$- (l^2-l+2\gamma+2)a^{\langle i_l}_{\rm B}\partial^{L-1\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) - lF^{j\langle i_l}_{\rm B}\partial^{L-1\rangle}\bar{U}^j(t,\boldsymbol{x}_{\rm B}) + X^L, \quad (l\geq 2) \tag{155}$$

where X^L represents a contribution of the local inertial forces to the gravitoelectric multipole,

$$X^L \equiv \begin{cases} 3a^{\langle i_1}_{\rm B}a^{i_2\rangle}_{\rm B} & \text{if } l=2; \\ 0 & \text{if } l\geq 3. \end{cases} \tag{156}$$

We point out that in spite of the fact that the term X^L appears in the expression (155) for the external multipoles, $\mathcal{Q}^L$, it is not a part of the curvature of spacetime manifold [71,87] and is exclusively associated with the local acceleration of worldline $\mathcal{W}$ of the origin of the body-adapted local coordinates. This is proved in Sec. XI D 2.

Gravitomagnetic multipoles.—External gravitomagnetic multipoles $\mathcal{C}_L \equiv \mathcal{C}_{\langle i_1 i_2\ldots i_l\rangle}$ for $l\geq 2$ are also obtained by solving (135) and given by Xie and Kopeikin [[234], Eq. 5.37][7]

$$\varepsilon_{ipk}\mathcal{C}_{pL} = 4(1+\gamma)\left[v^{[i}_{\rm B}\partial^{k]\langle L\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B}) + \partial^{\langle L\rangle[i}\bar{U}^{k]}(t,\boldsymbol{x}_{\rm B}) - \frac{l}{l+1}\delta^{\langle i_l[i}\partial^{k]L-1\rangle}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B})\right], \quad (l\geq 1) \tag{157}$$

where the dot denotes the time derivative with respect to time t, the angular brackets denote STF symmetry with respect to multi-index $L = i_1, i_2, \ldots, i_l$, and the square brackets denote antisymmetrization: $T^{[ij]} = (T^{ij} - T^{ji})/2$. The external multipoles $\mathcal{Q}_L$ and $\mathcal{C}_L$ are analogs of Dixon's multipoles $A_{\alpha_1\ldots\alpha_l\mu\nu}$ and $B_{\alpha_1\ldots\alpha_l\mu\nu}$ respectively; see (463) and (464) below. We shall use the above-given expressions for the external multipoles in derivation of the equations of motion of extended bodies in the next section.

VI. POST-NEWTONIAN EQUATIONS OF MOTION OF AN EXTENDED BODY IN THE LOCAL COORDINATES

Coordinate acceleration $a^i_{\rm B}$ of worldline $\mathcal{W}$ of the origin of the local coordinates adapted to body B with respect to the global coordinates is given by Eq. (152). It depends on the local acceleration $\mathcal{Q}_i$ of the origin of the local coordinates with respect to a timelike geodesic of the effective background metric $\bar{g}_{\alpha\beta}$. The acceleration $\mathcal{Q}_i$ cannot be determined by solving the matching Eqs. (134) and (135), and remains an arbitrary function of time. The center of mass of body B has not yet been defined but it certainly moves along worldline $\mathcal{Z}$ which is formally different from $\mathcal{W}$ in the most general case. However, we have enough freedom in choosing worldline $\mathcal{W}$ which we can use in order to make the two worldlines coincide. Mathematically, it means that the center of mass of body B remains at rest at the origin of the local coordinates adapted to body B as the body moves on a spacetime manifold. This condition imposes a functional constraint on the local acceleration $\mathcal{Q}_i$ which converts the translational equations of motion (152) of the origin of the local coordinates to those for the center of mass of body B with respect to the global coordinates. In order to put the center of mass of body B to the origin of the local coordinates and to hold it in there, we have to know the translational equations of motion of the body's center of mass in the local coordinates adapted to the body.

Derivation of translational equations of motion of the center of mass of body B in the local coordinates can be executed in three different ways, which are the following:

(1) the Fock-Papapetrou method of integration of microscopic equations of motion of matter over the body's volume [6,126,134,209,259];
(2) the Mathisson-Dixon method of integration of skeleton of the stress-energy tensor of matter of body B given in terms of distributions [4,5,11] and amended with some regularization technique [47,48,184];
(3) the Einstein-Infeld-Hoffmann (EIH) method of asymptotic surface integrals [30,49,58,84,85].

The Mathisson-Dixon and EIH methods consider the extended bodies in an $\mathbb{N}$-body system as singularities of a gravitational field endowed with a set of the internal multipoles which represent the internal structure of the bodies. The multipoles in these approaches are not given in terms of volume integrals from a smooth distribution of matter inside the bodies but are merely functions of time given on worldline $\mathcal{Z}$ of each body's center of mass. On the other hand, the Fock-Papapetrou method operates with a continuous distribution of matter inside the bodies and defines the internal multipoles of the bodies in terms of the volume integrals like in Sec. IV B 6 of the present paper. It is assumed that the Mathisson-Dixon and EIH methods should give the same equations of motion for extended, arbitrarily structured bodies as in the Fock-Papapetrou method. This is indeed true in case of pole-dipole particle approximation corresponding to rigidly rotating, spherically symmetric

[7]Formula (157) corrects a typo in [[17], Eq. 5.74] for the external gravitomagnetic multipole $\mathcal{C}_L$.

bodies. However, this correspondence has been never checked for higher-order internal multipoles. We use the Fock-Papapetrou method of derivation of translational equations of motion of extended bodies having all mass and spin internal multipoles, and compare them with similar equations derived by Racine and Flanagan [84] and Racine *et al.* [85] with the EIH technique [see Appendix B and with the covariant equations derived by Dixon [11] (see Appendix D)].

In this section we define a center of mass and a linear momentum of body B, derive the post-Newtonian microscopic equations of motion of matter of the body in the local coordinates and, then, integrate them over the body's volume in order to get the post-Newtonian equations of motion of the linear momentum and the center of mass of the body. As soon as the equations of motion for these quantities are established, the local acceleration $\mathcal{Q}_i$ is determined from the condition of vanishing of the linear momentum and the integral of the center of mass of the body which warrants that the center of mass of body B is always at the origin of the local coordinates. At the end of this section we give a post-Newtonian definition of the intrinsic angular momentum (spin) of body B and derive the spin's rotational equations of motion in the local coordinates.

A. Microscopic equations of motion of matter

The microscopic post-Newtonian equations of motion of matter of body B include the following:

(1) equation of continuity,

(2) thermodynamic equation relating the elastic energy, $\Pi = \Pi(u, \boldsymbol{w})$, to the stress tensor, $\mathfrak{s}_{\alpha\beta} = \mathfrak{s}_{\alpha\beta}(u, \boldsymbol{w})$,

(3) equation of conservation of the stress-energy tensor.

The equation of continuity of matter of body B in the body-adapted local coordinates $w^\alpha = (u, \boldsymbol{w})$ has the most simple form if we use the invariant density $\rho^* = \rho^*(u, \boldsymbol{w})$, defined in (35). It reads

$$\frac{\partial \rho^*}{\partial u} + \frac{\partial(\rho^* \nu^i)}{\partial w^i} = 0, \tag{158}$$

where $\nu^i = \nu^i(u, \boldsymbol{w}) = dw^i/du$ is a coordinate velocity of matter in the local coordinates. Equation (158) is exact in any order of the post-Newtonian approximations like (36).

The thermodynamic equation relating the internal elastic energy, Π, and the stress tensor, $\mathfrak{s}_{\alpha\beta}$, of body B is required only in a linearized approximation where the stress-energy tensor is completely characterized by its spatial (stress) components $\mathfrak{s}_{ij}$. After making this substitution to the covariant Eq. (17) we get the following thermodynamic equation in the local coordinates:

$$\rho^* \frac{d\Pi}{du} + \mathfrak{s}_{ij} \frac{\partial \nu^i}{\partial w^j} = 0, \tag{159}$$

where the operator of the total time derivative, $d/du \equiv \partial/\partial u + \nu^i \partial/\partial w^i$.

The covariant equation of conservation of the stress-energy tensor of matter of body B is (15). We need in the post-Newtonian approximation only the spatial component of this equation. Straightforward calculations with making use of the post-Newtonian components (31)–(34) of the stress-energy tensor of matter of body B yield the following form of the law of conservation (15) in the local coordinates:

$$\begin{aligned} \rho^* \frac{d}{du}\left[\left(1 + \frac{1}{2}\nu^2 + \Pi + \frac{1}{2}\hat{h}_{00} + \frac{1}{3}\hat{h}_{kk}\right)\nu^i + \hat{h}_{0i}\right] &= \frac{1}{2}\rho^* \frac{\partial(\hat{h}_{00} + \hat{l}_{00})}{\partial w^i} - \frac{\partial \mathfrak{s}_{ij}}{\partial w^j} \\ &+ \rho^* \left[\frac{1}{4}(\nu^2 + 2\Pi + \hat{h}_{00})\frac{\partial \hat{h}_{00}}{\partial w^i} + \frac{1}{6}\nu^2 \frac{\partial \hat{h}_{kk}}{\partial w^i} + \nu^k \frac{\partial \hat{h}_{0k}}{\partial w^i}\right] \\ &+ \frac{1}{2}\frac{\partial}{\partial w^j}\left[\mathfrak{s}_{ij}\left(\frac{\partial \hat{h}_{00}}{\partial w^k} - \frac{1}{3}\frac{\partial \hat{h}_{kk}}{\partial w^k}\right)\right] + \frac{1}{6}\mathfrak{s}_{kk}\frac{\partial \hat{h}_{jj}}{\partial w^i} + \frac{\partial(\mathfrak{s}_{ij}\nu^j)}{\partial u}, \end{aligned} \tag{160}$$

where the metric tensor perturbations $\hat{h}_{00}$, $\hat{l}_{00}$, $\hat{h}_{0i}$, $\hat{h}_{ij}$, and $\hat{h}_{ii}$ in the local coordinates have been defined above in Secs. IV B 3–IV B 5.

B. Post-Newtonian mass of a single body

There are two algebraically independent definitions of the post-Newtonian mass in the scalar-tensor theory—the *active* mass (Jordan's frame) and the *conformal* mass (Einstein's frame) which are defined respectively by equations (122) and (128) for multipolar index $l = 0$. More specifically, the active mass of body B is [17,87]

$$\begin{aligned} \mathcal{M} = \mathrm{M}_{\mathrm{GR}}[1 + (1 + \gamma - 2\beta)\mathcal{P}] + \frac{1}{6}(\gamma - 1)\ddot{\mathcal{N}} \\ - \frac{1}{2}\eta \int_{\mathcal{V}_{\mathrm{B}}} \rho^* \hat{U}_{\mathrm{B}} d^3 w \\ - \sum_{l=1}^{\infty} \frac{1}{l!}[(\gamma l + 1)\mathcal{Q}_L + 2(\beta - 1)\mathcal{P}_L]\mathcal{M}^L, \end{aligned} \tag{161}$$

where $\mathcal{P}_L$, $\mathcal{Q}_L$ are the scalar field and gravitoelectric external multipoles given in (153) and (155) respectively,

$$\mathrm{M}_{\mathrm{GR}} = \int_{\mathcal{V}_{\mathrm{B}}} \rho^* \left(1 + \frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}_{\mathrm{B}}\right) d^3 w \tag{162}$$

is a *bare* post-Newtonian mass of body B [88], $\mathcal{M}^L$ are *active* multipoles of the body defined in (122), $\mathcal{N}$ is the rotational moment of inertia defined in (125), and $\ddot{\mathcal{N}} = d^2\mathcal{N}/du^2$ denotes a second derivative of the moment of inertia with respect to time u.

Mass M_{GR} depends only on the internal distribution of mass, kinetic, thermal, and gravitational energy densities of body B. It coincides with the Tolman mass [260] of a single, isolated body residing in an asymptotically flat spacetime derived by volume integration of Tolman's superpotential [119], Eq. 1.4.32]. Had the body B been isolated, the mass

M_{GR} would be conserved. However, in an $\mathbb{N}$-body system gravitational interaction of body B with external bodies causes the body's tidal deformations which change the internal distribution of matter and shape of body B, thus, making M_{GR} dependent on time. The temporal change of M_{GR} is governed by the ordinary differential equation [17,75]

$$\dot{\mathrm{M}}_{\mathrm{GR}} = \sum_{l=1}^{\infty} \frac{1}{l!} \mathcal{Q}_L \dot{\mathcal{M}}^L, \tag{163}$$

where the overdot denotes a derivative with respect to coordinate time u.

The conformal mass of body B, $M \equiv \mathcal{I}$, is defined by equation (128) taken for $l = 0$, and is [17,87]

$$M = \mathrm{M}_{\mathrm{GR}}[1 + (\gamma - 1)\mathcal{P}] - \sum_{l=1}^{\infty} \frac{l+1}{l!} \mathcal{Q}_L \mathcal{M}^L. \tag{164}$$

The conformal mass M defines the inertial mass of a single body B in an $\mathbb{N}$-body system as we shall demonstrate in Sec. IX B. In case of a single isolated body the last term in the right-hand side of (164) is absent but it appears in the $\mathbb{N}$-body system (if the body under consideration is not spherically symmetric) and can be interpreted in the spirit of Mach's principle stating that the body's inertial mass originates from its gravitational interaction with an external Universe. Mach's idea is not completely right because the inertial mass of the body is primarily originating from the *bare* mass M_{GR} but it has a partial support as we cannot completely ignore the gravitational interaction of a single body with its external gravitational environment in the definition of the inertial mass of the body. This effect is important to take into account in inspiralling compact binaries as they are tidally distorted and, hence, the part of the inertial mass of each star associated with the very last term in (164) rapidly changes as the distance between them is decreasing. The overall time variation of the conformal mass M is given by equation,

$$\dot{M} = (\gamma - 1)\left(\mathcal{P}\sum_{l=1}^{\infty} \frac{1}{l!} \mathcal{Q}_L \dot{\mathcal{M}}^L + \dot{\mathcal{P}}\mathrm{M}_{\mathrm{GR}}\right) - \sum_{l=1}^{\infty} \frac{1}{(l-1)!}\left(\mathcal{Q}_L \dot{\mathcal{M}}^L + \frac{l+1}{l} \dot{\mathcal{Q}}_L \mathcal{M}^L\right), \tag{165}$$

where we have made use of (163).

Relation between the active and conformal masses is obtained by comparing (161) with (164)

$$M = \mathcal{M} + \frac{1}{2}\eta \int_{\mathcal{V}_{\mathrm{B}}} \rho^* \hat{U}_{\mathrm{B}} d^3 w - \frac{1}{6}(\gamma - 1)\ddot{\mathcal{N}} + 2(\beta - 1)\left(\mathcal{M}\mathcal{P} + \sum_{l=1}^{\infty} \frac{1}{l!} \mathcal{P}_L \mathcal{M}^L\right) + (\gamma - 1)\sum_{l=1}^{\infty} \frac{1}{(l-1)!} \mathcal{Q}_L \mathcal{M}^L, \tag{166}$$

where $\eta = 4\beta - \gamma - 3$ is called the Nordtvedt parameter [88]. We can see that the conformal mass M of body B differs from its active mass $\mathcal{M}$. This fact was noticed by Dicke [173,261], Will [88], and Nordtvedt [262] who found the integral term being proportional to the Nordtvedt parameter η in the right-hand side of (166). The actual difference between the masses turns out to be more complicated and includes a term with the second time derivative of the rotational moment of inertia of the body as well as the tidal contributions originating from gravitational interaction of the body's internal multipoles with the external multipoles. Had body B been completely isolated from the external gravitational field, the difference between the active and conformal masses would be caused only by the Dicke-Nordtvedt self-gravity term depending on parameter η, and the second time derivative of the body's rotational moment of inertia due to, e.g., radial oscillations of the body. In case of an $\mathbb{N}$-body system the gravitational field of $\mathbb{N} - 1$ external bodies cannot be ignored in the definition of the post-Newtonian mass of a single body due to the gravitational coupling of the external and internal multipoles of the body.

C. Post-Newtonian center of mass and linear momentum of a single body

The functional form of equations of motion of extended bodies in an $\mathbb{N}$-body system depends crucially on the choice of the reference point inside body B that defines its center of mass. There is a large freedom in choosing the definition of the center of mass beyond the Newtonian limit. Physically, any definition is allowed and makes a certain sense. However, the most optimal definition of the center of mass makes the equations of motion look simple and eliminates a number of spurious terms which would contaminate the equations of motion, like the *noncanonical* multipole moments $\mathcal{N}^L$ and $\mathcal{R}^L$ mentioned above, if the center of mass is not chosen properly. Damour *et al.* [74,75] have shown that in general relativity the position of the center of mass of body B, which is a member of the $\mathbb{N}$-body system, is the most optimally determined by picking up the zero value of the Blanchet-Damour mass dipole in the internal solution for the metric tensor perturbation. In scalar-tensor theory of gravity there are two possible definitions of the internal mass dipole depending on whether the Jordan or the Einstein frame is chosen for the multipole expansion of the metric tensor. The Jordan frame gives the *active* dipole moment $\mathcal{M}^i$, and the Einstein frame defines the *conformal* dipole $\mathcal{I}^i$. Before performing computations it is difficult to foresee which choice of the dipole is the best for positioning the center of mass of the body. Only after completing the derivation of the equations of motion does it become clear that it is the *conformal* mass dipole that yields the most optimal choice of the post-Newtonian center of mass of each body [17,87]. The physical reason for this is that the conformal dipole moment obeys the law of conservation of linear momentum, $\mathfrak{p}^i$, of each body in its own local coordinate chart while the post-Newtonian active dipole does not have such a property.

Thus, we define the post-Newtonian center of mass of each body B by making use of the conformal definition (128) of the internal multipoles of body B for a multipolar index $l = 1$. It yields

$$\mathcal{I}^i = \mathcal{I}^i_{\rm b} + \mathcal{I}^i_{\rm c}, \tag{167}$$

where

$$\mathcal{I}^i_{\rm b} = \int_{\mathcal{V}_{\rm B}} \varrho(u, \boldsymbol{w}) \left[1 - (1-\gamma)\mathcal{P} - \sum_{l=1}^{\infty} \frac{1}{l!} \mathcal{Q}_L w^L \right] w^i d^3w - \frac{2}{5}\left(3\dot{\mathcal{R}}^i - \frac{1}{4}\ddot{\mathcal{N}}^i\right) \tag{168}$$

is the *bare* conformal dipole of body B, and $\mathcal{I}^i_{\rm c}$ is a *complementary* post-Newtonian translation that is introduced in order to have freedom in a residual adjustment of worldline $\mathcal{Z}$ of the center of mass of the body in the process of derivation of equations of motion. At this stage the translation $\mathcal{I}^i_{\rm c}$ is left undetermined. It will be specified later on; see Eqs. (289) and (535).

The last two terms in the right-hand side of (168) can be written down more explicitly if we use a vector virial theorem,

$$\frac{2}{5}\left(3\dot{\mathcal{R}}^i - \frac{1}{4}\ddot{\mathcal{N}}^i\right) = \int_{\mathcal{V}_{\rm B}} \left(\rho^* \nu^2 + \mathfrak{s}_{kk} - \frac{1}{2}\rho^* \hat{U}_{\rm B}\right) w^i d^3w + \sum_{l=1}^{\infty} \frac{1}{(l-1)!} \mathcal{Q}_L \mathcal{M}^{iL} - \frac{1}{2}\sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} \mathcal{Q}_{iL} \mathcal{N}^L. \tag{169}$$

Replacing (169) to (168) brings the bare conformal dipole to the following form:

$$\mathcal{I}^i_{\rm b} = \int_{\mathcal{V}_{\rm B}} \rho^*(u, \boldsymbol{w}) \left[1 + \frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}_{\rm B} + (\gamma - 1)\mathcal{P}\right] w^i d^3w - \sum_{l=1}^{\infty} \frac{l+1}{l!} \mathcal{Q}_L \mathcal{M}^{iL} - \frac{1}{2}\sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} \mathcal{Q}_{iL} \mathcal{N}^L, \tag{170}$$

where the STF noncanonical multipole, $\mathcal{N}^L$, has been defined in (123).

We will also need the definition of the active dipole, $\mathcal{M}^i$, for it will appear in the equations of motion explicitly. The definition of the active mass dipole follows directly from the generic post-Newtonian formula for mass multipoles (122) taken for $l = 1$. After applying the virial theorem (169), we find out that the active dipole, $\mathcal{M}^i$, of body B relates to its bare conformal dipole, $\mathcal{I}^i_{\rm b}$ as follows:

$$\mathcal{M}^i = \mathcal{I}^i_{\rm b} + (\gamma - 1)\left(\frac{3}{5}\dot{\mathcal{R}}^i - \frac{1}{10}\ddot{\mathcal{N}}^i\right) - \frac{\eta}{2}\left(\int_{\mathcal{V}_{\rm B}} \rho^* \hat{U}_{\rm B} w^i d^3w + \sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} \mathcal{Q}_{iL} \mathcal{N}^L\right) - \sum_{l=1}^{\infty} \frac{(\gamma-1)l + 2(\beta-1)}{l!} \mathcal{Q}_L \mathcal{I}^{iL} - 2(\beta-1)(\mathcal{P}^k - \mathcal{Q}^k)\left(\mathcal{M}^{ik} + \frac{1}{3}\delta^{ik}\mathcal{N}\right). \tag{171}$$

The volume integrals entering definitions (170) and (171) of the conformal and active dipoles of body B are performed over hypersurface $\mathcal{H}_u$ of constant time u. All other terms entering these definitions are taken on worldline $\mathcal{W}$ of the origin of the local coordinates adapted to the body, at the point of intersection of $\mathcal{W}$ with hypersurface $\mathcal{H}_u$. Therefore, the dipole is a function of time u only.

The dipole defines a vector of displacement of the center of mass of body B from the origin of the local coordinates adapted to the body. If the origin of the local coordinates coincides with the center of mass of the body, the dipole vanishes. We draw to the attention of the reader that the post-Newtonian definition of the center of mass of body B depends (like in the case of the post-Newtonian definition of a body's mass) not only on the distribution of matter density, velocity, and stresses inside the body but also on the terms describing the coupling of the internal and external multipoles. Thorne and Hartle [58] were the first to notice the presence of such terms in the post-Newtonian definition of the center of mass (and other mass multipoles), but they did not provide their exact form that was found later by Damour *et al.* [74,75] in general relativity and by Kopeikin and Vlasov [87] in the scalar-tensor theory of gravity. We notice that dipole's definitions (170) and (171) contain *noncanonical* multipoles, $\mathcal{R}^L$ and $\mathcal{N}^L$, which do not appear in the canonical multipole decomposition of the metric tensor perturbation in vacuum [50,78,82]. Comprehensive calculations of equations of motion of extended bodies by the Fock-Papapetrou method have revealed [74,75,87] that if the noncanonical multipoles $\mathcal{R}^L$ and $\mathcal{N}^L$ are removed from the definition of the dipole, they appear explicitly in the equations of motion, thus, making them incompatible with the equations of motion in the Mathisson-Dixon or EIH approaches which cannot have the noncanonical multipoles, $\mathcal{R}^L$ and $\mathcal{N}^L$, at all. Therefore, it is natural to hold the noncanonical multipoles $\mathcal{R}^L$ and $\mathcal{N}^L$ in the definitions of the post-Newtonian mass, center of mass, and mass multipoles $\mathcal{M}^L$ of body B.

Definition (167) of the conformal dipole of body B is used to define the position of its center of mass with respect to the origin of the local coordinates adapted to body B. The center of mass, $w^i_{\rm cm}$, of the body is defined in its local coordinates by the overall value of its dipole,

$$Mw^i_{\rm cm} = \mathcal{I}^i, \tag{172}$$

where M is the post-Newtonian conformal mass of body B defined above in (164). The post-Newtonian linear momentum $\mathfrak{p}^i$ of body B is defined as the first derivative of the dipole (167) with respect to the local time u,

$$\mathfrak{p}^i \equiv \dot{\mathcal{I}}^i(u) = \mathfrak{p}^i_{\rm b} + \dot{\mathcal{I}}^i_{\rm c}, \tag{173}$$

where $\mathfrak{p}^i_{\rm b} \equiv \dot{\mathcal{I}}^i_{\rm b}$, and the overdot denotes the time derivative with respect to u. After taking the time derivative from the bare dipole (170) and using the local equations of motion of matter (160) to transform the integrand, we obtain [87]

$$\begin{aligned}\mathfrak{p}^i_{\rm b} = &\int_{\mathcal{V}_{\rm B}} \rho^* \nu^i \left(1 + \frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}_{\rm B}\right) d^3w \\ &+ \int_{\mathcal{V}_{\rm B}} \left(\mathfrak{s}_{ik}\nu^k - \frac{1}{2}\rho^* \hat{W}^i_{\rm B}\right) d^3w \\ &+ \frac{d}{du}\left[\mathcal{I}^i_{\rm c} - (1-\gamma)\mathcal{P}\mathcal{M}^i - \sum_{l=1}^{\infty}\frac{l+1}{l!}\mathcal{Q}_L\mathcal{M}^{iL}\right. \\ &\left. - \frac{1}{2}\sum_{l=0}^{\infty}\frac{1}{(2l+3)l!}\mathcal{Q}_{iL}\mathcal{N}^L\right] \\ &+ \sum_{l=1}^{\infty}\frac{1}{l!}\left[\mathcal{Q}_L\dot{\mathcal{M}}^{iL} + \frac{l}{2l+1}\mathcal{Q}_{iL-1}\dot{\mathcal{N}}^{L-1}\right. \\ &\left. - \mathcal{Q}_L\int_{\mathcal{V}_{\rm B}}\rho^*\nu^i w^L d^3w\right], \end{aligned} \tag{174}$$

where

$$\hat{W}^i_{\rm B} = \int_{\mathcal{V}_{\rm B}} \frac{\rho^*(u, \boldsymbol{w}')\nu'^k(w^k - w'^k)(w^i - w'^i)}{|\boldsymbol{w} - \boldsymbol{w}'|^3} d^3w' \tag{175}$$

is a new internal potential of gravitational field of body B; cf. [[88], Eq. 4.32].

We remind the reader now that the point $x^i_{\rm B}$ represents the position of the origin of the local coordinates adapted to body B in the global coordinates taken at instant of time t. It moves along worldline $\mathcal{W}$ which we want to make identical to worldline $\mathcal{Z}$ of the center of mass of body B. It can be achieved if we can retain the center of mass of body B at the origin of the local coordinates adapted to the body, that is to have for any instant of time, $w^i_{\rm cm} = 0$. This condition means that both functions of time—the conformal dipole $\mathcal{I}^i$ of the body and its linear momentum $\mathfrak{p}^i$—have to vanish,

$$\mathcal{I}^i = 0, \qquad \mathfrak{p}^i = 0. \tag{176}$$

These constraints imposed on the conformal dipole and linear momentum of body B can be satisfied if, and only if, the local equation of motion of the center of mass of the body can be reduced to equation

$$\dot{\mathfrak{p}}^i(u) = \dot{\mathfrak{p}}^i_{\rm b} + \ddot{\mathcal{I}}^i_{\rm c} = 0. \tag{177}$$

It is remarkable that Eq. (177) can be, indeed, fulfilled after making an appropriate choice of the external dipole $\mathcal{Q}_i$ that characterizes the acceleration of the origin of the local coordinates of body B with respect to a geodesic worldline of the effective external manifold $\bar{M}$. We prove this statement below in Sec. VI E.

D. Post-Newtonian spin of a single body

In the post-Newtonian approximation the spin multipoles of an extended body B appear in the multipolar decomposition of the metric tensor in the Newtonian form (131) where the body's spin corresponds to $l = 1$. The Newtonian definition of spin is insufficient for derivation of the post-Newtonian equations of rotational motion and must be extended to include the post-Newtonian terms. The post-Newtonian definition of spin of a single body residing in asymptotically flat spacetime can be extracted from the multipolar expansion of the metric tensor component $\hat{g}_{0i}(u, \boldsymbol{w})$ by taking into account terms of the post-post-Newtonian order [79]. The problem we face in the present paper is that we have to define the post-Newtonian spin of body B which is not residing in asymptotically flat spacetime but is a member of the $\mathbb{N}$-body system. We have also take into account the contribution of the scalar field as we work in scalar-tensor theory of gravity.

A post-Newtonian definition of the spin can be extracted from the local law of conservation of the stress-energy complex $\Theta^{\mu\nu}$

$$\Theta^{\mu\nu}{}_{,\nu} = 0, \tag{178}$$

which is used for building definitions of conserved quantities in metric theories of gravity [119]. The stress-energy complex is not unique and is defined up to a term whose divergence vanishes identically. One of the most convenient definitions of the symmetric stress-energy tensor in the scalar-tensor theory of gravity was found by Nutku [147]. It generalizes the Landau-Lifshitz stress-energy complex [42] and reads

$$\Theta^{\mu\nu} = -g(1+\phi)(T^{\mu\nu} + t^{\mu\nu}), \tag{179}$$

where $g = \det[g_{\mu\nu}]$, ϕ is the perturbation of the scalar field (20), $T^{\mu\nu}$ is the stress energy-tensor of matter, and $t^{\mu\nu}$ is an analog of the Landau-Lifshitz pseudotensor $t^{\mu\nu}_{\rm LL}$ of the gravitational field [42]. The pseudotensor has been determined by Nutku [147] and reads

$$\begin{aligned} t^{\mu\nu} = \frac{1}{16\pi}\Bigg[&(1+\phi^3)t^{\mu\nu}_{\rm LL} \\ &+ \frac{2\omega(\phi)+3}{1+\phi}\left(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi\right)\Bigg]. \end{aligned} \tag{180}$$

Let us now introduce the post-Newtonian definition of a bare spin of body B in the local coordinates adapted to the body, as follows:

$$\mathcal{S}^i_{\rm b} = \int_{\mathbb{R}^3} \varepsilon_{ijk} w^j [-\hat{g}(u,\boldsymbol{w})][1+(\gamma-1)\hat{\varphi}(u,\boldsymbol{w})] \times [\hat{T}^{0k}(u,\boldsymbol{w}) + \hat{t}^{0k}(u,\boldsymbol{w})] d^3w, \tag{181}$$

where ε_{ijk} is 3-dimensional symbol of Levi-Civita and the integration is performed over the entire 3-dimensional space $\mathbb{R}^3$. Special attention should be paid to the variables entering definition (181). Namely, the scalar field perturbation $\hat{\varphi}$ is given by (92) and includes both external and internal parts; the stress-energy tensor $\hat{T}^{\mu\nu}$ depends solely on matter variables of body B as defined in Eqs. (31)–(34) but it includes the overall—external and internal—post-Newtonian perturbations of the metric tensor (98) and scalar field (92), while the Nutku pseudotensor $\hat{t}^{\mu\nu}$ introduced in (180) depends only on the internal part of the post-Newtonian perturbations of the metric tensor (101)–(103) and scalar field (99). These limitations introduced to the definition of spin of body B prevents appearance of divergent terms that could emerge from the integration of a pseudotensor which is formally defined in the entire space $\mathbb{R}^3$.

Integrating by parts allows us to reduce (181) to the integral over the volume $\mathcal{V}_{\rm B}$ of body B only. Expanding it in the post-Newtonian series yields explicit expression for the bare post-Newtonian spin of body B in the following form [87]:

$$\mathcal{S}^i_{\rm b} = \int_{\mathcal{V}_{\rm B}} \rho^* \varepsilon_{ijk} w^j \nu^k \left[1 + \frac{1}{2}\nu^2 + \Pi + (2\gamma+1)\hat{U}_{\rm B} + (1-\gamma)\mathcal{P}\right] d^3w + \int_{\mathcal{V}_{\rm B}} \varepsilon_{ijk} w^j \mathfrak{s}^{kp} \nu^p d^3w$$
$$+ \sum_{l=1}^{\infty} \frac{1}{l!}[3\mathcal{Q}_L + 2(\gamma-1)\mathcal{P}_L] \int_{\mathcal{V}_{\rm B}} \rho^* \varepsilon_{ijk} w^j \nu^k w^L d^3w - \frac{1}{2}\int_{\mathcal{V}_{\rm B}} \rho^* \varepsilon_{ijk} w^j [\hat{W}^k_{\rm B} + (3+4\gamma)\hat{U}^k_{\rm B}] d^3w, \tag{182}$$

where $\nu^i = dw^i/du$ is velocity of matter of body B in the local coordinates, the integration is over volume of body B, and vector potential $\hat{W}^k_{\rm B}$ is defined in (175). The reader can notice that the spin of body B which is a member of the $\mathbb{N}$-body system depends not only on the internal structure of the body but also on the gravitational field of external bodies like in the case of the internal mass multipoles. We shall use definition (182) to derive the rotational equations of motion of the body's spin below in this section and in Sec. X.

E. Translational equation of motion of the center of mass of a single body

Translational equations of motion of the center of mass of body B with respect to the local coordinates w^α adapted to the body are derived by the Fock-Papapetrou method from the law of conservation (177) of the total linear momentum $\mathfrak{p}^i$ of the body. In order to implement this law we have to find out the time derivative of the bare linear momentum, $\mathfrak{p}^i_{\rm b}$ of the body. To this end, we differentiate both sides of Eq. (174) one time with respect to the local coordinate time u, make use of the microscopic equations of motion (158)–(160), and integrate by parts to rearrange a number of terms. One obtains [17,234]

$$\dot{\mathfrak{p}}^i_{\rm b} = \mathcal{M}\mathcal{Q}^i + \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_{iL}\mathcal{M}^L + \sum_{l=1}^{\infty}\frac{l}{(l+1)!}\mathcal{C}_{iL}\mathcal{S}^L - \sum_{l=1}^{\infty}\frac{1}{(l+1)!}[(l^2+l+4)\mathcal{Q}_L + 2(\gamma-1)\mathcal{P}_L]\ddot{\mathcal{M}}^{iL}$$
$$-\sum_{l=1}^{\infty}\frac{2l+1}{(l+1)(l+1)!}[(l^2+2l+5)\dot{\mathcal{Q}}_L + 2(\gamma-1)\dot{\mathcal{P}}_L]\dot{\mathcal{M}}^{iL}$$
$$-\sum_{l=1}^{\infty}\frac{2l+1}{(2l+3)(l+1)!}[(l^2+3l+6)\ddot{\mathcal{Q}}_L + 2(\gamma-1)\ddot{\mathcal{P}}_L]\mathcal{M}^{iL} - \sum_{l=1}^{\infty}\frac{1}{(l+1)!}\varepsilon_{ipq}\left[\mathcal{C}_{pL}\dot{\mathcal{M}}^{qL} + \frac{l+1}{l+2}\dot{\mathcal{C}}_{pL}\mathcal{M}^{qL}\right]$$
$$+2\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\left[(2\mathcal{Q}_{pL} + (\gamma-1)\mathcal{P}_{pL})\dot{\mathcal{S}}^{qL} + \frac{l+1}{l+2}(2\dot{\mathcal{Q}}_{pL} + (\gamma-1)\dot{\mathcal{P}}_{pL})\mathcal{S}^{qL}\right]$$
$$-(\mathcal{P}^i - \mathcal{Q}^i)\left[\frac{1}{2}\eta\int_{\mathcal{V}_{\rm B}}\rho^*\hat{U}^{(\rm B)}d^3w - \frac{1}{6}(\gamma-1)\ddot{\mathcal{N}} + 2(\beta-1)\left(\mathcal{M}\mathcal{P} + \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{P}_L\mathcal{M}^L\right)\right.$$
$$\left. + (\gamma-1)\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\mathcal{Q}_L\mathcal{M}^L\right], \tag{183}$$

where the spatial indices are raised and lowered with the Kronecker symbol, the active mass multipoles $\mathcal{M}^L$ are defined in (122) and include the post-Newtonian corrections, and the spin multipoles $\mathcal{S}^L$ are sufficient in the Newtonian limit (131). We have not shown in (183) a number of terms which are directly proportional to the internal conformal dipole, $\mathcal{I}^i$, and the linear momentum, $\mathfrak{p}^i$, of body B because these terms vanish if the origin of the local coordinates coincides with the center of mass of body B under condition (176) which we employ in the rest of the paper. The omitted dipole-dependent terms in (183) can be found in [[17], Eq. 6.19].

Equation (183) is the post-Newtonian generalization of the second Newton's law applied to body B and written down in the body-adapted local coordinates. Therefore, the right-hand side of (183) is the net force exerted on body B. This force does not include the self-action force as the scalar-tensor theory of gravity belongs to the class of conservative theories [88]. Formally, the self-action force terms appeared at different stages of the computation of the time derivative of the linear momentum but they all have mutually canceled out at the final expression (183). The external force standing in the right-hand side of (183) consists of three parts:

(1) the tidal gravitational force caused by the coupling of the internal active multipoles, $\mathcal{M}^L$, $\mathcal{S}^L$ of body B with the external multipoles $\mathcal{Q}_L$, $\mathcal{P}_L$, $\mathcal{C}_L$ for $l \geq 2$,
(2) the force of inertia consisting of $\mathcal{M}\mathcal{Q}_i$ and all other post-Newtonian terms being proportional to $\mathcal{Q}_i$, caused by the nongeodesic motion of the origin of the local coordinates adapted to body B;
(3) the Dicke-Nordtvedt force that is proportional to the difference $\mathcal{P}^i - \mathcal{Q}^i$ as shown by the very last term in the right-hand side of (183), caused by the violation of the strong principle of equivalence (SEP) in scalar-tensor theory of gravity.

In order to ensure vanishing of the total linear momentum of body B, $\dot{\mathfrak{p}}^i = 0$, we shall choose the local acceleration $\mathcal{Q}_i$ to compensate all terms in the right-hand side of (183) along with the complementary term $\ddot{\mathcal{I}}^i_{\rm c}$ that is used for small residual adjustment of the acceleration. This choice eliminates the relative acceleration of the worldline $\mathcal{Z}$ of the center of mass of body B with respect to worldline $\mathcal{W}$ of the origin of the body-adapted local coordinates. In this locally accelerated frame we can still have the center of mass of body B moving with respect to the origin of the local coordinates with constant velocity, but we impose further constraint (176) to eliminate this rectilinear motion and to put the center of mass of body B at the origin of its own local coordinates. It makes worldlines $\mathcal{Z}$ and $\mathcal{W}$ identical.

The solution of the law of conservation of the linear momentum (177), where $\dot{\mathfrak{p}}^i_{\rm b}$ is given by (183), with respect to $\mathcal{Q}_i$ yields

$$\mathcal{Q}_i = \mathcal{Q}_i^{\rm N} + \mathcal{Q}_i^{\rm pN} - \frac{\ddot{\mathcal{I}}^i_{\rm c}}{M}, \tag{184}$$

where the first term is the Newtonian part of acceleration, the second term is the post-Newtonian correction, and the third term is the complementary acceleration which allows us to make residual adjustments in the position of the center of mass of the body, if necessary. The residual freedom in choosing the position of the center of mass of body B is fixed at the last steps of derivation of translational equations of motion; see (289) and (535).

The Newtonian and post-Newtonian counterparts of the local acceleration of body B are defined by the following equations:

$$M\mathcal{Q}_i^{\rm N} = (M - \mathcal{M})\mathcal{P}_i - \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_{iL}\mathcal{M}^L, \tag{185}$$

$$\begin{aligned} M\mathcal{Q}_i^{\rm pN} =& \sum_{l=1}^{\infty}\frac{1}{(l+1)!}[(l^2+l+4)\mathcal{Q}_L + 2(\gamma-1)\mathcal{P}_L]\ddot{\mathcal{M}}^{iL} + \sum_{l=1}^{\infty}\frac{2l+1}{(l+1)(l+1)!}[(l^2+2l+5)\dot{\mathcal{Q}}_L + 2(\gamma-1)\dot{\mathcal{P}}_L]\dot{\mathcal{M}}^{iL} \\ &+ \sum_{l=1}^{\infty}\frac{2l+1}{(2l+3)(l+1)!}[(l^2+3l+6)\ddot{\mathcal{Q}}_L + 2(\gamma-1)\ddot{\mathcal{P}}_L]\mathcal{M}^{iL} + \sum_{l=1}^{\infty}\frac{1}{(l+1)!}\varepsilon_{ipq}\left[\mathcal{C}_{pL}\dot{\mathcal{M}}^{qL} + \frac{l+1}{l+2}\dot{\mathcal{C}}_{pL}\mathcal{M}^{qL}\right] \\ &- \sum_{l=1}^{\infty}\frac{l}{(l+1)!}\mathcal{C}_{iL}\mathcal{S}^L - 2\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\left[(2\mathcal{Q}_{pL} + (\gamma-1)\mathcal{P}_{pL})\dot{\mathcal{S}}^{qL} + \frac{l+1}{l+2}(2\dot{\mathcal{Q}}_{pL} + (\gamma-1)\dot{\mathcal{P}}_{pL})\mathcal{S}^{qL}\right], \end{aligned} \tag{186}$$

where M and $\mathcal{M}$ are the conformal and active gravitational masses of body B. The two masses, M and $\mathcal{M}$, are not equal according to (166). The difference between them plays a role of a scalar charge, $\mathfrak{q} \equiv \mathcal{M} - M$, of the scalar field ϕ which couples with the external dipole of the scalar field $\mathcal{P}^i = \bar{U}_{,i}$ and causes the Dicke-Nordtvedt anomalous acceleration, $\mathfrak{q}\mathcal{P}^i$, in (185) [88,173,261]. In general relativity, $\mathfrak{q} = 0$, and the Dicke-Nordtvedt acceleration in the right-hand side of (184) vanishes.

Equation (184) is a condition for the fulfillment of the law of conservation of linear momentum (177) in local coordinates. It ensures that the worldline $\mathcal{W}$ of the origin of local coordinates does not accelerate with respect to the worldline $\mathcal{Z}$ of the center of mass of body B.

Equation (184) does not guarantee, however, that $\mathcal{W}$ and $\mathcal{Z}$ coincide. The origin of the local coordinates still can move uniformly with respect to the center of mass of the body. To eliminate this uniform motion we impose condition, $\mathfrak{p}^i = 0$. The freedom which remains is a constant relative displacement of the origin of the local coordinates with respect to the center of mass of the body. This constant displacement is removed by an additional constraint imposed on the internal conformal dipole of the body, $\mathcal{I}^i = 0$. This procedure results in the constraint (176) and ensures that the worldlines $\mathcal{W}$ and $\mathcal{Z}$ coincide.

Acceleration $\mathcal{Q}_i$ given in (184) must be substituted to the equations of motion of the origin of the local coordinates (152) to convert them to the translational equations of motion of the center of mass of body B in the global coordinates. These equations still contain the external gravitational potentials $\bar{U}$, $\bar{\Psi}$, $\bar{U}^i$, and $\bar{\chi}$ defined in (68) and (91), which are given in the form of integrals expressed in the global coordinates. These integrals should be explicitly expanded with respect to the internal multipoles of the bodies of the $\mathbb{N}$-body system in order to complete the theory. We shall conduct this computation in Sec. VII and derive translational equations of motion of extended bodies in an $\mathbb{N}$-body system in terms of their internal multipoles as well as coordinates and velocities of their centers of mass.

F. Rotational equations of motion of spin of a single body

Rotational equations of motion of spin of an extended body are derived in the local coordinates by differentiating the bare spin of body B given by Eq. (182) with respect to the local coordinate time u. After taking the time derivative and making use of the microscopic equations of motion in the local coordinates given in Sec. VI A, we perform several transformations in the integrand to reduce similar terms, integrate the contributions from partial derivatives by parts, and simplify the final result. After long and tedious calculation we obtain the following expression for the first time derivative of the bare spin of body B in the local coordinates adapted to the body [87]

$$\frac{d\mathcal{S}^i_{\rm b}}{du} = \mathcal{T}^i_{\rm b} + \mathcal{T}^i_{\rm c} - \dot{\mathcal{S}}^i_{\rm c}, \tag{187}$$

where $\mathcal{T}^i_{\rm b}$ is the bare torque exerted on the body B due to the coupling of its internal multipoles with the external tidal multipoles, and $\mathcal{T}^i_{\rm c}$ is a post-Newtonian correction to the bare torque caused by the difference (171) between the active and conformal dipoles of body B, while $\dot{\mathcal{S}}^i_{\rm c} \equiv d\mathcal{S}^i_{\rm c}/du$ and $\mathcal{S}^i_{\rm c}$ is a linear combination of terms which can be treated as a complementary contribution to the bare spin of the body.

Gravitational bare torque, $\mathcal{T}^i_{\rm b}$, and the other terms in the right-hand side of (187) read as follows [87]:

$$\mathcal{T}^i_{\rm b} = [1 + (2\beta - \gamma - 1)\mathcal{P}] \sum_{l=0}^{\infty} \frac{1}{l!} \varepsilon_{ijk} \mathcal{Q}_{kL} \mathcal{M}^{jL} + \sum_{l=0}^{\infty} \frac{l+1}{(l+2)l!} \varepsilon_{ijk} \mathcal{C}_{kL} \mathcal{S}^{jL}, \tag{188}$$

$$\begin{aligned}\mathcal{T}^i_{\rm c} = \varepsilon_{ijk} a^j_{\rm B} \Bigg[&(1-\gamma)\left(\frac{3}{5}\dot{\mathcal{R}}^k - \frac{1}{10}\ddot{\mathcal{N}}^k\right) \\ &+ \frac{\eta}{2}\left(\int_{\mathcal{V}_{\rm B}} \rho^* \hat{U}_{\rm B} w^k d^3w + \sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} \mathcal{Q}_{kL} \mathcal{N}^L\right) \\ &+ \sum_{l=1}^{\infty} \frac{(\gamma-1)l + 2(\beta-1)}{l!} \mathcal{Q}_L \mathcal{M}^{kL} \\ &+ 2(\beta-1) a^p_{\rm B}\left(\mathcal{M}^{kp} + \frac{1}{3}\delta^{kp}\mathcal{N}\right)\Bigg],\end{aligned} \tag{189}$$

$$\begin{aligned}\mathcal{S}^i_{\rm c} = &-\sum_{l=1}^{\infty} \frac{l}{(l+1)!} \mathcal{C}_L \mathcal{M}^{iL} + \sum_{l=0}^{\infty} \frac{1}{(2l+3)l!} \mathcal{C}_{iL} \mathcal{N}^L \\ &+ \sum_{l=0}^{\infty} \frac{1}{(2l+5)l!} \varepsilon_{ijk} \Bigg[\frac{1}{2}\mathcal{Q}_{kL}\dot{\mathcal{N}}^{jL} - \frac{l+2(2\gamma+3)}{2(l+2)}\dot{\mathcal{Q}}_{kL}\mathcal{N}^{jL} \\ &- \frac{2(1+\gamma)(2l+3)}{l+2}\mathcal{Q}_{kL}\mathcal{R}^{jL}\Bigg] \\ &+ \frac{1-\gamma}{5}\varepsilon_{ijk}(3\mathcal{R}^j a^k_{\rm B} + \mathcal{N}^j \dot{a}^i_{\rm B}) + (\gamma-1)\mathcal{P}\mathcal{S}^i_{\rm b},\end{aligned} \tag{190}$$

where the noncanonical multipoles, $\mathcal{N}^L$ and $\mathcal{R}^L$ have been defined earlier in (123) and (124) respectively, and in all post-Newtonian terms the global acceleration, $a^i_{\rm B}$, is interpreted as the difference between the dipole of the scalar field and the local acceleration, $a^i_{\rm B} = \mathcal{P}_i - \mathcal{Q}_i$.

The bare torque, $\mathcal{T}^i_{\rm b}$, is caused by gravitational coupling of the internal and external multipoles of body B, and is rooted in general relativity. The complementary torque, $\mathcal{T}^i_{\rm c}$, is caused by the difference between the conformal and active dipoles of the body (171) and exists only in the scalar-tensor theory. Indeed, by comparison of (189) with (171) we can see that

$$\mathcal{T}^i_{\rm c} = \varepsilon_{ijk} a^j_{\rm B}(\mathcal{I}^k_{\rm b} - \mathcal{M}^k) = \varepsilon_{ijk}(\mathcal{P}^j - \mathcal{Q}^j)(\mathcal{I}^k_{\rm b} - \mathcal{M}^k), \tag{191}$$

where $\mathcal{I}^i_{\rm b}$ is the bare conformal dipole (170), and $\mathcal{M}^i$ is the active dipole of body B respectively. Equation (191) can be further transformed to yet another form by taking into account that the total conformal dipole (167) vanishes, $\mathcal{I}^i = 0$, due to our choice of the center of mass (176). After making use of this choice and implementing (167), the complementary torque takes on the following form:

$$\mathcal{T}^i_{\rm c} = -\varepsilon_{ijk}(\mathcal{P}^j - \mathcal{Q}^j)\mathcal{M}^k - \varepsilon_{ijk} a^j_{\rm B} \mathcal{I}^k_{\rm c}, \tag{192}$$

where the complementary vector function $\mathcal{I}^k_{\rm c}$ is still arbitrary. It will be fixed later by condition (289).

The complementary term $\dot{\mathcal{S}}_{\rm c}$ in (187) is a total time derivative which is naturally combined with the bare spin, thus, forming the total spin of body B,

$$\mathcal{S}^i \equiv \mathcal{S}^i_{\rm b} + \mathcal{S}^i_{\rm c}. \tag{193}$$

Defining the total torque in the local coordinates of body B by

$$\begin{aligned}\mathcal{T}^i &\equiv \mathcal{T}^i_{\rm b} + \mathcal{T}^i_{\rm c}\\ &= \varepsilon_{ijk}\Bigg[\mathcal{P}_k\mathcal{M}^j + \sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_{kL}\mathcal{M}^{jL} + a^k_{\rm B}\mathcal{I}^j_{\rm c}\\ &\quad + (2\beta-\gamma-1)\mathcal{P}\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{Q}_{kL}\mathcal{M}^{jL} + \sum_{l=1}^{\infty}\frac{l+1}{(l+2)l!}\mathcal{C}_{kL}\mathcal{S}^{jL}\Bigg]\end{aligned} \tag{194}$$

brings about the rotational equation of motion of spin of body B to its final form,

$$\frac{d\mathcal{S}^i}{du} = \mathcal{T}^i, \tag{195}$$

which includes all Newtonian and post-Newtonian corrections. Derivation of the rotational equations of motion given in this section follows the approach proposed by Damour *et al.* [76] in general relativity and by Kopeikin and Vlasov [87] in scalar-tensor theory of gravity.

VII. MULTIPOLAR EXPANSION OF EXTERNAL POTENTIALS IN THE GLOBAL COORDINATES

Equations of translational motion of each body B in the global coordinates are given in (152) where the local acceleration $\mathcal{Q}_i$ should be taken from (184)–(186). However, the external gravitational potentials of the body—$\bar{U}$, $\bar{\Psi}$, $\bar{U}^i$, $\bar{\chi}$—defined in (68) and (91) are represented in the form of volume integrals which have not yet been explicitly performed in terms of the configuration variables defining each body of the ℕ-body system—the internal multipoles, coordinates of the centers of mass, and their velocities. Computation of the integrals is rather straightforward and rendered by expanding an integrand in each integral defining the external potential, in a Taylor series around the point of the center of mass of body B with subsequent integration of the coefficients of the expansion over volume of body B. The resulting expansion of the external potentials is given in terms of the internal multipole moments of the bodies which are the integrals performed in the global coordinates, x^α. Additional transformation of the internal multipoles from the global to the body-adapted local coordinates is required. This section describes the details of the overall procedure of the multipolar expansion of the external potentials which are used, then, in the translational equations of motion.

We have built the local coordinates, $w^\alpha = (u, w^i) \equiv (u_{\rm B}, w^i_{\rm B})$, adapted to body $\mathrm{B} \in \{1, 2, \ldots, N\}$ by the matched asymptotic expansion technique. We have suppressed the subindex B in previous sections for all functions of the local coordinates adapted to body B to simplify notations. However, computations in this section involves the bodies of the ℕ-body system which are external with respect to body B, and we need to distinguish the local coordinates built around each body C from those adapted to body B. Therefore, we shall use a subindex $\mathrm{C} \in \{1, 2, \ldots, N\}$ to explicitly label the local coordinates adapted to body C along with all configuration variables associated with it.

A. Multipolar expansion of potential $\bar{U}$

The local coordinates adapted to body C are denoted $w^\alpha_{\rm C} = (u_{\rm C}, w^i_{\rm C})$ and the subindex C will appear explicitly in all computations associated with the body C. Post-Newtonian coordinate transformation between $w^\alpha_{\rm C}$ and the global coordinates x^α is identical to Eqs. (144) and (145) describing the transformation from the local coordinates adapted to body B to the global coordinates except that now we have to pin the label C to all quantities related to the local coordinates adapted to body C to distinguish them from the local coordinates adapted to body B. More specifically, the transformation reads

$$\begin{aligned}u_{\rm C} &= t + \frac{1}{c^2}(\mathcal{A}_{\rm C} - v^k_{\rm C}R^k_{\rm C})\\ &\quad + \frac{1}{c^4}\Bigg\{\left[\frac{1}{3}v^k_{\rm C}a^k_{\rm C} - \frac{1}{6}\dot{\bar{U}}_{\rm C}(t, \boldsymbol{x}_{\rm C}) - \frac{1}{10}\dot{a}^k_{\rm C}R^k_{\rm C}\right]R^2_{\rm C}\\ &\quad + \sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{B}^L_{\rm C}R^L_{\rm C}\Bigg\},\end{aligned} \tag{196}$$

$$w^i_{\rm C} = R^i_{\rm C} + \frac{1}{c^2}\left[\left(\frac{1}{2}v^i_{\rm C}v^k_{\rm C} + D^{ik}_{\rm C} + F^{ik}_{\rm C}\right)R^k_{\rm C} + D^{ijk}_{\rm C}R^j_{\rm C}R^k_{\rm C}\right], \tag{197}$$

where $R^i_{\rm C} = x^i - x^i_{\rm C}$, $x^i_{\rm C} = x^i_{\rm C}(t)$ marks the global spatial coordinates of the origin of the local coordinates adapted to body C, $v^i_{\rm C} = dx^i_{\rm C}/dt$ is velocity of the origin of the local coordinates of body C, $a^i_{\rm C} = dv^i_{\rm C}/dt$ is acceleration of the origin of the local coordinates, and we have made use of abbreviations,

$$D^{ik}_{\rm C} \equiv \delta^{ik}\gamma\bar{U}_{\rm C}(t, \boldsymbol{x}_{\rm C}), \tag{198}$$

$$D^{ijk}_{\rm C} \equiv \frac{1}{2}(a^j_{\rm C}\delta^{ik} + a^k_{\rm C}\delta^{ij} - a^i_{\rm C}\delta^{jk}), \tag{199}$$

that allows us to shorten formula (197) and is also useful in the computations which follow. Equations for functions like $\mathcal{A}_{\rm C} = \mathcal{A}_{\rm C}(t)$, $\mathcal{B}^L_{\rm C} = \mathcal{B}^L_{\rm C}(t)$, etc., in (196) and (197) repeat the corresponding equations for $\mathcal{A}$, $\mathcal{B}^L$, etc., in Sec. V C 1, after attaching the subindex C to all functions in (146)–(151). Notice that the potential $\bar{U}_{\rm C}(t, \boldsymbol{x}_{\rm C})$ in (198) denotes the Newtonian gravitational potential of all massive bodies being external to body C,

$$\bar{U}_{\rm C}(t, \boldsymbol{x}) = \sum_{\rm B\neq C} U_{\rm B}(t, \boldsymbol{x}). \tag{200}$$

We emphasize that the instant of time t that appears in (196) and which is also a time argument of all functions and functionals of body C in the global coordinate chart is the same as the instant of time t for functions and functionals of body B. This is because we consider dynamics of the entire $\mathbb{N}$-body system as a continuous past-to-future diffeomorphism of spatial coordinates of the bodies taken on a hypersurface of simultaneity $\mathcal{H}_t$ which points have the same value of a single parameter—time t.

The multipolar expansions of the external gravitational potentials $\bar{U}$, $\bar{U}^i$, $\bar{\Psi}$, $\bar{\chi}$ of body B defined in (68) and (91) are represented in the form of the multipolar expansions from a linear superposition of potentials $U_{\rm C}(t, \boldsymbol{x})$, $U^i_{\rm C}(t, \boldsymbol{x})$, $\Psi_{\rm C}(t, \boldsymbol{x})$, and $\chi_{\rm C}(t, \boldsymbol{x})$ correspondingly. Therefore, we focus on the multipolar expansions of the individual potentials.

Potentials $U_{\rm C}(t, \boldsymbol{x})$, $U^i_{\rm C}(t, \boldsymbol{x})$, $\Psi_{\rm C}(t, \boldsymbol{x})$ are given in the global coordinates as integrals (65), (75)–(79) with a kernel, $|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$, which is a Green function of the Laplace equation. This kernel is expanded into a multipolar series as follows:

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{|R_{\rm C} - R'_{\rm C}|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} R'^{\langle L\rangle}_{\rm C} \partial_L \left(\frac{1}{R_{\rm C}}\right), \tag{201}$$

where $R'^i_{\rm C} = x'^i - x^i_{\rm C}$ is the coordinate distance from the origin of the local coordinates $x^i_{\rm C}$ adapted to body C, $R^i_{\rm C} = x^i - x^i_{\rm C}$ is the coordinate distance from $x^i_{\rm C}$ to the field point, $R_C = (\delta_{ij} R^i_{\rm C} R^j_{\rm C})^{1/2}$, $\partial_L \equiv \partial_{i_1 \ldots i_l}$ denotes a partial derivative of lth order with respect to spatial global coordinates where each $\partial_i = \partial/\partial x^i$, the angular parentheses around indices indicate the STF projection, and the point x'^i lies inside volume with radius $R'_C < R_C$ so that the series (201) is convergent. Equation (201) yields the multipolar expansion of the Newtonian potential of body C in the global coordinates as follows:

$$U_{\rm C}(t, \boldsymbol{x}) = \int_{\mathcal{V}_{\rm C}} \frac{\rho^*(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathbb{I}^{\langle L\rangle}_{\rm C} \partial_L \left(\frac{1}{R_{\rm C}}\right), \tag{202}$$

where

$$\mathbb{I}^L_{\rm C} \equiv \mathbb{I}^L_{\rm C}(t) = \int_{\mathcal{V}_{\rm C}} \rho^*(t, \boldsymbol{x}') R'^{i_1}_{\rm C} R'^{i_2}_{\rm C} \ldots R'^{i_l}_{\rm C} d^3x' \tag{203}$$

are the Newtonian mass moments computed in the global coordinates. We preserve the prime in the notation of the spatial coordinates $R'^i_{\rm C} = x'^i - x^i_{\rm C}$ that appear in the integrand of (203) to prevent confusion of the point of integration x'^i with the field point x^i. Symmetric multipoles $\mathbb{I}^L_{\rm C}$ have to be transformed from the global to local coordinates adapted to body C in order to express them in terms of the internal STF mass and spin multipoles defined in Sec. IV B 6. The transformation procedure is somehow subtle and should be done with care as it involves not only a pointwise transformation of coordinates but a Lie transport of the integration points along worldlines of matter of body C [73,87]; see Fig. 1.

It starts from the post-Newtonian transformation of radius-vector $R^i_{\rm C} = x^i - x^i_{\rm C}$ from the global to local coordinates $w^i_{\rm C}$ adapted to body C. This is achieved by applying the inverse coordinate transformation of (197):

$$R^i_{\rm C} = w^i_{\rm C} - \frac{1}{c^2}\left[\left(\frac{1}{2} v^i_{\rm C} v^k_{\rm C} + D^{ik}_{\rm C} + F^{ik}_{\rm C}\right) w^k_{\rm C} + D^{ijk}_{\rm C} w^j_{\rm C} w^k_{\rm C}\right]. \tag{204}$$

However, we actually need a post-Newtonian transformation not $R^i_{\rm C}$ but a radius-vector $R'^i_{\rm C} = x'^i - x^i_{\rm C}$ from the global to the local coordinates because it is $R'^i_{\rm C}$ which appears in the definition of $\mathbb{I}^L_{\rm C}$ in (203) as a consequence of the Taylor expansion (202). This transformation is slightly different from (204) because in all integrals performed in the global coordinates the points x^i and x'^i are lying on hypersurface $\mathcal{H}_t$ of constant global coordinate time t, while the points $w^i_{\rm C}$ and $w'^i_{\rm C}$ are lying on hypersurfaces $\mathcal{H}_{u_{\rm C}}$ of constant local coordinate time $u_{\rm C}$ in all integrals defining the internal part of the metric tensor perturbation of body C. Hypersurface $\mathcal{H}_t$ differs from that $\mathcal{H}_{u_{\rm C}}$. Therefore, transformation of $\mathbb{I}^{\langle L\rangle}_{\rm C}$ from the global to local coordinates must include not only the transformations between the coordinate points but also a Lie transport of the integration point with coordinates x'^i from hypersurface $\mathcal{H}_t$ to hypersurface $\mathcal{H}_{u_{\rm C}}$ performed along the timelike worldlines of matter of body C. The magnitude of the Lie transport of each point of integration depends on the size of spatial separation of the integration point x'^i from the origin of the local coordinates adapted to body C, and is determined from the equation of time transformation (196), and a condition that all points on the hypersurface $\mathcal{H}_{u_{\rm C}}$ have the same value of the local coordinate time $u_{\rm C}$ as the field point P in Fig. 1. The Lie transport of the corresponding element of matter with coordinates x'^i is accompanied by the point-wise post-Newtonian transformation (196) applied to x'^i and the resulting transform was worked out by Brumberg and

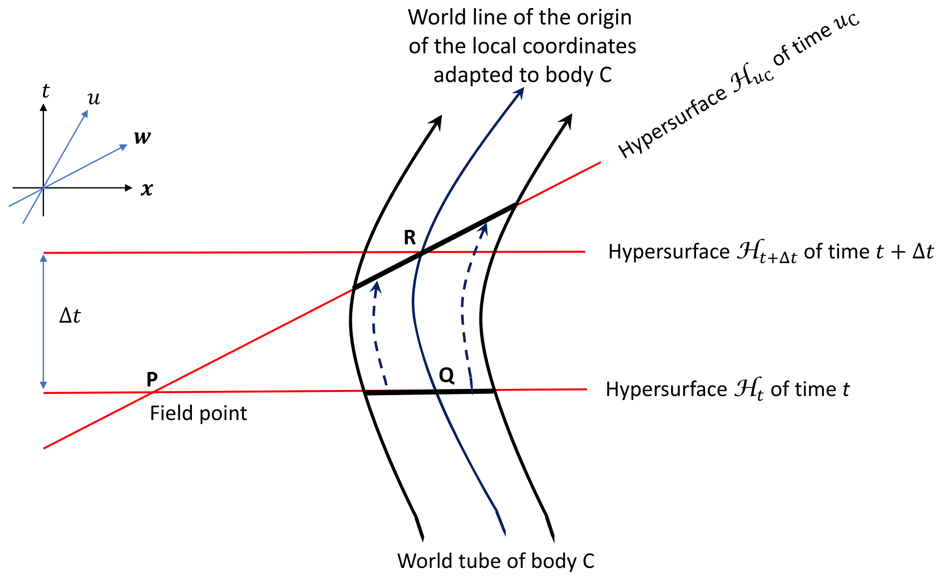

FIG. 1. World tube of matter of body C intersected by hypersurfaces of simultaneity in the global and local coordinates adapted to body C. Integration in the global coordinates goes over the hypersurface $\mathcal{H}_t$ of constant time t passing through points P and Q. Integration in the local coordinates goes over the hypersurface $\mathcal{H}_{u_C}$ of constant time u_C passing through points P and R. The two hypersurfaces intersect at the field point P having global coordinates $x^\alpha_P = \{t, \boldsymbol{x}\}$ and local coordinates $w^\alpha_P = \{u_C, \boldsymbol{w}_C\}$. The points Q and R are lying on the worldline $\mathcal{W}$ of the origin of the local coordinates adapted to body C. Lie transport of the elements of integration from $\mathcal{H}_t$ to $\mathcal{H}_{u_C}$ is shown by dotted lines and carried out along worldlines of matter particles forming the element of integration. Hypersurface $\mathcal{H}_{t+\Delta t}$ of constant time $t+\Delta t$ is passing through point R. Points Q and R have global coordinates $x^\alpha_Q = \{t, \boldsymbol{x}_C(t)\}$ and $x^\alpha_R = \{t+\Delta t, \boldsymbol{x}_C(t+\Delta t)\}$, respectively. Local coordinates of point R are $w^\alpha_R = \{u_C, 0\}$. Time shift Δt between hypersurfaces $\mathcal{H}_t$ and $\mathcal{H}_{t+\Delta t}$ is determined by the time transformation (196) applied to coordinates of two points, P and R which have the same value of the local time u_C. It is given by $\Delta t = v^k_C R^k_C$.

Kopejkin [73] and Kopejkin [263] and its comprehensive explanation is given in full detail in our textbook [[17], Secs. 5.2.3.1 and 6.3.2]. It yields for the post-Newtonian Lie transform of the spatial coordinate w'^i_C the following result [[17], Eq. 6.56]:

$$R'^i_C = w'^i_C - \frac{1}{c^2}\left[\left(\frac{1}{2}v^i_C v^k_C + D^{ik}_C + F^{ik}_C\right)w'^k_C + D^{ijk}_C w'^j_C w'^k_C + \nu'^i_C v^k_C (w'^k_C - w^k_C)\right], \tag{205}$$

where $\nu'^i_C = v'^i - v^i_C$ is the relative velocity of matter of body C located at point x'^i with respect to the origin of the local coordinates of the body, $v'^i = dx'^i/dt$, $v^i_C = dx^i_C/dt$. The difference between transformations (204) and (205) is in the presence of the very last term in (205) which is due to the Lie transport of an element of integration from the hypersurface $\mathcal{H}_t$ of constant time t to that $\mathcal{H}_{u_C}$ of u_C along worldlines of matter some of which are shown by dotted lines in Fig. 1. This term brings about a seemingly different appearance of our translational equations of motion for the center of mass of each body as compared with translational equations of motion derived by Racine and Flanagan [84] with corrections outlined in [85]. This is a matter of choice of the hypersurface of integration $\mathcal{H}_{u_C}$ in the local coordinates adapted to the body under consideration. We reconcile this issue in Appendix B; see discussion following Eq. (B17).

Equation (205) allows us to transform $\mathbb{I}^{\langle L\rangle}_C \equiv \mathbb{I}^{\langle L\rangle}_C(t)$ from the global to local coordinates as follows [[17], Eq. 6.60]:

$$\mathbb{I}^{\langle L\rangle}_C = \mathfrak{I}^{\langle L\rangle}_C - \frac{l}{2}v^k_C v^{\langle i_l}_C \mathfrak{I}^{L-1\rangle k}_C + lF^{k\langle i_l}_C \mathfrak{I}^{L-1\rangle k}_C - lD^{k\langle i_l}_C \mathfrak{I}^{L-1\rangle k}_C - l\mathfrak{I}^{jk\langle L-1}_C D^{i_l\rangle jk}_C - v^k_C \dot{\mathfrak{I}}^{k\langle L\rangle}_C + v^k_C R^k_C \dot{\mathfrak{I}}^{\langle L\rangle}_C + v^k_C \int_{\mathcal{V}_C} \rho^{*\prime}_C \nu'^k_C w'^{\langle L\rangle}_C d^3w'_C, \tag{206}$$

where a shorthand notation, $\rho^{*\prime}_C \equiv \rho^*_C(u_C, \boldsymbol{w}'_C)$, stands for the invariant density of matter at the integration point $\boldsymbol{w}'_C$ in the local coordinates, the moments

$$\mathfrak{I}^L_C \equiv \mathfrak{I}_C(u_C) = \int_{\mathcal{V}_C} \rho^{*\prime}_C w'^{i_1}_C w'^{i_2}_C \dots w'^{i_l}_C d^3w'_C, \tag{207}$$

are *symmetric* moments of body C depending on the local time u_C, and we have made use of the fact that the product of the mass density ρ^* with 3-dimensional coordinate volume is Lie invariant when transported from hypersurface $\mathcal{H}_t$ to hypersurface $\mathcal{H}_{u_C}$ along worldlines of matter, that is $\rho^*_C(t, \boldsymbol{x}')d^3x' = \rho^*_C(u_C, \boldsymbol{w}'_C)d^3w'_C$ [17]. Notice that formula (206) is not a pointwise transformation of the moments performed at the origin of the local coordinates adapted to body C because of the presence of the last but one term, $v^k_{\rm C}R^k_{\rm C}\dot{\mathfrak{I}}^{\langle L\rangle}_{\rm C}$, which depends on the coordinate distance $R^k_{\rm C}$ from the origin of the local coordinates to the field point (point P in Fig. 1). At first glance, the appearance of this term may look strange as by definition (203) the moments $\mathbb{I}^{\langle L\rangle}_{\rm C}$ are solely functions of time t alone. The reader should keep in mind that the moments $\mathfrak{I}^L_{\rm C}$ are functions of the local time $u_{\rm C}$ and, though both $\mathbb{I}^{\langle L\rangle}_{\rm C}$ and $\mathfrak{I}^L_{\rm C}$ are functions pinned down to the origin of the local coordinates adapted to body C, they are taken at different points on the worldline $\mathcal{W}$ of the origin because the field point (t, x^i) is considered as being fixed in the derivation of the transformation (206). Therefore, the transformation of the time arguments of the moments involves the time shift $\Delta t = v^k_{\rm C}R^k_{\rm C}$ of the moments along the worldline $\mathcal{W}$, which explains the origin of term $v^k_{\rm C}R^k_{\rm C}\dot{\mathfrak{I}}^{k\langle L\rangle}_{\rm C}$ in (206). It is worth noticing that the term $v^k_{\rm C}R^k_{\rm C}\dot{\mathfrak{I}}^{\langle L\rangle}_{\rm C}$ is not present in the transformation equations for multipole moments derived by Racine and Flanagan [84] as they have computed the multipoles of each body C at the value of the local time $u_{\rm C}$ taken at the center of mass of body C which is different from our convention. This leads to the translational equations of motion which look different from ours by several terms. This apparent difference is not an indicator of mistake but, as we show in Appendix B, a matter of computational approach and conventions.

It should be emphasized that the moments $\mathfrak{I}^L_{\rm C}$ are <u>*not*</u> the STF Cartesian tensors. Their STF projection is denoted as $\mathfrak{I}^{\langle L\rangle}_{\rm C}$ and, in general, $\mathfrak{I}^L_{\rm C} \neq \mathfrak{I}^{\langle L\rangle}_{\rm C}$. It means that after contraction of any two indices in (207) we get the trace $\mathfrak{I}^{kkL-2}_{\rm C} \neq 0$, and it must be taken into account in subsequent calculations. The STF part of the Newtonian-like moments (207) is related to the STF post-Newtonian internal mass multipoles, $\mathcal{M}^L_{\rm C} \equiv \mathcal{M}^L_{\rm C}(u_{\rm C})$, of body C as follows [234]:

$$\mathfrak{I}^{\langle L\rangle}_{\rm C} = \mathcal{M}^L_{\rm C}[1 + (2\beta-\gamma-1)\mathcal{P}_{\rm C}] - \int_{\mathcal{V}_{\rm C}} \rho^{*\prime}_{\rm C}\left[\left(\gamma+\frac{1}{2}\right)\nu'^2_{\rm C} + \Pi'_{\rm C} + \gamma\frac{\mathfrak{S}'^{kk}_{\rm C}}{\rho^{*\prime}_{\rm C}} - (2\beta-1)\hat{U}'_{\rm C}\right] w'^{\langle L\rangle}_{\rm C} d^3w'_{\rm C}$$
$$- \frac{1}{2(2l+3)}\left[\ddot{\mathcal{N}}^{\langle L\rangle}_{\rm C} - 4(1+\gamma)\frac{2l+1}{l+1}\dot{\mathcal{R}}^{\langle L\rangle}_{\rm C}\right] + \sum_{k=1}^{\infty}\frac{1}{k!}[\mathcal{Q}^K_{\rm C} + 2(\beta-1)\mathcal{P}^K_{\rm C}]\int_{\mathcal{V}_{\rm C}} \rho^{*\prime}_{\rm C} w'^{\langle K\rangle}_{\rm C} w'^{\langle L\rangle}_{\rm C} d^3w'_{\rm C}, \quad (208)$$

where a prime standing after function (like $\Pi'_{\rm C}$, etc.) in the integrand means that the function is taken at the point $w'^i_{\rm C}$, an overdot denotes a total time derivative with respect to the coordinate time $u_{\rm C}$ of the local coordinates adapted to body C,

$$\mathcal{P}^K_{\rm C} \equiv \sum_{\rm B\neq C}\partial_K U_{\rm B}(t, \boldsymbol{x}_{\rm C}) \quad (k \geq 0) \quad (209)$$

are monopole and higher-order external multipoles of the scalar field generated by all bodies being external to body C,

$$\mathcal{Q}^K_{\rm C} \equiv \sum_{\rm B\neq C}\partial_K U_{\rm B}(t, \boldsymbol{x}_{\rm C}) \quad (k \geq 2) \quad (210)$$

are higher-order gravitoelectric external multipoles of body C, and the local acceleration $\mathcal{Q}^i_{\rm C}$ is defined in (184) and must be referred to body C, and the noncanonical multipoles $\mathcal{N}^L_{\rm C}$ and $\mathcal{R}^L_{\rm C}$ are defined by equations similar to (123) and (124) where the integrals must be taken over a volume of body C,

$$\mathcal{N}^L_{\rm C} \equiv \int_{\mathcal{V}_{\rm C}} \rho^*_{\rm C}(u_C, \boldsymbol{w}_{\rm C}) w^2_{\rm C} w^{\langle L\rangle}_{\rm C} d^3w_{\rm C}, \quad (211)$$

$$\mathcal{R}^L_{\rm C} \equiv \int_{\mathcal{V}_{\rm C}} \rho^*_{\rm C}(u_C, \boldsymbol{w}_{\rm C}) \nu^k_{\rm C} w^{\langle kL\rangle}_{\rm C} d^3w_{\rm C}. \quad (212)$$

Now, we replace expression (206) for $\mathbb{I}^{\langle L\rangle}_{\rm C}$ to multipolar expansion (202) of the Newtonian potential of body C and use (208). It results in

$$U_{\rm C}(t,\boldsymbol{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\mathcal{M}^L_{\rm C}\partial_L\left(\frac{1}{R_{\rm C}}\right)[1+(2\beta-\gamma-1)\mathcal{P}_{\rm C}] - \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)$$
$$\times\left\{\int_{\mathcal{V}_{\rm C}} \rho^{*\prime}_{\rm C}\left[\left(\gamma+\frac{1}{2}\right)\nu'^2_{\rm C} + \Pi'_{\rm C} + \gamma\frac{\mathfrak{S}'^{kk}_{\rm C}}{\rho^{*\prime}_{\rm C}} - (2\beta-1)\hat{U}'_{\rm C}\right] w'^{\langle L\rangle}_{\rm C} d^3w'_{\rm C} + \frac{1}{2(2l+3)}\left[\ddot{\mathcal{N}}^{\langle L\rangle}_{\rm C} - 4(1+\gamma)\frac{2l+1}{l+1}\dot{\mathcal{R}}^{\langle L\rangle}_{\rm C}\right]\right.$$
$$- \sum_{k=1}^{\infty}\frac{1}{k!}[\mathcal{Q}^K_{\rm C} + 2(\beta-1)\mathcal{P}^K_{\rm C}]\int_{\mathcal{V}_{\rm C}} \rho^{*\prime}_{\rm C} w'^{\langle K\rangle}_{\rm C} w'^{\langle L\rangle}_{\rm C} d^3w'_{\rm C} + \frac{l}{2}v^k_{\rm C}v^{\langle i_l}_{\rm C}\mathfrak{I}^{L-1\rangle k}_{\rm C} - lF^{k\langle i_l}_{\rm C}\mathfrak{I}^{L-1\rangle k}_{\rm C} + lD^{k\langle i_l}_{\rm C}\mathfrak{I}^{L-1\rangle k}_{\rm C} + l\mathfrak{I}^{jk\langle L-1}_{\rm C}D^{i_l\rangle jk}_{\rm C}$$
$$\left. + v^k_{\rm C}\dot{\mathfrak{I}}^{k\langle L\rangle}_{\rm C} - v^k_{\rm C}R^k_{\rm C}\dot{\mathfrak{I}}^{\langle L\rangle}_{\rm C} - v^k_{\rm C}\int_{\mathcal{V}_{\rm C}} \rho^{*\prime}_{\rm C}\nu'^k_{\rm C} w'^{\langle L\rangle}_{\rm C} d^3w'_{\rm C}\right\}. \quad (213)$$

Neither the multipoles $\mathfrak{I}^L_{\rm C}$ nor the very last integral in (206) are the STF Cartesian tensors. Therefore, Eq. (213) must be further transformed to bring it to the form depending on the STF internal mass and spin multipoles, $\mathcal{M}^L_{\rm C}$ and $\mathcal{S}^L_{\rm C}$. This is achieved by making use of the following equations:

$$v^k_{\rm C}v^{\langle i_l}_{\rm C}\mathfrak{I}^{L-1\rangle k}_{\rm C}=v^k_{\rm C}v^{\langle i_l}_{\rm C}\mathcal{M}^{L-1\rangle k}_{\rm C}+\frac{l-1}{2l-1}v^{\langle i_1}_{\rm C}v^{i_2}_{\rm C}\mathcal{N}^{L-2\rangle}_{\rm C}, \quad (214)$$

$$D^{k\langle i_l}_{\rm C}\mathfrak{I}^{L-1\rangle k}_{\rm C}=\gamma\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C})\mathcal{M}^L_{\rm C}, \quad (215)$$

$$\mathfrak{I}^{jk\langle L-1}_{\rm C}D^{i_l\rangle jk}_{\rm C}=a^j_{\rm C}\mathcal{M}^{jL}_{\rm C}-\frac{1}{2(2l+1)}a^{\langle i_l}_{\rm C}\mathcal{N}^{L-1\rangle}_{\rm C}, \quad (216)$$

$$v^k_{\rm C}\dot{\mathfrak{I}}^{k\langle L\rangle}_{\rm C}=v^k_{\rm C}\dot{\mathcal{M}}^{kL}_{\rm C}+\frac{l}{2l+1}v^{\langle i_l}_{\rm C}\dot{\mathcal{N}}^{L-1\rangle}_{\rm C} \quad (217)$$

$$\int_{\mathcal{V}_{\rm C}}\rho^{*\prime}_{\rm C}\nu'^k_{\rm C}w'^{\langle L\rangle}_{\rm C}d^3w'_{\rm C}=\frac{1}{l+1}\dot{\mathcal{M}}^{kL}_{\rm C}+\frac{l}{l+1}\varepsilon^{kp\langle i_l}\mathcal{S}^{L-1\rangle p}_{\rm C}+\frac{2l-1}{2l+1}\delta^{k\langle i_l}\mathcal{R}^{L-1\rangle}_{\rm C}, \quad (218)$$

where the overdot denotes a total time derivative with respect to coordinate time $u_{\rm C}$ of the local coordinates adapted to body C, and we have used everywhere in the post-Newtonian terms $\mathfrak{I}^{\langle L\rangle}_{\rm C}=\mathcal{M}^L_{\rm C}$ which is valid in the approximation under consideration. Substituting (214)–(218) to Eq. (213) yields a multipolar post-Newtonian expansion of the Newtonian potential of body C given in terms of the internal active mass and spin multipoles of the body,

$$U_{\rm C}(t,\boldsymbol{x})=W_{\rm C}(t,\boldsymbol{x})+\Phi_{\rm C}(t,\boldsymbol{x}), \quad (219)$$

where

$$W_{\rm C}(t,\boldsymbol{x})=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm C}, \quad (220)$$

$$\begin{aligned}\Phi_{\rm C}(t,\boldsymbol{x})&=(2\beta-\gamma-1)\mathcal{P}_{\rm C}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm C}\\&-\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\left\{\int_{\mathcal{V}_{\rm C}}\rho^{*\prime}_{\rm C}\left[\left(\gamma+\frac{1}{2}\right)\nu'^2_{\rm C}+\Pi'_{\rm C}+\gamma\frac{\mathfrak{s}'^{kk}_{\rm C}}{\rho^{*\prime}_{\rm C}}-(2\beta-1)\hat{U}'_{\rm C}\right]w'^{\langle L\rangle}_{\rm C}d^3w'_{\rm C}\right.\\&+\frac{1}{2(2l+3)}\left[\ddot{\mathcal{N}}^L_{\rm C}-4(1+\gamma)\frac{2l+1}{l+1}\dot{\mathcal{R}}^L_{\rm C}\right]-\sum_{n=1}^{\infty}\frac{1}{n!}[\mathcal{Q}^N_{\rm C}+2(\beta-1)\mathcal{P}^N_{\rm C}]\int_C\rho^{*\prime}_{\rm C}w'^{\langle N\rangle}_{\rm C}w'^{\langle L\rangle}_{\rm C}d^3w'_{\rm C}\\&\left.+\frac{l}{2}v^k_{\rm C}v^{\langle i_l}_{\rm C}\mathcal{M}^{L-1\rangle k}_{\rm C}-lF^{k\langle i_l}_{\rm C}\mathcal{M}^{L-1\rangle k}_{\rm C}+l\gamma\bar{U}(t,\boldsymbol{x}_{\rm C})\mathcal{M}^L_{\rm C}+la^k_{\rm C}\mathcal{M}^{kL}_{\rm C}+v^k_{\rm C}\dot{\mathcal{M}}^{kL}_{\rm C}-v^k_{\rm C}R^k_{\rm C}\dot{\mathcal{M}}^L\right\}\\&+\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\partial_L\left(\frac{1}{R_{\rm C}}\right)v^k_{\rm C}\dot{\mathcal{M}}^{kL}_{\rm C}+\sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}\varepsilon_{kpq}v^k_{\rm C}\partial_{qL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}^{pL-1}_{\rm C}\\&+\sum_{l=1}^{\infty}\frac{(-1)^l(2l-1)}{(2l+1)l!}v^k_{\rm C}\partial_{kL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{R}^{L-1}_{\rm C}-\frac{1}{2}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+3)l!}\partial_{pqL}\left(\frac{1}{R_{\rm C}}\right)v^{\langle p}_{\rm C}v^q_{\rm C}\mathcal{N}^{L\rangle}_{\rm C}\\&-\frac{1}{2}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+3)l!}\partial_{pL}\left(\frac{1}{R_{\rm C}}\right)a^{\langle p}_{\rm C}\mathcal{N}^{L\rangle}_{\rm C}+\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+3)l!}\partial_{pL}\left(\frac{1}{R_{\rm C}}\right)v^{\langle p}_{\rm C}\dot{\mathcal{N}}^{L\rangle}_{\rm C}.\end{aligned} \quad (221)$$

The reader can notice that (221) includes explicitly a number of integrals depending on the intrinsic physical quantities of body C such as the internal velocity of matter $\nu^i_{\rm C}$, potential energy $\Pi_{\rm C}$, the stress tensor $\mathfrak{s}^{ij}_{\rm C}$, and self-gravity potential $\hat{U}_{\rm C}$, as well as the noncanonical multipoles, $\mathcal{N}^L_{\rm C}$ and $\mathcal{R}^L_{\rm C}$. The appearance of such terms is not expected in the final equations of motion if the principle of effacing of the internal structure is valid. Indeed, subsequent calculations demonstrate that the multipolar expansions of other gravitational potentials also contain similar terms depending on the internal structure of body C which are mutually canceled out in the final form of the post-Newtonian equations of motion.

Multipolar expansion of the Newtonian potential was rather cumbersome because we had to take into account the post-Newtonian corrections to the definitions of the internal multipoles $\mathcal{M}^L$ and to implement the post-Newtonian transformation from the global to local coordinates. Multipolar expansions of other external potentials are less laborious as they show up only in the post-Newtonian terms in definition of the external gravitoelectric multipoles $\mathcal{Q}_L$. Thus, their multipolar expansions can be performed by operating merely with the Newtonian part of the coordinate transformations and taking the leading (Newtonian-order) terms in the definition of the active internal multipoles $\mathcal{M}^L$.

B. Multipolar expansion of potential $\bar{U}^i$

The external vector potential $U^i_{\rm C}$ is defined in the global coordinates by Eq. (75) and depends on the velocity of matter v^i of the body C taken with respect to the origin of the global coordinates. This velocity is a linear sum of two pieces,

$$v^i = v^i_{\rm C} + \nu^i_{\rm C}, \tag{222}$$

where $v^i_{\rm C} = dx^i_{\rm C}/dt$ is velocity of the origin of local coordinates adapted to body C with respect to the global coordinates, and $\nu^i_{\rm C}$ is velocity of matter of body C with respect to the origin of the local coordinates. After accounting for the linear decomposition of the velocity, the vector potential $U^i_{\rm C}$ is expanded in terms of the internal multipoles as follows:

$$\begin{aligned}U^i_{\rm C}(t,\boldsymbol{x}) &= \int_{\mathcal{V}_{\rm C}} \frac{\rho^*(t,\boldsymbol{x}')v'^i}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm C}v^i_{\rm C} \\ &\quad + \sum_{l=1}^{\infty}\frac{(-1)^l}{(l+1)!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^{iL}_{\rm C} \\ &\quad + \sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}\varepsilon_{ipq}\partial_{qL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}^{pL-1}_{\rm C} \\ &\quad + \sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\frac{2l-1}{2l+1}\partial_{iL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{R}^{L-1}_{\rm C},\end{aligned} \tag{223}$$

where $\mathcal{M}^L_{\rm C}$ and $\mathcal{S}^L_{\rm C}$ are the canonical internal mass and spin multipoles of body C defined in (122) and (131) respectively, and $\mathcal{R}^L$ are the noncanonical multipoles of body C defined in (124).

C. Multipolar expansion of potential $\bar{\Psi}$

Multipolar expansion of the external potential $\bar{\Psi}$ entering the definition of the external tidal potential $\mathcal{Q}_L$ for body B is a sum of gravitational potentials of the bodies being external with respect to body B,

$$\bar{\Psi}(t,\boldsymbol{x}) = \sum_{\rm C\neq B}\Psi_{\rm C}(t,\boldsymbol{x}), \tag{224}$$

where

$$\begin{aligned}\Psi_{\rm C}(t,\boldsymbol{x}) \equiv{}& \left(\gamma+\frac{1}{2}\right)\Psi_{\rm C1}(t,\boldsymbol{x}) + (1-2\beta)\Psi_{\rm C2}(t,\boldsymbol{x}) \\ &+ \Psi_{\rm C3}(t,\boldsymbol{x}) + \gamma\Psi_{\rm C4}(t,\boldsymbol{x})\end{aligned} \tag{225}$$

is a linear superposition of potentials $\Psi_{\rm C1}$, $\Psi_{\rm C2}$, $\Psi_{\rm C3}$, $\Psi_{\rm C4}$ defined in (76)–(79) respectively.

Potential $\Psi_{\rm C1}$ is a quadratic functional of matter's velocity with respect to the global coordinates. The square of the velocity is split in three pieces in accordance with decomposition (222),

$$v^2 = v^2_{\rm C} + 2v^k_{\rm C}\nu^k_{\rm C} + \nu^2_{\rm C}. \tag{226}$$

Replacing v^2 with the right-hand side of (226) in (76), and performing multipolar decomposition of each integral with the help of (201), we obtain

$$\begin{aligned}&\Psi_{\rm C1}(t,\boldsymbol{x}) \\ &= \int_{\mathcal{V}_{\rm C}}\frac{\rho^*(t,\boldsymbol{x}')v'^2}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\rm C}}\right)\left(\mathcal{M}^L_{\rm C}v^2_{\rm C} + \int_{\mathcal{V}_{\rm C}}\rho^{*\prime}_{\rm C}\nu'^2_{\rm C}w'^{\langle L\rangle}_{\rm C}d^3w'_{\rm C}\right) \\ &\quad + 2\sum_{l=1}^{\infty}\frac{(-1)^l}{(l+1)!}\partial_L\left(\frac{1}{R_{\rm C}}\right)v^p_{\rm C}\dot{\mathcal{M}}^{pL}_{\rm C} \\ &\quad + 2\sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}v^k_{\rm C}\varepsilon_{kpq}\partial_{qL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}^{pL-1}_{\rm C} \\ &\quad + 2\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\frac{2l-1}{2l+1}v^k_{\rm C}\partial_{kL-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{R}^{L-1}_{\rm C},\end{aligned} \tag{227}$$

where a prime after a function means that the function is taken at the integration point with coordinates $w'^i_{\rm C}$ in the local coordinates adapted to body C.

Potential $\Psi_{\rm C2}$ depends on the total Newtonian potential U of all bodies in an $\mathbb{N}$-body system. It is split in two pieces,

$$U(t,\boldsymbol{x}) = U_{\rm C}(t,\boldsymbol{x}) + \bar{U}_{\rm C}(t,\boldsymbol{x}), \tag{228}$$

where $U_{\rm C}(t,\boldsymbol{x})$ is the Newtonian potential of body C, and $\bar{U}_{\rm C}(t,\boldsymbol{x}) = \sum_{\rm B\neq C}U_{\rm B}(t,\boldsymbol{x})$ is the Newtonian potential of all other bodies of the $\mathbb{N}$-body system. Transformation of the Newtonian potential from the global to local coordinates of body C is sufficient in the Newtonian approximation: $U_{\rm C}(t,\boldsymbol{x}) = U_{\rm C}(u_C,\boldsymbol{w}_{\rm C})$. The external Newtonian potential is decomposed in a Taylor series around the origin $x^i_{\rm C}$ of the local coordinates of body C, which is also transformed from the global to local coordinates,

$$\bar{U}_{\rm C}(t,\boldsymbol{x}) = \bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C}) + \sum_{k=1}^{\infty}\frac{1}{k!}\partial_K\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C})w^K_{\rm C}, \tag{229}$$

where we have used notations

$$\begin{aligned}\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C}) &\equiv \sum_{\rm B\neq C}U_{\rm B}(t,\boldsymbol{x}_{\rm C}), \\ \partial_K\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C}) &\equiv \lim_{\boldsymbol{x}\to\boldsymbol{x}_{\rm C}}\sum_{\rm B\neq C}\partial_{\langle i_1\ldots i_k\rangle}U_{\rm B}(t,\boldsymbol{x}).\end{aligned} \tag{230}$$

Taking the above considerations into account, and performing calculations of integrals, we get a multipolar decomposition of potential Ψ_{C2} in the following form:

$$\Psi_{C2}(t,\boldsymbol{x}) = \int_{\mathcal{V}_C} \frac{\rho^*(t,\boldsymbol{x}')U(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{R_C}\right) \int_{\mathcal{V}_C} \rho_C^{*\prime} U'_C w_C'^{\langle L\rangle} d^3w'_C$$
$$+ \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{R_C}\right) \left[\bar{U}_C(t,\boldsymbol{x}_C)\mathcal{M}_C^L\right.$$
$$\left. + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_K \bar{U}_C(t,\boldsymbol{x}_C) \int_{\mathcal{V}_C} \rho_C^{*\prime} w_C'^{\langle K\rangle} w_C'^{\langle L\rangle} d^3w'_C\right]. \tag{231}$$

Multipolar decompositions of potentials Ψ_{C3}, Ψ_{C4} are straightforward, and result in

$$\Psi_{C3}(t,\boldsymbol{x}) = \int_{\mathcal{V}_C} \frac{\rho^*(t,\boldsymbol{x}')\Pi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{R_C}\right) \int_{\mathcal{V}_C} \rho_C^{*\prime} \Pi'_C w_C'^{\langle L\rangle} d^3w'_C, \tag{232}$$

$$\Psi_{C4}(t,\boldsymbol{x}) = \int_{\mathcal{V}_C} \frac{\mathfrak{s}^{kk}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'$$
$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{R_C}\right) \int_{\mathcal{V}_C} \mathfrak{s}_C'^{kk} w_C'^{\langle L\rangle} d^3w'_C. \tag{233}$$

D. Multipolar expansion of potential $\bar{\chi}$

Multipolar expansion of external potential $\chi_C(t,\boldsymbol{x})$ defined by Eq. (81), is based on the multipolar expansion of coordinate distance $|\boldsymbol{x}-\boldsymbol{x}'| = |R_C - R'_C|$ that is a kernel of the integral in (81), near the origin of the local coordinates that is the point with coordinates $\boldsymbol{x}_C$. Taylor's expansion of the kernel $|\boldsymbol{x}-\boldsymbol{x}'|$ with respect to $\boldsymbol{x}'$ is given in terms of the Gegenbauer polynomials, $C_l^{(-\frac{1}{2})}(\boldsymbol{x})$, [[264], Sec. 8.93], and its STF expansion near the origin of the local coordinates of body B reads

$$|\boldsymbol{x}-\boldsymbol{x}'| = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} R_C'^{\langle L\rangle} \partial_L R_C$$
$$+ \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+3)l!} R_C'^2 R_C'^{\langle L\rangle} \partial_L \left(\frac{1}{R_C}\right). \tag{234}$$

Therefore, the multipolar expansion of external potential $\chi_C(t,\boldsymbol{x})$ has the following form:

$$\chi_C(t,\boldsymbol{x}) = -\int_{\mathcal{V}_C} \rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'| d^3x'$$
$$= -\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L R_C \mathcal{M}_C^L$$
$$- \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+3)l!} \partial_L \left(\frac{1}{R_C}\right) \mathcal{N}_C^L, \tag{235}$$

which is a direct consequence of integration of (234).

In what follows, we will need the multipolar expansion of the second partial derivative of the potential χ_C with respect to the global coordinate time, $\partial_t^2\chi_C(t,\boldsymbol{x})$, because it is this quantity that enters definition of the external gravitoelectric multipoles, $\mathcal{Q}_L$. The partial time derivative of χ_C with respect to the global coordinate time, t, should be transformed to the time derivative taken with respect to the local coordinate time u_C of body C which allow us to separate the internal, time-dependent physical processes inside body C from the temporal changes caused by motion of body C with respect to the global coordinates. The law of transformation of the first time derivative is derived directly from the coordinate transformation (196), (197) and is given by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u_C}\frac{\partial u_C}{\partial t} + \frac{\partial}{\partial w^i}\frac{\partial w^i}{\partial t} = \frac{\partial}{\partial u_C} - v_C^k \frac{\partial}{\partial R_C^k}, \tag{236}$$

where we have neglected all terms of the post-Newtonian order because they contribute only to the post-post-Newtonian approximation which we do not consider. Applying (236) one more time, we get for the second partial derivative

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial u_C^2} - 2v_C^k \frac{\partial^2}{\partial R_C^k \partial u_C} + v_C^k v_C^p \frac{\partial^2}{\partial R_C^k \partial R_C^p} - a_C^k \frac{\partial}{\partial R_C^k}. \tag{237}$$

Now, we employ (237) to calculate the second time derivative from expansion (235). In doing this, we remind the reader that the internal potentials, $\mathcal{M}_C^L$ and $\mathcal{N}_C^L$, are functions of the local coordinate time u_C only, and the partial derivative $\partial/\partial R_C^i = \partial/\partial x^i \equiv \partial_i$. Therefore, taking the second time derivative from χ_C results in

$$\frac{\partial^2\chi_C}{\partial t^2} = -\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} [\ddot{\mathcal{M}}_C^L \partial_L R_C - 2\dot{\mathcal{M}}_C^L v_C^k \partial_{kL} R_C$$
$$+ \mathcal{M}_C^L v_C^k v_C^p \partial_{kpL} R_C - \mathcal{M}_C^L a_C^k \partial_{kL} R_C]$$
$$- \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+3)l!} \left[\partial_L \left(\frac{1}{R_C}\right) \ddot{\mathcal{N}}_C^L - 2v_C^k \partial_{kL} \left(\frac{1}{R_C}\right) \dot{\mathcal{N}}_C^L\right.$$
$$\left. + v_C^k v_C^p \partial_{kpL} \left(\frac{1}{R_C}\right) \mathcal{N}_C^L - a_C^k \partial_{kL} \left(\frac{1}{R_C}\right) \mathcal{N}_C^L\right], \tag{238}$$

where we have discarded all post-Newtonian terms as they contribute only to the second post-Newtonian approximation which we do not consider; $v^i_{\rm C} = dx^i_{\rm C}/dt$ and $a^i_{\rm C} = dv^i_{\rm C}/dt$ are, respectively, velocity and acceleration of the origin of the local coordinates of body C with respect to the global coordinates.

It is worth noticing that the partial derivatives from function $1/R_{\rm C}$ like $\partial_L(1/R_{\rm C})$, $\partial_{kL}(1/R_{\rm C})$, etc., are STF derivatives with respect to all indices. At the same time the partial derivatives from $R_{\rm C}$, like $\partial_L R_{\rm C}$, $\partial_{kL} R_{\rm C}$, etc., are not STF derivatives with respect to their indices; only that part of indices in the derivatives which is contracted with STF multipoles becomes symmetric and trace free. Transformation of the partial derivatives from $R_{\rm C}$ to their STF counterpart will be required in derivation of the equations of motion and is given below in (268).

VIII. MULTIPOLAR EXPANSION OF EXTERNAL MULTIPOLES IN THE GLOBAL COORDINATES

The external tidal multipoles $\mathcal{P}_L$, $\mathcal{Q}_L$, and $\mathcal{C}_L$ of body B have been introduced in Sec. V C 4 in the form of the STF partial derivatives from the external potentials. We need explicit expressions of the external multipoles in terms of the multipolar series with respect to the internal multipoles of the extended bodies for calculating equations of motion of an $\mathbb{N}$-body system in the global coordinates. The present section provides this multipolar decomposition.

A. Scalar-field multipoles $\mathcal{P}_L$

Multipolar decomposition of the external scalar-field multipoles $\mathcal{P}_L$ of body B is obtained from (153) where the scalar field $\bar{\varphi}(t,\boldsymbol{x}_{\rm B}) = \bar{W}(t,\boldsymbol{x}_{\rm B})$ of external bodies and $\bar{W}(t,\boldsymbol{x}_{\rm B}) = \sum_{{\rm C}\neq{\rm B}} W_C(t,\boldsymbol{x}_{\rm B})$ is the external Newtonian potential. Multipolar decomposition of the potential $W_{\rm C}(\boldsymbol{x}_B)$ of body C is given in (220). Making use of it, we get the external scalar-field multipoles

$$\mathcal{P}_L = \partial_L \bar{W}(t,\boldsymbol{x}_{\rm B}) = \sum_{{\rm C}\neq{\rm B}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{M}^N_{\rm C} \partial_{LN}\left(\frac{1}{R_{\rm C}}\right)_{\boldsymbol{x}=\boldsymbol{x}_{\rm B}}, \tag{239}$$

where the STF index N should not be confused with the number $\mathbb{N}$ of the extended bodies in the $\mathbb{N}$-body system. Expression (239) will be used later for calculating the post-Newtonian part of the gravitational force depending on the external scalar-field multipoles.

B. Gravitoelectric multipoles $\mathcal{Q}_L$

Gravitoelectric multipoles $\mathcal{Q}_L$ are defined by Eq. (155). It is instructive to introduce potentials $\bar{W}$, $\bar{V}$, and $\bar{V}^i$ as linear combinations of potentials $W_{\rm C}$, $V_{\rm C}$, $V^i_{\rm C}$ of individual bodies from the $\mathbb{N}$-body system,

$$\bar{W}(t,\boldsymbol{x}) \equiv \sum_{{\rm C}\neq{\rm B}} W_{\rm C}(t,\boldsymbol{x}), \qquad \bar{V}(t,\boldsymbol{x},l) \equiv \sum_{{\rm C}\neq{\rm B}} V_{\rm C}(t,\boldsymbol{x},l), \qquad \bar{V}_i(t,\boldsymbol{x},l) \equiv \sum_{{\rm C}\neq{\rm B}} V^i_{\rm C}(t,\boldsymbol{x},l), \tag{240}$$

where the scalar potential $W_{\rm C}$ has been defined earlier in (220), the scalar potential

$$V_{\rm C}(t,\boldsymbol{x},l) \equiv \Phi_{\rm C}(t,\boldsymbol{x}) + \Psi_{\rm C}(t,\boldsymbol{x}) - \frac{1}{2}\partial_{tt}\chi_{\rm C}(t,\boldsymbol{x}) - 2(1+\gamma)v^k_{\rm B} U^k_{\rm C}(t,\boldsymbol{x}) + (1+\gamma)v^2_{\rm B} U_{\rm C}(t,\boldsymbol{x}) + (2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})U_{\rm C}(t,\boldsymbol{x}), \tag{241}$$

and the vector potential

$$V^i_{\rm C}(t,\boldsymbol{x},l) \equiv 2(1+\gamma)\dot{U}^i_{\rm C}(t,\boldsymbol{x}) + (l-2-2\gamma)v^i_{\rm B}\dot{U}_{\rm C}(t,\boldsymbol{x}) - \frac{l}{2}v^i_{\rm B}v^k_{\rm B}\partial_k U_{\rm C}(t,\boldsymbol{x}) - (l^2-l+2+2\gamma)a^i_{\rm B}U_{\rm C}(t,\boldsymbol{x}) - lF^{ki}_{\rm B}\partial_k U_{\rm C}(t,\boldsymbol{x}). \tag{242}$$

Notice that potentials $V_{\rm C}(t,\boldsymbol{x},l)$ and $V^i_{\rm C}(t,\boldsymbol{x},l)$ depend explicitly on the multipolar index l. In terms of the new potentials the gravitoelectric multipole $\mathcal{Q}_L$ takes on a simpler expression,

$$\mathcal{Q}_L = \partial_{\langle L\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B}) + \partial_{\langle L\rangle}\bar{V}(\boldsymbol{x}_{\rm B},l) + \partial_{\langle L-1}\bar{V}_{i_l\rangle}(\boldsymbol{x}_{\rm B},l) + X_{\langle L\rangle} \quad (l\geq 2). \tag{243}$$

Multipolar expansion of potential $W_{\rm C}$ is given in (220). Multipolar expansion of two other gravitational potentials are obtained form the results of Sec. VIII

$$\begin{aligned}V_{\rm C}(t,\boldsymbol{x},l) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\left[(1+\gamma)v_{\rm B}^2+\left(\gamma+\frac{1}{2}\right)v_{\rm C}^2\right]\mathcal{M}_{\rm C}^N\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)[(2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})-\gamma(n+1)\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C})]\mathcal{M}_{\rm C}^N\\ &-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\left[\frac{n}{2}v_{\rm C}^pv_{\rm C}^{i_n}\mathcal{M}_{\rm C}^{pN-1}-nF_{\rm C}^{pi_n}\mathcal{M}_{\rm C}^{pN-1}+(n+1)a_{\rm C}^p\mathcal{M}_{\rm C}^{pN}+v_{\rm C}^p\dot{\mathcal{M}}_{\rm C}^{pN}-v_{\rm C}^pR_{\rm C}^p\dot{\mathcal{M}}_{\rm C}^N\right]\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{1}{2}\ddot{\mathcal{M}}_{\rm C}^N\partial_NR_{\rm C}-\dot{\mathcal{M}}_{\rm C}^Nv_{\rm C}^p\partial_{pN}R_{\rm C}+\frac{1}{2}\mathcal{M}_{\rm C}^Nv_{\rm C}^pv_{\rm C}^q\partial_{pqN}R_{\rm C}-\frac{1}{2}\mathcal{M}_{\rm C}^Na_{\rm C}^p\partial_{pN}R_{\rm C}\right]\\ &+2(1+\gamma)\left[\sum_{n=1}^{\infty}\frac{(-1)^nn}{(n+1)!}\varepsilon_{kpq}\partial_{pN-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}_{\rm C}^{qN-1}v_{\rm BC}^k-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}_{\rm C}^{pN}v_{\rm BC}^p\right.\\ &-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)v_{\rm B}^pv_{\rm C}^p\mathcal{M}_{\rm C}^N-\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{2n-1}{2n+1}\partial_{pN-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{R}_{\rm C}^{N-1}v_{\rm BC}^p\\ &\left.-\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{2n-1}{2n+1}\partial_{N-1}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{R}}_{\rm C}^{N-1}\right],\end{aligned}\tag{244}$$

where

$$v_{\rm BC}^i\equiv v_{\rm B}^i-v_{\rm C}^i\tag{245}$$

is the relative coordinate velocity between the bodies B and C, and the external potentials $\bar{U}$ and $\bar{U}_{\rm C}$ have been defined in (68) and (230).

Expression (242) for $V_{\rm C}^i$ contains the total time derivatives from the potentials taken on the worldline, $x_{\rm B}^i(t)$, of the origin of the local coordinates adapted to body B. It is expressed in terms of the partial time and spatial derivatives as follows:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_{\rm B}^i\frac{\partial}{\partial x^i},\tag{246}$$

where $v_{\rm B}^i=dx_{\rm B}^i/dt$ is velocity of the origin of the local coordinates adapted to body B with respect to the global coordinates. The partial time derivative in (246) is taken with respect to the variables associated with each body C that is external to the body B. It is related to the partial time derivative taken with respect to the local coordinate time $u_{\rm C}$ of body C by Eq. (236). Hence, the total time derivative from the external potentials associated with body C taken on the worldline of body B reads

$$\frac{d}{dt}=\frac{\partial}{\partial u_{\rm C}}+v_{\rm BC}^i\frac{\partial}{\partial x^i}.\tag{247}$$

where again $v_{\rm BC}^i\equiv v_{\rm B}^i-v_{\rm C}^i$ is the relative velocity between two bodies, B and C. After employing (247) for taking the total time derivatives in (242) and the multipolar expansions of other potentials entering the definition of $\bar{V}_{\rm C}^i$, we get

$$\begin{aligned}V_{\rm C}^i(t,\boldsymbol{x},l) &= 2(1+\gamma)\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\ddot{\mathcal{M}}_{\rm C}^{iN}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}_{\rm C}^Nv_{\rm C}^i+\mathcal{M}_{\rm C}^Na_{\rm C}^i)\right.\\ &-\sum_{n=1}^{\infty}\frac{(-1)^nn}{(n+1)!}\varepsilon_{ipq}\partial_{pN-1}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{S}}_{\rm C}^{qN-1}-\sum_{n=1}^{\infty}\frac{(-1)^nn}{(n+1)!}\varepsilon_{ipq}\partial_{kpN-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}_{\rm C}^{qN-1}v_{\rm BC}^k\\ &+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}_{\rm C}^{iN}v_{\rm BC}^p+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^Nv_{\rm BC}^pv_{\rm C}^i\\ &\left.+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{2n-1}{2n+1}\partial_{iN-1}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{R}}_{\rm C}^{N-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{2n-1}{2n+1}\partial_{ipN-1}\left(\frac{1}{R_{\rm C}}\right)\mathcal{R}_{\rm C}^{N-1}v_{\rm BC}^p\right]\\ &+(l-2-2\gamma)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}_{\rm C}^Nv_{\rm B}^i+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^Nv_{\rm BC}^pv_{\rm B}^i\right]\\ &-(l^2-l+2+2\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^Na_{\rm B}^i-\frac{l}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^Nv_{\rm B}^pv_{\rm B}^i-lF_{\rm B}^{ki}\partial_k\bar{U}(t,\boldsymbol{x}_{\rm B}).\end{aligned}\tag{248}$$

Multipolar expansion of the external gravitoelectric multipole $\mathcal{Q}_L$ is obtained by substituting (244) and (248) to definition (243). It is remarkable that each potential, $V(t,\boldsymbol{x},l)$ and $\bar{V}_i(t,\boldsymbol{x},l)$, entering (243) depends separately on the noncanonical multipoles $\mathcal{R}^L$ and $\mathcal{N}^L$ but they are mutually canceled out in the linear combination $\partial_{\langle L\rangle}\bar{V}(t,\boldsymbol{x},l)+\partial_{\langle L-1}\bar{V}_{i\rangle}(t,\boldsymbol{x},l)$ so that the gravitoelectric multipoles $\mathcal{Q}_L$ depend exclusively on the canonical internal active mass, $\mathcal{M}^L_{\rm C}$, and spin, $\mathcal{S}^L_{\rm C}$, multipoles. We do not provide here the explicit expression for the multipolar decomposition of $\mathcal{Q}_L$. It will be given below in Sec. IX A 1.

C. Gravitomagnetic multipoles $\mathcal{C}_L$

Gravitomagnetic external multipoles, $\mathcal{C}_L$, have been defined in (157). They represent a linear combination of the gravitomagnetic multipoles, $H^{ikL}_{\rm C}$, generated by all bodies of the $\mathbb{N}$-body system which are external with respect to body C. More specifically, we reformulate (157) as follows:

$$\varepsilon_{ipk}\mathcal{C}_{pL}\equiv\bar{H}_{ikL}, \tag{249}$$

where

$$\bar{H}_{ikL}=\sum_{{\rm C}\neq{\rm B}}H^{ikL}_{\rm C}(t,\boldsymbol{x}_{\rm B}), \tag{250}$$

and $H^{ikL}_{\rm C}$ is a skew-symmetric tensor with respect to the first two indices and STF tensor with respect to the multi-index L, that is $H^{ikL}_{\rm C}\equiv H^{[ik]\langle L\rangle}_{\rm C}$. The same property naturally holds for $\bar{H}_{ikL}$.

For each body C Eq. (157) yields

$$\begin{aligned}H^{ikL}_{\rm C}(t,\boldsymbol{x})&=4(1+\gamma)\{v^{[i}_{\rm B}\partial^{k]L}U_{\rm C}(t,\boldsymbol{x})+\partial^{L[i}U^{k]}_{\rm C}(t,\boldsymbol{x})\}\\&\quad-2(1+\gamma)\frac{l}{l+1}\{\delta^{i\langle i_l}\partial^{L-1\rangle k}\dot{U}_{\rm C}(t,\boldsymbol{x})\\&\quad-\delta^{k\langle i_l}\partial^{L-1\rangle i}\dot{U}_{\rm C}(t,\boldsymbol{x})\}.\end{aligned} \tag{251}$$

According to definition (249) we have $\varepsilon_{ipk}\mathcal{C}_{pkL-1}\equiv 0$ due to the antisymmetry of the Levi-Civita symbol and the STF symmetry of $\mathcal{C}_L$. It follows, then, that $H^{ik\langle kL-1\rangle}_{\rm C}=0$ as well. This property can be confirmed by inspection after contracting the corresponding indices in the right-hand side of (251), and remembering that according to equation of continuity (36), we have in the global coordinates, $\partial_k U^k_{\rm C}+\partial_t U_{\rm C}=0$.

Multipolar expansion of $H^{ikL}_{\rm C}$ is obtained after making use of multipolar decomposition of potentials $U_{\rm C}$ and $U^i_{\rm C}$ given above in Secs. VII A and VII B,

$$\begin{aligned}H^{ikL}_{\rm C}(t,\boldsymbol{x})&=2(1+\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[v^i_{\rm BC}\partial^{kLN}\left(\frac{1}{R_{\rm C}}\right)-v^k_{\rm BC}\partial^{iLN}\left(\frac{1}{R_{\rm C}}\right)\right]\mathcal{M}^N_{\rm C}\\&\quad-2(1+\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\left[\dot{\mathcal{M}}^{iN}_{\rm C}\partial^{kLN}\left(\frac{1}{R_{\rm C}}\right)-\dot{\mathcal{M}}^{kN}_{\rm C}\partial^{iLN}\left(\frac{1}{R_{\rm C}}\right)\right]\\&\quad+2(1+\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)n!}\left[\varepsilon^{pqi}\partial^{kqLN}\left(\frac{1}{R_{\rm C}}\right)-\varepsilon^{pqk}\partial^{iqLN}\left(\frac{1}{R_{\rm C}}\right)\right]\mathcal{S}^{pN}_{\rm C}\\&\quad-2(1+\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n l}{(l+1)n!}v^p_{\rm BC}\left[\delta^{i\langle i_l}\partial^{L-1\rangle Npk}\left(\frac{1}{R_{\rm C}}\right)-\delta^{k\langle i_l}\partial^{L-1\rangle Npi}\left(\frac{1}{R_{\rm C}}\right)\right]\mathcal{M}^N_{\rm C}\\&\quad-2(1+\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n l}{(l+1)n!}\left[\delta^{i\langle i_l}\partial^{L-1\rangle Nk}\left(\frac{1}{R_{\rm C}}\right)-\delta^{k\langle i_l}\partial^{L-1\rangle Ni}\left(\frac{1}{R_{\rm C}}\right)\right]\dot{\mathcal{M}}^N_{\rm C}.\end{aligned} \tag{252}$$

It is worth noticing that the noncanonical multipoles $\mathcal{R}_L$ which are present in the multipolar expansion (223) of the external gravitomagnetic potential $\bar{U}^i_{\rm C}$ are canceled out in (252) after taking the skew-symmetric partial derivative, $\bar{U}^{[i,k]}$. Therefore, the external gravitomagnetic multipoles, $\mathcal{C}_L$, do not depend on the noncanonical multipoles $\mathcal{R}_L$.

IX. TRANSLATIONAL EQUATIONS OF MOTION OF BODIES IN THE GLOBAL COORDINATES

The aim of this section is to derive the post-Newtonian equations of translational motion of extended bodies in the global coordinates while taking into account all possible gravitational interactions taking place between mass and spin internal multipoles of the bodies in an $\mathbb{N}$-body system. Our derivation is based on the Fock-Papapetrou method along with the matched asymptotic expansions technique and significantly extends the post-Newtonian equations of motion of extended bodies in gravitationally bound systems beyond the pole-dipole approximation. A similar task was set forth and solved in the post-Newtonian approximation of general relativity by Racine and Flanagan [84] and Racine *et al.* [85] who used the EIH technique of surface integration along with the post-Newtonian transformations of asymptotic expansions of the metric tensors and Blanchet-Damour multipole formalism. We shall compare our translational equations with those derived previously by Racine and Flanagan [84] and Racine *et al.* [85] in Appendix B.

A. Computation of gravitational force

1. Reduction of similar terms

Translational equations of motion of the center of mass of body B in the global coordinates follow directly from the equations of motion (152) of the origin of the local coordinates adapted to body B after making use of the specific value of the local acceleration $\mathcal{Q}_i$ defined in (184)–(186) and the multipolar decomposition of the external multipoles $\mathcal{P}_L$, $\mathcal{Q}_L$, $\mathcal{C}_L$ provided in Sec. VIII. This makes the worldline $\mathcal{W}$ of the origin of the local coordinates of body B identical with the worldline $\mathcal{Z}$ of the body's center of mass.

It is instrumental to rewrite the right-hand side of (152) in terms of the gravitational potentials $\bar{V}(t,\boldsymbol{x},l)$ and $\bar{V}_i(t,\boldsymbol{x},l)$ introduced above in (241) and (242). We have

$$a^i_{\rm B} = \partial_i\bar{W}(t,\boldsymbol{x}_{\rm B}) - \mathcal{Q}^{\rm N}_i + \partial_i\bar{V}(t,\boldsymbol{x}_{\rm B},1) + \bar{V}_i(t,\boldsymbol{x}_{\rm B},1) - \mathcal{Q}^{\rm pN}_i + \frac{\ddot{\mathcal{I}}^i_{\rm c}}{M_{\rm B}} - \frac{1}{2}v^i_{\rm B}v^k_{\rm B}a^k_{\rm B} - F^{ik}_{\rm B}a^k_{\rm B} - v^2_{\rm B}a^i_{\rm B} + \gamma a^i_{\rm B}\bar{U}(t,\boldsymbol{x}_{\rm B}), \tag{253}$$

where accelerations $\mathcal{Q}^{\rm N}_i$ and $\mathcal{Q}^{\rm pN}_i$ are determined by (185) and (186), the external gravitational potentials

$$\bar{V}(t,\boldsymbol{x},1) \equiv \sum_{\rm C\neq B} V_{\rm C}(t,\boldsymbol{x},1), \quad \bar{V}_i(t,\boldsymbol{x},1) \equiv \sum_{\rm C\neq B} V^i_{\rm C}(t,\boldsymbol{x},1), \tag{254}$$

and gravitational potentials of body C are given respectively by (241) and (242) for the value of multipole index $l=1$,

$$V_{\rm C}(t,\boldsymbol{x},1) \equiv \Phi_{\rm C}(t,\boldsymbol{x}) + \Psi_{\rm C}(t,\boldsymbol{x}) - \frac{1}{2}\partial_{tt}\chi_{\rm C}(t,\boldsymbol{x}) - 2(1+\gamma)v^k_{\rm B}U^k_{\rm C}(t,\boldsymbol{x}) + (1+\gamma)v^2_{\rm B}U_{\rm C}(t,\boldsymbol{x}) + (2-2\beta-\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})U_{\rm C}(t,\boldsymbol{x}), \tag{255}$$

$$V^i_{\rm C}(t,\boldsymbol{x},1) \equiv 2(1+\gamma)\dot{U}^i_{\rm C}(t,\boldsymbol{x}) - (1+2\gamma)v^i_{\rm B}\dot{U}_{\rm C}(t,\boldsymbol{x}) - \frac{1}{2}v^i_{\rm B}v^k_{\rm B}\partial_k U_{\rm C}(t,\boldsymbol{x}) - 2(1+\gamma)a^i_{\rm B}U_{\rm C}(t,\boldsymbol{x}) - F^{ki}_{\rm B}\partial_k U_{\rm C}(t,\boldsymbol{x}). \tag{256}$$

Local acceleration $\mathcal{Q}^{\rm N}_i$ of the center of mass of body B is given by (185) where the external gravitoelectric multipoles $\mathcal{Q}_L$ are defined in (243) in terms of the derivatives from the potentials $\bar{W}(t,\boldsymbol{x})$, $\bar{V}(t,\boldsymbol{x},l)$, and $\bar{V}_i(t,\boldsymbol{x},l)$. Taking into account in the definition (185) of $\mathcal{Q}^{\rm N}_i$ that, according to (239), the external scalar-field dipole $\mathcal{P}_i = \partial_i\bar{W}(t,\boldsymbol{x}_{\rm B})$, we can reduce relativistic equation of motion (253) to the form of the second Newton's law,

$$M_{\rm B}a^i_{\rm B} = F^i, \tag{257}$$

where $M_{\rm B}$ is the conformal mass of body B, and

$$F^i = \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B} - \sum_{l=0}^{\infty}\frac{1}{l!}\left\{[v^2_{\rm B} - \gamma\bar{U}(t,\boldsymbol{x}_{\rm B})]\partial_{\langle iL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B} + \frac{1}{2}v^i_{\rm B}v^k_{\rm B}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B}\right\} + \sum_{l=0}^{\infty}\frac{1}{l!}\{\partial_{\langle iL\rangle}\bar{V}(t,\boldsymbol{x}_{\rm B},l+1)\mathcal{M}^L_{\rm B} + \partial_{\langle L}\bar{V}_{i\rangle}(t,\boldsymbol{x}_{\rm B},l+1)\mathcal{M}^L_{\rm B} - M_{\rm B}\mathcal{Q}^{\rm pN}_i + \ddot{\mathcal{I}}^i_{\rm c} - F^{ik}_{\rm B}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_B)\mathcal{M}^L_{\rm B}\} \tag{258}$$

is a relativistic force exerted on body B by external bodies of the $\mathbb{N}$-body system. It depends explicitly on the active internal multipoles, $\mathcal{M}^L_{\rm B}$ of body B, and we identify, here and everywhere else, the active mass $\mathcal{M}_{\rm B}$ of body B with a monopole value ($l=0$) of the active mass multipole of body B, that is $\mathcal{M}\equiv\mathcal{M}_{\rm B}$. It is instructive to emphasize that the Newtonian part of the force, given by the first term in the right-hand side of (258), depends on the active dipole, $\mathcal{M}^i_{\rm B}$, of body B which does not vanish in scalar-tensor theory of gravity because the position of the center of mass of body B is defined by the condition (176) of vanishing of the conformal dipole moment, $\mathcal{I}^i_{\rm B}$ of body B.[8]

Before proceeding to the explicit calculation of the gravitational force, we notice that there are some cancellations of similar terms in (258). More specifically, we note the following:

The very last term in the third line of (244) can be transformed to

[8]We remind the reader that in general case, $\mathcal{M}^i_{\rm B}\neq\mathcal{I}^i_{\rm B}$. The difference has a post-Newtonian order of magnitude.

$$\dot{\mathcal{M}}_{\mathrm{C}}^{N} v_{\mathrm{C}}^{p} R_{\mathrm{C}}^{p} \partial_{N}\left(\frac{1}{R_{\mathrm{C}}}\right) = \dot{\mathcal{M}}_{\mathrm{C}}^{a_1\ldots a_n} v_{\mathrm{C}}^{p}\left[\partial_{\langle pa_1\ldots a_n\rangle} R_{\mathrm{C}} - n\frac{2n-1}{2n+1}\delta_{p\langle a_1}\partial_{a_2\ldots a_n\rangle}\left(\frac{1}{R_{\mathrm{C}}}\right)\right]$$
$$= \dot{\mathcal{M}}_{\mathrm{C}}^{a_1\ldots a_n} v_{\mathrm{C}}^{p}\left[\partial_{pa_1\ldots a_n} R_{\mathrm{C}} - \frac{2}{2n+1}\delta_{\{pa_1}\partial_{a_2\ldots a_n\}}\left(\frac{1}{R_{\mathrm{C}}}\right) - n\frac{2n-1}{2n+1}\delta_{p\langle a_1}\partial_{a_2\ldots a_n\rangle}\left(\frac{1}{R_{\mathrm{C}}}\right)\right]$$
$$= \dot{\mathcal{M}}_{\mathrm{C}}^{N} v_{\mathrm{C}}^{p}\partial_{pN} R_{\mathrm{C}} - n v_{\mathrm{C}}^{p}\dot{\mathcal{M}}_{\mathrm{C}}^{pN-1}\partial_{N-1}\left(\frac{1}{R_{\mathrm{C}}}\right). \tag{259}$$

The first and second terms in the very last line of (259) cancel out, respectively, the second term in the forth line of (244) and the forth term in the third line of (244) which all depend on the time derivative $\dot{\mathcal{M}}_{\mathrm{C}}^{L}$.
The very last term in (248) enters Eq. (258) in STF form $(l+1)F_{\mathrm{B}}^{k\langle i}\partial^{L\rangle k}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})$ which can be decomposed with the help of peeling formula (A1) separating the STF index i from that L, so that we get

$$\sum_{l=0}^{\infty}\frac{l+1}{l!}F_{\mathrm{B}}^{k\langle i}\partial^{L\rangle k}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{L} = \sum_{l=0}^{\infty}\frac{1}{l!}F_{\mathrm{B}}^{ki}\partial_{kL}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{L} + \sum_{l=0}^{\infty}\frac{1}{l!}F_{\mathrm{B}}^{kp}\partial_{ikL}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{pL}. \tag{260}$$

The first term in the right-hand side of (260) cancels the very last (precessional) term in (258).

Each potential $V(t,\boldsymbol{x},l+1)$ and $\bar{V}_i(t,\boldsymbol{x},l+1)$ entering (258) depends on the noncanonical multipoles $\mathcal{R}^{L}$ and $\mathcal{N}^{L}$ but they are mutually canceled out in the linear combination $\partial_{\langle iL\rangle}\bar{V}(t,\boldsymbol{x},l+1)+\partial_{\langle L}\bar{V}_{i\rangle}(t,\boldsymbol{x},l+1)$ so that the right side of the translational equations of motion (258) depends only on the active internal mass and spin multipoles $\mathcal{M}_{\mathrm{C}}^{L}$ and $\mathcal{S}_{\mathrm{C}}^{L}$ of the bodies.

Finally, we notice that the post-Newtonian term, X_L, which is a part of $\mathcal{Q}_L$, does not appear in (258). The term X_L would appear in (258) only in the form of the quadrupole-dipole coupling, $X_{ip}\mathcal{M}_{\mathrm{B}}^{p}$, as a consequence of its definition (156). However, with sufficient accuracy the active mass dipole $\mathcal{M}_{\mathrm{B}}^{p}=\mathcal{I}_{\mathrm{B}}^{p}=0$ in the post-Newtonian approximation due to the choice of the center of mass (176).
The above-mentioned cancellations simplify (258) and recast it to

$$F^{i} = \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{W}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{L} - \left(v_{\mathrm{B}}^{2}\delta^{ik}+\frac{1}{2}v_{\mathrm{B}}^{i}v_{\mathrm{B}}^{k}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{L}$$
$$-F_{\mathrm{B}}^{pk}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle ipL\rangle}\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})\mathcal{M}_{\mathrm{B}}^{kL} + \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{\Omega}(t,\boldsymbol{x}_{\mathrm{B}},l)\mathcal{M}_{\mathrm{B}}^{L}$$
$$+\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle L}\bar{\Omega}_{i\rangle}(t,\boldsymbol{x}_{\mathrm{B}},l)\mathcal{M}_{\mathrm{B}}^{L} - M_{\mathrm{B}}\mathcal{Q}_{i}^{\mathrm{pN}} + \ddot{M}_{\mathrm{c}}^{i}, \tag{261}$$

where the external potentials

$$\bar{W}(t,\boldsymbol{x}) \equiv \sum_{\mathrm{C}\neq\mathrm{B}} W_{\mathrm{C}}(t,\boldsymbol{x}), \qquad \bar{\Omega}(t,\boldsymbol{x},l) \equiv \sum_{\mathrm{C}\neq\mathrm{B}}\Omega_{\mathrm{C}}(t,\boldsymbol{x},l), \qquad \bar{\Omega}_i(t,\boldsymbol{x},l) \equiv \sum_{\mathrm{C}\neq\mathrm{B}}\Omega_{\mathrm{C}}^{i}(t,\boldsymbol{x},l), \tag{262}$$

represent the linear superposition of gravitational potentials W_{C}, Ω_{C}, and Ω_{C}^{i} generated by body $\mathrm{C}\neq\mathrm{B}$. Multipolar expansion of potential $W_{\mathrm{C}}(t,\boldsymbol{x})$ is given in (220). The new potentials $\Omega_{\mathrm{C}}(t,\boldsymbol{x},l)$ and $\Omega_{\mathrm{C}}^{i}(t,\boldsymbol{x},l)$ are modifications of $V_{\mathrm{C}}(t,\boldsymbol{x},l+1)$ and $V_{\mathrm{C}}^{i}(t,\boldsymbol{x},l+1)$ respectively after taking into account the above-mentioned cancellations of similar terms in (258). They read

$$\Omega_{\mathrm{C}}(t,\boldsymbol{x},l) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{N}\left(\frac{1}{R_{\mathrm{C}}}\right)\left[(\gamma+1)v_{\mathrm{B}}^{2}+\left(\gamma+\frac{1}{2}\right)v_{\mathrm{C}}^{2}\right]\mathcal{M}_{\mathrm{C}}^{N}$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{N}\left(\frac{1}{R_{\mathrm{C}}}\right)[(2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\mathrm{B}})-\gamma(n+1)\bar{U}_{\mathrm{C}}(t,\boldsymbol{x}_{\mathrm{C}})]\mathcal{M}_{\mathrm{C}}^{N}$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\mathrm{C}}}\right)\left(\frac{1}{2}v_{\mathrm{C}}^{p}v_{\mathrm{C}}^{k}\mathcal{M}_{\mathrm{C}}^{kN}-F_{\mathrm{C}}^{kp}\mathcal{M}_{\mathrm{C}}^{kN}\right)-\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{n!}\partial_{N}\left(\frac{1}{R_{\mathrm{C}}}\right)a_{\mathrm{C}}^{p}\mathcal{M}_{\mathrm{C}}^{pN}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}[\ddot{\mathcal{M}}_{\mathrm{C}}^{N}\partial_{N}R_{\mathrm{C}}+\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{C}}^{p}v_{\mathrm{C}}^{q}\partial_{pq\langle N\rangle}R_{\mathrm{C}}-\mathcal{M}_{\mathrm{C}}^{N}a_{\mathrm{C}}^{p}\partial_{p\langle N\rangle}R_{\mathrm{C}}]$$

$$
-2(1+\gamma)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)n!}\varepsilon_{kpq}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}^{qN}_{\rm C}v^k_{\rm BC}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^{pN}_{\rm C}v^p_{\rm BC}\right.
$$
$$
\left.+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)v^p_{\rm B}v^p_{\rm C}\mathcal{M}^N_{\rm C}\right], \tag{263}
$$

$$
\Omega^i_{\rm C}(t,\boldsymbol{x},l)=2(1+\gamma)\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\ddot{\mathcal{M}}^{iN}_{\rm C}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}^N_{\rm C}v^i_{\rm C}+\mathcal{M}^N_{\rm C}a^i_{\rm C})\right.
$$
$$
+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)n!}\varepsilon_{ipq}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{S}}^{qN}_{\rm C}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)n!}\varepsilon_{ipq}\partial_{kpN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}^{qN}_{\rm C}v^k_{\rm BC}
$$
$$
\left.+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^{iN}_{\rm C}v^p_{\rm BC}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^i_{\rm C}\right]
$$
$$
+(l-1-2\gamma)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^N_{\rm C}v^i_{\rm B}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^i_{\rm B}\right]
$$
$$
-\frac{l+1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{pN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^N_{\rm C}v^p_{\rm B}v^i_{\rm B}-(l^2+l+2+2\gamma)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_N\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^N_{\rm C}a^i_{\rm B}. \tag{264}
$$

Notice that both potentials $\Omega_{\rm C}(t,\boldsymbol{x},l)$ and $\Omega^i_{\rm C}(t,\boldsymbol{x},l)$ depend on the multipolar index l explicitly which should be taken into account when rendering summation in (258).

2. STF derivatives from the scalar potentials $\bar{W}$ and $\bar{U}$

The force contains the STF derivatives from the scalar potentials $\bar{W}$ and $\bar{U}$ that appear in the first line of (261). The derivatives are computed with the help of expansions (220) and observation that we can equate $U_{\rm C}$ to $W_{\rm C}$ in the post-Newtonian terms. Accounting for the fact that the partial derivative of any order from the inverse distance, $R_{\rm C}^{-1}$, is automatically STF Cartesian tensor because this function is a fundamental solution of the Laplace equation, we get

$$
\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{W}(t,\boldsymbol{x})\mathcal{M}^L_{\rm B}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}, \tag{265}
$$

$$
\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x})\mathcal{M}^L_{\rm B}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}, \tag{266}
$$

$$
\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle ipL\rangle}\bar{U}(t,\boldsymbol{x})\mathcal{M}^{kL}_{\rm B}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)\times\mathcal{M}^{kL}_{\rm B}\mathcal{M}^N_{\rm C}, \tag{267}
$$

where we have dropped the angular (STF) brackets around spatial indices of the partial derivatives from $R_{\rm C}^{-1}$ as they are redundant because the partial and STF derivatives of $R_{\rm C}^{-1}$ are identical, $\partial_{iLN}R_{\rm C}^{-1}=\partial_{\langle iLN\rangle}R_{\rm C}^{-1}$, etc.

3. STF derivatives from the scalar potential $\bar{\Omega}$

Computation of the STF partial derivative from $\bar{\Omega}$ in the second line of Eq. (261) for the force F^i involves taking the partial derivatives from the coordinate distance $R_{\rm C}$. We already know that all the partial derivatives taken from the inverse distance, $R_{\rm C}^{-1}$, are automatically STF derivatives in the sense that $\partial_L R_{\rm C}^{-1}=\partial_{\langle L\rangle}R_{\rm C}^{-1}$ for any number l of indices. On the other hand, the partial derivatives from $R_{\rm C}$ are not the STF derivatives, that is $\partial_L R_{\rm C}\neq\partial_{\langle L\rangle}R_{\rm C}$. The partial derivatives from $R_{\rm C}$ enter the forth line of formula (263) for $\Omega_{\rm C}(t,\boldsymbol{x},l)$, and additional partial derivatives from $R_{\rm C}$ are taken in (261) in the form of $\partial_{\langle iL\rangle}\bar{\Omega}(t,\boldsymbol{x},l)$. The derivatives from $R_{\rm C}$ have to be converted to the STF partial derivatives in order to represent all terms in the equations of motion as expansions with respect to the STF Cartesian tensors. This is achieved by making use of the following complementary relation allowing us to transform a partial derivative of order p from $R_{\rm C}$ to its STF counterpart [[50], Eq. (A21b)]:

$$
\partial_{a_1a_2\ldots a_p}R_{\rm C}=\partial_{<a_1a_2\ldots a_p>}R_{\rm C}+\frac{2}{2p-1}\delta_{\{a_1a_2}\partial_{a_3\ldots a_p\}}\left(\frac{1}{R_{\rm C}}\right), \tag{268}
$$

where the curly brackets around tensor indices denote a full symmetrization with respect to the smallest set of permutations $(1,2,\ldots,p)$ of the indices.

Let us consider a transformation of the partial derivatives from $R_{\rm C}$ to the STF derivatives more explicitly. The first term in $\partial_{\langle iL\rangle}\bar{\Omega}$ with the partial derivatives from $R_{\rm C}$ is proportional to $\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle iL\rangle N}R_{\rm C}$. It is converted to the STF derivative by applying (268) in two steps. First, we use (268) in reverse order,

$$\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle iL\rangle N}R_{\rm C} = \mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\left[\partial_{iLN}R_{\rm C} - \frac{2}{2l+1}\delta_{\{ia_1}\partial_{a_2\ldots a_l\}b_1\ldots b_n}\left(\frac{1}{R_{\rm C}}\right)\right], \tag{269}$$

with the purpose of getting the symmetric partial derivative $\partial_{iLN}R_{\rm C}$ from the partial derivative $\partial_{\langle iL\rangle N}R_{\rm C}$ which contains a mixture of the STF and symmetric derivatives. Second, we apply (268) in direct order for converting the symmetric derivative $\partial_{iLN}R_{\rm C}$ to its STF counterpart,

$$\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{iLN}R_{\rm C} = \mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\left[\partial_{\langle iLN\rangle}R_{\rm C} + \frac{2}{2l+2n+1}\delta_{\{ia_1}\partial_{a_2\ldots a_l b_1\ldots b_n\}}\left(\frac{1}{R_{\rm C}}\right)\right]. \tag{270}$$

Expanding the symmetric permutation symbol in the second term of (270) to a corresponding number of terms and remembering that the Laplacian from $R_{\rm C}^{-1}$ vanishes, $\Delta R_{\rm C}^{-1} = 0$, we eventually get

$$\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle iL\rangle N}R_{\rm C} = \mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle iLN\rangle}R_{\rm C} + \left[\frac{2l}{2l+2n+1} - \frac{2l}{2l+1}\right]\mathcal{M}^{iL-1}_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle NL-1\rangle}\left(\frac{1}{R_{\rm C}}\right) + \frac{2n}{2l+2n+1}\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^{iN-1}_{\rm C}\partial_{\langle LN-1\rangle}\left(\frac{1}{R_{\rm C}}\right) + \frac{2ln}{2l+2n+1}\mathcal{M}^{pL-1}_{\rm B}\ddot{\mathcal{M}}^{pN-1}_{\rm C}\partial_{\langle iL-1N-1\rangle}\left(\frac{1}{R_{\rm C}}\right). \tag{271}$$

Proceeding in a similar way, we get for two other partial derivatives from R_C

$$\begin{aligned}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle iL\rangle pqN}R_{\rm C} &= \mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle ipqLN\rangle}R_{\rm C} + \frac{2}{2l+2n+5}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^2_{\rm C}\partial_{\langle iLN\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{4}{2l+2n+5}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^i_{\rm C}v^p_{\rm C}\partial_{\langle pLN\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \left[\frac{2l}{2l+2n+5} - \frac{2l}{2l+1}\right]\mathcal{M}^{iL-1}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle pqNL-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{2n}{2l+2n+5}\mathcal{M}^L_{\rm B}\mathcal{M}^{iN-1}_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle pqLN-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{4l}{2l+2n+5}\mathcal{M}^{qL-1}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle ipNL-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{4n}{2l+2n+5}\mathcal{M}^{qN-1}_{\rm C}\mathcal{M}^L_{\rm B}v^p_{\rm C}v^q_{\rm C}\partial_{\langle ipLN-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{2ln}{2l+2n+5}\mathcal{M}^{kL-1}_{\rm B}\mathcal{M}^{kN-1}_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle ipqL-1N-1\rangle}\left(\frac{1}{R_{\rm C}}\right),\end{aligned} \tag{272}$$

$$\begin{aligned}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}a^p_{\rm C}\partial_{\langle iL\rangle pN}R_{\rm C} &= \mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}a^p_{\rm C}\partial_{\langle ipLN\rangle}R_{\rm C} + \frac{2}{2l+2n+3}a^i_{\rm C}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}\partial_{\langle LN\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \left[\frac{2l}{2l+2n+3} - \frac{2l}{2l+1}\right]\mathcal{M}^{iL-1}_{\rm B}\mathcal{M}^N_{\rm C}a^p_{\rm C}\partial_{\langle pNL-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{2n}{2l+2n+3}\mathcal{M}^L_{\rm B}\mathcal{M}^{iN-1}_{\rm C}a^p_{\rm C}\partial_{\langle pLN-1\rangle}\left(\frac{1}{R_{\rm C}}\right) + \frac{2l}{2l+2n+3}a^p_{\rm C}\mathcal{M}^{pL-1}_{\rm B}\mathcal{M}^N_{\rm C}\partial_{\langle iNL-1\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+ \frac{2n}{2l+2n+3}a^p_{\rm C}\mathcal{M}^{pN-1}_{\rm C}\mathcal{M}^L_{\rm B}\partial_{\langle iLN-1\rangle}\left(\frac{1}{R_{\rm C}}\right) + \frac{2ln}{2l+2n+3}\mathcal{M}^{qL-1}_{\rm B}\mathcal{M}^{qN-1}_{\rm C}a^p_{\rm C}\partial_{\langle ipL-1N-1\rangle}\left(\frac{1}{R_{\rm C}}\right).\end{aligned} \tag{273}$$

Employing these relations to present all terms in $\partial_{\langle iL\rangle}\bar{\Omega}$ in the STF form, we compute its contribution to the force (261) as follows:

$$
\begin{aligned}
\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\Omega_{\rm C}(t,\boldsymbol{x},l)\mathcal{M}_{\rm B}^{L} = &\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\left[(1+\gamma)v_{\rm BC}^2-\frac{1}{2}\frac{2l+2n+3}{2l+2n+5}v_{\rm C}^2\right]\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{N}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)[(2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})-\gamma(n+1)\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C})]\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{N}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)\left(\frac{1}{2}v_{\rm C}^{p}v_{\rm C}^{k}-F_{\rm C}^{kp}\right)\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{kN}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}(n+1)\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)a_{\rm C}^{p}\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{pN}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\left[\frac{1}{2}\ddot{\mathcal{M}}_{\rm C}^{N}\partial_{\langle iLN\rangle}R_{\rm C}+\frac{1}{2}\mathcal{M}_{\rm C}^{N}v_{\rm C}^{p}v_{\rm C}^{q}\partial_{\langle ipqLN\rangle}R_{\rm C}-\frac{1}{2}\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p}\partial_{\langle ipLN\rangle}R_{\rm C}\right]\mathcal{M}_{\rm B}^{L}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\left[\frac{1}{2l+2n+3}-\frac{1}{2l+3}\right]\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\ddot{\mathcal{M}}_{\rm C}^{N}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+3}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{L}\ddot{\mathcal{M}}_{\rm C}^{iN}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\ddot{\mathcal{M}}_{\rm C}^{pN}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{2}{2l+2n+5}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{N}v_{\rm C}^{p}v_{\rm C}^{i}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\left[\frac{1}{2l+2n+7}-\frac{1}{2l+3}\right]\partial_{pqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}v_{\rm C}^{p}v_{\rm C}^{q}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+7}\partial_{pqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{iN}v_{\rm C}^{p}v_{\rm C}^{q}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{2}{2l+2n+7}\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)(\mathcal{M}_{\rm B}^{qL}\mathcal{M}_{\rm C}^{N}-\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{qN})v_{\rm C}^{p}v_{\rm C}^{q}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+9}\partial_{ipqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{kN}v_{\rm C}^{p}v_{\rm C}^{q}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+3}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{N}a_{\rm C}^{i}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\left[\frac{1}{2l+2n+5}-\frac{1}{2l+3}\right]\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{iN}a_{\rm C}^{p}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)(\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}-\mathcal{M}_{\rm B}^{L}\mathcal{M}_{\rm C}^{pN})a_{\rm C}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+7}\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{qL}\mathcal{M}_{\rm C}^{qN}a_{\rm C}^{p}
\end{aligned}
$$

$$
\begin{aligned}
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{n+2}\varepsilon_{kpq}\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{S}^{qN}_{\rm C}v^k_{\rm BC}\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{n+1}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^{pN}_{\rm C}v^p_{\rm BC},
\end{aligned}
\tag{274}
$$

where, in the fifth line of this long formula, we keep the angular brackets around indices of the spatial derivatives from $R_{\rm C}$ to make clear that these are the STF partial derivatives from $R_{\rm C}$.

4. STF derivatives from the vector potential $\bar{\Omega}^i$

Our next step is the computation of the STF derivative $\partial_{\langle L}\bar{\Omega}_{i\rangle}(t,\boldsymbol{x},l)$ that appears in the third line of equation (261) for force F^i. Calculation of this term is based on application of the index peeling-off formula (A1) which yields

$$
\begin{aligned}
\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle L}\bar{\Omega}_{i\rangle}(t,\boldsymbol{x},l)\mathcal{M}^L_{\rm B} = {}&\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\partial_{\langle L\rangle}\bar{\Omega}_i(t,\boldsymbol{x},l)\mathcal{M}^L_{\rm B} + \sum_{l=0}^{\infty}\frac{l}{(l+1)!}\partial_{i\langle L-1}\bar{\Omega}_{p\rangle}(t,\boldsymbol{x},l)\mathcal{M}^{pL-1}_{\rm B}\\
&-\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\frac{2l}{2l+1}\partial_{p\langle L-1\rangle}\bar{\Omega}_p(t,\boldsymbol{x},l)\mathcal{M}^{iL-1}_{\rm B},
\end{aligned}
\tag{275}
$$

that helps to disentangle one index from the remaining STF indices and simplifies computation of the partial derivative. Vector potential $\bar{\Omega}_i$ is given by the last term in (262) as a linear superposition of vector-potentials $\Omega^i_{\rm C}$ of bodies with index $\rm C \neq B$ that are external with respect to body B. Applying (275) to the individual $\Omega^i_{\rm C}$ defined in (264), we obtain the first term in the right-hand side of (275),

$$
\begin{aligned}
\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\partial_L\Omega^i_{\rm C}(t,\boldsymbol{x},l)\mathcal{M}^L_{\rm B} = {}&2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+1}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^{iN}_{\rm C}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}^N_{\rm C}v^i_{\rm C}+\mathcal{M}^N_{\rm C}a^i_{\rm C})\mathcal{M}^L_{\rm B}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+2}\varepsilon_{ipq}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\dot{\mathcal{S}}^{qN}_{\rm C}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+2}\varepsilon_{ipq}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{S}^{qN}_{\rm C}v^k_{\rm BC}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+1}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^{iN}_{\rm C}v^p_{\rm BC}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^i_{\rm C}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l-1-2\gamma)\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C}v^i_{\rm B}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l-1-2\gamma)\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^i_{\rm B}\\
&-\frac{1}{2}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm B}v^i_{\rm B}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l^2+l+2+2\gamma)\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}a^i_{\rm B}.
\end{aligned}
\tag{276}
$$

In order to compute the second and third terms in the right-hand side of (275) it is useful to reformulate them by changing the index of summation, $l\to l+1$, which also replaces STF index $L-1\to L$. This procedure is convenient for reduction of similar terms in the final equation for the force which consists of the contributions of many separate pieces. We have

$$\sum_{l=0}^{\infty}\frac{l}{(l+1)!}\partial_{iL-1}\bar{\Omega}_p(t,\boldsymbol{x},l)\mathcal{M}_{\rm B}^{pL-1}=\sum_{l=0}^{\infty}\frac{1}{l!(l+2)}\partial_{iL}\bar{\Omega}_p(t,\boldsymbol{x},l+1)\mathcal{M}_{\rm B}^{pL} \tag{277}$$

$$\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\frac{2l}{2l+1}\partial_{pL-1}\bar{\Omega}_p(t,\boldsymbol{x},l)\mathcal{M}_{\rm B}^{iL-1}=\sum_{l=0}^{\infty}\frac{1}{(l+2)!}\frac{2(l+1)}{2l+3}\partial_{pL}\bar{\Omega}_p(t,\boldsymbol{x},l+1)\mathcal{M}_{\rm B}^{iL}, \tag{278}$$

and the STF derivatives are

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{1}{l!(l+2)}\partial_{iL}\Omega_{\rm C}^{p}(t,\boldsymbol{x},l+1)\mathcal{M}_{\rm B}^{pL}=&\,2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!(n+1)!}\frac{1}{l+2}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\ddot{\mathcal{M}}_{\rm C}^{pN}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{l+2}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm C}^{p}+\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p})\mathcal{M}_{\rm B}^{pL}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(l+1)}{(l+2)!(n+2)!}\varepsilon_{pkq}\partial_{ikLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\dot{\mathcal{S}}_{\rm C}^{qN}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(l+1)}{(l+2)!(n+2)!}\varepsilon_{pkq}\partial_{ijkLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\mathcal{S}_{\rm C}^{qN}v_{\rm BC}^{j}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!(n+1)!}\frac{1}{l+2}\partial_{ijLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\dot{\mathcal{M}}_{\rm C}^{pN}v_{\rm BC}^{j}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{l+2}\partial_{ijLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}v_{\rm BC}^{j}v_{\rm C}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l-2\gamma}{l+2}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm B}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l-2\gamma}{l+2}\partial_{ijLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}v_{\rm BC}^{j}v_{\rm B}^{p}\\
&-\frac{1}{2}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{ijLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}v_{\rm B}^{j}v_{\rm B}^{p}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l^2+3l+4+2\gamma}{l+2}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}a_{\rm B}^{p},\end{aligned} \tag{279}$$

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{1}{(l+2)!}\frac{2(l+1)}{2l+3}\partial_{pL}\Omega_{\rm C}^{p}(t,\boldsymbol{x},l+1)\mathcal{M}_{\rm B}^{iL}=&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{4(1+\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\ddot{\mathcal{M}}_{\rm C}^{N}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{4(1+\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm C}^{p}+\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p})\mathcal{M}_{\rm B}^{iL}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{4(1+\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{jLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm BC}^{j}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{4(1+\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{pjLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}v_{\rm BC}^{j}v_{\rm C}^{p}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{2(l-2\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm B}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{2(l-2\gamma)(-1)^n}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{pjLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}v_{\rm BC}^{j}v_{\rm B}^{p}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+3}\partial_{pjLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}v_{\rm B}^{j}v_{\rm B}^{p}\\
&-\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(2l^2+6l+8+4\gamma)}{(l+2)!n!}\frac{l+1}{2l+3}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{iL}\mathcal{M}_{\rm C}^{N}a_{\rm B}^{p},
\end{aligned}\tag{280}
$$

5. Post-Newtonian local acceleration $\mathcal{Q}_i^{\rm pN}$

We also need to express the post-Newtonian part $\mathcal{Q}_i^{\rm pN}$ of the local acceleration (186) of body B explicitly as a function of the STF mass and spin internal multipoles of all bodies in the $\mathbb{N}$-body system. For completing this task we, first of all, transform the terms in the first three lines of expression (186) for $\mathcal{Q}_i^{\rm pN}$ by making use of the fact that the external gravitoelectric multipoles $\mathcal{Q}_L=\mathcal{P}_L$ for $l\geq 2$ in all post-Newtonian terms. After accounting for this equality, Eq. (186) can be reshuffled to the following form:

$$
\begin{aligned}
M_{\rm B}\mathcal{Q}_i^{\rm pN}=&\,3(\mathcal{Q}_k\ddot{\mathcal{M}}_{\rm B}^{ik}+2\dot{\mathcal{Q}}_k\dot{\mathcal{M}}_{\rm B}^{ik}+\ddot{\mathcal{Q}}_k\mathcal{M}_{\rm B}^{ik})+(\gamma-1)\left(\mathcal{P}_k\ddot{\mathcal{M}}_{\rm B}^{ik}+\frac{3}{2}\dot{\mathcal{P}}_k\dot{\mathcal{M}}_{\rm B}^{ik}+\frac{3}{5}\ddot{\mathcal{P}}_k\mathcal{M}_{\rm B}^{ik}\right)\\
&+\sum_{l=2}^{\infty}\frac{l^2+l+2+2\gamma}{(l+1)!}\mathcal{P}_L\ddot{\mathcal{M}}_{\rm B}^{iL}+\sum_{l=2}^{\infty}\frac{2l+1}{l+1}\frac{l^2+2l+3+2\gamma}{(l+1)!}\dot{\mathcal{P}}_L\dot{\mathcal{M}}_{\rm B}^{iL}\\
&+\sum_{l=2}^{\infty}\frac{2l+1}{2l+3}\frac{l^2+3l+4+2\gamma}{(l+1)!}\ddot{\mathcal{P}}_L\mathcal{M}_{\rm B}^{iL}-\sum_{l=1}^{\infty}\frac{l}{(l+1)!}\mathcal{C}_{iL}\mathcal{S}_{\rm B}^{L}\\
&+\sum_{l=1}^{\infty}\frac{1}{(l+1)!}\varepsilon_{ipq}\left[\mathcal{C}_{pL}\dot{\mathcal{M}}^{\langle qL\rangle}+\frac{l+1}{l+2}\dot{\mathcal{C}}_{pL}\mathcal{M}^{\langle qL\rangle}\right]\\
&-2(1+\gamma)\sum_{l=1}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\left[\mathcal{P}_{pL}\dot{\mathcal{S}}_{\rm B}^{qL}+\frac{l+1}{l+2}\dot{\mathcal{P}}_{pL}\mathcal{S}_{\rm B}^{qL}\right]\\
&-\frac{1}{2}\varepsilon_{ikq}[(4\mathcal{Q}_k+2(\gamma-1)\mathcal{P}_k)\dot{\mathcal{S}}_{\rm B}^{q}+(2\dot{\mathcal{Q}}_k+(\gamma-1)\dot{\mathcal{P}}_k)\mathcal{S}_{\rm B}^{q}].
\end{aligned}\tag{281}
$$

At the second step of the computation, we take advantage of equation of motion (152) in the global coordinates to replace the local acceleration $\mathcal{Q}_i$ everywhere in (281) with its global counterpart $a_{\rm B}^i$. The Newtonian approximation is sufficient, $\mathcal{Q}_i=\partial_i\bar{U}(t,\boldsymbol{x}_{\rm B})-a_{\rm B}^i=\mathcal{P}^i-a_{\rm B}^i$, where we employed $\partial_i\bar{U}(t,\boldsymbol{x}_{\rm B})=\mathcal{P}_i$ in accordance with the definition of the external scalar-field multipoles provided in Sec. V C 4. Proceeding in this way, we obtain

$$
\begin{aligned}
M_{\rm B}\mathcal{Q}_i^{\rm pN}=&\sum_{l=0}^{\infty}\frac{l^2+l+2+2\gamma}{(l+1)!}\mathcal{P}_L\ddot{\mathcal{M}}_{\rm B}^{iL}+\sum_{l=0}^{\infty}\frac{2l+1}{l+1}\frac{l^2+2l+3+2\gamma}{(l+1)!}\dot{\mathcal{P}}_L\dot{\mathcal{M}}_{\rm B}^{iL}\\
&+\sum_{l=0}^{\infty}\frac{2l+1}{2l+3}\frac{l^2+3l+4+2\gamma}{(l+1)!}\ddot{\mathcal{P}}_L\mathcal{M}_{\rm B}^{iL}-\sum_{l=0}^{\infty}\frac{l}{(l+1)!}\mathcal{C}_{iL}\mathcal{S}_{\rm B}^{L}\\
&+\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\varepsilon_{ipq}\left[\mathcal{C}_{pL}\dot{\mathcal{M}}_{\rm B}^{qL}+\frac{l+1}{l+2}\dot{\mathcal{C}}_{pL}\mathcal{M}_{\rm B}^{qL}\right]\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ipq}\left[\mathcal{P}_{pL}\dot{\mathcal{S}}_{\rm B}^{qL}+\frac{l+1}{l+2}\dot{\mathcal{P}}_{pL}\mathcal{S}_{\rm B}^{qL}\right]\\
&+\varepsilon_{ikq}(2a_{\rm B}^k\dot{\mathcal{S}}_{\rm B}^q+\dot{a}_{\rm B}^k\mathcal{S}_{\rm B}^q)-3(a_{\rm B}^k\ddot{\mathcal{M}}_{\rm B}^{ik}+2\dot{a}_{\rm B}^k\dot{\mathcal{M}}_{\rm B}^{ik}+\ddot{a}_{\rm B}^k\mathcal{M}_{\rm B}^{ik}),
\end{aligned}\tag{282}
$$

where we have formally extended summation to value $l = 0$ in all sums by taking into account that in terms of the post-Newtonian order of magnitude the active dipole of each body vanishes, $\mathcal{M}^i_{\rm B} = 0$.

The external multipoles, $\mathcal{P}_L$ and $\mathcal{C}_L$ are expressed in terms of the external gravitational potentials, $\bar{U}$ and $\bar{U}^i$ of the body B with the help of (153), (154), and (157) respectively. Particular attention should be paid to the term $\mathcal{C}_{iL}\mathcal{S}^L_{\rm B}$. After a few algebraic transformations it becomes

$$\mathcal{C}_{iL}\mathcal{S}^L_{\rm B} = \delta_{pq}\mathcal{C}_{ipL-1}\mathcal{S}^{qL-1}_{\rm B} = \frac{1}{2}\varepsilon_{jpk}\varepsilon_{jqk}\mathcal{C}_{piL-1}\mathcal{S}^{qL-1}_{\rm B} = \frac{1}{2}\varepsilon_{jqk}\bar{H}_{jk\langle iL-1\rangle}\mathcal{S}^{qL-1}, \tag{283}$$

where, at the last step, we have used (249). After substituting $\bar{H}_{jk\langle iL-1\rangle}$ from (250) and (251) to the above expression and noticing that contraction of any two indices in STF multipole $\mathcal{S}^L_{\rm B}$ vanishes, we get

$$\mathcal{C}_{iL}\mathcal{S}^L_{\rm B} = 2(1+\gamma)\varepsilon_{jqk}[v^j_{\rm B}\partial_{ikL-1}\bar{U}(t,\boldsymbol{x}_{\rm B}) - \partial_{ikL-1}\bar{U}^j(t,\boldsymbol{x}_{\rm B})]\mathcal{S}^{qL-1}_{\rm B} + \frac{2(1+\gamma)}{l+1}\varepsilon_{ijq}\partial_{jL-1}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B})\mathcal{S}^{qL-1}_{\rm B}. \tag{284}$$

After implementing this and other replacements of the external multipoles in (282) with the corresponding external global potentials, and reducing similar terms, the post-Newtonian local acceleration takes on the following form:

$$M_{\rm B}\mathcal{Q}^{\rm pN}_i = \bar{\Xi}^i_{\rm C}(t,\boldsymbol{x}_{\rm B}) + \varepsilon_{ikq}(2a^k_{\rm B}\dot{\mathcal{S}}^q_{\rm B} + \dot{a}^k_{\rm B}\mathcal{S}^q_{\rm B}) - 3(a^k_{\rm B}\ddot{\mathcal{M}}^{ik}_{\rm B} + 2\dot{a}^k_{\rm B}\dot{\mathcal{M}}^{ik}_{\rm B} + \ddot{a}^k_{\rm B}\mathcal{M}^{ik}_{\rm B}), \tag{285}$$

where the first term

$$\bar{\Xi}^i_{\rm C}(t,\boldsymbol{x}) = \sum_{\rm C\neq B}\Xi^i_{\rm C}(t,\boldsymbol{x}) \tag{286}$$

is a linear superposition of vectors

$$\begin{aligned}\Xi^i_{\rm C}(t,\boldsymbol{x}) = {} & \sum_{l=0}^{\infty}\frac{l^2+l+2+2\gamma}{(l+1)!}\partial_L U_{\rm C}(t,\boldsymbol{x})\ddot{\mathcal{M}}^{iL}_{\rm B} + \sum_{l=0}^{\infty}\frac{2l^2+3l+3+2\gamma}{(l+1)!}\partial_L\dot{U}_{\rm C}(t,\boldsymbol{x})\dot{\mathcal{M}}^{iL}_{\rm B} \\ & + \sum_{l=0}^{\infty}\frac{2l^3+9l^2+12l+8+4\gamma}{(l+2)(2l+3)l!}\partial_L\ddot{U}_{\rm C}(t,\boldsymbol{x})\mathcal{M}^{iL}_{\rm B} \\ & + 4(1+\gamma)\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\left[v^{[i}_{\rm B}\partial^{k]L}U_{\rm C}(t,\boldsymbol{x}) + \partial^{L[i}U^{k]}_{\rm C}(t,\boldsymbol{x})\right]\dot{\mathcal{M}}^{kL}_{\rm B} \\ & + 4(1+\gamma)\sum_{l=0}^{\infty}\frac{1}{(l+2)l!}\left[a^{[i}_{\rm B}\partial^{k]L}U_{\rm C}(t,\boldsymbol{x}) + v^{[i}_{\rm B}\partial^{k]L}\dot{U}_{\rm C}(t,\boldsymbol{x}) + \partial^{L[i}\dot{U}^{k]}_{\rm C}(t,\boldsymbol{x})\right]\mathcal{M}^{kL}_{\rm B} \\ & + 2(1+\gamma)\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{jkq}\left[v^j_{\rm B}\partial_{ikL}U_{\rm C}(t,\boldsymbol{x}) - \partial_{ikL}\bar{U}^j_{\rm C}(t,\boldsymbol{x})\right]\mathcal{S}^{qL}_{\rm B} \\ & - 2(1+\gamma)\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ikq}\partial_{kL}\dot{U}_{\rm C}(t,\boldsymbol{x})\mathcal{S}^{qL}_{\rm B} - 2(1+\gamma)\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{ikq}\partial_{kL}U_{\rm C}(t,\boldsymbol{x})\dot{\mathcal{S}}^{qL}_{\rm B}.\end{aligned} \tag{287}$$

Finally, after making use of multipolar decompositions of potentials $U_{\rm C} = W_{\rm C}$ and $U^i_{\rm C}$ given in (220) and (223) respectively, vector $\Xi^i_{\rm C}$ becomes

$$\begin{aligned}\Xi^i_{\rm C}(t,\boldsymbol{x}) = {} & \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l^2+l+2+2\gamma}{(l+1)!}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\ddot{\mathcal{M}}^{iL}_{\rm B}\mathcal{M}^N_{\rm C} \\ & + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2l^2+3l+3+2\gamma}{(l+1)!}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^{iL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C} \\ & + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2l^2+3l+3+2\gamma}{(l+1)!}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}^{iL}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm BC} \\ & + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2l^3+9l^2+12l+8+4\gamma}{(l+2)(2l+3)l!}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{iL}_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{4l^3+18l^2+24l+16+8\gamma}{(l+2)(2l+3)l!}\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{iL}\dot{\mathcal{M}}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2l^3+9l^2+12l+8+4\gamma}{(l+2)(2l+3)l!}\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{iL}\mathcal{M}_{\mathrm{C}}^{N}a_{\mathrm{BC}}^{p}\\
&+\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2l^3+9l^2+12l+8+4\gamma}{(l+2)(2l+3)l!}\partial_{pqLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{iL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}v_{\mathrm{BC}}^{q}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\left[\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{i}-\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!(n+1)!}\left[\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{pN}-\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{iN}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(l+1)!(n+2)!}\left[\varepsilon_{kpq}\partial_{ipLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{qN}-\varepsilon_{ipq}\partial_{kpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{M}}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{qN}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(l+1)}{(l+2)!}\left[\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}a_{\mathrm{BC}}^{i}-\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}a_{\mathrm{BC}}^{p}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(l+1)}{(l+2)!}\left[\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{i}-\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(l+1)}{(l+2)!}\left[\partial_{kpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{i}v_{\mathrm{BC}}^{k}-\partial_{ikLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}v_{\mathrm{BC}}^{k}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\frac{(l+1)}{(l+2)!}\left[\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\ddot{\mathcal{M}}_{\mathrm{C}}^{pN}-\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\ddot{\mathcal{M}}_{\mathrm{C}}^{iN}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\frac{(l+1)}{(l+2)!}\left[\partial_{ikLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{pN}v_{\mathrm{BC}}^{k}-\partial_{pkLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{pL}\dot{\mathcal{M}}_{\mathrm{C}}^{iN}v_{\mathrm{BC}}^{k}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(n+2)!}\frac{l+1}{(l+2)!}\left[\varepsilon_{kpq}\partial_{ipLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{kL}\dot{\mathcal{S}}_{\mathrm{C}}^{qN}-\varepsilon_{ipq}\partial_{kpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{kL}\dot{\mathcal{S}}_{\mathrm{C}}^{qN}\right]\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(n+2)!}\frac{l+1}{(l+2)!}\left[\varepsilon_{kpq}\partial_{ijpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{qN}v_{\mathrm{BC}}^{j}-\varepsilon_{ipq}\partial_{jkpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{qN}v_{\mathrm{BC}}^{j}\right]\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l+1}{(l+2)!}\varepsilon_{kpq}\partial_{ikLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{S}_{\mathrm{B}}^{qL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\frac{l+1}{(l+2)!}\varepsilon_{kpq}\partial_{ikLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{S}_{\mathrm{B}}^{qL}\dot{\mathcal{M}}_{\mathrm{C}}^{pN}\\
&+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(n+2)!}\frac{l+1}{(l+2)!}\partial_{ikqLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{S}_{\mathrm{B}}^{qL}\mathcal{S}_{\mathrm{C}}^{kN}\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l+1}{(l+2)!}\varepsilon_{ikq}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{S}_{\mathrm{B}}^{qL}\dot{\mathcal{M}}_{\mathrm{C}}^{N}\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l+1}{(l+2)!}\varepsilon_{ikq}\partial_{kpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{S}_{\mathrm{B}}^{qL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{p}\\
&-2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l+1}{(l+2)!}\varepsilon_{ikq}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\dot{\mathcal{S}}_{\mathrm{B}}^{qL}\mathcal{M}_{\mathrm{C}}^{N}.
\end{aligned}
\tag{288}
$$

6. Complementary vector function $\mathcal{I}^i_{\rm c}$. Adjustment of the center of mass

We notice that the last three terms in (285) represent a second time derivative from the product, $a^k_{\rm B}\mathcal{M}^{ik}_{\rm B}$. These terms can be eliminating from the net force (261) by choosing the complementary vector function $\mathcal{I}^i_{\rm c}$ in definition (167) of the center of mass of body B, as follows:

$$\mathcal{I}^i_{\rm c} = -3a^k_{\rm B}\mathcal{M}^{ik}_{\rm B}. \tag{289}$$

This choice slightly simplifies equations of translational motion and makes a small adjustment of the worldline $\mathcal{Z}$ of the center of mass of body B as compared with the choice $\mathcal{I}^i_{\rm c} = 0$ which was used, for example, in [84,85].

The terms which are proportional to spin $\mathcal{S}^i_{\rm B}$ of body B in the right-hand side of (285) do not represent a second time derivative and will be left in the equations of motion. In principle, we can always group some terms in the net force (261) to form a second time derivative that can be eliminated from the force. This procedure can make sense for simplifying the translational equations of motion of body B. However, it brings additional terms to the rotational equations of motion for the body's spin and, overall, may be not so effective. Therefore, we do not implement it beyond applying Eq. (289).

B. Explicit formula for gravitational force

After making adjustment (289) of the worldline of the center of mass of body B, translational equations of motion (257) take on the following form:

$$M_{\rm B}a^i_{\rm B} = F^i_{\rm N} + F^i_{\rm pN}, \tag{290}$$

where $M_{\rm B}$ is the inertial (conformal) mass of the body, and the net gravitational force F^i is split in two components: $F^i_{\rm N}$ is the Newtonian gravitational force, and $F^i_{\rm pN}$ is the post-Newtonian gravitational force. The force components read

$$F^i_{\rm N} = \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B}, \tag{291}$$

$$\begin{aligned} F^i_{\rm pN} = {}& \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle iL\rangle}\bar{\Omega}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B} + \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle L}\bar{\Omega}_{i\rangle}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B} - \bar{\Xi}^i(t,\boldsymbol{x}_{\rm B}) \\ & - \left(v^2_{\rm B}\delta^{ik} + \frac{1}{2}v^i_{\rm B}v^k_{\rm B}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B} - F^{pk}_{\rm B}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle ipL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^{kL}_{\rm B} \\ & - M^{-1}_{\rm B}\varepsilon_{ikq}\left(2\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B}\dot{\mathcal{S}}^q_{\rm B} + \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B})\mathcal{M}^L_{\rm B}\mathcal{S}^q_{\rm B} + \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{U}(t,\boldsymbol{x}_{\rm B})\dot{\mathcal{M}}^L_{\rm B}\mathcal{S}^q_{\rm B}\right), \end{aligned} \tag{292}$$

where the very last two terms with spin multipoles originate from the middle group of the spin-dependent terms in (285) after replacing acceleration $a^i_{\rm B}$ with the Newtonian equations of motion of body B.

Computation of the explicit form of the force is now achieved by substituting to (291) and (292) the STF derivatives of gravitational potentials obtained in Secs. IX A 2–IX A 5, and employing relations (A7) and (A8) for computations of partial derivatives from $R_{\rm C} = |\boldsymbol{x} - \boldsymbol{x}_{\rm C}|$,

$$\partial_{\langle L\rangle}R^{-1}_{\rm BC} \equiv \lim_{\boldsymbol{x}\to\boldsymbol{x}_{\rm B}}\partial_{\langle L\rangle}R^{-1}_{\rm C} = (-1)^l(2l-1)!!\frac{R^{\langle L\rangle}_{\rm BC}}{R^{2l+1}_{\rm BC}}, \tag{293}$$

$$\partial_{\langle L\rangle}R_{\rm BC} \equiv \lim_{\boldsymbol{x}\to\boldsymbol{x}_{\rm B}}\partial_{\langle L\rangle}R_{\rm C} = (-1)^{l+1}(2l-3)!!\frac{R^{\langle L\rangle}_{\rm BC}}{R^{2l-1}_{\rm BC}}, \tag{294}$$

which are taken at point $\boldsymbol{x}_{\rm B}$—the center of mass of body B. It is worth noticing that $\partial_{\langle L\rangle}R^{-1}_{\rm BC} = \partial_L R^{-1}_{\rm BC}$ due to the fact that function $R^{-1}_{\rm C}$ is a fundamental solution of the Laplace equation, $\triangle R^{-1}_{\rm C} = 0$, everywhere but the point $x^i = x^i_{\rm C}$.

1. Newtonian force

The total Newtonian gravitational force, $F^i_{\rm N}$, is given by a linear superposition of gravitational forces exerted on the body B by all other bodies of the $\mathbb{N}$-body system. Using (240) and (265) and taking the partial derivative in (291) with the help of (293), we get

$$\begin{aligned} F^i_{\rm N} &= \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}\partial_{iLN}R^{-1}_{\rm BC} \\ &= -\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+2n+1)!!}{l!n!}\frac{\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}}{R^{2l+2n+3}_{\rm BC}}R^{\langle iLN\rangle}_{\rm BC}, \end{aligned} \tag{295}$$

where $\mathcal{M}^L_{\rm B} = \mathcal{M}^{\langle a_1\ldots a_l\rangle}_{\rm B}$ are active STF multipoles of body B, $\mathcal{M}^N_{\rm C} = \mathcal{M}^{\langle b_1\ldots b_n\rangle}_{\rm C}$ are active STF multipoles of the external body C, $R_{\rm BC} = |\boldsymbol{R}_{\rm BC}| = (\delta_{ij}R^i_{\rm BC}R^j_{\rm BC})^{1/2}$,

$$R^i_{\rm BC} \equiv x^i_{\rm B} - x^i_{\rm C} = x^i_{\rm B}(t) - x^i_{\rm C}(t) \tag{296}$$

is the coordinate distance between the centers of mass of the bodies, $R_{\rm BC}^{\langle iLN\rangle}=R_{\rm BC}^{\langle ia_1\ldots a_l b_1\ldots b_n\rangle}$, and the repeated indices mean the Einstein summation rule.

We draw to the attention of the reader that the coordinates of the centers of mass of all bodies are computed at the same instant of global time t that is $x^i_{\rm B}=x^i_{\rm B}(t)$, $x^i_{\rm C}=x^i_{\rm C}(t)$, and so on. On the other hand, each body STF multipole is a function of the coordinate time of the corresponding local coordinates adapted to the body. According to the procedure of derivation of the equations of motion adopted in the present paper, the numerical values of all local coordinate times are computed

enters the left-hand side of (290). In other words, we have $\mathcal{M}^L_{\rm B}\equiv\mathcal{M}^L_{\rm B}(u^*_{\rm B})$ and $\mathcal{M}^L_{\rm C}\equiv\mathcal{M}^L_{\rm C}(u^*_{\rm C})$ (and similar convention is applied to the spin multipoles) where the local times

$$u^*_{\rm B}=u_{\rm B}|_{\boldsymbol{x}=\boldsymbol{x}_{\rm B}}=t+\frac{1}{c^2}\mathcal{A}_{\rm B}(t)+\mathcal{O}\left(\frac{1}{c^4}\right),\qquad(297)$$

$$u^*_{\rm C}=u_{\rm C}|_{\boldsymbol{x}=\boldsymbol{x}_{\rm B}}=t+\frac{1}{c^2}[\mathcal{A}_{\rm C}(t)-v^k_{\rm C}(t)R^k_{\rm BC}]+\mathcal{O}\left(\frac{1}{c^4}\right),\qquad(298)$$

where time dilation functions $\mathcal{A}_{\rm B}$ and $\mathcal{A}_{\rm C}$ are defined by solutions of the ordinary differential equations

$$\frac{d\mathcal{A}_{\rm B}}{dt}=-\frac{1}{2}v^2_{\rm B}(t)-\bar{U}_{\rm B}(t,\boldsymbol{x}_{\rm B})\qquad(299)$$

$$\frac{d\mathcal{A}_{\rm C}}{dt}=-\frac{1}{2}v^2_{\rm C}(t)-\bar{U}_{\rm C}(t,\boldsymbol{x}_{\rm C}),\qquad(300)$$

which constitute a part of the coordinate transformation between the local and global coordinates of the corresponding massive body. The mass $M_{\rm B}$ in the left-hand side of Eq. (290) is computed at the time u^*_B defined above in (297).

We emphasize that the Newtonian gravitational force (295) in scalar-tensor theory of gravity depends on the active multipoles which include the post-Newtonian corrections as shown in (122). The force also has a post-Newtonian contribution from the active mass dipole $\mathcal{M}^i$ of the bodies (terms with $l=1$ and $n=1$) which do not vanish because the center of mass of the body is defined by the condition of vanishing conformal mass dipole, $\mathcal{I}^i=0$ of each body. The active dipole $\mathcal{M}^i\neq\mathcal{I}^i$ according to (171).

Additional notice is that the inertial mass, $M_{\rm B}$, in the left side of (290) is the conformal mass (164) of body B while the gravitational force in the right side of (290) depends on the active mass $\mathcal{M}_{\rm B}$—see (161)—of body B and the active masses of other bodies, which corresponds, for example, to the terms with $l=0$ in the right-hand side of (295). The active and conformal masses do not coincide as follows from (166). It violates the strong principle of equivalence in scalar-tensor theory of gravity [88,172,262].

2. Post-Newtonian force

The post-Newtonian gravitational force can be represented in the form of a linear superposition of STF partial derivatives taken from functions $R^{-1}_{\rm BC}$ and $R_{\rm BC}$,

$$\begin{aligned}F^i_{\rm pN}=&\frac{1}{2}\sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}[\ddot{\mathcal{M}}^N_{\rm C}\partial_{\langle iLN\rangle}-\mathcal{M}^N_{\rm C}a^p_{\rm C}\partial_{\langle ipLN\rangle}+\mathcal{M}^N_{\rm C}v^p_{\rm C}v^q_{\rm C}\partial_{\langle ipqLN\rangle}]R_{\rm BC}\\&+\sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}[(\alpha^{iLN}_{\rm F}+\beta^{iLN}_{\rm F})\partial_{\langle LN\rangle}+(\alpha^{ipLN}_{\rm F}+\beta^{ipLN}_{\rm F})\partial_{\langle pLN\rangle}+\alpha^{ipqLN}_{\rm F}\partial_{\langle pqLN\rangle}\\&+(\alpha^{LN}_{\rm F}+\beta^{LN}_{\rm F}+\gamma^{LN}_{\rm F})\partial_{\langle iLN\rangle}+(\mu^{pLN}_{\rm F}+\nu^{pLN}_{\rm F}+\rho^{pLN}_{\rm F})\partial_{\langle ipLN\rangle}+\sigma^{pqLN}_{\rm F}\partial_{\langle ipqLN\rangle}]R^{-1}_{\rm BC},\end{aligned}\qquad(301)$$

where all partial derivatives are understood in the sense of Eqs. (293) and (294) and the coefficients of the differential operator are

$$\begin{aligned}\alpha^{iLN}_{\rm F}=&[v^i_{\rm B}-2(1+\gamma)v^i_{\rm BC}]\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C}+\left[\frac{2(1+\gamma)}{n+1}-\frac{1}{2l+2n+3}\right]\mathcal{M}^L_{\rm B}\ddot{\mathcal{M}}^{iN}_{\rm C}\\&+2(1+\gamma)\left[\frac{1}{n+1}\dot{\mathcal{M}}^L_{\rm B}\dot{\mathcal{M}}^{iN}_{\rm C}-\dot{\mathcal{M}}^L_{\rm B}\mathcal{M}^N_{\rm C}v^i_{\rm BC}\right]-\frac{2l^2+3l+3+2\gamma}{l+1}\dot{\mathcal{M}}^{iL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C}\\&-\frac{1}{2l+3}\left[(l+2)(2l+1)+\frac{2n}{2l+2n+3}\right]\mathcal{M}^{iL}_{\rm B}\ddot{\mathcal{M}}^N_{\rm C}-\frac{l^2+l+2+2\gamma}{l+1}\ddot{\mathcal{M}}^{iL}_{\rm B}\mathcal{M}^N_{\rm C},\end{aligned}\qquad(302)$$

$$\beta^{iLN}_{\rm F}=\left[\left(2+2\gamma-\frac{1}{2l+2n+3}\right)a^i_{\rm C}-(l+2+2\gamma)a^i_{\rm B}\right]\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C},\qquad(303)$$

$$\alpha_{\rm F}^{ipLN} = \left[\frac{2}{2l+2n+5} v_{\rm C}^i v_{\rm C}^p - 2(1+\gamma) v_{\rm BC}^i v_{\rm BC}^p - v_{\rm B}^i v_{\rm C}^p - \frac{2}{M_{\rm B}} \varepsilon_{ipq} \dot{\mathcal{S}}_{\rm B}^q\right] \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^N$$
$$+ \frac{2(1+\gamma)}{n+1} \mathcal{M}_{\rm B}^L \dot{\mathcal{M}}_{\rm C}^{iN} v_{\rm BC}^p + 2\left[\frac{(l+2)(2l+1)}{2l+3} v_{\rm C}^p - (l+1) v_{\rm B}^p\right] \mathcal{M}_{\rm B}^{iL} \dot{\mathcal{M}}_{\rm C}^N$$
$$- \frac{2l^2+3l+3+2\gamma}{l+1} \dot{\mathcal{M}}_{\rm B}^{iL} \mathcal{M}_{\rm C}^N v_{\rm BC}^p - \frac{1}{M_{\rm B}} \varepsilon_{ipq} (\mathcal{M}_{\rm B}^L \dot{\mathcal{M}}_{\rm C}^N \mathcal{S}_{\rm B}^q + \dot{\mathcal{M}}_{\rm B}^L \mathcal{M}_{\rm C}^N \mathcal{S}_{\rm B}^q)$$
$$+ \frac{2(1+\gamma)}{n+2} \varepsilon_{ipq} (\mathcal{M}_{\rm B}^L \dot{\mathcal{S}}_{\rm C}^{qN} + \dot{\mathcal{M}}_{\rm B}^L \mathcal{S}_{\rm C}^{qN}) + \frac{2(1+\gamma)}{l+2} \varepsilon_{ipq} (\dot{\mathcal{S}}_{\rm B}^{qN} \mathcal{M}_{\rm C}^L + \mathcal{S}_{\rm B}^{qN} \dot{\mathcal{M}}_{\rm C}^L), \tag{304}$$

$$\beta_{\rm F}^{ipLN} = \left[\left(l+1-\frac{1}{2l+2n+5}\right) a_{\rm C}^p - l a_{\rm B}^p\right] \mathcal{M}_{\rm B}^{iL} \mathcal{M}_{\rm C}^N + \frac{1}{2l+2n+5} a_{\rm C}^p \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{iN}, \tag{305}$$

$$\alpha_{\rm F}^{ipqLN} = \frac{1}{2l+2n+7} (\mathcal{M}_{\rm B}^{iL} \mathcal{M}_{\rm C}^N - \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{iN}) v_{\rm C}^p v_{\rm C}^q - (l+1) \mathcal{M}_{\rm B}^{iL} \mathcal{M}_{\rm C}^N v_{\rm BC}^p v_{\rm BC}^q$$
$$+ \frac{2(1+\gamma)}{l+2} \varepsilon^{ipk} \mathcal{S}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N v_{\rm BC}^q + \frac{2(1+\gamma)}{n+2} \varepsilon^{ipk} \mathcal{M}_{\rm B}^L \mathcal{S}_{\rm C}^{kN} v_{\rm BC}^q - \frac{1}{M_{\rm B}} \varepsilon^{ipk} \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^N \mathcal{S}_{\rm C}^k v_{\rm BC}^q, \tag{306}$$

$$\alpha_{\rm F}^{LN} = \left[(1+\gamma) v_{\rm BC}^2 - \frac{1}{2}\frac{2l+2n+3}{2l+2n+5} v_{\rm C}^2 - v_{\rm B}^2\right] \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^N + \mathcal{M}_{\rm B}^{kL} \dot{\mathcal{M}}_{\rm C}^N v_{\rm B}^k - \frac{2(1+\gamma)}{n+1} \mathcal{M}_{\rm B}^L \dot{\mathcal{M}}_{\rm C}^{kN} v_{\rm BC}^k$$
$$- \frac{1}{2l+2n+5} \mathcal{M}_{\rm B}^{kL} \ddot{\mathcal{M}}_{\rm C}^{kN} + \frac{2(1+\gamma)}{l+1} \left(\dot{\mathcal{M}}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N v_{\rm BC}^k - \frac{1}{n+1} \dot{\mathcal{M}}_{\rm B}^{kL} \dot{\mathcal{M}}_{\rm C}^{kN}\right), \tag{307}$$

$$\beta_{\rm F}^{LN} = -(l+1) \mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N a_{\rm B}^k - (n+1) \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{kN} a_{\rm C}^k - \frac{1}{2l+2n+5} (\mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N - \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{kN}) a_{\rm C}^k, \tag{308}$$

$$\gamma_{\rm F}^{LN} = [(2-2\beta-l\gamma) \bar{U}(t,\boldsymbol{x}_{\rm B}) - \gamma(n+1) \bar{U}(t,\boldsymbol{x}_{\rm C})] \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^N, \tag{309}$$

$$\mu_{\rm F}^{pLN} = \frac{1}{2}\frac{2l+2n+3}{2l+2n+7} \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{kN} v_{\rm C}^p v_{\rm C}^k + \frac{2(1+\gamma)}{n+2} \varepsilon_{pkq} \mathcal{M}_{\rm B}^L \mathcal{S}_{\rm C}^{qN} v_{\rm BC}^k$$
$$+ \left(v_{\rm B}^k v_{\rm BC}^p - \frac{1}{2} v_{\rm B}^k v_{\rm B}^p + \frac{2}{2l+2n+7} v_{\rm C}^p v_{\rm C}^k\right) \mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N \frac{2(1+\gamma)}{(l+1)(n+2)} \varepsilon_{pkq} \dot{\mathcal{M}}_{\rm B}^{kL} \mathcal{S}_{\rm C}^{qN}$$
$$- \frac{2(1+\gamma)}{l+2} \varepsilon_{pkq} \mathcal{S}_{\rm B}^{kL} \left(\mathcal{M}_{\rm C}^N v_{\rm BC}^q - \frac{1}{n+1} \dot{\mathcal{M}}_{\rm C}^{qN}\right), \tag{310}$$

$$\nu_{\rm F}^{pLN} = \frac{1}{2l+2n+7} \mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^{kN} a_{\rm C}^p, \tag{311}$$

$$\rho_{\rm F}^{pLN} = -F_{\rm C}^{kp} \mathcal{M}_{\rm B}^L \mathcal{M}_{\rm C}^{kN} - F_{\rm B}^{pk} \mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^N, \tag{312}$$

$$\sigma_{\rm F}^{pqLN} = -\frac{1}{2l+2n+9} \mathcal{M}_{\rm B}^{kL} \mathcal{M}_{\rm C}^{kN} v_{\rm C}^p v_{\rm C}^q - \frac{2(1+\gamma)}{(n+2)(l+2)} \mathcal{S}_{\rm B}^{pL} \mathcal{S}_{\rm C}^{qN}. \tag{313}$$

The coefficients (302)–(313) depend on the active mass and spin multipoles of the bodies of the ℕ-body system and their time derivatives. They also depend on velocities of the centers of mass and their accelerations with respect to the origin of the global coordinates. Coefficient (312) describes dependence of the force on the matrix of relativistic precession for each body which is a solution of the equation of relativistic precession (151). Post-Newtonian force for arbitrarily structured extended bodies with accounting for all mass and spin multipoles of the bodies has been derived in general relativity by Racine and Flanagan [84] and Racine *et al.* [85]. We compare their result with our expression (301) for the force in Appendix B.

C. Reduced post-Newtonian force

The post-Newtonian force (301) depends explicitly on the coordinate accelerations $a^i_{\rm B}$ and $a^i_{\rm C}$ of the centers of mass of extended bodies. In case, when velocities of bodies are significantly smaller than the fundamental speed c, we can use the Newtonian equations of motion of bodies, $M_{\rm B}a^i_{\rm B} = F^i_{\rm N}$, in order to replace the accelerations, $a^i_{\rm B}$, with the explicit form of the Newtonian force, $F^i_{\rm N}$, taken from (295). It gives us the reduced post-Newtonian force which depends on three types of interaction between multipoles of the extended bodies in the ℕ-body system which are due to mass-mass, spin-mass, and spin-spin gravitational couplings. In order to set in order the different types of the multipole-multipole interactions, which enter different coefficients (302)–(313), we split the post-Newtonian gravitational force in three main constituents,

$$F^i_{\rm pN} = F^i_{\rm M} + F^i_{\rm S} + F^i_{\rm P}, \tag{314}$$

where $F^i_{\rm M}$ is the force caused by the gravitational interaction between the mass multipoles of extended bodies, $F^i_{\rm S}$ is the force caused by the spin-mass and spin-spin multipole interactions, and the force $F^i_{\rm P}$ is due to the relativistic precession of the body-adapted local coordinates with respect to the spatial axes of the global coordinates. We describe the structure of each of the three components of (314) below.

1. Mass multipole coupling force

The mass multipole coupling force $F^i_{\rm M}$ consists of a number of terms describing mutual gravitational interaction between the mass multipoles of two, three, and four bodies comprising the ℕ-body system. Besides, the force includes terms depending on the first and second time-derivatives of the mass multipoles as well. The force can be represented as a sum of vectorial components,

$$F^i_{\rm M} = F^i_{\mathcal{M}\mathcal{M}} + F^i_{\mathcal{M}\dot{\mathcal{M}}} + F^i_{\mathcal{M}\ddot{\mathcal{M}}} + F^i_{\dot{\mathcal{M}}\dot{\mathcal{M}}} + F^i_{\mathcal{M}\mathcal{M}\mathcal{M}} + F^i_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}, \tag{315}$$

where each particular term in the right hand-side of (315) is labeled in correspondence with the number of the mass multipoles and/or their time derivatives participating in the multipole-to-multipole coupling. Specific expressions for different terms in (315) are given in the form of products of the coupling coefficients $A^{LN}_{\mathcal{M}\mathcal{M}}$, $A^{LN}_{\mathcal{M}\dot{\mathcal{M}}}$, etc., with the explicit expressions of STF derivatives (293) and (294). The components of the mass-mass multipole coupling force, $F_{\rm M}$, are as follows:

$$F^i_{\mathcal{M}\mathcal{M}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[A^{LN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + A^{ijLN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle jLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + A^{jLN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle ijLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + A^{ijpLN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle jpLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + B^{jpLN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle ijpLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + C^{jpLN}_{\mathcal{M}\mathcal{M}}\frac{R^{\langle ijpLN\rangle}_{\rm BC}}{R^{2l+2n+7}_{\rm BC}}\Bigg], \tag{316}$$

$$F^i_{\mathcal{M}\dot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[A^{iLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle LN\rangle}_{\rm BC}}{R^{2l+2n+1}_{\rm BC}} + A^{LN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + A^{ijLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle jLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + B^{jLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle ijLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + C^{jLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle ijLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}\Bigg], \tag{317}$$

$$F^i_{\mathcal{M}\ddot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[A^{iLN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R^{\langle LN\rangle}_{\rm BC}}{R^{2l+2n+1}_{\rm BC}} + A^{LN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+1}_{\rm BC}} + B^{LN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}\Bigg], \tag{318}$$

$$F^i_{\dot{\mathcal{M}}\dot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[A^{iLN}_{\dot{\mathcal{M}}\dot{\mathcal{M}}}\frac{R^{\langle LN\rangle}_{\rm BC}}{R^{2l+2n+1}_{\rm BC}} + A^{LN}_{\dot{\mathcal{M}}\dot{\mathcal{M}}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}\Bigg], \tag{319}$$

$$F^i_{\mathcal{M}\mathcal{M}\mathcal{M}} = \sum_{C\neq B}\sum_{D\neq C}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A^{LNK}_{\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle iLN\rangle}_{\rm BC}R^{\langle K\rangle}_{\rm CD}}{R^{2l+2n+3}_{\rm BC}R^{2k+1}_{\rm CD}} + \sum_{C\neq B}\sum_{D\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B^{LNK}_{\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle iLN\rangle}_{\rm BC}R^{\langle K\rangle}_{\rm BD}}{R^{2l+2n+3}_{\rm BC}R^{2k+1}_{\rm BD}}, \tag{320}$$

$$
\begin{aligned}
F^i_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \sum_{C\neq B}\sum_{D\neq C}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\Bigg[& A^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle LN\rangle}_{\mathrm{BC}}R^{\langle iKS\rangle}_{\mathrm{CD}}}{R^{2l+2n+1}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{CD}}} \\
& + B^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle ijLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{CD}}}{R^{2l+2n+3}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{CD}}} + \mathcal{C}^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle ijLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{CD}}}{R^{2l+2n+5}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{CD}}} \\
& + (A^{jLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} + B^{jLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}})\frac{R^{\langle iLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{CD}}}{R^{2l+2n+3}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{CD}}} \\
& + (\mathcal{C}^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} + D^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}})\frac{R^{\langle jLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{CD}}}{R^{2l+2n+3}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{CD}}}\Bigg] \\
+ \sum_{C\neq B}\sum_{D\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\Bigg[& D^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle LN\rangle}_{\mathrm{BC}}R^{\langle iKS\rangle}_{\mathrm{BD}}}{R^{2l+2n+1}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{BD}}} \\
& + E^{jLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle iLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{BD}}}{R^{2l+2n+3}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{BD}}} + H^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R^{\langle jLN\rangle}_{\mathrm{BC}}R^{\langle jKS\rangle}_{\mathrm{BD}}}{R^{2l+2n+3}_{\mathrm{BC}}R^{2k+2s+3}_{\mathrm{BD}}}\Bigg],
\end{aligned}
\tag{321}
$$

where the coupling coefficients $A^{LN}_{\mathcal{M}\mathcal{M}}$, $A^{ijLN}_{\mathcal{M}\mathcal{M}}$, etc., are given by

$$
A^{LN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\left[2(\gamma+1)v^k_{\mathrm{B}}v^k_{\mathrm{C}} - \gamma v^2_{\mathrm{B}} - \left(\gamma+\frac{1}{2}+\frac{1}{2l+2n+5}\right)v^2_{\mathrm{C}}\right]\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}, \tag{322}
$$

$$
A^{LN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\left[2v^k_{\mathrm{BC}}\left(\frac{\gamma+1}{n+1}\mathcal{M}^L_{\mathrm{B}}\dot{\mathcal{M}}^{kN}_{\mathrm{C}} - \frac{\gamma+1}{l+1}\dot{\mathcal{M}}^{kL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\right) + \left(\frac{2\gamma+1}{l+2}v^k_{\mathrm{C}} - v^k_{\mathrm{B}}\right)\mathcal{M}^{kL}_{\mathrm{B}}\dot{\mathcal{M}}^N_{\mathrm{C}}\right], \tag{323}
$$

$$
A^{LN}_{\dot{\mathcal{M}}\dot{\mathcal{M}}} = 2(\gamma+1)\frac{(-1)^l(2l+2n+1)!!}{(l+1)!(n+1)!}\dot{\mathcal{M}}^{kL}_{\mathrm{B}}\dot{\mathcal{M}}^{kN}_{\mathrm{C}}, \tag{324}
$$

$$
A^{LN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{2l!n!}\mathcal{M}^L_{\mathrm{B}}\ddot{\mathcal{M}}^N_{\mathrm{C}}, \tag{325}
$$

$$
\begin{aligned}
A^{iLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{l!n!}\Bigg[& \frac{1}{2}\frac{2l+2n+3}{2l+2n+7}v^i_{\mathrm{C}}v^p_{\mathrm{C}}\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^{pN}_{\mathrm{C}} \\
& + \left(\frac{1}{2}v^i_{\mathrm{B}}v^p_{\mathrm{B}} - v^i_{C}v^p_{\mathrm{B}} + \frac{2}{2l+2n+7}v^i_{\mathrm{C}}v^p_{\mathrm{C}}\right)\mathcal{M}^{pL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\Bigg],
\end{aligned}
\tag{326}
$$

$$
A^{iLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{l!n!}\{[v^i_{\mathrm{B}} - 2(\gamma+1)v^i_{\mathrm{BC}}]\mathcal{M}^L_{\mathrm{B}}\dot{\mathcal{M}}^N_{\mathrm{C}} - 2(\gamma+1)\dot{\mathcal{M}}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}v^i_{\mathrm{BC}}\}, \tag{327}
$$

$$
A^{iLN}_{\dot{\mathcal{M}}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{l!n!}\left[\frac{2(\gamma+1)}{n+1}\dot{\mathcal{M}}^L_{\mathrm{B}}\dot{\mathcal{M}}^{iN}_{\mathrm{C}} - \frac{2l^2+3l+2\gamma+3}{l+1}\dot{\mathcal{M}}^{iL}_{\mathrm{B}}\dot{\mathcal{M}}^N_{\mathrm{C}}\right], \tag{328}
$$

$$
\begin{aligned}
A^{iLN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{l!n!}\Bigg\{ & \left[\frac{2(\gamma+1)}{n+1} - \frac{1}{2l+2n+3}\right]\mathcal{M}^L_{\mathrm{B}}\ddot{\mathcal{M}}^{iN}_{\mathrm{C}} \\
& + \left[\frac{1}{2l+2n+3} - \frac{(l+2)(2l+1)}{(2l+3)}\right]\mathcal{M}^{iL}_{\mathrm{B}}\ddot{\mathcal{M}}^N_{\mathrm{C}} - \frac{l^2+l+2\gamma+2}{l+1}\ddot{\mathcal{M}}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\Bigg\},
\end{aligned}
\tag{329}
$$

$$
A^{ijLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\left[v^i_{\mathrm{B}}v^j_{C} - \frac{2}{2l+2n+5}v^i_{\mathrm{C}}v^j_{\mathrm{C}} + 2(\gamma+1)v^i_{\mathrm{BC}}v^j_{\mathrm{BC}}\right]\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}, \tag{330}
$$

$$A^{ijLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\left\{2\left[(l+1)v^j_{\mathrm{BC}} + \frac{1}{2l+3}v^j_{\mathrm{C}}\right]\mathcal{M}^{iL}_{\mathrm{B}}\dot{\mathcal{M}}^N_{\mathrm{C}} - \frac{2(\gamma+1)}{n+1}\mathcal{M}^L_{\mathrm{B}}\dot{\mathcal{M}}^{iN}_{\mathrm{C}}v^j_{\mathrm{BC}} + \frac{2l^2+3l+2\gamma+3}{l+1}\dot{\mathcal{M}}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}v^j_{\mathrm{BC}}\right\}, \tag{331}$$

$$A^{ijpLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{l!n!}\left\{\frac{1}{2l+2n+7}[\mathcal{M}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}} - \mathcal{M}^L_{\mathrm{B}}\mathcal{M}^{iN}_{\mathrm{C}}]v^j_{\mathrm{C}}v^p_{\mathrm{C}} - \frac{1}{2l+3}[2v^j_{\mathrm{B}}v^p_{\mathrm{B}} - 3v^j_{\mathrm{B}}v^p_{\mathrm{C}} + (l+2)(2l+1)v^j_{\mathrm{BC}}v^p_{\mathrm{BC}}]\mathcal{M}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\right\}, \tag{332}$$

$$A^{LNK}_{\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+k}(2l+2n+1)!!(2k-1)!!}{l!n!k!}[\gamma(n+1)]\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^K_D, \tag{333}$$

$$A^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = -\frac{(-1)^{l+s}(2l+2n-1)!!(2k+2s+1)!!}{l!n!k!s!}\left[2(\gamma+1) - \frac{1}{2l+2n+3}\right]\frac{\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{334}$$

$$A^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = -\frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}\frac{1}{2l+2n+5}\frac{\mathcal{M}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{335}$$

$$B^{LN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{(2l+2n+5)l!n!}\mathcal{M}^{kL}_{\mathrm{B}}\ddot{\mathcal{M}}^{kN}_{\mathrm{C}}, \tag{336}$$

$$B^{iLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\mathcal{M}^L_{\mathrm{B}}\dot{\mathcal{M}}^N_{\mathrm{C}}v^i_{\mathrm{C}}, \tag{337}$$

$$B^{ijLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{2l!n!}\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}v^i_{\mathrm{C}}v^j_{\mathrm{C}}, \tag{338}$$

$$B^{LNK}_{\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+k}(2l+2n+1)!!(2k-1)!!}{l!n!k!}[\gamma l + 2(\beta-1)]\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^K_D, \tag{339}$$

$$B^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = -\frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}\frac{\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{2\mathcal{M}_{\mathrm{C}}}, \tag{340}$$

$$B^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}\left(n+1+\frac{1}{2l+2n+5}\right)\frac{\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^{iN}_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{341}$$

$$C^{ijLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+5)!!}{l!n!(2l+2n+9)}\mathcal{M}^{pL}_{\mathrm{B}}\mathcal{M}^{pN}_{\mathrm{C}}v^i_{\mathrm{C}}v^j_{\mathrm{C}}, \tag{342}$$

$$C^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n+3)!!(2k+2s+1)!!}{l!n!k!s!}\frac{1}{2l+2n+7}\frac{\mathcal{M}^{pL}_{\mathrm{B}}\mathcal{M}^{pN}_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{343}$$

$$C^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}\frac{1}{2l+2n+5}\frac{\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^{iN}_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{344}$$

$$D^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n-1)!!(2k+2s+1)!!}{l!n!k!s!}[l+2(\gamma+1)]\frac{\mathcal{M}^L_{\mathrm{B}}\mathcal{M}^S_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{B}}}, \tag{345}$$

$$D^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}\left[\frac{(l+2)(2l+1)}{2l+3} - \frac{1}{2l+2n+5}\right]\frac{\mathcal{M}^{iL}_{\mathrm{B}}\mathcal{M}^N_{\mathrm{C}}\mathcal{M}^S_{\mathrm{C}}\mathcal{M}^K_D}{\mathcal{M}_{\mathrm{C}}}, \tag{346}$$

$$E^{iLNSK}_{\mathcal{MMMM}} = -\frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{l!n!k!s!}(l+1)\frac{\mathcal{M}^{iL}_{\rm B}\mathcal{M}^{S}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{K}_{D}}{\mathcal{M}_{\rm B}}, \tag{347}$$

$$H^{iLNSK}_{\mathcal{MMMM}} = -\frac{(-1)^{l+s}(2l+2n+1)!!(2k+2s+1)!!}{(l-1)!n!k!s!}\frac{\mathcal{M}^{iL}_{\rm B}\mathcal{M}^{S}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{K}_{D}}{\mathcal{M}_{\rm B}}. \tag{348}$$

2. Spin multipole coupling force

The spin multipole post-Newtonian force entering the translational equations of motion has the following structure:

$$F^i_{\rm S} = F^i_{\mathcal{SM}} + F^i_{\dot{\mathcal{S}}\mathcal{M}} + F^i_{\mathcal{S}\dot{\mathcal{M}}} + F^i_{\mathcal{SS}} + F^i_{s\mathcal{MM}} + F^i_{s\mathcal{M}\dot{\mathcal{M}}} + F^i_{\dot{s}\mathcal{MM}}, \tag{349}$$

where each component of the force is expressed in terms of the corresponding coupling coefficients $A^{pLN}_{\mathcal{SI}}$, $A^{ipLN}_{\dot{\mathcal{S}}\mathcal{I}}$, etc., and the STF Cartesian tensors made out of the tensor products of the relative coordinate distances (296) between the bodies. Forces $F^i_{\mathcal{SM}}$, $F^i_{\dot{\mathcal{S}}\mathcal{M}}$, and $F^i_{\mathcal{S}\dot{\mathcal{M}}}$ describe gravitational interaction between the spin and mass multipoles of the bodies. The force $F^i_{\mathcal{SS}}$ describes the spin-spin multipole interaction between the bodies. It generalizes to higher multipoles the known spin-spin gravitational force of interaction between spins of rigidly rotating, spherically symmetric bodies given by Brumberg [96][page 275, Eq. (19)], and Barker and O'Connell [[265], Eq. (54)]. The last three terms in the right-hand side of (349) labeled with a small Roman letter s take their origin from the last three terms in (292). They describe gravitational interaction of spin of body B and its first time derivative with the mass multipoles of other bodies.

The spin coupling force components are

$$F^i_{\mathcal{SM}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[A^{pLN}_{\mathcal{SM}}\frac{R^{\langle ipLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + A^{ipqLN}_{\mathcal{SM}}\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}\right], \tag{350}$$

$$F^i_{\mathcal{S}\dot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[A^{ipLN}_{\mathcal{S}\dot{\mathcal{M}}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + A^{pLN}_{\mathcal{S}\dot{\mathcal{M}}}\frac{R^{\langle ipLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}\right], \tag{351}$$

$$F^i_{\dot{\mathcal{S}}\mathcal{M}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}A^{ipLN}_{\dot{\mathcal{S}}\mathcal{M}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}, \tag{352}$$

$$F^i_{\mathcal{SS}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}A^{pqLN}_{\mathcal{SS}}\frac{R^{\langle ipqLN\rangle}_{\rm BC}}{R^{2l+2n+7}_{\rm BC}}, \tag{353}$$

$$F^i_{s\mathcal{MM}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}A^{ipqLN}_{s\mathcal{MM}}\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}, \tag{354}$$

$$F^i_{s\mathcal{M}\dot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}A^{ipLN}_{s\mathcal{M}\dot{\mathcal{M}}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}, \tag{355}$$

$$F^i_{\dot{s}\mathcal{MM}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}A^{ipLN}_{\dot{s}\mathcal{MM}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}, \tag{356}$$

where the coupling coefficients entering the various members of the spin coupling force are

$$A^{pLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!}\varepsilon_{kpq}v^q_{\rm BC}\left[\frac{\mathcal{S}^{kL}_{\rm B}\mathcal{M}^{N}_{\rm C}}{l+2} + \frac{\mathcal{S}^{kN}_{\rm C}\mathcal{M}^{L}_{\rm B}}{n+2}\right], \tag{357}$$

$$A^{ipqLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!}\varepsilon_{ipk}v^q_{\rm BC}\left[\frac{\mathcal{S}^{kL}_{\rm B}\mathcal{M}^{N}_{\rm C}}{l+2} + \frac{\mathcal{S}^{kN}_{\rm C}\mathcal{M}^{L}_{\rm B}}{n+2}\right], \tag{358}$$

$$A^{pLN}_{\mathcal{S}\dot{\mathcal{M}}} = -2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!}\varepsilon_{kpq}\left[\frac{\mathcal{S}^{kL}_{\rm B}\dot{\mathcal{M}}^{qN}_{\rm C}}{(l+2)(n+1)} + \frac{\mathcal{S}^{qN}_{\rm C}\dot{\mathcal{M}}^{kL}_{\rm B}}{(l+1)(n+2)}\right], \tag{359}$$

$$A^{ipLN}_{\mathcal{S}\dot{\mathcal{M}}} = -2(\gamma+1)\frac{(-1)^l(2l+2n+1)!!}{l!n!}\varepsilon_{ipq}\left[\frac{\mathcal{S}^{qN}_{\rm B}\dot{\mathcal{M}}^L_{\rm C}}{(l+2)} + \frac{\dot{\mathcal{M}}^L_{\rm B}\mathcal{S}^{qN}_{\rm C}}{n+2}\right], \tag{360}$$

$$A^{ipLN}_{\dot{\mathcal{S}}\mathcal{M}} = -2(1+\gamma)\frac{(-1)^l(2l+2n+1)!!}{l!n!}\varepsilon_{ipq}\left[\frac{\dot{\mathcal{S}}^{qN}_{\rm B}\mathcal{M}^L_{\rm C}}{l+2} + \frac{\mathcal{M}^L_{\rm B}\dot{\mathcal{S}}^{qN}_{\rm C}}{n+2}\right], \tag{361}$$

$$A^{pqLN}_{\mathcal{S}\mathcal{S}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+5)!!}{l!n!(l+2)(n+2)}\mathcal{S}^{pL}_{\rm B}\mathcal{S}^{qN}_{\rm C}, \tag{362}$$

$$A^{ipqLN}_{s\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{l!n!}\varepsilon_{ikp}\frac{\mathcal{S}^k_{\rm C}}{\mathcal{M}_{\rm B}}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}v^q_{\rm BC}, \tag{363}$$

$$A^{ipLN}_{s\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\varepsilon_{ipq}\frac{\mathcal{S}^q_{\rm B}}{\mathcal{M}_{\rm B}}(\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} + \dot{\mathcal{M}}^L_{\rm B}\mathcal{M}^N_{\rm C}), \tag{364}$$

$$A^{ipLN}_{\dot{s}\mathcal{M}\mathcal{M}} = 2\frac{(-1)^l(2l+2n+1)!!}{l!n!}\varepsilon_{ipq}\frac{\dot{\mathcal{S}}^q_{\rm B}}{\mathcal{M}_{\rm B}}\mathcal{M}^L_{\rm B}\mathcal{M}^N_{\rm C}. \tag{365}$$

3. Precession multipole coupling force

Finally, the force caused by the relativistic precession of spatial axes of the local coordinates adapted to each body is

$$F^i_{\rm P} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+2n+3)!!}{l!n!} \times [F^{pk}_{\rm B}\mathcal{M}^{kL}_{\rm B}\mathcal{M}^N_{\rm C} + F^{kp}_{\rm C}\mathcal{M}^L_{\rm B}\mathcal{M}^{kN}_{\rm C}]\frac{R^{\langle ikLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}. \tag{366}$$

This completes derivation of the translational equations of motion of extended bodies in the global coordinates.

D. Comments

The post-Newtonian force in translational equations of motion has been calculated in this paper for the system of the ℕ-extended bodies with an arbitrary internal structure, shape and density distribution. It includes the Newtonian and post-Newtonian forces due to the gravitational coupling between all internal mass and spin multipoles of extended bodies in an ℕ-body system. The force (316), denoted as $F^i_{\mathcal{M}\mathcal{M}}$, converges in monopole approximation to Einstein-Infeld-Hoffman (EIH) equations of motion [49,126,259] of pointlike particles. The force (350), denoted as $F^i_{\mathcal{S}\mathcal{M}}$, yields the correct analytic expression for the Lense-Thirring (gravitomagnetic) force due to the gravitational coupling of a body's intrinsic spin to orbital angular momentum of the body [96,101]. The force (353), denoted as $F^i_{\mathcal{S}\mathcal{S}}$, is reduced to the known spin-spin coupling force [96,265–267] when higher-order multipoles ($l \geq 1$) are neglected.

Calculation of the post-Newtonian force in quadrupole approximation ($l = 2$) were completed by Xu *et al.* [62] in general relativity. Their result disagrees by a sufficiently large number of terms with our expression for the post-Newtonian force (314) in the quadrupole approximation. We could not identify the mathematical reason of this disagreement which origin has yet to be clarified. On the other hand, the complete post-Newtonian force for the quadrupole and all other higher-order multipoles taken into account, derived in general relativity by Racine, Vine, and Flanagan (RVF) [84,85] by means of a different mathematical technique [30,58,59,268] nicely coincides (in case of the PPN parameters $\gamma = \beta = 1$) with our expression (314) in spite of different appearance of a few extra terms. Mathematical origin of this discrepancy is due to the different convention in the definition of time moments at which the numerical value of the body multipoles are to be computed on their worldlines. This is explained in more detail in Appendix B.

In particular, the term that had been missed in [[84], Eq. (6.12c)] and recovered in [[85], Eq. (1.1)] is given by our coupling coefficient $B^{LNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}$ in Eq. (340) which enters our expression (321) for the post-Newtonian force component $F^i_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}$. Notice also that we give our coupling coefficients for the expansion of force while Racine and Flanagan [84] provide their coupling coefficients for acceleration of body B. Therefore, our tensor coupling coefficients must be divided by the inertial mass $M_{\rm B}$ of body B in order to get the RVF coefficients. It is also worth noticing that, contrary to our choice of *dynamically* nonrotating local coordinates, Racine and Flanagan [84] had chosen the body-adapted local coordinates as being *kinematically* nonrotating with respect to the global coordinates. For this reason the force (366) caused by the relativistic precession of the local frame is absent in the RVF equations of motion. The present paper generalizes translational equations of motion derived by Racine and

Flanagan [84] and Racine *et al.* [85] to the realm of scalar-tensor theory of gravity parametrized with two covariantly defined parameters, β and γ. This generalization is important for testing scalar-tensor theories of gravity with gravitational wave detectors and for developing more comprehensive experiments within the Solar System.

It is instructive to better understand the correspondence between the post-Newtonian force (314) for spherically symmetric bodies and the EIH force [49]. The EIH equations of motion are traditionally viewed as equations of motion of pointlike test particles which are modeled as nonrotating solid spheres having spherically symmetric distribution of mass. The post-Newtonian force (314) depends on the STF internal multipoles and it is reduced to the EIH force if we neglect all STF multipoles except of monopole ($l = 0$) that corresponds to the relativistic (Tolman) mass of the body [42,165,269] if the body is fully isolated from the external gravitational environment preventing its tidal deformations. However, the spherically symmetric distribution of matter does not ensure vanishing internal multipoles of the body. Indeed, the post-Newtonian definition of the mass multipoles (122) includes the terms depending on volume integral, $\mathcal{Q}_K \int_{\mathcal{V}_B} \sigma w^{\langle K\rangle} w^{\langle L\rangle} d^3w$, which does not vanish after integration over the unit sphere making the post-Newtonian force depending on the rotational moments of inertia of the spherically symmetric bodies. Thus, the post-Newtonian force of interaction between rigid, spherically symmetric bodies in an $\mathbb{N}$-body system is not completely reduced to the EIH force but includes additional terms depending on the size of extended bodies. It makes clear that spherical bodies of finite size do not move like massive point particles and the effacing principle is violated [185].

Finite-size post-Newtonian effects in general-relativistic equations of motion of spherically symmetric bodies were discussed previously by Brumberg [96], Spyrou [127,270–272], Caporali [273,274], Dallas [275], Vincent [276], and, more recently, by Arminjon [128]. The post-Newtonian correction to the EIH force obtained by these authors depends on the second-order rotational moments of inertia $\mathcal{N}$ defined in (125). We have shown in [[17], Sec. 6.3.4] that this correction is not physical and represents a spurious, coordinate-dependent effect which can be removed by adjusting position of the center of mass and transforming the body's quadrupole moment from the global to the body-adapted local coordinates. This fact was also noticed by Nordtvedt [277]. Nonetheless, the post-Newtonian force of interaction between spherically symmetric bodies can depend on the rotational moments of inertia of the second order in scalar-tensor theory of gravity; see [[17], Eq. (6.85)].

X. ROTATIONAL EQUATIONS OF MOTION OF SPIN IN THE GLOBAL COORDINATES

Translational equations of motion of the centers of mass of extended, arbitrarily structured bodies are not sufficient to describe gravitational dynamics of an $\mathbb{N}$-body system. This is because the translational equations depend on the mass and spin multipoles of all bodies which are complicated functions of time. Therefore, they must be complemented with equations describing temporal evolution of the multipoles in order to close the system of differential equations for the configuration variables characterizing dynamics of an $\mathbb{N}$-body system. Derivation of the complete system of the evolution equations for configuration variables is a daunting task as it includes among other issues, solution of the post-Newtonian problem of the elastic response of an extended body to the tidal perturbations caused by the presence of external bodies and calculation of rotational deformations of the body due to its rotation. Calculation of the tidal and rotational responses requires a corresponding development of the post-Newtonian theory of elastic deformations of extended, self-gravitating bodies [278–280] with its further dissemination to treat more subtle effects of viscosity and multi-layer structure of stars in astrophysical systems emitting gravitational waves. The overall task seems to be very complicated and will be discussed somewhere else. The present paper centers on the developing of equation of temporal evolution of the most important configuration variable in gravitational dynamics of an $\mathbb{N}$-body system—the intrinsic angular momentum or spin of the bodies. Spin is closely related to three rotational d.o.f. of a rigidly rotating extended body characterized by the vector of angular velocity. Therefore, we call the equation of temporal evolution for spin as rotational equations of motion.

Rotational equations of motion of spin of body B in the body-adapted local coordinates, $w^\alpha = (u, \boldsymbol{w})$, have been already derived in Sec. VI F. The rotational equations of motion are parametrized with the local coordinate time u_B of the body-adpated coordinates and describe the force precession of body's spin, $\mathcal{S}^i_B$, caused by gravitational coupling of the internal mass and spin multipoles of body B with the external multipoles. In its own turn, the body-adapated local frame is subject to the Fermi-Walker transport [165] describing the relativistic precession of the spatial axes of the local coordinates with respect to the global coordinates in accordance with Eq. (151). It is convenient from a computational point of view to transform the rotational equations of motion of each body from the local to global coordinates to parametrize them with a single parameter—the global coordinate time t and to include the Fermi-Walker transport to the evolution equation of spin. Moreover, we want to express all external multipoles in the rotational equations in the form of explicit functions of the global coordinates and multipole moments of the bodies. This procedure will formulate the rotational equations of motion in terms of the same set of configuration variables as that in the translational equations of motion of bodies.

Let us define the spin components of body B measured with respect to the global coordinates as S^i. They are related to the spin components, $\mathcal{S}^i$, measured with respect to the body-adapted local coordinates by means of a post-Newtonian rotational transformation,

$$S^i = \mathcal{S}^j(\delta^{ij} - F_{\rm B}^{ij}), \tag{367}$$

where $F_{\rm B}^{ij}$ is the matrix of the Fermi-Walker precession of the local coordinates of body B. Then, rotational equations of motion of spin S^i in the global coordinates are

$$\frac{dS^i}{dt} = \frac{d\mathcal{S}^i}{du}\frac{du}{dt} - \frac{dF_{\rm B}^{ij}}{dt}\mathcal{S}^j - F_{\rm B}^{ij}\frac{d\mathcal{S}^j}{du}, \tag{368}$$

where all derivatives are taken along the worldline $\mathcal{Z}$ of the center of mass of body B. Using Eqs. (146), (151), (195) for computing the time derivatives in (368), we get the rotational equations of spin of body B in the form

$$\frac{dS^i}{dt} = T^i \tag{369}$$

where the spin S^i is considered now as a function of time t that is $S^i = S^i(u)|_{u=t}$. The total torque $T^i = T_{\rm B}^i + T_{\rm FW}^i$ is a linear combination of a torque $T_{\rm B}^i$ caused by the gravitational interaction of the internal multipoles of body B with the external multipoles, and a torque $T_{\rm FW}^i$ stemming from the Fermi-Walker precession,

$$T_{\rm B}^i = \left\{1 + \frac{1}{2}v_{\rm B}^2 - \bar{U}(t,\boldsymbol{x}_{\rm B})\right\}\mathcal{T}^i - F_{\rm B}^{ij}\mathcal{T}^j, \tag{370}$$

$$T_{\rm FW}^i = \{v_{\rm B}^{[i}a_{\rm B}^{j]} - 2(1+\gamma)\partial^{[i}\bar{U}^{j]}(t,\boldsymbol{x}_{\rm B}) - 2(1+\gamma)v_{\rm B}^{[i}\partial^{j]}\bar{U}(t,\boldsymbol{x}_{\rm B})\}S_{\rm B}^j. \tag{371}$$

Torque $\mathcal{T}^i$ in (370) was introduced earlier in (194). The next step in derivation of the rotational equations of motion is to compute the torque in the right hand-side of (369) in an explicit analytic form as a function of common configuration variables—the global coordinates of the center of mass of the bodies and their internal mass and spin multipole moments.

A. Computation of torque

Torque $T_{\rm B}^i$ in (370) is proportional to torque $\mathcal{T}^i$ given by Eq. (194) that is computed by accounting for (176), (191) and (243). It yields

$$\begin{aligned} T_{\rm B}^i = {} & \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\left[1 + \frac{1}{2}v_{\rm B}^2 + (2\beta - \gamma - 2)\bar{U}(t,\boldsymbol{x}_{\rm B})\right]\partial_{\langle kL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jL} \\ & + \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\left[\partial_{\langle kL\rangle}\bar{V}(t,\boldsymbol{x}_{\rm B},l+1)\mathcal{M}_{\rm B}^{jL} + \partial_{\langle L}\bar{V}_{k\rangle}(t,\boldsymbol{x}_{\rm B},l+1)\mathcal{M}_{\rm B}^{jL} + \frac{l+1}{l+2}\mathcal{C}_{kL}\mathcal{S}_{\rm B}^{jL}\right] \\ & - \varepsilon_{jpk}F_{\rm B}^{ij}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{pL} + \varepsilon_{ijk}a_{\rm B}^k(3a_{\rm B}^p\mathcal{M}_{\rm B}^{jp} + \mathcal{I}_{\rm c}^j), \end{aligned} \tag{372}$$

where we have taken into account that the active dipole moment $\mathcal{M}_{\rm B}^i$ can be neglected in the post-Newtonian terms. Gravitational potentials $\bar{V}(t,\boldsymbol{x}_{\rm B},l+1)$ and $\bar{V}^i(t,\boldsymbol{x}_{\rm B},l+1)$ are defined in (240) as sums taken over all bodies of an $\mathbb{N}$-body system from potentials $V_{\rm C}$ and $V_{\rm C}^i$ given in (241) and (242) along with (244) and (248). The linear sum of the STF derivatives from potentials $\bar{V}(t,\boldsymbol{x}_{\rm B},l+1)$ and $\bar{V}^i(t,\boldsymbol{x}_{\rm B},l+1)$ that appear in (372) does not contain the noncanonical potentials $\mathcal{R}^L$ and $\mathcal{N}^L$ which are mutually canceled out. We also notice that the acceleration-dependent terms in the third line of (372) actually vanish because of the adjustment of the position of the center of mass of body B given by the complementary dipole function $I_{\rm c}^i$ defined in (289). After summing up all terms in (372) and accounting for the index peeling-off formula (275), we reduce the torque to a simpler form

$$\begin{aligned} T_{\rm B}^i = {} & \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\left[1 + \frac{1}{2}v_{\rm B}^2 + 2(\beta - \gamma - 1)\bar{U}(t,\boldsymbol{x}_{\rm B})\right]\partial_{\langle kL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jL} \\ & + \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{\Omega}(t,\boldsymbol{x}_{\rm B},l)\mathcal{M}_{\rm B}^{jL} + \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\partial_L\bar{\Omega}_k(t,\boldsymbol{x}_{\rm B},l)\mathcal{M}_{\rm B}^{jL} \\ & + \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{l}{(l+1)!}\partial_{kL-1}\bar{\Omega}_p(t,\boldsymbol{x}_{\rm B},l)\mathcal{M}_{\rm B}^{pjL-1} + \sum_{l=0}^{\infty}\frac{l+1}{(l+2)l!}\bar{H}_{jiL}\mathcal{S}_{\rm B}^{jL} \\ & - \varepsilon_{jpk}F_{\rm B}^{ij}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{pL} - \varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}[F_{\rm B}^{pk}\partial_{\langle pL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jL} + F_{\rm B}^{qp}\partial_{\langle kqL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jpL}], \end{aligned} \tag{373}$$

where the potentials $\bar{W}$, $\bar{\Omega}$, $\bar{\Omega}_k$ are given in (262)–(264) and tensor $\bar{H}_{jiL}$ is explained in (249)–(252). The STF derivatives from $\bar{W}$, $\bar{\Omega}$, and $\bar{\Omega}_k$ have been computed in (265), (274), and (275)–(280) respectively.

The torque depends on the contraction of the STF derivatives of the potentials with the Levi-Civita symbol ε_{ijk}. For computational convenience of the reader we provide their exact form below in order to facilitate tracking down the process of the computation. Because each of the barred potential is a linear superposition of the corresponding potentials of each body labeled with a letter C, we write down the corresponding formulas of contraction of the Levi-Civita symbol with the STF derivatives for the single-body potentials. Contraction of the derivatives from potentials $W_{\rm C}$ and $\Omega_{\rm C}$ with the Levi-Civita symbol are

$$\varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}W_{\rm C}(t,\boldsymbol{x})\mathcal{M}_{\rm B}^{jL}=\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}, \tag{374}$$

$$\begin{aligned}
&\varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\Omega_{\rm C}(t,\boldsymbol{x},l)\mathcal{M}_{\rm B}^{jL}\\
&=\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)\left[(1+\gamma)v_{\rm BC}^2-\frac{1}{2}\frac{2l+2n+3}{2l+2n+5}v_{\rm C}^2\right]\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)[(2-2\beta-l\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})-\gamma(n+1)\bar{U}(t,\boldsymbol{x}_{\rm C})]\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)\left(\frac{1}{2}v_{\rm C}^pv_{\rm C}^q-F_{\rm C}^{qp}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{qN}+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}(n+1)\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)a_{\rm C}^p\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{pN}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{2l!n!}[\ddot{\mathcal{M}}_{\rm C}^{N}\partial_{\langle kLN\rangle}R_{\rm C}+\mathcal{M}_{\rm C}^{N}v_{\rm C}^pv_{\rm C}^q\partial_{\langle kpqLN\rangle}R_{\rm C}-\mathcal{M}_{\rm C}^{N}a_{\rm C}^p\partial_{\langle kpLN\rangle}R_{\rm C}]\mathcal{M}_{\rm B}^{jL}\\
&-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+3}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\ddot{\mathcal{M}}_{\rm C}^{kN}-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jpL}\ddot{\mathcal{M}}_{\rm C}^{pN}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{2}{2l+2n+5}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}v_{\rm C}^pv_{\rm C}^k-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+7}\partial_{pqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{kN}v_{\rm C}^pv_{\rm C}^q\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{2}{2l+2n+7}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)(\mathcal{M}_{\rm B}^{jqL}\mathcal{M}_{\rm C}^{N}-\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{qN})v_{\rm C}^pv_{\rm C}^q\\
&-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+9}\partial_{kpqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jmL}\mathcal{M}_{\rm C}^{mN}v_{\rm C}^pv_{\rm C}^q-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+3}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}a_{\rm C}^k\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{kN}a_{\rm C}^p\\
&-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+5}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)(\mathcal{M}_{\rm B}^{jpL}\mathcal{M}_{\rm C}^{N}-\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{pN})a_{\rm C}^p\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{2l+2n+7}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jqL}\mathcal{M}_{\rm C}^{qN}a_{\rm C}^p-2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{n+1}\partial_{kLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\dot{\mathcal{M}}_{\rm C}^{pN}v_{\rm BC}^p\\
&-2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{n+2}\varepsilon_{mpq}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm B}^{jL}\mathcal{S}_{\rm C}^{qN}v_{\rm BC}^m.
\end{aligned} \tag{375}$$

The very last term in the right-hand side of (375) contains a product of two Levi-Civita symbols which can be expressed as a linear combination of the Kronecker delta symbols [[165], Exercise 3.13],

$$\varepsilon_{ijk}\varepsilon_{mpq}\equiv\left\|\begin{matrix}\delta_{im} & \delta_{ip} & \delta_{iq}\\ \delta_{jm} & \delta_{jp} & \delta_{jq}\\ \delta_{km} & \delta_{kp} & \delta_{kq}\end{matrix}\right\| = \delta_{im}\delta_{jp}\delta_{kq}+\delta_{ip}\delta_{jq}\delta_{km}+\delta_{iq}\delta_{jm}\delta_{kp}-\delta_{jm}\delta_{ip}\delta_{kq}-\delta_{jp}\delta_{iq}\delta_{km}-\delta_{jq}\delta_{im}\delta_{kp}. \tag{376}$$

It allows us to recast the term with two Levi-Civita symbols to a more transparent form

$$\varepsilon_{ijk}\varepsilon_{mpq}\partial_{kpLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{S}^{qN}_{\rm C}v^m_{\rm BC} = 2\partial_{ipLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{qL}_{\rm B}\mathcal{S}^{N[q}_{\rm C}v^{p]}_{\rm BC} - 2\partial_{pqLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{qL}_{\rm B}\mathcal{S}^{N[i}_{\rm C}v^{p]}_{\rm BC}. \tag{377}$$

The two terms in (373) depending on the contraction of the Levi-Civita symbol with the STF derivatives of the vector potential $\Omega^i_{\rm C}$ are

$$\begin{aligned}\varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{(l+1)!}\partial_L\Omega^k_{\rm C}(t,\boldsymbol{x},l)\mathcal{M}^{jL}_{\rm B} &= 2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+1}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\ddot{\mathcal{M}}^{kN}_{\rm C}\\
&+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)(\dot{\mathcal{M}}^N_{\rm C}v^k_{\rm C}+\mathcal{M}^N_{\rm C}a^k_{\rm C})\mathcal{M}^{jL}_{\rm B}\\
&+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+2}\varepsilon_{kpq}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\dot{\mathcal{S}}^{qN}_{\rm C}\\
&+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+2}\varepsilon_{kpq}\partial_{mpLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{S}^{qN}_{\rm C}v^m_{\rm BC}\\
&+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(l+1)!n!}\frac{1}{n+1}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\dot{\mathcal{M}}^{kN}_{\rm C}v^p_{\rm BC}\\
&+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^k_{\rm C}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l-1-2\gamma)\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C}v^k_{\rm B}\\
&+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l-1-2\gamma)\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm BC}v^k_{\rm B}\\
&-\frac{1}{2}\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{pLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{M}^N_{\rm C}v^p_{\rm B}v^k_{\rm B}\\
&-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(l+1)!n!}(l^2+l+2+2\gamma)\partial_{LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}^{jL}_{\rm B}\mathcal{M}^N_{\rm C}a^k_{\rm B},\end{aligned} \tag{378}$$

$$
\begin{aligned}
&\varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{l}{(l+1)!}\partial_{kL-1}\Omega^{p}_{\mathrm{C}}(t,\boldsymbol{x},l)\mathcal{M}^{jpL-1}_{\mathrm{B}}\\
&=2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!(n+1)!}\frac{1}{l+2}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\ddot{\mathcal{M}}^{pN}_{\mathrm{C}}\\
&\quad+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{l+2}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)(\dot{\mathcal{M}}^{N}_{\mathrm{C}}v^{p}_{\mathrm{C}}+\mathcal{M}^{N}_{\mathrm{C}}a^{p}_{\mathrm{C}})\mathcal{M}^{jpL}_{\mathrm{B}}\\
&\quad+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(l+1)}{(l+2)!(n+2)!}\varepsilon_{pmq}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\dot{\mathcal{S}}^{qN}_{\mathrm{C}}\\
&\quad+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(l+1)}{(l+2)!(n+2)!}\varepsilon_{pmq}\partial_{kbmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{S}^{qN}_{\mathrm{C}}v^{b}_{\mathrm{BC}}\\
&\quad+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!(n+1)!}\frac{1}{l+2}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\dot{\mathcal{M}}^{pN}_{\mathrm{C}}v^{m}_{\mathrm{BC}}\\
&\quad+2(1+\gamma)\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{1}{l+2}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{M}^{N}_{\mathrm{C}}v^{m}_{\mathrm{BC}}v^{p}_{\mathrm{C}}\\
&\quad+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l-2\gamma}{l+2}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\dot{\mathcal{M}}^{N}_{\mathrm{C}}v^{p}_{\mathrm{B}}\\
&\quad+\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l-2\gamma}{l+2}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{M}^{N}_{\mathrm{C}}v^{m}_{\mathrm{BC}}v^{p}_{\mathrm{B}}\\
&\quad-\frac{1}{2}\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{M}^{N}_{\mathrm{C}}v^{m}_{\mathrm{B}}v^{p}_{\mathrm{B}}\\
&\quad-\varepsilon_{ijk}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\frac{l^2+3l+4+2\gamma}{l+2}\partial_{kLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{M}^{N}_{\mathrm{C}}a^{p}_{\mathrm{B}}.
\end{aligned}\tag{379}
$$

Again, we use (376) in order to simplify those terms in (378) and (379) which contain the product of two Levi-Civita symbols. More specifically, the two terms in Eq. (378) are simplified to

$$\varepsilon_{ijk}\varepsilon_{kpq}\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jL}_{\mathrm{B}}\dot{\mathcal{S}}^{qN}_{\mathrm{C}}=2\partial_{iLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{qL}_{\mathrm{B}}\dot{\mathcal{S}}^{qN}_{\mathrm{C}}-2\partial_{pLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{pL}_{\mathrm{B}}\dot{\mathcal{S}}^{iN}_{\mathrm{C}},\tag{380}$$

$$\varepsilon_{ijk}\varepsilon_{kpq}\partial_{mpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jL}_{\mathrm{B}}\mathcal{S}^{qN}_{\mathrm{C}}v^{m}_{\mathrm{BC}}=2\partial_{ipLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{qL}_{\mathrm{B}}\mathcal{S}^{qN}_{\mathrm{C}}v^{p}_{\mathrm{BC}}-2\partial_{pqLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{pL}_{\mathrm{B}}\mathcal{S}^{iN}_{\mathrm{C}}v^{q}_{\mathrm{BC}},\tag{381}$$

and the two other terms in (379) are

$$\varepsilon_{ijk}\varepsilon_{pmq}\partial_{kmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\dot{\mathcal{S}}^{qN}_{\mathrm{C}}=2\partial_{jkLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jL[i}_{\mathrm{B}}\dot{\mathcal{S}}^{k]N}_{\mathrm{C}}+\partial_{ikLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{qkL}_{\mathrm{B}}\dot{\mathcal{S}}^{qN}_{\mathrm{C}},\tag{382}$$

$$\varepsilon_{ijk}\varepsilon_{pmq}\partial_{kbmLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jpL}_{\mathrm{B}}\mathcal{S}^{qN}_{\mathrm{C}}v^{b}_{\mathrm{BC}}=2\partial_{jkpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{jL[i}_{\mathrm{B}}\mathcal{S}^{k]N}_{\mathrm{C}}v^{p}_{\mathrm{BC}}+\partial_{ikpLN}\left(\frac{1}{R_{\mathrm{C}}}\right)\mathcal{M}^{qkL}_{\mathrm{B}}\mathcal{S}^{qN}_{\mathrm{C}}v^{p}_{\mathrm{BC}},\tag{383}$$

Multipolar expansion of the term in (373) containing the product of the STF derivative of $H_{\rm C}^{jiL}$ with the spin multipoles, reads

$$\sum_{l=0}^{\infty}\frac{l+1}{(l+2)l!}H_{\rm C}^{jiL}\mathcal{S}_{\rm B}^{jL}=4(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{l+1}{(l+2)l!}v_{\rm BC}^{[j}\partial^{i]LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^{N}\mathcal{S}_{\rm B}^{jL}$$
$$-4(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\frac{l+1}{(l+2)l!}\dot{\mathcal{M}}_{\rm C}^{N[j}\partial^{i]LN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}_{\rm B}^{jL}$$
$$+4(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)n!}\frac{l+1}{(l+2)l!}\varepsilon^{pq[j}\partial^{i]qLN}\left(\frac{1}{R_{\rm C}}\right)\mathcal{S}_{\rm C}^{pN}\mathcal{S}_{\rm B}^{jL}$$
$$+2(1+\gamma)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(l+3)l!}\left[v_{\rm BC}^{p}\partial^{pjNL}\left(\frac{1}{R_{\rm C}}\right)\mathcal{M}_{\rm C}^{N}+\partial^{jNL}\left(\frac{1}{R_{\rm C}}\right)\dot{\mathcal{M}}_{\rm C}^{N}\right]\mathcal{S}_{\rm B}^{ijL}. \tag{384}$$

B. Explicit formula for torque

The total torque T^i governing precession of spin of body B in the global coordinates is given in the right-hand side of the rotational equations of motion (369) as a sum of two terms, $T_{\rm B}^i+T_{\rm FW}^i$, where the pure gravitational torque, $T_{\rm B}^i$, has been defined in (370) and (373) and the Fermi-Walker torque, $T_{\rm FW}^i$, is given in (371). After substituting Eqs. (374)–(384) into (373) and reducing similar terms the gravitational torque can be represented as a sum of the Newtonian and post-Newtonian terms, $T_{\rm B}^i=T_{\rm N}^i+T_{\rm pN}^i$. Hence, the total torque T^i is given by

$$T^i=T_{\rm N}^i+T_{\rm pN}^i+T_{\rm FW}^i, \tag{385}$$

where $T_{\rm N}^i$ is the Newtonian part of the torque, $T_{\rm pN}^i$ is its post-Newtonian counterpart, and $T_{\rm FW}^i$ is the Fermi-Walker torque. We provide explicit multipolar expressions for the gravitational torque in Secs. X B 1 and X B 2 below. Explicit multipolar expansion of the Fermi-Walker torque is given in Sec. X B 3.

1. Newtonian torque

The Newtonian torque, $T_{\rm N}^i$, is defined by the very first term in Eq. (373),

$$T_{\rm N}^i=\varepsilon_{ijk}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}\bar{W}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jL}$$
$$=\varepsilon_{ijk}\sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\langle kL\rangle}W_{\rm C}(t,\boldsymbol{x}_{\rm B})\mathcal{M}_{\rm B}^{jL} \tag{386}$$

where $\partial_{\langle kL\rangle}W_{\rm C}(t,\boldsymbol{x}_{\rm B})=\lim_{\boldsymbol{x}\to\boldsymbol{x}_{\rm B}}\partial_{\langle kL\rangle}W_{\rm C}(t,\boldsymbol{x})$, and multipolar expansion of gravitational potential $W_{\rm C}(t,\boldsymbol{x})$ has been defined in (220). After taking the partial STF derivatives from the potential $W_{\rm C}$, the Newtonian torque takes on the following explicit form:

$$T_{\rm N}^i=\varepsilon_{ijk}\sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}\partial_{\langle kLN\rangle}R_{\rm BC}^{-1}. \tag{387}$$

Applying (293) yields the Newtonian torque in its final form,

$$T_{\rm N}^i=-\varepsilon_{ijk}\sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+2n+1)!!}{l!n!}$$
$$\times\mathcal{M}_{\rm B}^{jL}\mathcal{M}_{\rm C}^{N}\frac{R_{\rm BC}^{\langle kLN\rangle}}{R_{\rm BC}^{2l+2n+3}}, \tag{388}$$

where, here and everywhere else, all multipoles of body B are taken at the time $u_{\rm B}^*$ given by (297), and all multipoles of body C ≠ B are taken at time $u_{\rm C}^*$ given by (298). Formula of the multipolar expansion for the Newtonian torque has been also derived by Racine [129] in general relativity. Torque (388) depends on the active mass multipoles in the right-hand side of this equation and generalizes the results of [129] to scalar-tensor theory of gravity. Equation (388) reduces to the expression derived by Racine [129] in case of the PPN parameters $\beta=\gamma=1$.

We draw to the attention of the reader the fact that the active multipoles in (388) are defined with taking into account all post-Newtonian contributions from the stress-energy tensor of the extended bodies in accordance with their definition (122). It is also worth noticing that the active dipole $\mathcal{M}_{\rm B}^i$ of each body is explicitly included in the right-hand side of the Newtonian torque (388) as it does not vanish because the center of mass of each body B is defined by the condition of vanishing conformal dipole, $\mathcal{I}_{\rm B}^i=0$, in accordance with (176). It means that in contrast to general theory of relativity (cf. [[129], Eq. (91)]), the dipole-monopole gravitational torque that is the term with $l=0$, $n=0$ in (388) is present in the scalar-tensor theory of gravity even if the origin of the local coordinates is fixed exactly at the center of mass of the body. The dipole-monopole torque in the rotational equation of motion of spin causes an *anomalous* precession of the body's spin as compared with general relativity. The anomalous precession of the spin is caused by the difference between the active, $\mathcal{M}^i_{\rm B}$, and conformal, $\mathcal{I}^i_{\rm B}$, dipole moments of the body B in scalar-tensor theory of gravity. This resembles the Dicke-Nordtvedt effect of violation of strong principle of equivalence in translational motion of the bodies, which is caused by the difference between active, $\mathcal{M}_{\rm B}$, and conformal, $M_{\rm B}$, masses of the body, to the case of rotational motion of the bodies. Measurement of the anomalous pole-dipole torque can help to set a direct experimental limitation on the PPN parameter β which is currently measured only indirectly through the measurement of the Nordtvedt parameter $\eta = 4\beta - \gamma - 3$, primarily by the lunar laser ranging technique [192,281,282], after subtracting the best numerical estimate of the parameter γ obtained, for example, from the measurement of gravitational bending of light [283–285].

2. Post-Newtonian torque

Multipolar expansion of the post-Newtonian gravitational torque, $T^i_{\rm pN}$, can be represented in the form of a linear operator from the STF partial derivatives with respect to spatial coordinates similarly to the presentation of the post-Newtonian force in the translational equations of motion,

$$\begin{aligned}
T^i_{\rm pN} = {} & \sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}[(\alpha_{\rm T}^{iLN}+\beta_{\rm T}^{iLN})\partial_{\langle LN\rangle}+(\alpha_{\rm T}^{ipLN}+\beta_{\rm T}^{ipLN}+\gamma_{\rm T}^{ipLN})\partial_{\langle pLN\rangle}\\
& +(\alpha_{\rm T}^{ipqLN}+\beta_{\rm T}^{ipqLN})\partial_{\langle pqLN\rangle}+\alpha_{\rm T}^{ikpqLN}\partial_{\langle kpqLN\rangle}]R_{\rm C}^{-1}\\
& +\sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}[\alpha_{\rm T}^{LN}\partial_{\langle iLN\rangle}+\mu_{\rm T}^{pLN}\partial_{\langle ipLN\rangle}+\sigma_{\rm T}^{pqLN}\partial_{\langle ipqLN\rangle}]R_{\rm C}^{-1}\\
& +\frac{1}{2}\varepsilon_{ijk}\sum_{{\rm C}\neq{\rm B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}_{\rm B}^{jL}[\ddot{\mathcal{M}}_{\rm C}^{N}\partial_{\langle kLN\rangle}-\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p}\partial_{\langle kpLN\rangle}+\mathcal{M}_{\rm C}^{N}v_{\rm C}^{p}v_{\rm C}^{q}\partial_{\langle kpqLN\rangle}]R_{\rm C},
\end{aligned} \tag{389}$$

where the STF derivatives from $R_{\rm BC}^{-1}$ and $R_{\rm BC}$ are understood in the sense of Eqs. (293) and (294). The coefficients of operator (389) are

$$\alpha_{\rm T}^{iLN} = \varepsilon_{ipk}\left[\frac{2(1+\gamma)}{(l+1)(n+1)}-\frac{1}{2l+2n+3}\right]\mathcal{M}_{\rm B}^{pL}\ddot{\mathcal{M}}_{\rm C}^{kN}+\varepsilon_{ipk}\left[v_{\rm B}^{k}-\frac{2(1+\gamma)}{l+1}v_{\rm BC}^{k}\right]\mathcal{M}_{\rm B}^{pL}\dot{\mathcal{M}}_{\rm C}^{N}, \tag{390}$$

$$\beta_{\rm T}^{iLN} = \varepsilon_{ikp}\left[\left(\frac{1}{2l+2n+3}-2\frac{1+\gamma}{l+1}\right)a_{\rm C}^{k}+\left(l+2\frac{1+\gamma}{l+1}\right)a_{\rm B}^{k}\right]\mathcal{M}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}, \tag{391}$$

$$\begin{aligned}
\alpha_{\rm T}^{ipLN} = {} & \varepsilon_{ikp}\left[\frac{1}{2}v_{\rm B}^{2}+(1+\gamma)v_{\rm BC}^{2}-\frac{1}{2}\frac{2l+2n+3}{2l+2n+5}v_{\rm C}^{2}\right]\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{N}-\varepsilon_{jkp}F_{\rm B}^{ij}\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{N}\\
& +\varepsilon_{ikq}\left[v_{\rm BC}^{p}v_{B}^{q}-\frac{1}{2}v_{B}^{p}v_{B}^{q}-\frac{2(1+\gamma)}{l+1}v_{\rm BC}^{p}v_{\rm BC}^{q}+\frac{2}{2l+2n+5}v_{\rm C}^{p}v_{\rm C}^{q}-F_{\rm B}^{pq}\right]\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{N}\\
& +\varepsilon_{ikp}\left[\frac{2(1+\gamma)}{(l+2)(n+1)}-\frac{1}{2l+2n+5}\right]\mathcal{M}_{\rm B}^{kqL}\ddot{\mathcal{M}}_{\rm C}^{qN}+\frac{2(1+\gamma)}{n+1}\left[\frac{\varepsilon_{ikq}}{l+1}v_{\rm BC}^{p}-\varepsilon_{ikp}v_{\rm BC}^{q}\right]\mathcal{M}_{\rm B}^{kL}\dot{\mathcal{M}}_{\rm C}^{qN}\\
& +\varepsilon_{ikp}\left[v_{\rm B}^{q}-\frac{2(1+\gamma)}{l+2}v_{\rm BC}^{q}\right]\mathcal{M}_{\rm B}^{kqL}\dot{\mathcal{M}}_{\rm C}^{N}-2(1+\gamma)\frac{l+1}{l+2}\left[\mathcal{S}_{\rm B}^{pL}\mathcal{M}_{\rm C}^{N}v_{\rm BC}^{i}-\frac{1}{n+1}\mathcal{S}_{\rm B}^{pL}\dot{\mathcal{M}}_{\rm C}^{iN}\right]\\
& +\frac{2(1+\gamma)}{l+3}\mathcal{S}_{\rm B}^{ipL}\dot{\mathcal{M}}_{\rm C}^{N}-\frac{4(1+\gamma)}{(l+1)(n+2)}\mathcal{M}_{\rm B}^{pL}\dot{\mathcal{S}}_{\rm C}^{iN},
\end{aligned} \tag{392}$$

$$\begin{aligned}
\beta_{\rm T}^{ipLN} = {} & \varepsilon_{ikp}\left[n+1+\frac{1}{2l+2n+5}\right]\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{qN}a_{\rm C}^{q}+\frac{1}{2l+2n+5}\varepsilon_{ikq}\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{qN}a_{\rm C}^{p}\\
& +\varepsilon_{ikp}\left[\frac{2(1+\gamma)}{l+2}a_{\rm C}^{q}-\frac{1}{2l+2n+5}a_{\rm C}^{q}-\frac{l^2+3l+4+2\gamma}{l+2}a_{\rm B}^{q}\right]\mathcal{M}_{\rm B}^{kqL}\mathcal{M}_{\rm C}^{N},
\end{aligned} \tag{393}$$

$$\gamma_{\rm T}^{ipLN} = -\gamma\varepsilon_{ikp}[(l+2)\bar{U}(t,\boldsymbol{x}_{\rm B})+(n+1))\bar{U}(t,\boldsymbol{x}_{\rm C})]\mathcal{M}_{\rm B}^{kL}\mathcal{M}_{\rm C}^{N}, \tag{394}$$

$$
\begin{aligned}
\alpha_{\mathrm{T}}^{ipqLN} &= \varepsilon_{ijq}\left[\frac{1}{2}-\frac{2}{2l+2n+7}\right]\mathcal{M}_{\mathrm{B}}^{jL}\mathcal{M}_{\mathrm{C}}^{kN}v_{\mathrm{C}}^{k}v_{\mathrm{C}}^{p}-\frac{1}{2l+2n+7}\varepsilon_{ijk}\mathcal{M}_{\mathrm{B}}^{jL}\mathcal{M}_{\mathrm{C}}^{kN}v_{\mathrm{C}}^{p}v_{\mathrm{C}}^{q}\\
&+\varepsilon_{ijq}\left[-\frac{1}{2}v_{\mathrm{B}}^{k}v_{\mathrm{B}}^{p}+v_{\mathrm{BC}}^{p}v_{\mathrm{B}}^{k}-\frac{2(1+\gamma)}{l+2}v_{\mathrm{BC}}^{p}v_{\mathrm{BC}}^{k}+\frac{2}{2l+2n+7}v_{\mathrm{C}}^{k}v_{\mathrm{C}}^{p}\right]\mathcal{M}_{\mathrm{B}}^{jkL}\mathcal{M}_{\mathrm{C}}^{N}\\
&+\frac{2(1+\gamma)}{(l+2)(n+1)}\varepsilon_{ijq}\mathcal{M}_{\mathrm{B}}^{jkL}\dot{\mathcal{M}}_{\mathrm{C}}^{kN}v_{\mathrm{BC}}^{p}-\varepsilon_{ijq}(F_{\mathrm{C}}^{kp}\mathcal{M}_{\mathrm{B}}^{jL}\mathcal{M}_{\mathrm{C}}^{kN}+F_{\mathrm{B}}^{pk}\mathcal{M}_{\mathrm{B}}^{jkL}\mathcal{M}_{\mathrm{C}}^{N})\\
&+\frac{2(1+\gamma)}{n+2}\left[\frac{l-1}{l+1}\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{S}_{\mathrm{C}}^{iN}v_{\mathrm{BC}}^{q}-\mathcal{M}_{\mathrm{B}}^{pL}\mathcal{S}_{\mathrm{C}}^{qN}v_{\mathrm{BC}}^{i}\right]+\frac{2(1+\gamma)}{l+3}\mathcal{S}_{\mathrm{B}}^{ipL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{q}\\
&+\frac{2(1+\gamma)}{(l+2)(n+2)}[\mathcal{M}_{\mathrm{B}}^{ipL}\dot{\mathcal{S}}_{\mathrm{C}}^{qN}-\mathcal{M}_{\mathrm{B}}^{pqL}\dot{\mathcal{S}}_{\mathrm{C}}^{iN}-(l+1)\varepsilon_{ijq}\mathcal{S}_{\mathrm{B}}^{pL}\mathcal{S}_{\mathrm{C}}^{jN}],
\end{aligned}\tag{395}
$$

$$
\beta_{\mathrm{T}}^{ipqLN}=\frac{1}{2l+2n+7}\varepsilon_{ijq}\mathcal{M}_{\mathrm{B}}^{jkL}\mathcal{M}_{\mathrm{C}}^{kN}a_{\mathrm{C}}^{p},\tag{396}
$$

$$
\alpha_{\mathrm{T}}^{ikpqLN}=\frac{2(1+\gamma)}{(l+2)(n+2)}[\mathcal{M}_{\mathrm{B}}^{iqL}\mathcal{S}_{\mathrm{C}}^{kN}v_{\mathrm{BC}}^{p}-\mathcal{M}_{\mathrm{B}}^{qkL}\mathcal{S}_{\mathrm{C}}^{iN}v_{\mathrm{BC}}^{p}]-\frac{1}{2l+2n+9}\varepsilon_{ijk}\mathcal{M}_{\mathrm{B}}^{jnL}\mathcal{M}_{\mathrm{C}}^{nN}v_{\mathrm{C}}^{p}v_{\mathrm{C}}^{q},\tag{397}
$$

$$
\alpha_{\mathrm{T}}^{LN}=2(1+\gamma)\left[\frac{l+1}{l+2}\mathcal{S}_{\mathrm{B}}^{jL}\mathcal{M}_{\mathrm{C}}^{N}v_{\mathrm{BC}}^{j}-\frac{l+1}{(l+2)(n+1)}\mathcal{S}_{\mathrm{B}}^{jL}\dot{\mathcal{M}}_{\mathrm{C}}^{jN}+\frac{2}{(l+1)(n+2)}\mathcal{M}_{\mathrm{B}}^{qL}\dot{\mathcal{S}}_{\mathrm{C}}^{qN}\right],\tag{398}
$$

$$
\mu_{\mathrm{T}}^{pLN}=\frac{2(1+\gamma)}{n+2}\left[\frac{l+1}{l+2}\varepsilon_{qpk}\mathcal{S}_{\mathrm{B}}^{kN}\mathcal{S}_{\mathrm{C}}^{qL}+\mathcal{M}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{pN}v_{\mathrm{BC}}^{k}-\frac{l-1}{l+1}\mathcal{M}_{\mathrm{B}}^{kL}\mathcal{S}_{\mathrm{C}}^{kN}v_{\mathrm{BC}}^{p}+\frac{1}{l+2}\mathcal{M}_{\mathrm{B}}^{kpL}\dot{\mathcal{S}}_{\mathrm{C}}^{kN}\right],\tag{399}
$$

$$
\sigma_{\mathrm{T}}^{pqLN}=\frac{2(1+\gamma)}{(l+2)(n+2)}\mathcal{M}_{\mathrm{B}}^{kpL}\mathcal{S}_{\mathrm{C}}^{kN}v_{\mathrm{BC}}^{q}.\tag{400}
$$

3. Fermi-Walker torque

The Fermi-Walker torque (371) can be easily calculated by making use of Eqs. (213) and (223) and replacing acceleration of the center of mass $a_{\mathrm{B}}^{i}=F_{\mathrm{N}}^{i}/M_{\mathrm{B}}$, where the Newtonian force F_{N}^{i} is shown in (295). Taking the STF derivatives from the corresponding expressions we get

$$
\begin{aligned}
T_{\mathrm{FW}}^{i} &= 2(1+\gamma)\sum_{C\neq\mathrm{B}}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!}\mathcal{S}_{\mathrm{B}}^{j}\left(\mathcal{M}_{\mathrm{C}}^{L}v_{\mathrm{BC}}^{[i}\partial^{j]L}+\frac{1}{l+1}\dot{\mathcal{M}}_{\mathrm{C}}^{L[i}\partial^{j]L}-\frac{1}{l+2}\mathcal{S}_{\mathrm{C}}^{pL}\varepsilon^{pq[i}\partial^{j]L}\right)R_{\mathrm{BC}}^{-1}\\
&+\frac{1}{M_{\mathrm{B}}}\sum_{C\neq\mathrm{B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{l!n!}\mathcal{M}_{\mathrm{B}}^{L}\mathcal{M}_{\mathrm{C}}^{N}\mathcal{S}_{\mathrm{B}}^{j}v_{\mathrm{B}}^{[i}\partial^{j]LN}R_{\mathrm{BC}}^{-1}.
\end{aligned}\tag{401}
$$

Taking the STF derivatives from R_{BC}^{-1} defined in (293) we obtain the multipolar expansion of the Fermi-Walker torque,

$$
\begin{aligned}
T_{\mathrm{FW}}^{i} &= -2(1+\gamma)\sum_{C\neq\mathrm{B}}\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\left(\mathcal{M}_{\mathrm{C}}^{L}v_{\mathrm{BC}}^{[i}R_{\mathrm{BC}}^{\langle j]L\rangle}+\frac{1}{l+1}\dot{\mathcal{M}}_{\mathrm{C}}^{L[i}R_{\mathrm{BC}}^{\langle j]L\rangle}-\frac{1}{l+2}\mathcal{S}_{\mathrm{C}}^{pL}\varepsilon^{pq[i}R_{\mathrm{BC}}^{\langle j]L\rangle}\right)\frac{\mathcal{S}_{\mathrm{B}}^{j}}{R_{\mathrm{BC}}^{2l+3}}\\
&-\frac{1}{M_{\mathrm{B}}}\sum_{C\neq\mathrm{B}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{l}(2l+2n+1)}{l!n!}\frac{v_{\mathrm{B}}^{[i}R_{\mathrm{BC}}^{\langle j]LN\rangle}}{R_{\mathrm{BC}}^{2l+2n+3}}\mathcal{M}_{\mathrm{B}}^{L}\mathcal{M}_{\mathrm{C}}^{N}\mathcal{S}_{\mathrm{B}}^{j}.
\end{aligned}\tag{402}
$$

C. Reduced post-Newtonian torque

It is instructive to represent the post-Newtonian torque T_{pN}^{i} in yet another form by splitting up coefficients (390)–(400) into various terms describing different types of gravitational coupling between the internal multipoles of extended bodies like mass-mass, mass-spin, spin-spin multipole interaction as well as the geometric coupling due to the Fermi-Walker precession. This requires one to reduce the coefficients depending on the acceleration a_{B}^{i} of the center of mass of body B by making use of the Newtonian equations of translational motion, $M_{\mathrm{B}}a_{\mathrm{B}}^{i}=F_{\mathrm{N}}^{i}$, with the explicit form of the Newtonian force

$F_{\rm N}^{i}$ given in (295). We perform this procedure and split the post-Newtonian torque in three main constituents,

$$T_{pN}^{i}=T_{\rm M}^{i}+T_{\rm S}^{i}+T_{\rm P}^{i}, \tag{403}$$

where $T_{\rm M}^{i}$ is caused by the gravitational coupling between the mass multipoles of extended bodies, $T_{\rm S}^{i}$ describes gravitational interaction between the spin and mass multipoles, and $T_{\rm P}^{i}$ originates from the Fermi-Walker precession of the spatial axes of the body-adapted local coordinates. Specific expressions for each terms in the right-hand side of (403) are given below.

1. Mass multipole coupling torque

The mass-mass multipole coupling torque $T_{\rm M}^{i}$ consists of various terms describing two-, three-, and four-body gravitational interactions between the internal mass multipoles of the bodies comprisin an $\mathbb{N}$-body system. The torque depends on the interaction between the first and second time derivatives of the mass multipoles as well. It has the following schematic structure:

$$T_{\rm M}^{i}=T^{i}_{\mathcal{M}\mathcal{M}}+T^{i}_{\mathcal{M}\dot{\mathcal{M}}}+T^{i}_{\mathcal{M}\ddot{\mathcal{M}}}+T^{i}_{\mathcal{M}\mathcal{M}\mathcal{M}}+T^{i}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}, \tag{404}$$

where each particular term denotes the number of the gravitationally coupled multipoles. Specific expressions for different terms in (404) are given below in terms of the coordinate distances (296) between the bodies and the corresponding coupling coefficients $\mathcal{K}^{LN}_{\mathcal{I}\mathcal{I}},\mathcal{K}^{iLN}_{\mathcal{M}\dot{\mathcal{M}}},\mathcal{K}^{iLN}_{\mathcal{I}\dot{\mathcal{I}}}$, etc., which are shown explicitly in Eqs. (410)–(428). The torque components read

$$T^{i}_{\mathcal{M}\mathcal{M}}=\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ipLN}_{\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}}{R_{\rm BC}^{2l+2n+3}}+\mathcal{K}^{ipqLN}_{\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pqLN\rangle}}{R_{\rm BC}^{2l+2n+5}}\right]+\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ikpqLN}_{\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle kpqLN\rangle}}{R_{\rm BC}^{2l+2n+7}}+\mathcal{L}^{ikpqLN}_{\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle kpqLN\rangle}}{R_{\rm BC}^{2l+2n+5}}\right], \tag{405}$$

$$T^{i}_{\mathcal{M}\dot{\mathcal{M}}}=\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{iLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle LN\rangle}}{R_{\rm BC}^{2l+2n+1}}+\mathcal{K}^{ipLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle pLN\rangle}}{R_{\rm BC}^{2l+2n+3}}+\mathcal{K}^{ipqLN}_{\mathcal{M}\dot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle pqLN\rangle}}{R_{\rm BC}^{2l+2n+5}}\right], \tag{406}$$

$$T^{i}_{\mathcal{M}\ddot{\mathcal{M}}}=\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{iLN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle LN\rangle}}{R_{\rm BC}^{2l+2n+1}}+\mathcal{K}^{ipLN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle pLN\rangle}}{R_{\rm BC}^{2l+2n+3}}+\mathcal{L}^{ipLN}_{\mathcal{M}\ddot{\mathcal{M}}}\frac{R_{\rm BC}^{\langle pLN\rangle}}{R_{\rm BC}^{2l+2n+1}}\right], \tag{407}$$

$$T^{i}_{\mathcal{M}\mathcal{M}\mathcal{M}}=\sum_{C\neq B}\sum_{D\neq C}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{K}^{ipLNK}_{\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}R_{\rm CD}^{\langle K\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm CD}^{2k+1}}+\sum_{C\neq B}\sum_{D\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{L}^{ipLNK}_{\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}R_{\rm BD}^{\langle K\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm BD}^{2k+1}}, \tag{408}$$

$$\begin{aligned}T^{i}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}=&\sum_{C\neq B}\sum_{D\neq C}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\left[\mathcal{K}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle LN\rangle}R_{\rm CD}^{\langle pKS\rangle}}{R_{\rm BC}^{2l+2n+1}R_{\rm CD}^{2k+2s+3}}+\mathcal{K}^{ipqLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}R_{\rm CD}^{\langle qKS\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm CD}^{2k+2s+3}}\right.\\&\left.+\mathcal{K}^{iLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}R_{\rm CD}^{\langle pKS\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm CD}^{2k+2s+3}}+\mathcal{L}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pqLN\rangle}R_{\rm CD}^{\langle qKS\rangle}}{R_{\rm BC}^{2l+2n+5}R_{\rm CD}^{2k+2s+3}}+\mathcal{M}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pqLN\rangle}R_{\rm CD}^{\langle qKS\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm CD}^{2k+2s+3}}\right]\\&+\sum_{C\neq B}\sum_{D\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\left[\mathcal{N}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle LN\rangle}R_{\rm BD}^{\langle pKS\rangle}}{R_{\rm BC}^{2l+2n+1}R_{\rm BD}^{2k+2s+3}}+\mathcal{N}^{ipqLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}}\frac{R_{\rm BC}^{\langle pLN\rangle}R_{\rm BD}^{\langle qKS\rangle}}{R_{\rm BC}^{2l+2n+3}R_{\rm BD}^{2k+2s+3}}\right].\end{aligned} \tag{409}$$

The coupling coefficients of the mass-mass multipole interaction that appear in (405)–(409) are

$$\begin{aligned}\mathcal{K}^{ipLN}_{\mathcal{M}\mathcal{M}}=\frac{(-1)^{l}(2l+2n+1)!!}{l!n!}&\left\{\varepsilon_{ipq}\left[\frac{1}{2}v_{\rm B}^{2}+(1+\gamma)v_{\rm BC}^{2}-\frac{1}{2}\frac{2l+2n+3}{2l+2n+5}v_{\rm C}^{2}\right]\mathcal{M}_{\rm B}^{qL}\mathcal{M}_{\rm C}^{N}\right.\\&\left.+\varepsilon_{ikq}\left[v_{B}^{k}v_{\rm BC}^{p}-\frac{1}{2}v_{B}^{k}v_{B}^{p}-\frac{2(1+\gamma)}{l+1}v_{\rm BC}^{k}v_{\rm BC}^{p}+\frac{2}{2l+2n+5}v_{\rm C}^{k}v_{\rm C}^{p}\right]\mathcal{M}_{\rm B}^{qL}\mathcal{M}_{\rm C}^{N}\right\},\end{aligned} \tag{410}$$

$$\mathcal{K}^{ipqLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{l!n!} \times \left\{ \varepsilon_{ijq}\left[\frac{1}{2} - \frac{2}{2l+2n+7}\right]\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{kN}_{\rm C}v^k_{\rm C}v^p_{\rm C} - \frac{1}{2l+2n+7}\varepsilon_{ijk}\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{kN}_{\rm C}v^p_{\rm C}v^q_{\rm C} + \varepsilon_{ijq}\left[-\frac{1}{2}v^k_{\rm B}v^p_{\rm B} + v^p_{\rm BC}v^k_{\rm B} - \frac{2(1+\gamma)}{l+2}v^k_{\rm BC}v^p_{\rm BC} + \frac{2}{2l+2n+7}v^k_{\rm C}v^p_{\rm C}\right]\mathcal{M}^{jkL}_{\rm B}\mathcal{M}^{N}_{\rm C}\right\}, \tag{411}$$

$$\mathcal{K}^{ikpqLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l}{l!n!}\frac{(2l+2n+5)!!}{2l+2n+9}\varepsilon_{ijk}\mathcal{M}^{jaL}_{\rm B}\mathcal{M}^{aN}_{\rm C}v^p_{\rm C}v^q_{\rm C}, \tag{412}$$

$$\mathcal{L}^{ikpqLN}_{\mathcal{M}\mathcal{M}} = \frac{(-1)^l(2l+2n+3)!!}{2l!n!}\varepsilon_{ijk}\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{N}_{\rm C}v^p_{\rm C}v^q_{\rm C}, \tag{413}$$

$$\mathcal{K}^{iLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{l!n!}\varepsilon_{ijk}\left[v^k_{\rm B} - \frac{2(1+\gamma)}{l+1}v^k_{\rm BC}\right]\mathcal{M}^{jL}_{\rm B}\dot{\mathcal{M}}^{N}_{\rm C}, \tag{414}$$

$$\mathcal{K}^{ipLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\left\{\frac{2(1+\gamma)}{n+1}\left[\frac{1}{l+1}\varepsilon_{ikq}v^p_{\rm BC} - \varepsilon_{ipq}v^k_{\rm BC}\right]\mathcal{M}^{qL}_{\rm B}\dot{\mathcal{M}}^{kN}_{\rm C} - \varepsilon_{ikp}\left[v^q_{\rm B} - \frac{2(1+\gamma)}{l+2}v^q_{\rm BC}\right]\mathcal{M}^{kqL}_{\rm B}\dot{\mathcal{M}}^{N}_{\rm C}\right\} \tag{415}$$

$$\mathcal{K}^{ipqLN}_{\mathcal{M}\dot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+3)!!}{l!(n+1)!}\left[\frac{2(1+\gamma)}{l+2}\right]\varepsilon_{ijq}\mathcal{M}^{jkL}_{\rm B}\dot{\mathcal{M}}^{kN}_{\rm C}v^p_{\rm BC}, \tag{416}$$

$$\mathcal{K}^{iLN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{l!n!}\varepsilon_{ipk}\left[\frac{2(1+\gamma)}{(l+1)(n+1)} - \frac{1}{2l+2n+3}\right]\mathcal{M}^{pL}_{\rm B}\ddot{\mathcal{M}}^{kN}_{\rm C}, \tag{417}$$

$$\mathcal{K}^{ipLN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n+1)!!}{l!n!}\varepsilon_{ipq}\left[\frac{2(1+\gamma)}{(l+2)(n+1)} + \frac{1}{2l+2n+5}\right]\mathcal{M}^{kqL}_{\rm B}\ddot{\mathcal{M}}^{kN}_{\rm C}, \tag{418}$$

$$\mathcal{L}^{ipLN}_{\mathcal{M}\ddot{\mathcal{M}}} = \frac{(-1)^l(2l+2n-1)!!}{2l!n!}\varepsilon_{ijp}\mathcal{M}^{jL}_{\rm B}\ddot{\mathcal{M}}^{N}_{\rm C}, \tag{419}$$

$$\mathcal{K}^{ipLNK}_{\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+k}(2l+2n+1)!!(2k-1)!!}{l!n!k!}[\gamma(n+1)]\varepsilon_{ijp}\mathcal{M}^{jL}_{B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{K}_{D}, \tag{420}$$

$$\mathcal{L}^{ipLNK}_{\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+k}(2l+2n+1)!!(2k-1)!!}{l!n!k!}[\gamma(l+2)]\varepsilon_{ijp}\mathcal{M}^{jL}_{B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{K}_{D}, \tag{421}$$

$$\mathcal{K}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n-1)!!(2s+2k+1)!!}{l!n!s!k!}\varepsilon_{ijp}\left[\frac{1}{2l+2n+3} - 2\frac{1+\gamma}{l+1}\right]\frac{\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}, \tag{422}$$

$$\mathcal{N}^{ipLNSK}_{\mathcal{M}\mathcal{M}\mathcal{M}\mathcal{M}} = \frac{(-1)^{l+s}(2l+2n-1)!!(2s+2k+1)!!}{l!n!s!k!}\varepsilon_{ijp}\left[l + 2\frac{1+\gamma}{l+1}\right]\frac{\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{S}_{\rm B}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm B}}, \tag{423}$$

$$\mathcal{K}^{ipqLNSK}_{\mathcal{MMMM}} = \frac{(-1)^{l+s}(2l+2n+1)!!(2s+2k+1)!!}{l!n!s!k!}\varepsilon_{ijp}\left\{\left[n+1+\frac{1}{2l+2n+5}\right]\frac{\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{qN}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}\right.$$
$$\left. + \varepsilon_{ijp}\left[\frac{2(1+\gamma)}{l+2} - \frac{1}{2l+2n+5}\right]\frac{\mathcal{M}^{jqL}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}\right\}, \tag{424}$$

$$\mathcal{K}^{iLNSK}_{\mathcal{MMMM}} = \frac{(-1)^{l+s}(2l+2n+1)!!(2s+2k+1)!!}{l!n!s!k!(2l+2n+5)}\varepsilon_{ijq}\frac{\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{qN}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}, \tag{425}$$

$$\mathcal{L}^{ipLNSK}_{\mathcal{MMMM}} = \frac{(-1)^{l+s+1}(2l+2n+3)!!(2s+2k+1)!!}{l!n!s!k!(2l+2n+7)}\varepsilon_{ijp}\frac{\mathcal{M}^{jkL}_{\rm B}\mathcal{M}^{kN}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}, \tag{426}$$

$$\mathcal{M}^{ipLNSK}_{\mathcal{MMMM}} = \frac{(-1)^{l+s+1}(2l+2n+1)!!(2s+2k+1)!!}{2l!n!s!k!}\varepsilon_{ijp}\frac{\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{S}_{\rm C}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm C}}, \tag{427}$$

$$\mathcal{N}^{ipqLNSK}_{\mathcal{MMMM}} = \frac{(-1)^{l+s+1}(2l+2n+1)!!(2s+2k+1)!!}{l!n!s!k!}\frac{l^2+3l+4+2\gamma}{l+2}\varepsilon_{ijp}\frac{\mathcal{M}^{jqL}_{\rm B}\mathcal{M}^{N}_{\rm C}\mathcal{M}^{S}_{\rm B}\mathcal{M}^{K}_{\rm D}}{\mathcal{M}_{\rm B}}. \tag{428}$$

2. Spin multipole coupling torque

The post-Newtonian torque describing the spin-mass and spin-spin coupling between the internal multipoles of the extended bodies consists of four terms,

$$T^i_{\rm S} = T^i_{\mathcal{SM}} + T^i_{\dot{\mathcal{S}}\mathcal{M}} + T^i_{\mathcal{S}\dot{\mathcal{M}}} + T^i_{\mathcal{SS}}, \tag{429}$$

where each component of the torque is expressed in terms of the corresponding coupling coeffcients $\mathcal{K}_{\mathcal{SI}}$, $\mathcal{K}_{\dot{\mathcal{S}}\mathcal{I}}$, etc. The components of the spin multipole coupling torque are

$$T^i_{\mathcal{SM}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ipLN}_{\mathcal{SM}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + \mathcal{K}^{ipqLN}_{\mathcal{SM}}\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + \mathcal{K}^{ikpqLN}_{\mathcal{SM}}\frac{R^{\langle kpqLN\rangle}_{\rm BC}}{R^{2l+2n+7}_{\rm BC}}\right.$$
$$\left. + \mathcal{K}^{LN}_{\mathcal{SM}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + \mathcal{K}^{pLN}_{\mathcal{SM}}\frac{R^{\langle ipLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + \mathcal{L}^{pqLN}_{\mathcal{SM}}\frac{R^{\langle ipqLN\rangle}_{\rm BC}}{R^{2l+2n+7}_{\rm BC}}\right], \tag{430}$$

$$T^i_{\mathcal{S}\dot{\mathcal{M}}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ipLN}_{\mathcal{S}\dot{\mathcal{M}}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + \mathcal{K}^{LN}_{\mathcal{S}\dot{\mathcal{M}}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}}\right], \tag{431}$$

$$T^i_{\dot{\mathcal{S}}\mathcal{M}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ipLN}_{\dot{\mathcal{S}}\mathcal{M}}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + \mathcal{K}^{ipqLN}_{\dot{\mathcal{S}}\mathcal{M}}\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + \mathcal{K}^{LN}_{\dot{\mathcal{S}}\mathcal{M}}\frac{R^{\langle iLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} + \mathcal{K}^{pLN}_{\dot{\mathcal{S}}\mathcal{M}}\frac{R^{\langle ipLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}\right], \tag{432}$$

$$T^i_{\mathcal{SS}} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\left[\mathcal{K}^{ipqLN}_{\mathcal{SS}}\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}} + \mathcal{K}^{pLN}_{\mathcal{SS}}\frac{R^{\langle ipLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}\right]. \tag{433}$$

The coupling coefficients that appear in (430)–(433) are

$$\mathcal{K}^{ipLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+1)!!}{l!n!}\frac{l+1}{l+2}\mathcal{S}^{pL}_{\rm B}\mathcal{M}^{N}_{\rm C}v^i_{\rm BC}, \tag{434}$$

$$\mathcal{K}^{ipqLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!} \times\left[\frac{1}{n+2}\left(\frac{l-1}{l+1}\mathcal{M}^{pL}_{\rm B}\mathcal{S}^{iN}_{\rm C}v^q_{\rm BC} - \mathcal{M}^{pL}_{\rm B}\mathcal{S}^{qN}_{\rm C}v^i_{\rm BC}\right) + \frac{1}{l+3}\mathcal{S}^{ipL}_{\rm B}\mathcal{M}^{N}_{\rm C}v^q_{\rm BC}\right], \tag{435}$$

$$\mathcal{K}^{ikpqLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+5)!!}{l!n!(l+2)(n+2)}[\mathcal{M}^{qkL}_{\rm B}\mathcal{S}^{iN}_{\rm C} - \mathcal{M}^{qiL}_{\rm B}\mathcal{S}^{kN}_{\rm C}]v^p_{\rm BC}, \tag{436}$$

$$\mathcal{K}^{LN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^{l+1}(2l+2n+1)!!}{l!n!}\frac{l+1}{l+2}\mathcal{S}^{jL}_{\rm B}\mathcal{M}^{N}_{\rm C}v^j_{\rm BC}, \tag{437}$$

$$\mathcal{K}^{pLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!(n+2)}\left[\mathcal{M}^{kL}_{\rm B}\mathcal{S}^{pN}_{\rm C}v^k_{\rm BC} - \frac{l-1}{l+1}\mathcal{M}^{kL}_{\rm B}\mathcal{S}^{kN}_{\rm C}v^p_{\rm BC}\right], \tag{438}$$

$$\mathcal{L}^{pqLN}_{\mathcal{SM}} = 2(1+\gamma)\frac{(-1)^{l+1}(2l+2n+5)!!}{l!n!(l+2)(n+2)}\mathcal{M}^{kpL}_{\rm B}\mathcal{S}^{kN}_{\rm C}v^q_{\rm BC}, \tag{439}$$

$$\mathcal{K}^{ipLN}_{\mathcal{S}\dot{\mathcal{M}}} = 2(1+\gamma)\frac{(-1)^{l+1}(2l+2n+1)!!}{l!n!}\left[\frac{l+1}{(l+2)(n+1)}\mathcal{S}^{pL}_{\rm B}\dot{\mathcal{M}}^{iN}_{\rm C} - \frac{1}{l+3}\mathcal{S}^{ipL}_{\rm B}\dot{\mathcal{M}}^{N}_{\rm C}\right], \tag{440}$$

$$\mathcal{K}^{LN}_{\mathcal{S}\dot{\mathcal{M}}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+1)!!}{l!(n+1)!}\frac{l+1}{l+2}\mathcal{S}^{jL}_{\rm B}\dot{\mathcal{M}}^{jL}_{\rm C}, \tag{441}$$

$$\mathcal{K}^{ipLN}_{\dot{\mathcal{S}}\mathcal{M}} = 4(1+\gamma)\frac{(-1)^l(2l+2n+1)!!}{(l+1)!n!(n+2)}\mathcal{M}^{pL}_{\rm B}\dot{\mathcal{S}}^{iN}_{\rm C}, \tag{442}$$

$$\mathcal{K}^{ipqLN}_{\dot{\mathcal{S}}\mathcal{M}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!(l+2)(n+2)}[\mathcal{M}^{ipL}_{\rm B}\dot{\mathcal{S}}^{qN}_{\rm C} - \mathcal{M}^{pqL}_{\rm B}\dot{\mathcal{S}}^{iN}_{\rm C}], \tag{443}$$

$$\mathcal{K}^{LN}_{\dot{\mathcal{S}}\mathcal{M}} = 4(1+\gamma)\frac{(-1)^{l+1}(2l+2n+1)!!}{(l+1)!n!(n+2)}\mathcal{M}^{jL}_{\rm B}\dot{\mathcal{S}}^{jN}_{\rm C}, \tag{444}$$

$$\mathcal{K}^{pLN}_{\dot{\mathcal{S}}\mathcal{M}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!(l+2)(n+2)}\mathcal{M}^{jpL}_{\rm B}\dot{\mathcal{S}}^{jN}_{\rm C}, \tag{445}$$

$$\mathcal{K}^{ipqLN}_{\mathcal{SS}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!(n+2)}\frac{l+1}{l+2}\varepsilon_{iqj}\mathcal{S}^{pL}_{\rm B}\mathcal{S}^{jN}_{\rm C}, \tag{446}$$

$$\mathcal{K}^{pLN}_{\mathcal{SS}} = 2(1+\gamma)\frac{(-1)^l(2l+2n+3)!!}{l!n!(n+2)}\frac{l+1}{l+2}\varepsilon_{pkq}\mathcal{S}^{kL}_{\rm B}\mathcal{S}^{qN}_{\rm C}. \tag{447}$$

3. Precession-multipole coupling torque

The Fermi-Walker precession causes a spatial rotation of each body-adapted local coordinates with respect to the distant observers at spatial infinity which is interpreted in the global coordinates as torque $T^i_{\rm P}$ caused by the geometric coupling of the matrix of relativistic precession to the internal mass multipoles of extended bodies. Picking up the precessional terms in the coupling coefficients $\alpha_{\rm T}^{ipLN}$ and $\alpha_{\rm T}^{ipqLN}$ in (392) and (395), we get for the torque

$$T^i_{\rm P} = \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+2n+1)!!}{l!n!}(\varepsilon_{kpq}F^{iq}_{\rm B}+\varepsilon_{ikq}F^{pq}_{\rm B})\mathcal{M}^{kL}_{\rm B}\mathcal{M}^{N}_{\rm C}\frac{R^{\langle pLN\rangle}_{\rm BC}}{R^{2l+2n+3}_{\rm BC}} - \sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^l(2l+2n+3)!!}{l!n!}\varepsilon_{ijq}(F^{kp}_{\rm C}\mathcal{M}^{jL}_{\rm B}\mathcal{M}^{kN}_{\rm C}+F^{pk}_{\rm B}\mathcal{M}^{jkL}_{\rm B}\mathcal{M}^{N}_{\rm C})\frac{R^{\langle pqLN\rangle}_{\rm BC}}{R^{2l+2n+5}_{\rm BC}}. \quad (448)$$

Racine [129] analyzed spin evolution equations for a wide class of extended bodies and gave a surface integral derivation of the leading-order evolution equations for the spin of a relativistic body interacting with other bodies. He expanded the spin evolution equations in the multipolar series but was unable to obtain the torque beyond the Newtonian formula (388). The present section significantly extends the result of paper [129] and provides the multipolar expansion of the torque in the post-Newtonian approximation which has been never published before.

XI. COVARIANT EQUATIONS OF MOTION OF EXTENDED BODIES WITH ALL MULTIPOLES

This section formulates the translational and rotational equations of motion derived in the previous sections, in the covariant form in the spirit of the "covariantization" approach worked out by Thorne and Hartle [58] who followed earlier developments outlined in [42,165]. The covariantization procedure allows us to relax the slow-motion limitation of the first post-Newtonian approximation as the covariant equations of motion are apparently Lorentz invariant and are applicable at both slow- and ultrarelativistic speeds. However, it should be understood that such covariant equations are still missing gravity-field potentials from the second- and higher-order post-Newtonian approximations and their application is limited by the weak-field, first post-Newtonian approximation. Nonetheless, the covariant equations of motion derived in this section may be instrumental in order to get a glimpse of the relativistic dynamics of the very last several orbits of an inspiralling binary system emitting gravitational waves before the bodies in the binary merge.

Before discussing our own formalism we introduce the reader to the theory of Mathisson-Papapetrou-Dixon (MPD) equations of motion of extended bodies with higher-order multipoles that is considered as one of the most comprehensive and rigorous approaches for solving the fundamental problem of derivation of equations of motion of extended bodies in general relativity [11,135,136] and in the affine-metric theories of gravity [143,145]. The original MPD theory has been developed mainly in the test-body approximation and had a number of other issues which made the domain of its astrophysical application fairly limited [58,247]. In order to circumvent this issue, Harte [141,142,244,286,287] has developed a solid theoretical platform for stretching out the domain of applicability of the MPD theory to extended bodies with a strong self-gravity field. The concrete results obtained in this section are fully consistent with the basic principles of Harte's general formalism and confirm validity of its predictions in the framework of the post-Newtonian dynamics of extended self-gravitating bodies possessing the entire collection of mass and spin multipoles.

A. The Mathisson variational dynamics

The goal to build a covariant post-Newtonian theory of motion of extended bodies and to find out the relativistic corrections to the equations of motion of a pointlike particle which account for *all* multipoles characterizing the interior structure of the extended bodies was put forward by Mathisson [4,5] and further explored by Taub [137], Tulczyjew [207], Tulczyjew and Tulczyjew [208], and Madore [138]. However, the most significant advance in tackling this problem was achieved by Dixon [7–11] who elaborated on mathematically rigorous derivation of multipolar covariant equations of motion of extended bodies from the microscopic law of conservation of matter,

$$\nabla_\alpha T^{\alpha\beta} = 0, \quad (449)$$

where ∇_α denotes a covariant derivative on spacetime manifold M with metric $g_{\alpha\beta}$, and $T^{\alpha\beta}$ is the stress-energy tensor of matter composing the extended bodies. Mathisson has dubbed this approach to the derivation of covariant equations of motion as *variational dynamics* [4]. Comprehensive reviews of the historical development and current status of the variational dynamics can be found in papers by Dixon [135,136] and Sauer and Trautman [288].

Dixon has significantly improved the Mathisson variational dynamics by employing a novel method of integration of the linear connection in general relativity as well as other innovations which allowed him to advance the original Mathisson's theory of variational dynamics. The generic mathematical technique used by Dixon to achieve this goal was the formalism of two-point world function, $\sigma(z,x)$, and its partial derivatives (called sometimes bi-tensors) introduced by Synge [164], the distributional theory of multipoles stemmed from the theory of generalized functions [212,289], and the horizontal and vertical (or Ehresmann's [290]) covariant derivatives of two-point tensors defined on a vector bundle formed by the direct product of the reference timelike worldline $\mathcal{Z}$ and a spacelike hypersurface consisting of geodesics emitted at each instant of time from point z on $\mathcal{Z}$ in all directions being orthogonal to $\mathcal{Z}$.

An extended body in Dixon's approach is idealized as a timelike world tube filled with continuous matter whose stress-energy tensor $T^{\alpha\beta}$ vanishes outside the tube. By making use of the bi-tensor propagators, $K^{\alpha}{}_{\mu} \equiv K^{\alpha}{}_{\mu}(z,x)$ and $H^{\alpha}{}_{\mu} \equiv H^{\alpha}{}_{\mu}(z,x)$, composed out of the inverse matrices of the first-order partial derivatives of the world function $\sigma(z,x)$ with respect to z and x, Dixon defined the total linear momentum, $\mathfrak{p}^{\alpha} \equiv \mathfrak{p}^{\alpha}(z)$, and the total angular momentum, $S^{\alpha\beta} \equiv S^{\alpha\beta}(z)$, of the extended body by integrals over a spacelike hypersurface Σ, [[11], Eqs. (66–67)]

$$\mathfrak{p}^{\alpha} \equiv \int_{\Sigma} K^{\alpha}{}_{\mu} T^{\mu\nu}\sqrt{-g}\,d\Sigma_{\nu}, \tag{450}$$

$$S^{\alpha\beta} \equiv -2\int_{\Sigma} X^{[\alpha} H^{\beta]}{}_{\mu} T^{\mu\nu}\sqrt{-g}\,d\Sigma_{\nu}, \tag{451}$$

where $z \equiv z^{\alpha}(\tau)$ is a reference worldline $\mathcal{Z}$ of a representative point that is associated with the center of mass of the body with τ being the proper time on this worldline, and vector

$$X^{\alpha} = -g^{\alpha\beta}(z)\frac{\partial\sigma(z,x)}{\partial z^{\beta}} \tag{452}$$

is tangent to a geodesic emitted from the point z and passing through point x. The oriented element of integration on the hypersurface,

$$d\Sigma_{\alpha} = \frac{1}{3!}E_{\alpha\mu\nu\sigma}dX^{\mu}\wedge dX^{\nu}\wedge dX^{\sigma}, \tag{453}$$

where $E_{\alpha\mu\nu\sigma}$ is 4-dimensional, fully ant-symmetric symbol of Levi-Chivita, and the symbol $\wedge$ denotes the wedge product [[165], § 3.5] of the 1-forms dX^{α}. Notice that Dixon's definition (451) of $S^{\alpha\beta}$ yields (after a duality transformation) spin of the body that has an opposite sign as compared to our definition (182) of spin.

It is further assumed in Dixon's formalism that the linear momentum, $\mathfrak{p}^{\alpha}$, is proportional to the dynamic velocity, $\mathfrak{n}^{\alpha}$, of the body [[11], Eq. (83)]

$$\mathfrak{p}^{\alpha} \equiv M\mathfrak{n}^{\alpha}, \tag{454}$$

where $M = M(\tau)$ is the total mass of the body which, in general, can depend on time. The dynamic velocity is a unit vector, $\mathfrak{n}_{\alpha}\mathfrak{n}^{\alpha} = -1$. The kinematic 4-velocity of the body moving along worldline $\mathcal{Z}$ is tangent to this worldline, $u^{\alpha} = dz^{\alpha}/d\tau$. It relates to the dynamic 4-velocity by condition, $\mathfrak{n}_{\alpha}u^{\alpha} = -1$, while the normalization condition of the kinematic 4-velocity is $u_{\alpha}u^{\alpha} = -1$. Notice that in the most general case the dynamic and kinematic velocities are not equal due to the gravitational interaction between the bodies of the $\mathbb{N}$-body system; see [[11], Eq. (88)] and [139] for more detail.

Dixon defines the mass dipole, $m^{\alpha} = m^{\alpha}(z,\Sigma)$, of the body [[11], Eq. (78)],

$$m^{\alpha} \equiv S^{\alpha\beta}\mathfrak{n}_{\beta}, \tag{455}$$

and chooses the worldline $z = z^{\alpha}(\tau)$ of the center of mass of the body by condition, $m^{\alpha} = 0$ This condition is equivalent due to (454) and (455), to

$$\mathfrak{p}_{\beta}S^{\alpha\beta} = 0, \tag{456}$$

which is known as Dixon's supplementary condition [[11], Eq. (81)].

Dixon builds the body-adapted, local coordinates at each point z on worldline $\mathcal{Z}$ as a set of the Riemann normal coordinates [[291], Chapter III, Sec. 7] denoted by X^{α} with the time coordinate X^0 along a timelike geodesic in the direction of the dynamic velocity $\mathfrak{n}^{\alpha}$, and the spatial coordinates $X^i = \{X^1, X^2, X^3\}$ lying on the hypersurface $\Sigma = \Sigma(z)$ consisting of all spacelike geodesics passing through z orthogonal to the unit vector $\mathfrak{n}^{\alpha}$ so that,

$$\mathfrak{n}_{\alpha}X^{\alpha} = 0. \tag{457}$$

It is important to understand that the Fermi normal coordinates (FNC) of an observer moving along a timelike geodesic do not coincide with the Riemann normal coordinates (RNC) used by Dixon [11,135]. The FNC are constructed under the condition that the Christoffel symbols vanish at *every* point along the geodesic [[291], Chapter III, Sec. 8] while the Christoffel symbols of the RNC vanish only at a single event on a spacetime manifold. The correspondence between the RNC and the FNC is discussed, for example, in [[292], Chapter 5], [293] and generalization of the FNC for the case of accelerated and locally rotating observers is given in [[165], Sec. 13.6] and [257]. The present paper uses the conformal-harmonic gauge (39) to build the body-adapted local coordinates which coincide with the FNC of accelerated observer only in the linearized approximation of the Taylor expansion of the metric tensor with respect to the spatial coordinates around the worldline of the observer.

Further development of the variational dynamics requires a clear separation of the matter and field variables in the solution of the full Einstein's field equations. This problem has not been solved in the MPD approach explicitly.[9] It was replaced with the solution of a simpler problem of the separation of the matter and field variables in the equations of motion (449) by introducing a symmetric tensor distribution $\hat{T}^{\mu\nu}$ known as the stress-energy *skeleton* of the body [4,5,11]. Effectively, it means that the variational dynamics of each body is described on the effective background manifold $\bar{M}$ that is equivalent to the full manifold M from which the self-field effects of the body have been removed. We denote the geometric quantities and fields defined on the effective background manifold with a bar above the corresponding object. Mathematical construction of the effective background manifold in our formalism is given below in Sec. XI B.

Dixon [[11], Eq. (140)] defined high-order multipoles of an extended body in the normal Riemann coordinates, X^α, by means of a tensor integral

$$I^{\alpha_1\ldots\alpha_l\mu\nu}(z) = \int X^{\alpha_1}\ldots X^{\alpha_l}\hat{T}^{\mu\nu}(z,X)\sqrt{-\bar{g}(z)}DX \quad (l \geq 2) \tag{458}$$

where $X^\alpha \equiv X^\alpha(z,x)$ is the same vector as in (452), $\hat{T}^{\mu\nu}$ is the stress-energy skeleton of the body, and the integration is performed over the tangent space of the point z with the volume element of integration $DX = dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3$. Definition (458) implies the following symmetries:

$$I^{\alpha_1\ldots\alpha_l\mu\nu} = I^{(\alpha_1\ldots\alpha_l)(\mu\nu)}, \tag{459}$$

where the round parentheses around the tensor indices denote a full symmetrization. Microscopic equation of motion (449) also tells us that

$$I^{(\alpha_1\ldots\alpha_l\mu)\nu} = 0, \tag{460}$$

and a similar relation holds after exchanging indices μ and ν due to symmetry (459). Dixon's multipoles have a number of interesting symmetries which are discussed in [10,136] and summarized in Appendix C of the present paper. Appendix D 1 compares the Dixon multipoles (458) with the Blanchet-Damour multipoles (122) and (131) and establishes a relationship between them in the post-Newtonian approximation of general relativity when the effects of the hypothetical scalar field are ignored.

Dixon [11] presented a number of theoretical arguments suggesting that the covariant equations of motion of the extended body have the following covariant form [[9], Eqs. (4.9–4.10)]:

$$\frac{\mathcal{D}\mathfrak{p}_\alpha}{\mathcal{D}\tau} = \frac{1}{2}\bar{u}^\beta S^{\mu\nu}\bar{R}_{\mu\nu\beta\alpha} + \frac{1}{2}\sum_{l=2}^{\infty}\frac{1}{l!}\bar{\nabla}_\alpha A_{\beta_1\ldots\beta_l\mu\nu}I^{\beta_1\ldots\beta_l\mu\nu} \tag{461}$$

$$\frac{\mathcal{D}S^{\alpha\beta}}{\mathcal{D}s} = 2\mathfrak{p}^{[\alpha}\bar{u}^{\beta]} + \sum_{l=1}^{\infty}\frac{1}{l!}B_{\gamma_1\ldots\gamma_l\sigma\mu\nu}\bar{g}^{\sigma[\alpha}I^{\beta]\gamma_1\ldots\gamma_l\mu\nu}, \tag{462}$$

where $\mathcal{D}/\mathcal{D}\tau \equiv \bar{u}^\alpha\bar{\nabla}_\alpha$ is the covariant derivative taken along the reference line $z = z(\tau)$, the moments $I^{\alpha_1\ldots\alpha_l\mu\nu}$ are defined in (458), $A_{\beta_1\ldots\beta_l\mu\nu}$ and $B_{\gamma_1\ldots\gamma_l\sigma\mu\nu}$ are the symmetric tensors computed at point z, and the bar above any tensor indicates that it belongs to the background spacetime manifold $\bar{M}$.

Thorne and Hartle [58] call the body's multipoles $I^{\alpha_1\ldots\alpha_l\mu\nu}$ the *internal* multipoles. Tensors $A_{\beta_1\ldots\beta_l\mu\nu}$ and $B_{\gamma_1\ldots\gamma_l\mu\nu\sigma}$ are called the *external* multipoles of the background spacetime. The external multipoles are the *normal* tensors in the sense of Veblen and Thomas [294]. They are reduced to the repeated partial derivatives of the metric tensor, $\bar{g}_{\mu\nu}$, and the Christoffel symbols, $\bar{\Gamma}_{\sigma\mu\nu}$, in the Riemann normal coordinates taken at the origin of the coordinate $X = 0$ (corresponding to the point z in coordinates x^α) [11,291],

$$A_{\beta_1\ldots\beta_l\mu\nu} = \lim_{X\to 0}\partial_{\beta_1\ldots\beta_l}\bar{g}_{\mu\nu}(X), \tag{463}$$

$$\begin{aligned} B_{\beta_1\ldots\beta_l\sigma\mu\nu} &= 2\lim_{X\to 0}\partial_{\beta_1\ldots\beta_l}\Gamma_{\sigma\mu\nu}(X) \\ &= \lim_{X\to 0}[\partial_{\beta_1\ldots\beta_l\sigma}\bar{g}_{\mu\nu}(X) + \partial_{\beta_1\ldots\beta_l\mu}\bar{g}_{\nu\sigma}(X) \\ &\quad - \partial_{\beta_1\ldots\beta_l\nu}\bar{g}_{\sigma\mu}(X)]. \end{aligned} \tag{464}$$

In arbitrary coordinates x^α, the normal tensors are expressed in terms of the Riemann tensor, $\bar{R}^\alpha{}_{\mu\beta\nu}$, and its covariant derivatives [[291], Chapter III, Sec. 7]. More specifically, if the terms being quadratic with respect to the Riemann tensor are neglected, the external Dixon multipoles read

$$A_{\beta_1\ldots\beta_l\mu\nu} = 2\frac{l-1}{l+1}\bar{\nabla}_{(\beta_1\ldots\beta_{l-2}}\bar{R}_{|\mu|\beta_{l-1}\beta_l)\nu}, \tag{465}$$

$$\begin{aligned} B_{\beta_1\ldots\beta_l\sigma\mu\nu} = \frac{2l}{l+2}[&\bar{\nabla}_{(\beta_1\ldots\beta_{l-1}}\bar{R}_{|\mu|\sigma\beta_l)\nu} \\ &+ \bar{\nabla}_{(\beta_1\ldots\beta_{l-1}}\bar{R}_{|\sigma|\mu\beta_l)\nu} - \bar{\nabla}_{(\beta_1\ldots\beta_{l-1}}\bar{R}_{|\sigma|\nu\beta_l)\mu}] \end{aligned} \tag{466}$$

where the vertical bars around an index means that it is excluded from the symmetrization denoted by the round

[9] In the present paper the separation of the matter and field variables in the metric tensor is achieved by means of the matched asymptotic expansion technique.

parentheses. Notice that each term with the Riemann tensor in (465) and (466) is symmetric with respect to the first and forth indices of the Riemann tensor. This tells us that $A_{\beta_1\ldots\beta_l\mu\nu} = A_{(\beta_1\ldots\beta_l)(\mu\nu)}$ and $B_{\gamma_1\ldots\gamma_l\sigma\mu\nu} = B_{(\gamma_1\ldots\gamma_l)(\sigma\mu)\nu}$ in accordance with the symmetries of (463) and (464).

Substituting these expressions to (461) and (462) yields the Dixon equations of motion in the following form:

$$\frac{\mathcal{D}\mathfrak{p}_\alpha}{\mathcal{D}\tau} = \frac{1}{2}\bar{u}^\beta S^{\mu\nu}\bar{R}_{\mu\nu\beta\alpha} + \sum_{l=2}^{\infty}\frac{l-1}{(l+1)!}\bar{\nabla}_{\alpha(\beta_1\ldots\beta_{l-2}}\bar{R}_{|\mu|\beta_{l-1}\beta_l)\nu}J^{\beta_1\ldots\beta_{l-1}\mu\beta_l\nu}, \tag{467}$$

$$\frac{\mathcal{D}S^{\alpha\beta}}{\mathcal{D}\tau} = 2\mathfrak{p}^{[\alpha}\bar{u}^{\beta]} + 2\sum_{l=1}^{\infty}\frac{l(l+1)}{(l+2)!}\bar{\nabla}_{(\gamma_1\ldots\gamma_{l-1}}\bar{R}_{|\mu|\sigma\gamma_l)\nu}\bar{g}^{\sigma[\alpha}J^{\beta]\gamma_1\ldots\gamma_{l-1}\mu\gamma_l\nu}, \tag{468}$$

where

$$J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu} \equiv I^{\alpha_1\ldots\alpha_p[\lambda[\sigma\mu]\nu]} \tag{469}$$

denotes the internal multipoles with a skew symmetry with respect to two pairs of indices, $[\lambda\mu]$ and $[\sigma\nu]$. The Dixon I and J multipoles are compared in Appendix C of the present paper. Comparison of Dixon's equations of motion (467), (468) with our covariant equations is given in Appendix D.

Mathematical elegance and apparently covariant nature of the variational dynamics has been attracting researchers to work on improving various aspects of derivation of the MPD equations of motion [13,105,139,140,144,145,247, 295,296]. From an astrophysical point of view Dixon's formalism is viewed as being of considerable importance for the modeling of the gravitational waves emitted by the extreme mass-ratio inspirals (EMRIs) which are binary black holes consisting of a supermassive black hole and a stellar mass black hole. EMRIs form a key science goal for the planned space based gravitational wave observatory LISA and the equations of motion of the black holes in those systems must be known with unprecedented accuracy [28,252]. Nonetheless, in spite of the power of Dixon's mathematical apparatus, there are several issues which make the MPD theory of the variational dynamics yet unsuitable for relativistic celestial mechanics, astrophysics, and gravitational wave astronomy which have been pointed out by Dixon himself [11] and by Thorne and Hartle [58].

The main problem is that the variational dynamics is too generic and does not engage any particular theory of gravity. It tacitly assumes that some valid theory of gravity is chosen, gravitational field equations are solved, and the metric tensor is known. However, the field equations and the equations of motion of matter are closely tied up—matter generates gravity while gravity governs motion of matter. Due to this coupling the definition of the center of mass, linear momentum, spin, and other body's internal multipoles depend on the metric tensor which, in its own turn, depends on the multipoles through the nonlinearity of the field equations. It complicates the problem of interpretation of the gravitational stress-energy skeleton in the nonlinear regime of a gravitational field and makes the MPD equations (461), (462) valid solely in the linearized approximation of general relativity. For the same reason it is difficult to evaluate the residual terms in the existing derivations of the MPD equations and their multipolar extensions. One more serious difficulty relates to the lack of prescription for separation of self-gravity effects of a moving body from the external gravitational environment. The MPD equations of motion are valid on the background effective manifold $\bar{M}$ but its exact mathematical formulation remains unclear in the framework of the variational dynamics alone [247]. Because of these shortcomings the MPD variational dynamics has not been commonly used in real astrophysical applications in spite of the fact that it is sometime claimed as a "standard theory" of the equations of motion of massive bodies in relativistic gravity [210].

In order to complete the MPD approach to variational dynamics and make it applicable in astrophysics several critical ingredients have to be added. More specifically, what we need includes the following:

(1) the procedure of unambiguous characterization and determination of the gravitational self-force and self-torque exerted by the body on itself, and the proof that they are actually vanishing;
(2) the procedure of building the effective background spacetime manifold $\bar{M}$ with the background metric $\bar{g}_{\alpha\beta}$ used to describe the motion of the body which is a member of the $\mathbb{N}$-body system;
(3) the precise algorithm for calculating the body's internal multipoles (458) and their connection to the gravitational field of the body;
(4) the relationship between the Blanchet-Damour mass and spin body's multipoles, $\mathcal{M}^{\alpha_1\cdots\alpha_l}$ and $\mathcal{S}^{\alpha_1\cdots\alpha_l}$, the Dixon internal multipoles (458), and the gravitational stress-energy skeleton.

In this section we implement the formalism of derivation of covariant equations of motion of massive bodies proposed by Thorne and Hartle [58] which yields a complete set of the covariant equations of translational and rotational motion. It relies upon the construction of the effective background manifold $\bar{M}$ by solving the field equations of scalar-tensor theory of gravity and applying the asymptotic matching technique which separates the self-field effects from the external gravitational environment, defines all external multipoles, and establishes the local equations of motion of the body in the body-adapted local coordinates. The body's internal multipoles are defined in the conformal harmonic gauge by solving the field equations in the body-adapted local coordinates as proposed by Blanchet and Damour [78]. The covariant equations of motion follow immediately from the local equations of motion by applying the Einstein equivalence principle [58]. We compare our covariant equations of motion, derived in this section, with the MPD equations in Appendix D.

B. The effective background manifold

Equations of translational motion (290) of an extended body B in the global coordinate chart depend on an infinite set of configuration variables—the internal mass and spin multipoles of the body, $\mathcal{M}^L_{\rm B}$ and $\mathcal{S}^L_{\rm B}$, and the external gravitoelectric and gravitomagnetic multipoles—$\mathcal{Q}_L$ and $\mathcal{C}_L$—all are pinned down to the worldline $\mathcal{Z}$ of the center of mass of the body. The same equations in the local coordinate chart adapted to the body B are given by (183) after applying the law of conservation of the linear momentum of the body (177). These equations in two different coordinate charts are interconnected by the spacetime coordinate transformation (144), (145)—the proof is given below in Sec. XI C. It points out that the equations of motion derived in the local coordinates can be lifted to the generic covariant form by making use of the Einstein equivalence principle applied to body B that can be treated as a massive particle endowed with the internal multipoles $\mathcal{M}^L_{\rm B}$ and $\mathcal{S}^L_{\rm B}$, and moving along the worldline $\mathcal{Z}$ on the effective background spacetime manifold $\bar{M}$ whose properties are characterized by the external multipoles $\mathcal{Q}_L$ and $\mathcal{C}_L$ that presumably depend on the curvature tensor on $\bar{M}$ and its covariant derivatives. The covariant form of the equations is independent of a particular realization of the conformal-harmonic coordinates but we hold on the gauge conditions (39) to prevent the appearance of gauge-dependent, nonphysical multipoles of gravitational field in the equations of motion.

The power of our approach to the covariant equations of motion is that the effective background manifold $\bar{M}$ for each body B is not postulated or introduced *ad hoc*. It is constructed by solving the field equations in the local and global charts and separating the field variables—scalar field and metric tensor perturbations—in the internal and external parts. The separation is fairly straightforward in the local chart. The internal part of the metric tensor, $\hat{h}^{\rm int}_{\alpha\beta}$ and scalar field $\hat{\varphi}^{\rm int}$, are determined by matter of body B and is expanded in the multipolar series outside the body which are singular at the origin of the body-adapted local coordinates. The external part of the metric tensor $\hat{h}^{\rm ext}_{\alpha\beta}$ and scalar field $\hat{\varphi}^{\rm ext}$ are solutions of vacuum field equations and, hence, are regular at the origin of the local chart. There is also an internal-external coupling component $\hat{l}^{\rm int}_{00}$ of the metric tensor perturbation but it is a nonlinear functional of the internal solution and its multipolar series is also singular at the origin of the local chart of body B.

The effective background manifold is regular at the origin of the local coordinates and its geometry is entirely determined by the external part of the metric tensor, $\bar{g}_{\alpha\beta} = \eta_{\alpha\beta} + h^{\rm ext}_{\alpha\beta}$. This is fully consistent with the result of matching of the asymptotic expansions of the metric tensor and scalar field in the global and local coordinates described in Sec. V. All terms whose multipolar expansions are singular at the origin of the local chart are canceled out identically in the matching Eqs. (134) and (135). This establishes a one-to-one correspondence between the external metric perturbation $h^{\rm ext}_{\alpha\beta}$ in the local chart and its counterpart in the global coordinate chart which is uniquely defined by the external gravitational potentials $\bar{U}, \bar{U}^i, \bar{\Psi}, \bar{\chi}$ given in (91). In the rest of this section we demonstrate that translational equations of motion of body B are equations of a perturbed timelike geodesic of a massive particle on the effective background manifold with the metric $\bar{g}_{\alpha\beta}$. The particle has mass $M = M_{\rm B}$ and internal multipoles $\mathcal{M}^L = \mathcal{M}^L_{\rm B}$ and $\mathcal{S}^L = \mathcal{S}^L_{\rm B}$. The perturbation of the geodesic is the local acceleration $\mathcal{Q}_i$ caused by the interaction of the particle's multipoles with the external gravitoelectric and gravitomagnetic multipoles, $\mathcal{Q}_L$ and $\mathcal{C}_L$, which are fully expressed in terms of the covariant derivatives of the Riemann tensor, $\bar{R}_{\alpha\beta\mu\nu}$ and scalar field $\bar{\varphi}$ of the background manifold. Covariant equations of rotational motion of the body spin are described by the Fermi-Walker transport with the external torques caused by the coupling of the internal and external multipoles of the body.

The effective background metric $\bar{g}_{\alpha\beta}$ is given in the global coordinates by the following equations (cf. [58]):

$$\bar{g}_{00}(t,\boldsymbol{x}) = -1 + 2\bar{U}(t,\boldsymbol{x}) + 2\left[\bar{\Psi}(t,\boldsymbol{x}) - \beta\bar{U}^2(t,\boldsymbol{x}) - \frac{1}{2}\partial_{tt}\bar{\chi}(t,\boldsymbol{x})\right], \quad (470)$$

$$\bar{g}_{0i}(t,\boldsymbol{x}) = -2(1+\gamma)\bar{U}^i(t,\boldsymbol{x}), \quad (471)$$

$$\bar{g}_{ij}(t,\boldsymbol{x}) = \delta_{ij} + 2\gamma\delta_{ij}\bar{U}(t,\boldsymbol{x}), \quad (472)$$

where the potentials in the right-hand side of (470)–(472) are defined in (68) and (91) as functions of the global coordinates $x^\alpha = (t,\boldsymbol{x})$. The background metric in arbitrary coordinates can be obtained from (470)–(472) by performing a corresponding coordinate transformation. It is worth emphasizing that the effective metric $\bar{g}_{\alpha\beta}$ is constructed for each body of the $\mathbb{N}$-body system separately and is a function of the external gravitational potentials which depend on which body is chosen. It means that we have a collection of $\mathbb{N}$-effective manifolds $\bar{M}$—one for each extended body. Another prominent point to draw to the attention of the reader is the fact that the effective metric of the extended-body B depends on the gravitational field of the body itself through the nonlinear interaction term Ψ_{C2} in the potential $\bar{\Psi}$; see (77) and its multipolar expansion (231). This dependence of the background metric tensor on the gravitational field of the body itself is known as the *back-action* effect of a gravitational field [58,156]. It was first noticed by Fichtenholtz [218] who pointed out that derivation of the post-Newtonian equations of motion of bodies of comparable masses, given in the first edition of the "Classical Theory of Fields" by Landau and Lifshitz, is erroneous as they missed the backaction term in the effective metric. This error was corrected and did not appear in the subsequent editions of the Landau-Lifshitz textbook [42].

The background metric, $\bar{g}_{\alpha\beta}$, is a starting point of the covariant development of the equations of motion. It has the Christoffel symbols

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\beta}(\partial_\nu \bar{g}_{\beta\mu} + \partial_\mu \bar{g}_{\beta\nu} - \partial_\beta \bar{g}_{\mu\nu}), \tag{473}$$

which can be directly calculated in the global coordinates, x^α, by taking partial derivatives from the metric components (470)–(472). In what follows, we shall make use of a covariant derivative defined on the background manifold $\bar{M}$ with the help of the Christoffel symbols $\bar{\Gamma}^{\alpha}_{\mu\nu}$. The covariant derivative on the background manifold, $\bar{M}$, is denoted $\bar{\nabla}_\alpha$ in order to distinguish it from the covariant derivative defined on the original spacetime manifold, M, denoted ∇_α. For example, the covariant derivative of vector field V^α is defined on the background manifold by the following equation:

$$\bar{\nabla}_\beta V^\alpha = \partial_\beta V^\alpha + \bar{\Gamma}^{\alpha}_{\mu\beta} V^\mu, \tag{474}$$

which is naturally extended to tensor fields of arbitrary type and rank in a standard way [17]. It is straightforward to define other geometric objects on the background manifold like the Riemann tensor (4),

$$\bar{R}^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \bar{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu \bar{\Gamma}^{\alpha}_{\mu\beta} + \bar{\Gamma}^{\alpha}_{\sigma\beta}\bar{\Gamma}^{\sigma}_{\mu\nu} - \bar{\Gamma}^{\alpha}_{\sigma\nu}\bar{\Gamma}^{\sigma}_{\mu\beta}, \tag{475}$$

and its contractions—the Ricci tensor $\bar{R}_{\mu\nu} = \bar{R}^{\alpha}{}_{\mu\alpha\nu}$, and the Ricci scalar $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$. Tensor indices on the background manifold are raised and lowered with the help of the metric $\bar{g}_{\alpha\beta}$.

The background metric tensor $\bar{\mathrm{g}}_{\alpha\beta}(u, \boldsymbol{w})$ in the local coordinates $w^\alpha = (u, w^i)$ adapted to body B is given by

$$\bar{\mathrm{g}}_{\alpha\beta}(u, \boldsymbol{w}) = \eta_{\alpha\beta} + \hat{h}^{\mathrm{ext}}_{\alpha\beta}(u, \boldsymbol{w}), \tag{476}$$

where the perturbation, $\hat{h}^{\mathrm{ext}}_{\alpha\beta}$, is given by the polynomial expansions (117)–(119) of the external gravitational field with respect to the local spatial coordinates. Notice that at the origin of the local coordinates, where $w^i = 0$, the background metric $\bar{g}_{\alpha\beta}$ is reduced to the Minkowski metric $\eta_{\alpha\beta}$. It means that on the effective background manifold $\bar{M}$ the coordinate time u is identical to the proper time τ measured on the worldline $\mathcal{W}$ of the origin of the local coordinates adapted to body B,

$$\tau = u. \tag{477}$$

Post-Newtonian transformation from the global to local coordinates, $w^\alpha = w^\alpha(x^\beta)$, has been provided in Sec. V C. It smoothly matches the two forms of the background metric $\bar{g}_{\alpha\beta}(t, \boldsymbol{x})$ and $\bar{\mathrm{g}}_{\alpha\beta}(u, \boldsymbol{w})$ on the background manifold $\bar{M}$ in the sense that

$$\bar{g}_{\mu\nu}(t, \boldsymbol{x}) = \bar{\mathrm{g}}_{\alpha\beta}(u, \boldsymbol{w})\frac{\partial w^\alpha}{\partial x^\mu}\frac{\partial w^\beta}{\partial x^\nu}. \tag{478}$$

This should be compared with the law of transformation (135) applied to the full metric $g_{\alpha\beta}$ on spacetime manifold M which includes besides the external part also the internal and internal-external coupling components of the metric tensor perturbations but they are mutually canceled out in (135) leaving only the external terms, thus, converting (135) to (478) without making any additional assumptions about the structure of the effective background manifold. The cancellation of the internal and internal-external components of the metric tensor perturbations in (135) is a manifestation of the *effacing* principle [185] that excludes the internal structure of body B from the definition of the effective background manifold $\bar{M}$ used for the description of motion of the body [99]. Compatibility of Eqs. (135) and (478) confirms that the internal and external problems of the relativistic celestial mechanics in an $\mathbb{N}$-body system are completely decoupled regardless of the structure of the extended bodies and can be extrapolated to compact astrophysical objects like neutron stars and black holes.

In what follows, we will need a matrix of transformation taken on the worldline of the origin of the local coordinates,

$$\Lambda^\alpha{}_\beta \equiv \Lambda^\alpha{}_\beta(\tau) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_\mathrm{B}} \frac{\partial w^\alpha}{\partial x^\beta}. \tag{479}$$

The components of this matrix can be easily computed from equations of coordinate transformation (144) and (145) and its complete post-Newtonian form is shown in [[17], Sec. 5.1.3]. With an accuracy being sufficient for derivation of the covariant post-Newtonian equations of motion in the present paper, it reads

$$\Lambda^0{}_0 = 1 + \frac{1}{2}v_\mathrm{B}^2 - \bar{U}(t, \boldsymbol{x}_\mathrm{B}), \tag{480}$$

$$\Lambda^0{}_i = -v^i_\mathrm{B}\left(1 + \frac{1}{2}v^2_\mathrm{B}\right) + 2(1+\gamma)\bar{U}^i(t, \boldsymbol{x}_\mathrm{B}) - (1+2\gamma)v^i_\mathrm{B}\bar{U}(t, \boldsymbol{x}_\mathrm{B}), \tag{481}$$

$$\Lambda^i{}_0 = -v^i_\mathrm{B}\left[1 + \frac{1}{2}v^2_\mathrm{B} + \gamma\bar{U}(t, \boldsymbol{x}_\mathrm{B})\right] - F^{ij}_\mathrm{B} v^j_\mathrm{B}, \tag{482}$$

$$\Lambda^i{}_j = \delta^{ij}[1 + \gamma\bar{U}(t, \boldsymbol{x}_\mathrm{B})] + \frac{1}{2}v^i_\mathrm{B} v^j_\mathrm{B} + F^{ij}_\mathrm{B}, \tag{483}$$

where $F^{ij}_{\rm B}$ is the skew-symmetric matrix of the Fermi-Walker precession of the spatial axes of the local frame adapted to body B, with respect to the global coordinates; see (151).

We will also need the inverse matrix of transformation between the local and global coordinates taken on the worldline $\mathcal{W}$ of the origin of the local coordinates. We shall denote this matrix as

$$\Omega^\alpha{}_\beta \equiv \Omega^\alpha{}_\beta(\tau) = \lim_{w\to 0} \frac{\partial x^\alpha}{\partial w^\beta}. \tag{484}$$

In accordance with the definition of the inverse matrix we have

$$\Lambda^\alpha{}_\beta \Omega^\beta{}_\gamma = \delta^\alpha_\gamma, \qquad \Omega^\alpha{}_\beta \Lambda^\beta{}_\gamma = \delta^\alpha_\gamma. \tag{485}$$

Solving (485) with respect to the components of $\Omega^\alpha{}_\beta$, we get

$$\Omega^0{}_0 = 1 + \frac{1}{2} v^2_{\rm B} + \bar{U}(t, \boldsymbol{x}_{\rm B}), \tag{486}$$

$$\Omega^0{}_i = v^i_{\rm B}\left(1 + \frac{1}{2} v^2_{\rm B}\right) + F^{ij}_{\rm B} v^j_{\rm B} - 2(1+\gamma)\bar{U}^i(t, \boldsymbol{x}_{\rm B}) + (2+\gamma) v^i_{\rm B} \bar{U}(t, \boldsymbol{x}_{\rm B}), \tag{487}$$

$$\Omega^i{}_0 = v^i_{\rm B}\left[1 + \frac{1}{2} v^2_{\rm B} + \bar{U}(t, \boldsymbol{x}_{\rm B})\right], \tag{488}$$

$$\Omega^i{}_j = \delta^{ij}[1 - \gamma \bar{U}(t, \boldsymbol{x}_{\rm B})] + \frac{1}{2} v^i_{\rm B} v^j_{\rm B} - F^{ij}_{\rm B}. \tag{489}$$

As we shall see below, the matrices $\Lambda^\alpha{}_\beta$ and $\Omega^\alpha{}_\beta$ are instrumental in lifting the geometric objects pinned down to the worldline $\mathcal{W}$ and residing on 3-dimensional hypersurface $\mathcal{H}_u$ of constant time u of the tangent space to the background manifold, from $\mathcal{H}_u$ up to 4-dimensional spacetime manifold $\bar{M}$.

In order to arrive to the covariant formulation of the translational and rotational equations of motion, we take the equations of motion derived in the local coordinates of body B, and prolongate them to the 4-dimensional, covariant form with the help of the transformation matrices and replacing the partial derivatives with the covariant ones. This is in accordance with the Einstein principle of equivalence which establishes a correspondence between spacetime manifold and its tangent space [165]. It turns out that, eventually, all direct and inverse transformation matrices cancel out due to (485) and the equations acquire a final, covariant 4-dimensional form without any reference to the original coordinate charts that were used in the intermediate transformations. In what follows, we carry out this type of calculations.

C. Geodesic worldline and 4-force on the background manifold

Our algorithm of derivation of equations of motion defines the center of mass of body B by equating the conformal dipole of the body to zero, $\mathcal{I}^i = 0$. The linear momentum, $\mathfrak{p}^i$ also vanishes $\mathfrak{p}^i = d\mathcal{I}^i/du = 0$, as explained in Sec. VI C. We have shown that these two conditions can be always satisfied by choosing the appropriate value (184)–(186) of the local acceleration, $\mathcal{Q}_i$, of the origin of the local coordinates adapted to body B in such a way that the worldline $\mathcal{W}$ of the origin of the local coordinates coincides with the worldline $\mathcal{Z}$ of the center of mass of the body. This specific choice of $\mathcal{Q}_i$ converts the equations of motion of the origin of the local coordinates of body B (152) to the equations of motion of its center of mass in the global coordinates. Below we prove that this equation can be interpreted on the background manifold $\bar{M}$ as the equation of timelike geodesic of a massive particle with the conformal mass, $M = M_{\rm B}$, of body B that is perturbed by the force of inertia produced by the local acceleration $\mathcal{Q}_i$ of the origin of the local coordinates. This is in concordance with the effacing principle [99,154,185] which determines dynamics in general relativity and scalar-tensor theory of gravity and suggests that the laws governing the motion of self-interacting masses are structurally identical to the laws governing the motion of test bodies [142].

Let us introduce a 4-velocity $\bar{u}^\alpha$ of the center of mass of body B. In the global coordinates, x^α, the worldline $\mathcal{Z}$ of the body's center of mass is described parametrically by $x^0_{\rm B} = t$, and $x^i_{\rm B}(t)$. The 4-velocity is defined by

$$\bar{u}^\alpha = \frac{dx^\alpha_{\rm B}}{d\tau}, \tag{490}$$

where τ is the proper time along the worldline $\mathcal{Z}$. The increment $d\tau$ of the proper time is related to the increments dx^α of the global coordinates by equation,

$$d\tau^2 = -\bar{g}_{\alpha\beta} dx^\alpha dx^\beta, \tag{491}$$

which tells us that the 4-velocity (490) is normalized to unity, $\bar{u}_\alpha \bar{u}^\alpha = \bar{g}_{\alpha\beta}\bar{u}^\alpha \bar{u}^\beta = -1$. In the local coordinates the worldline $\mathcal{Z}$ is given by equations, $w^\alpha = (\tau, w^i = 0)$, and the 4-velocity has components $\bar{\mathrm{u}}^\alpha = (1,0,0,0)$. In the global coordinates the components of the 4-velocity are $u^\alpha = (dt/d\tau, dx^i_{\rm B}/d\tau)$, which yields 3-dimensional velocity of the body's center of mass, $v^i_{\rm B} = \bar{u}^i/\bar{u}^0 = dx^i_{\rm B}/dt$. Components of the 4-velocity are transformed from the local to global coordinates in accordance to the transformation equation, $\bar{u}^\alpha = \Omega^\alpha{}_\beta \bar{\mathrm{u}}^\beta$, which points out that in the global coordinates $\bar{u}^\alpha = \Omega^\alpha{}_0$. On the other hand, a covector of 4-velocity obeys the transformation equation, $\bar{u}_\alpha = \Lambda^\beta{}_\alpha \bar{\mathrm{u}}_\beta$, where $\bar{\mathrm{u}}_\alpha = (-1,0,0,0)$ are components of the covector of 4-velocity in the local coordinates. Thus, in the global coordinates $\bar{u}_\alpha = -\Lambda^0{}_\alpha$. The above presentation of the components of 4-velocity in terms of the matrices of transformation along with Eq. (485) makes it evident that the 4-velocity is subject to two reciprocal conditions of orthogonality,

$$\Lambda^i{}_\alpha \bar{u}^\alpha = 0, \qquad \bar{u}_\alpha \Omega^\alpha{}_i = 0. \tag{492}$$

Equations (492) will be used later on in the procedure of lifting the spatial components of the internal and external multipoles to the covariant form.

In the covariant description of the equations of motion, an extended body B from the $\mathbb{N}$-body system is treated as a particle having a conformal mass, $M = M_{\rm B}$, the active mass $\mathcal{M} \equiv \mathcal{M}_{\rm B}$, the active mass multipoles $\mathcal{M}^L \equiv \mathcal{M}^L_{\rm B}$, and the active spin multipoles $\mathcal{S}^L \equiv \mathcal{S}^L_{\rm B}$ attached to the particle, in other words, to the center of mass of the body. This set of the internal multipoles fully characterizes the internal structure of the body. The multipoles, in general, depend on time including the mass of the body which is not constant due to the temporal change of the multipoles (163) caused by tidal interaction. The mass and spin multipoles are fully determined by their spatial components in the body-adapted local coordinates in terms of integrals from the stress-energy distribution of matter through the solution of the field equations; see Sec. IV B 6. Covariant generalization of the multipoles from the spatial to spacetime components is provided by the condition of orthogonality of the multipoles to the 4-velocity $\bar{u}^\alpha$ of the center of mass of the body as explained below in Sec. XI D.

We postulate that the covariant definition of the linear momentum of the body is

$$\mathfrak{p}^\alpha = M\bar{u}^\alpha, \tag{493}$$

where $\mathfrak{p}^\alpha$ is a covariant generalization of 3-dimensional linear momentum $\mathfrak{p}^i$ of body B introduced in (173) where, for the time being, we do not specify the complementary part $\dot{\mathcal{I}}^i_{\rm c}$. We are looking for the covariant translational equations of motion of body B in the following form:

$$\frac{\mathcal{D}\mathfrak{p}^\alpha}{\mathcal{D}\tau} \equiv \bar{u}^\beta \bar{\nabla}_\beta \mathfrak{p}^\alpha = \frac{d\mathfrak{p}^\alpha}{d\tau} + \bar{\Gamma}^\alpha_{\mu\nu}\mathfrak{p}^\mu \bar{u}^\nu = F^\alpha, \tag{494}$$

where F^α is a 4-force that causes the worldline $\mathcal{Z}$ of the center of mass of the body to deviate from the geodesic worldline of the background manifold $\bar{M}$. We introduce this force to Eq. (494) because the body's center of mass experiences a local acceleration $\mathcal{Q}_i$ given by (184) which means that it is not in a state of a free fall and does not move on the geodesic of the background manifold. In order to establish the mathematical form of the force F^α it is more convenient to rewrite (494) in terms of a 4-acceleration $a^\alpha \equiv \mathcal{D}\bar{u}^\alpha/\mathcal{D}\tau = \bar{u}^\beta \bar{\nabla}_\beta \bar{u}^\alpha$

$$M\left(\frac{d\bar{u}^\alpha}{d\tau} + \bar{\Gamma}^\alpha_{\mu\nu}\bar{u}^\mu \bar{u}^\nu\right) = F^\alpha - \dot{M}\bar{u}^\alpha, \tag{495}$$

where $\dot{M}$ is given in (165).

In what follows, it is more convenient to operate with a 4-force per unit mass defined by $f^\alpha \equiv F^\alpha/M$. Equation of motion (495) is reduced to

$$\frac{d\bar{u}^\alpha}{d\tau} + \bar{\Gamma}^\alpha_{\mu\nu}\bar{u}^\mu \bar{u}^\nu = f^\alpha - \frac{\dot{M}}{M}\bar{u}^\alpha. \tag{496}$$

The force f^α is orthogonal to 4-velocity, $u_\alpha f^\alpha = 0$ as a consequence of (494) and the 4-velocity normalization condition. Hence, in the global coordinates the time component of the force is related to its spatial components as $f_0 = -v^i_{\rm B} f_i$. The condition of the orthogonality also yields the contravariant time component of the force in terms of its spatial components,

$$f^0 = -\frac{1}{\bar{g}_{00}}\bar{g}_{ij} v^i_{\rm B} f^j. \tag{497}$$

Our task is to prove that the covariant equation of motion (496) is exactly the same as the equation of motion (152) of the center of mass of body B derived in the global coordinates that was obtained by asymptotic matching of the external and internal solutions of the field equations. To this end we reparametrize Eq. (496) by coordinate time t instead of the proper time τ, which yields

$$\begin{aligned} a^i_{\rm B} = &-\bar{\Gamma}^i_{00} - 2\bar{\Gamma}^i_{0p} v^p_{\rm B} - \bar{\Gamma}^i_{pq} v^p_{\rm B} v^q_{\rm B} \\ &+ (\bar{\Gamma}^0_{00} + 2\bar{\Gamma}^0_{0p} v^p_{\rm B} + \bar{\Gamma}^0_{pq} v^p_{\rm B} v^q_{\rm B}) v^i_{\rm B} \\ &+ (f^i - f^0 v^i_{\rm B})\left(\frac{d\tau}{dt}\right)^2, \end{aligned} \tag{498}$$

where $v^i_{\rm B} = dx^i_{\rm B}/dt$ and $a^i_{\rm B} = dv^i_{\rm B}/dt$ are the coordinate velocity and acceleration of the body's center of mass with respect to the global coordinates.

We calculate the Christoffel symbols, $\bar{\Gamma}^\alpha_{\mu\nu}$, the derivative $d\tau/dt$, substitute them to (498) along with (497), and retain only the Newtonian and post-Newtonian terms. Equation (498) takes on the following form:

$$\begin{aligned} a^i_{\rm B} = &\,\partial^i \bar{U}(t,\boldsymbol{x}_{\rm B}) + \partial^i \bar{\Psi}(t,\boldsymbol{x}_{\rm B}) \\ &-\frac{1}{2}\partial_{tt}\partial^i \bar{\chi}(t,\boldsymbol{x}_{\rm B}) + 2(\gamma+1)\dot{\bar{U}}^i(t,\boldsymbol{x}_{\rm B}) \\ &- 2(\gamma+1) v^j_{\rm B}\partial^i \bar{U}^j(t,\boldsymbol{x}_{\rm B}) - (2\gamma+1) v^i_{\rm B}\dot{\bar{U}}(t,\boldsymbol{x}_{\rm B}) \\ &- 2(\beta+\gamma)\bar{U}(t,\boldsymbol{x}_{\rm B})\partial^i \bar{U}(t,\boldsymbol{x}_{\rm B}) \\ &+ \gamma v^2_{\rm B}\partial^i \bar{U}(t,\boldsymbol{x}_{\rm B}) - v^i_{\rm B} v^j_{\rm B}\partial^j \bar{U}(t,\boldsymbol{x}_{\rm B}) \\ &+ f^i - v^i_{\rm B} v^k_{\rm B} f^k - [2\bar{U}(t,\boldsymbol{x}_{\rm B}) + v^2_{\rm B}] f^i. \end{aligned} \tag{499}$$

This equation exactly matches the translational equation of motion (152) if we make the following identification of the spatial components f^i of the force per unit mass with the local acceleration $\mathcal{Q}^i$:

$$f^i \equiv -\mathcal{Q}^i - \frac{1}{2} v_{\rm B}^i v_{\rm B}^j \mathcal{Q}_j + F_{\rm B}^{ij}\mathcal{Q}_j + \gamma \bar{U}(t, \boldsymbol{x}_{\rm B})\mathcal{Q}^i. \quad (500)$$

By simple inspection we reveal that the right-hand side of the post-Newtonian force (500) can be written down in a covariant form

$$f^\alpha = -\bar{g}^{\alpha\beta}\Lambda^i{}_\beta \mathcal{Q}_i = \bar{g}^{\alpha\beta}\mathcal{Q}_\beta = -\mathcal{Q}^\alpha, \quad (501)$$

where $\Lambda^i{}_\beta$ is given above in (480)–(483), and $\mathcal{Q}_i$ is a vector of 4-acceleration in the local coordinates. The quantity $\mathcal{Q}_\alpha = \Lambda^i{}_\alpha \mathcal{Q}_i$ defines the covariant form of the local acceleration in the global coordinates with $\mathcal{Q}_\alpha$ being orthogonal to 4-velocity, $\bar{u}^\alpha \mathcal{Q}_\alpha = 0$, which is a direct consequence of the condition (492). Explicit form of $\mathcal{Q}_i$ in the local coordinates is given in (184) and should be used in (501) along with the covariant form of the external —$\mathcal{Q}_L$, $\mathcal{C}_L$, $\mathcal{P}_L$ and internal—$\mathcal{M}^L$, $\mathcal{S}^L$ multipoles in order to get $f^\alpha = -\bar{g}^{\alpha\beta}\mathcal{Q}_\beta$. The covariant form of the multipoles is a matter of discussion in the next subsection.

D. Four-dimensional form of multipoles

1. Internal multipoles

The mathematical procedure that was used in construction of the local coordinates adapted to an extended body B in an $\mathbb{N}$-body system indicates that all type of multipoles are defined at the origin of the local coordinates as the STF Cartesian tensors having only spatial components with their time components being identically nil. It means that the multipoles are projections of 4-dimensional tensors on a hyperplane passing through the origin of the local coordinates orthogonally to 4-velocity $\bar{u}^\alpha$ of the worldline $\mathcal{Z}$ of the center of mass of the body. The 4-dimensional form of the internal multipoles can be established by making use of the law of transformation from the local to global coordinates,

$$\mathcal{M}^{\alpha_1\ldots\alpha_l} \equiv \Omega^{\alpha_1}{}_{i_1}\ldots\Omega^{\alpha_l}{}_{i_l}\mathcal{M}^{i_1 i_2\ldots i_l},$$
$$\mathcal{S}^{\alpha_1\ldots\alpha_l} \equiv \Omega^{\alpha_1}{}_{i_1}\ldots\Omega^{\alpha_l}{}_{i_l}\mathcal{S}^{i_1 i_2\ldots i_l}, \quad (502)$$

where the matrix of transformation $\Omega^\alpha{}_i$ is given in (486)–(489). Transforming 3-dimensional STF Cartesian tensors to 4-dimensional form does not change the property of the tensors to be symmetric and trace-free in the sense that we have for any pair of spacetime (Greek) indices

$$\bar{g}_{\alpha_1\alpha_2}\mathcal{M}^{\alpha_1\alpha_2\ldots\alpha_l} = 0, \qquad \bar{g}_{\alpha_1\alpha_2}\mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_l} = 0. \quad (503)$$

The 4-dimensional form (502) of the multipoles along with Eq. (492) confirms that the multipoles are orthogonal to 4-velocity, that is

$$\bar{u}_{\alpha_1}\mathcal{M}^{\alpha_1\ldots\alpha_l} = 0, \qquad \bar{u}_{\alpha_1}\mathcal{S}^{\alpha_1\ldots\alpha_l} = 0, \quad (504)$$

and due to the symmetry of the internal multipoles, Eq. (504) is valid to each index.

Notice that the matrix of transformation (484) has been used in making up the contravariant components of the multipoles (502) which are tensors of type $[^l_0]$. Tensor components of the multipoles, $\mathcal{M}_{\alpha_1\ldots\alpha_l}$ and $\mathcal{S}_{\alpha_1\ldots\alpha_l}$, which are of the type $[^0_l]$ are obtained by lowering each index of $\mathcal{M}^{\alpha_1\ldots\alpha_l}$ and $\mathcal{S}^{\alpha_1\ldots\alpha_l}$ respectively with the help of the background metric tensor $\bar{g}_{\alpha\beta}$. It is worth emphasizing that we have introduced 4-dimensional definitions of the internal multipoles as tensors of type $[^l_0]$ on the ground of transformation equations (502) because we defined the spatial components of $\mathcal{M}^{i_1\ldots i_l}$ and $\mathcal{S}^{i_1\ldots i_l}$ as integrals (122) and (131) taken from the STF products of the components of 3-dimensional coordinate w^i which behaves as a vector under the linear coordinate transformations. Another reason to use the contravariant components $\mathcal{M}^{i_1\ldots i_l}$ and $\mathcal{S}^{i_1\ldots i_l}$ as a starting point for their 4-dimensional prolongation is that the internal multipoles are the coefficients of the Cartesian tensors of type $[^l_0]$ in the Taylor expansions (220), (221), and (223) of the gravitational potentials $U_{\rm B}(t, \boldsymbol{x})$ and $U^i_{\rm B}(t, \boldsymbol{x})$ with respect to the components of the partial derivatives $\partial_{i_1\ldots i_l} r_{\rm B}^{-1}$ which are considered as the STF Cartesian tensors of type $[^0_l]$.

2. External multipoles

The external multipoles, $\mathcal{P}_{i_1\ldots i_l}$, $\mathcal{Q}_{i_1\ldots i_l}$ and $\mathcal{C}_{i_1\ldots i_l}$, have been defined at the origin of the local coordinates of body B by external solutions of the field equations for the metric tensor and scalar field in such a way that they are purely spatial STF Cartesian tensors of type $[^0_l]$; see Sec. IV B 4. It means that 4-dimensional tensor extensions of the external multipoles must be orthogonal to 4-velocity of the origin of the local coordinates which is, by construction, identical to 4-velocity $\bar{u}^\alpha$ of the worldline $\mathcal{Z}$ of the center of mass of the body B,

$$\bar{u}^{\alpha_1}\mathcal{Q}_{\alpha_1\alpha_2\ldots\alpha_l} = 0,$$
$$\bar{u}^{\alpha_1}\mathcal{P}_{\alpha_1\alpha_2\ldots\alpha_l} = 0,$$
$$\bar{u}^{\alpha_1}\mathcal{C}_{\alpha_1\alpha_2\ldots\alpha_l} = 0. \quad (505)$$

These orthogonality conditions suggests that the 4-dimensional components of the external multipoles are obtained from their 3-dimensional counterparts by making use of the matrix of transformation (479) which yields

$$\mathcal{Q}_{\alpha_1\ldots\alpha_l} \equiv \Lambda^{i_1}{}_{\alpha_1}\ldots\Lambda^{i_l}{}_{\alpha_l}\mathcal{Q}_{i_1\ldots i_l},$$
$$\mathcal{C}_{\alpha_1\ldots\alpha_l} \equiv \Lambda^{i_1}{}_{\alpha_1}\ldots\Lambda^{i_l}{}_{\alpha_l}\mathcal{C}_{i_1\ldots i_l},$$
$$\mathcal{P}_{\alpha_1\ldots\alpha_l} \equiv \Lambda^{i_1}{}_{\alpha_1}\ldots\Lambda^{i_l}{}_{\alpha_l}\mathcal{P}_{i_1\ldots i_l}. \quad (506)$$

We have used in here the matrix of transformation (479) because the external multipoles are defined originally as tensor coefficients of the Taylor expansions of the external potentials $\bar{U}$, $\bar{\Psi}$, etc., which are expressed in terms of partial derivatives from these potentials and behave under coordinate transformations like tensors of type $\left[{0 \atop l}\right]$. Definitions (506) and the properties of the matrices of transformation suggest that 4-dimensional tensors $\mathcal{Q}_{\alpha_1\ldots\alpha_l}$, $\mathcal{C}_{\alpha_1\ldots\alpha_l}$ and $\mathcal{P}_{\alpha_1\ldots\alpha_l}$ are STF tensors in the sense of (503) that is $\bar{g}^{\alpha_1\alpha_2}\mathcal{Q}_{\alpha_1\ldots\alpha_l}=0$, etc.

It is known that in general relativity the external multipoles, $\mathcal{Q}_{i_1\ldots i_l}$ and $\mathcal{C}_{i_1\ldots i_l}$ are defined in the local coordinates by partial derivatives of the Riemann tensor, $\bar{R}^{\alpha}{}_{\mu\beta\nu}$, of the background metric (476) taken at the origin of the local coordinates [47,58,297,298]. This definition remains valid with some modification in the scalar-tensor theory of gravity which is explained below. The external multipoles, $\mathcal{P}_{i_1\ldots i_l}$, of the scalar field are not related in any way to the Riemann tensor because they depend merely on derivatives of the background scalar field $\bar{\varphi}$.

As we show below, the 4-dimensional tensor formulation of the external multipoles is achieved by contracting the Riemann tensor with vectors of 4-velocity, $\bar{u}^\alpha$, and taking the covariant derivatives $\bar{\nabla}_\alpha$ projected on the hyperplane being orthogonal to the 4-velocity. The projection is fulfilled with the help of the operator of projection,

$$\pi^\alpha_\beta \equiv \delta^\alpha_\beta + \bar{u}^\alpha\bar{u}_\beta,$$
$$\pi^{\alpha\beta} = \bar{g}^{\alpha\beta} + \bar{u}^\alpha\bar{u}^\beta,$$
$$\pi_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{u}_\alpha\bar{u}_\beta. \tag{507}$$

The operator of projection satisfies the following relations: $\pi^\alpha_\gamma\pi^\gamma_\beta=\pi^\alpha_\beta$, $\pi^{\alpha\beta}=\bar{g}^{\alpha\gamma}\pi^\beta_\gamma$, $\pi_{\alpha\beta}=\bar{g}_{\alpha\gamma}\pi^\gamma_\beta$, and $\pi^\alpha_\alpha=3$. The latter property points out that π^α_β has only three algebraically independent components which are reduced to the Kronecker symbol when π^α_β is computed in the local coordinates of body B, that is in the local coordinates $\pi^0_0=0, \pi^i_0=\pi^0_i=0, \pi^i_j=\delta^i_j$. In other words, the projection operator is a 3-dimensional Kronecker symbol δ^i_j lifted up to 4-dimensional effective background manifold $\bar{M}$. We notice that the operator of the projection has some additional algebraic properties. Namely,

$$\pi^\alpha_\beta\Lambda^i{}_\alpha = \Lambda^i{}_\beta, \qquad \pi^\beta_\alpha\Omega^\alpha{}_i = \Omega^\beta{}_i, \tag{508}$$

that are in accordance with the condition of orthogonality (492). They point out that π^α_β can be also represented as a product of two reciprocal transformation matrices,

$$\pi^\alpha_\beta = \Omega^\alpha{}_i\Lambda^i{}_\beta. \tag{509}$$

The projection operator is required to extend 3-dimensional spatial derivatives of geometric objects to their 4-dimensional counterparts. Indeed, in the local coordinates the external multipoles are purely spatial Cartesian tensors which are expressed in terms of the partial spatial derivatives of the external perturbations of the metric tensor and/or scalar field. It means that the extension of a spatial partial derivative to its 4-dimensional form must preserve its orthogonality to the 4-velocity $\bar{u}^\alpha$ of the worldline $\mathcal{Z}$ which is achieved by coupling the spatial derivatives with the projection operator. For example, 4-dimensional STF form of the external STF scalar multipole $\mathcal{P}_L\equiv\mathcal{P}_{i_1\ldots i_l}=\mathcal{P}_{\langle i_1\ldots i_l\rangle}$ introduced in (153) in terms of the spatial derivatives of the external scalar field, reads

$$\begin{aligned}\mathcal{P}_{\alpha_1\ldots\alpha_l} &= \Lambda^{i_1}{}_{\alpha_1}\ldots\Lambda^{i_l}{}_{\alpha_l}\mathcal{P}_{i_1\ldots i_l} = \Lambda^{\langle i_1}{}_{\alpha_1}\ldots\Lambda^{i_l\rangle}{}_{\alpha_l}\bar{\nabla}_{\langle i_1\ldots i_l\rangle}\bar{\varphi}\\ &= \Lambda^{\langle i_1}{}_{\alpha_1}\ldots\Lambda^{i_l\rangle}{}_{\alpha_l}\Omega^{\beta_1}{}_{\langle i_1}\ldots\Omega^{\beta_l}{}_{i_l\rangle}\bar{\nabla}_{\beta_1\ldots\beta_l}\bar{\varphi}\\ &= \pi^{\beta_1}_{\langle\alpha_1}\cdots\pi^{\beta_l}_{\alpha_l\rangle}\bar{\nabla}_{\beta_1\cdots\beta_l}\bar{\varphi},\end{aligned} \tag{510}$$

where $\bar{\varphi}$ is the background scalar field perturbation, and the angular brackets around Greek indices indicate 4-dimensional generalization of 3-dimensional STF tensor defined earlier in (2). Extending 3-dimensional Kronecker symbol and other 3-tensors to 4-dimensional form we get

$$\begin{aligned}T_{\langle\alpha_1\ldots\alpha_l\rangle} \equiv \sum_{n=0}^{[l/2]}\frac{(-1)^n}{2^n n!}\frac{l!}{(l-2n)!}\frac{(2l-2n-1)!!}{(2l-1)!!}\\ \times\pi_{(\alpha_1\alpha_2\ldots}\pi_{\alpha_{2n-1}\alpha_{2n}}S_{\alpha_{2n+1}\ldots\alpha_l)\beta_1\gamma_1\ldots\beta_n\gamma_n}\pi^{\beta_1\gamma_1}\ldots\pi^{\beta_n\gamma_n}.\end{aligned} \tag{511}$$

We also notice that the projection operator can be effectively used to rise and/or to lower 4-dimensional (Greek) indices of the internal and external multipoles like the metric tensor $\bar{g}_{\alpha\beta}$. This is because all multipoles are orthogonal to the 4-velocity $\bar{u}^\alpha$. Thus, for example, $\mathcal{Q}_{\alpha\beta}\bar{g}^{\beta\gamma}=\mathcal{Q}_{\alpha\beta}\pi^{\beta\gamma}=\mathcal{Q}_\alpha{}^\gamma$, etc.

The external multipoles $\mathcal{Q}_{\alpha_1\ldots\alpha_l}$ and $\mathcal{C}_{\alpha_1\ldots\alpha_l}$ are directly connected to the Riemann tensor of the background manifold and its covariant derivatives. In order to establish this connection we work in the local coordinates and employ a covariant definition of the Riemann tensor (4) of the background manifold where the background metric tensor in the local coordinates is

$$\bar{g}_{\alpha\beta} = \eta_{\alpha\beta} + \hat{h}^{\rm ext}_{\alpha\beta}(u,\boldsymbol{w}) + \hat{l}^{\rm ext}_{\alpha\beta}(u,\boldsymbol{w}), \tag{512}$$

with the perturbations $\hat{h}^{\rm ext}_{\alpha\beta}$ and $\hat{l}^{\rm ext}_{\alpha\beta}$ defined in (117)–(120) respectively. The products of the connections entering (4) at the post-Newtonian level of approximation requires the following components of the Christoffel symbols:

$$\bar{\Gamma}^i_{00} = \bar{\Gamma}^0_{0i} = -\frac{1}{2}\partial_i \hat{h}^{\rm ext}_{00},$$
$$\bar{\Gamma}^i_{jk} = \frac{1}{2}(\partial_j \hat{h}^{\rm ext}_{ik} + \partial_k \hat{h}^{\rm ext}_{ik} - \partial_i \hat{h}^{\rm ext}_{jk}). \tag{513}$$

Substituting (512) and (513) to (4) and taking into account all post-Newtonian terms we get the STF part of the Riemann tensor component $[\bar{R}_{0i0j}]^{\rm STF} \equiv \bar{R}_{0\langle i|0|j\rangle}$ in the following form:

$$\begin{aligned}[\bar{R}_{0i0j}]^{\rm STF} = &-D_{\langle ij\rangle} + 3D_{\langle i}D_{j\rangle} + 2DD_{\langle ij\rangle} + 2(\gamma-1)D_{\langle i}H_{j\rangle} + 2(\beta-1)\times[H_{\langle i}H_{j\rangle} + (H-\mathcal{P})H_{\langle ij\rangle}]\\ &+2\sum_{l=0}^{\infty}\frac{(l-1)(l+1)}{(2l+5)(l+2)!}\ddot{\mathcal{Q}}_{L\langle i}w_{j\rangle L} + (\gamma-1)\sum_{l=0}^{\infty}\frac{(2l+1)(l+1)}{(2l+5)(l+2)!}\ddot{\mathcal{P}}_{L\langle i}w_{j\rangle L}\\ &-\frac{1}{2}\sum_{l=0}^{\infty}\frac{l+7}{(2l+7)(l+3)l!}\ddot{\mathcal{Q}}_{\langle ij\rangle L}w^L w^2 - (\gamma-1)\sum_{l=0}^{\infty}\frac{1}{(2l+7)(l+3)l!}\ddot{\mathcal{P}}_{\langle ij\rangle L}w^L w^2 + \sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon_{pq\langle i}\dot{\mathcal{C}}_{j\rangle pL}w^{qL},\end{aligned} \tag{514}$$

where we have discarded all terms of the post-post-Newtonian order and introduced the shorthand notations

$$D \equiv D(u,\boldsymbol{w}) = \sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{Q}_K(u)w^K, \tag{515}$$

$$H \equiv H(u,\boldsymbol{w}) = \sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{P}_K(u)w^K, \tag{516}$$

$$D_{i_1\ldots i_l} \equiv D_{i_1\ldots i_l}(u,\boldsymbol{w}) = \partial_{i_1\ldots i_l}D = \sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{Q}_{i_1\ldots i_l K}(u)w^K, \tag{517}$$

$$H_{i_1\ldots i_l} \equiv H_{i_1\ldots i_l}(u,\boldsymbol{w}) = \partial_{i_1\ldots i_l}H = \sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{P}_{i_1\ldots i_l K}(u)w^K. \tag{518}$$

Notice that at the origin of the local coordinates where $w^i = 0$, we have $D(u,0) = 0$, $H(u,0) = \mathcal{P}$, $D_{i_1\ldots i_p}(u,0) = \mathcal{Q}_{i_1\ldots i_p}$ and $H_{i_1\ldots i_p}(u,0) = \mathcal{P}_{i_1\ldots i_p}$. Therefore, at the origin of the local coordinates, that is on the worldline $\mathcal{Z}$, the value of the STF Riemann tensor (514) is simplified to

$$\begin{aligned}[\bar{R}_{0i0j}]^{\rm STF}_{\mathcal{Z}} = &-\mathcal{Q}_{\langle ij\rangle} + 3\mathcal{Q}_{\langle i}\mathcal{Q}_{j\rangle} + 2(\gamma-1)\mathcal{Q}_{\langle i}\mathcal{P}_{j\rangle}\\ &+2(\beta-1)\mathcal{P}_{\langle i}\mathcal{P}_{j\rangle}.\end{aligned} \tag{519}$$

This relationship establishes the connection between the external mass quadrupole $\mathcal{Q}_{ij}$ and the STF Riemann tensor. The reader should notice that (519) includes terms depending on acceleration $\mathcal{Q}_i$ of the worldline of the center of mass of body B. This may look strange as the curvature of spacetime (the Riemann tensor) does not depend on the choice of the worldline of the local coordinates. Indeed, it can be verified that the acceleration-dependent terms in (519) are mutually canceled out with the similar terms coming out of the explicit expression for $\mathcal{Q}_{ij}$ taken from (155), and obtained by the asymptotic matching technique.

Relationship between the STF covariant derivative of lth order from the Riemann tensor and the external gravitoelectric multipole of the same order is derived by taking covariant derivatives l times from both sides of (514). Covariant derivative of the order l from the Riemann tensor is a linear operator on the background manifold that involves the products of the Christoffel symbols and the covariant derivatives of the order $l-1$ from the Riemann tensor. They can be calculated by iterations starting from $l = 1$. Straightforward but tedious calculation shows that at the post-Newtonian level of approximation the covariant derivative of the order $l-2$ combined with the Riemann tensor to STF tensor of the order l, reads

$$\begin{aligned}&[\bar{\nabla}_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF}\\ &=[\partial_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF} + 2\sum_{k=0}^{l-3}(k+1)\partial_{\langle i_1\ldots i_{l-k-3}}[D_{i_{l-k-2}\ldots i_{l-1}}D_{i_l\rangle}]\\ &\quad+2(\gamma-1)\sum_{k=0}^{l-3}(k+2)\partial_{\langle i_1\ldots i_{l-k-3}}[D_{i_{l-k-2}\ldots i_{l-1}}H_{i_l\rangle}].\end{aligned} \tag{520}$$

Applying the Leibniz rule of differentiation to the product of two functions [[264], Eq. (0.42)] standing in the right-hand side of (520), we obtain a more simple expression,

$$\begin{aligned}&[\bar{\nabla}_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF}\\ &\quad= [\partial_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF}\\ &\qquad+2\sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-2)k!}{s!(k-s)!}D_{\langle i_1\ldots i_{s+1}}D_{i_{s+2}\ldots i_l\rangle}\\ &\qquad+2(\gamma-1)\sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-1)k!}{s!(k-s)!}H_{\langle i_1\ldots i_{s+1}}D_{i_{s+2}\ldots i_l\rangle}.\end{aligned} \tag{521}$$

The $l-2$th order partial derivatives from terms $D_{\langle i}D_{j\rangle}$, $DD_{\langle ij\rangle}$, etc., entering $[\partial_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF}$, are also calculated with the help of the Leibniz rule, yielding

$$\partial_{\langle i_1\ldots i_{l-2}}[D_{i_{l-1}}D_{i_l\rangle}] = \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}D_{\langle i_1\ldots i_{k+1}}D_{i_{k+2}\ldots i_l\rangle}, \tag{522}$$

$$\partial_{\langle i_1\ldots i_{l-2}}[D_{i_{l-1}i_l\rangle}D] = \sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}D_{\langle i_1\ldots i_k}D_{i_{k+1}\ldots i_l\rangle}, \tag{523}$$

$$\partial_{\langle i_1\ldots i_{l-2}}[D_{i_{l-1}}H_{i_l\rangle}] = \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}D_{\langle i_1\ldots i_{k+1}}H_{i_{k+2}\ldots i_l\rangle}, \tag{524}$$

$$\partial_{\langle i_1\ldots i_{l-2}}[H_{i_{l-1}}H_{i_l\rangle}] = \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}H_{\langle i_1\ldots i_{k+1}}H_{i_{k+2}\ldots i_l\rangle}, \tag{525}$$

$$\partial_{\langle i_1\ldots i_{l-2}}[D_{i_{l-1}i_l\rangle}(H-\mathcal{P})] = \sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}H_{\langle i_1\ldots i_k}H_{i_{k+1}\ldots i_l\rangle}. \tag{526}$$

Actually, we need the covariant derivatives of the STF part of the Riemann tensor only at the origin of the local coordinates adapted to body B. Therefore, after taking the STF covariant derivatives from the Riemann tensor we take the value of the local spatial coordinates $w^i = 0$, which eliminates all terms depending on the time derivatives of the external multipoles in the right hand side of (514) for the STF part of the Riemann tensor. Hence, the STF covariant derivative of the Riemann tensor taken on the worldline of the center of mass of body B reads

$$\begin{aligned}
[\bar{\nabla}_{i_1\ldots i_{l-2}}\bar{R}_{0i_{l-1}0i_l}]^{\rm STF}_{\mathcal{Z}} = &-\mathcal{Q}_{\langle i_1\ldots i_l\rangle} + 3\sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{Q}_{\langle i_1\ldots i_{k+1}}\mathcal{Q}_{i_{k+2}\ldots i_l\rangle}\\
&+2\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{Q}_{\langle i_1\ldots i_k}\mathcal{Q}_{i_{k+1}\ldots i_l\rangle} + \sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-2)k!}{s!(k-s)!}\mathcal{Q}_{\langle i_1\ldots i_{s+1}}\mathcal{Q}_{i_{s+2}\ldots i_l\rangle}\right]\\
&+2(\gamma-1)\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{Q}_{\langle i_1\ldots i_k}\mathcal{P}_{i_{k+1}\ldots i_l\rangle} + \sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-1)k!}{s!(k-s)!}\mathcal{P}_{\langle i_1\ldots i_{s+1}}\mathcal{Q}_{i_{s+2}\ldots i_l\rangle}\right]\\
&+2(\beta-1)\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{P}_{\langle i_1\ldots i_k}\mathcal{P}_{i_{k+1}\ldots i_l\rangle} + \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{P}_{\langle i_1\ldots i_{k+1}}\mathcal{P}_{i_{k+2}\ldots i_l\rangle}\right].
\end{aligned} \tag{527}$$

It is rather straightforward now to convert (527) to 4-dimensional form valid in arbitrary coordinates on the effective manifold $\bar{M}$ by making use of the transformation matrices and the operator of projection as it was explained above. We introduce a new notation for the covariant STF derivative of the Riemann tensor taken on the worldline $\mathcal{Z}$,

$$\mathcal{E}_{\alpha_1\ldots\alpha_l} \equiv \pi^{\beta_1}_{\langle\alpha_1}\pi^{\beta_2}_{\alpha_2}\ldots\pi^{\beta_l}_{\alpha_l\rangle}[\bar{\nabla}_{\beta_1\ldots\beta_{l-2}}\bar{R}_{\mu\beta_{l-1}\beta_l\nu}u^\mu u^\nu]^{\rm STF}_{\mathcal{Z}}, \tag{528}$$

and use it for transformation of (527) to arbitrary coordinates. It yields a covariant expression for the external gravitoelectric multipoles $\mathcal{Q}_{\alpha_1\ldots\alpha_l}$ in terms of the STF covariant derivatives from the Riemann tensor,

$$\begin{aligned}
\mathcal{Q}_{\alpha_1\ldots\alpha_l} = &\,\mathcal{E}_{\langle\alpha_1\ldots\alpha_l\rangle} + 3\sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{E}_{\langle\alpha_1\ldots\alpha_{k+1}}\mathcal{E}_{\alpha_{k+2}\ldots\alpha_l\rangle}\\
&+2\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{E}_{\langle\alpha_1\ldots\alpha_k}\mathcal{E}_{\alpha_{k+1}\ldots\alpha_l\rangle} + \sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-2)k!}{s!(k-s)!}\mathcal{E}_{\langle\alpha_1\ldots\alpha_{s+1}}\mathcal{E}_{\alpha_{s+2}\ldots\alpha_l\rangle}\right]\\
&+2(\gamma-1)\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\mathcal{E}_{\langle\alpha_1\ldots\alpha_k}\Phi_{\alpha_{k+1}\ldots\alpha_l\rangle} + \sum_{k=0}^{l-3}\sum_{s=0}^{k}\frac{(l-k-1)k!}{s!(k-s)!}\Phi_{\langle\alpha_1\ldots\alpha_{s+1}}\mathcal{E}_{\alpha_{s+2}\ldots\alpha_l\rangle}\right]\\
&+2(\beta-1)\left[\sum_{k=1}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\Phi_{\langle\alpha_1\ldots\alpha_k}\Phi_{\alpha_{k+1}\ldots\alpha_l\rangle} + \sum_{k=0}^{l-2}\frac{(l-2)!}{k!(l-k-2)!}\Phi_{\langle\alpha_1\ldots\alpha_{k+1}}\Phi_{\alpha_{k+2}\ldots\alpha_l\rangle}\right],
\end{aligned} \tag{529}$$

where we have made identification: $\mathcal{E}_a \equiv \mathcal{Q}_\alpha$. At this stage of calculation, it is worth noticing that 4-acceleration of the center of mass of body B, $a_\alpha \equiv \bar{u}^\beta \bar{\nabla}_\beta \bar{u}^\alpha$, is not exactly equal to $\mathcal{E}_\alpha$ because of a term depending on the time derivative of body's mass, $\dot{M}$, in the right-hand side of (495). Only in case when the mass is conserved, $a^\alpha = \mathcal{E}^\alpha$.

Similar, but less tedious procedure allows us to calculate 4-dimensional form of the external gravitomagnetic multipoles $\mathcal{C}_{\alpha_1\ldots\alpha_l}$ in terms of the STF covariant derivative of the Riemann tensor. We get

$$\mathcal{C}_{\alpha_1\ldots\alpha_l} \equiv \pi^{\beta_1}_{\langle\alpha_1}\pi^{\beta_2}_{\alpha_2}\ldots\pi^{\beta_l}_{\alpha_l\rangle}[\bar{\nabla}_{\beta_1\ldots\beta_{l-2}}\bar{R}_{\sigma\mu\nu\beta_{l-1}}\varepsilon_{\beta_l}{}^{\sigma\mu}\bar{u}^\nu]^{\rm STF}_{\mathcal{Z}}. \quad (530)$$

where we have utilized 3-dimensional covariant tensor of Levi-Civita $\varepsilon_{\alpha\beta\gamma}$ which is a projection of 4-dimensional, fully antisymmetric Levi-Civita symbol $E_{\alpha\mu\nu\rho}$ [[165], § 3.5] on the hyperplane being orthogonal to 4-velocity $\bar{u}^\alpha$,

$$\varepsilon_{\alpha\beta\gamma} \equiv (-\bar{g})^{1/2}\bar{u}^\mu\pi^\nu_\alpha\pi^\rho_\beta\pi^\sigma_\gamma E_{\mu\nu\rho\sigma}. \quad (531)$$

It can be checked by inspection that in the global coordinates the right-hand sides of (529) and (530) are reduced to $\mathcal{Q}_L$ and $\mathcal{C}_L$ respectively as it must be.

Four-dimensional definitions of the external multipoles given in this section allow us to transform products of the multipoles given in the local coordinates to their covariant counterparts, for example, $\mathcal{Q}_L\mathcal{M}^L \equiv \mathcal{Q}_{i_1\ldots i_l}\mathcal{M}^{i_1\ldots i_l} = \mathcal{Q}_{\alpha_1\ldots\alpha_l}\mathcal{M}^{\alpha_1\ldots\alpha_l}$, etc. In all such products the matrices of transformation cancel out giving rise to covariant expressions being independent of a particular choice of coordinates.

E. Covariant translational equations of motion

A generic form of the covariant translational equations of motion have been formulated in (495). Substituting to these equations the force $F^\alpha = -M\mathcal{Q}^\alpha$ where $\mathcal{Q}^\alpha$ was introduced in (501), yields

$$M\frac{\mathcal{D}\bar{u}^\mu}{\mathcal{D}\tau} = F^\mu - \dot{M}\bar{u}^\alpha, \quad (532)$$

where the force

$$F^\mu = F^\mu_{\mathfrak{q}} + F^\mu_{\mathcal{Q}} + F^\mu_{\mathcal{C}} + F^\mu_{\mathcal{P}}, \quad (533)$$

and the second term in the right-hand side of (532) is due to the nonconservation of mass (165) having the following covariant form:

$$\begin{aligned}\dot{M} = (\gamma-1)\left(\mathcal{P}\sum_{l=1}^\infty\frac{1}{l!}\mathcal{Q}_{\alpha_1\ldots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{M}^{\alpha_1\ldots\alpha_l}}{\mathcal{D}\tau} + \frac{\mathcal{D}_{\rm F}\mathcal{P}}{\mathcal{D}\tau}\mathcal{M}\right) \\ -\sum_{l=1}^\infty\frac{1}{(l-1)!}\mathcal{Q}_{\alpha_1\ldots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{M}^{\alpha_1\ldots\alpha_l}}{\mathcal{D}\tau} \\ -\sum_{l=1}^\infty\frac{l+1}{l!}\mathcal{M}^{\alpha_1\ldots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{E}_{\alpha_1\ldots\alpha_l}}{\mathcal{D}\tau},\end{aligned} \quad (534)$$

where we have used the covariant Fermi-Walker derivative of the multipole moments which is a covariant generalization of the total time derivative in the local coordinates. The Fermi-Walker derivative is explained in more detail at the end of this section; see Eq. (542).

Gravitational force F^μ in the right-hand side of (532) is the 4-dimensional extension of 3-dimensional force (501) with the local 4-acceleration $\mathcal{Q}_i$ defined in (184) where the complementary function $\mathcal{I}^i_{\rm c}$ is chosen as $\mathcal{I}^i_{\rm c} = 3\mathcal{Q}_k\mathcal{M}^{ik}$, or, in 4-dimensional form

$$\mathcal{I}^\alpha_{\rm c} = 3\mathcal{Q}_\beta\mathcal{M}^{\alpha\beta}. \quad (535)$$

This form of $\mathcal{I}^\alpha_{\rm c}$ eliminates the terms depending on the local acceleration $\mathcal{Q}_\alpha$ coupled with the quadrupole moment $\mathcal{M}^{\alpha\beta}$ of the body from the force F^α, and delivers a covariant definition of the center of mass of body B. It is similar but not exactly equal to the choice (289) of this function in the global coordinates.

The first term in the right side of (533) describes the Dicke-Nordtvedt anomalous force caused by the violation of the *strong* principle of equivalence (SEP)

$$F^\alpha_{\mathfrak{q}} = \mathfrak{q}\mathcal{P}^\alpha, \quad (536)$$

where

$$\mathcal{P}^\alpha = \pi^{\alpha\beta}\bar{\nabla}_\beta\bar{\varphi} \quad (537)$$

is an external scalar-field dipole and $\mathfrak{q} \equiv \mathcal{M} - M$ is the difference between the active—$\mathcal{M}$, and conformal—M, masses of body B. The quantity $\mathfrak{q}$ can be interpreted as an effective scalar charge of body B interacting with the external scalar field and causing the body to accelerate with respect to a body having negligible self-gravity but the same set of internal multipole moments. The anomalous scalar-field gravitational force $F^\mu_{\mathfrak{q}}$ was predicted by Dicke and its effect in three body system (Earth-Moon-Sun) was studied by Nordtvedt in the framework of PPN formalism [[88], § 8.1]. Explicit expression for the scalar charge $\mathfrak{q}$ is obtained from (166) and reads

$$\mathfrak{q}=\frac{1}{2}\eta\int_{\mathcal{V}_{\rm B}}\rho^*\hat{U}_{\rm B}d^3w-\frac{1}{6}(\gamma-1)\frac{\mathcal{D}^2_{\rm F}\mathcal{N}}{\mathcal{D}\tau^2}+2(\beta-1)\mathcal{M}\mathcal{P}+2(\beta-1)\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{P}_{\alpha_1\dots\alpha_l}\mathcal{M}^{\alpha_1\dots\alpha_l}+(\gamma-1)\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\mathcal{Q}_{\alpha_1\dots\alpha_l}\mathcal{M}^{\alpha_1\dots\alpha_l}. \tag{538}$$

The first and second terms in the right-hand side of (538) compose a bare part of the scalar charge being proportional to self-gravitational energy of the body and the second time derivative of the body's moment of inertia $\mathcal{N}$. Standard treatment of the Nordtvedt effect [[88], § 8.1] takes into account only the very first term in the right-hand side of (538) which is proportional to the Nordtvedt parameter η assuming that the time derivative of the moment of inertia is either negligibly small or that its average value vanishes for periodic motions and/or stationary rotation of celestial bodies. This assumption may be sufficient in case of slow-motion and weak gravitational field approximation. However, it is not true in strongly gravitating $\mathbb{N}$-body systems like coalescing binary neutron stars and/or black holes. The remaining terms in the right-hand side of (538) describe gravitational coupling of the internal multipoles of body B and external multipoles of gravitational field. The dominant term, $2(\beta-1)\mathcal{M}\mathcal{P}$, is usually included to the Einstein-Infeld-Hoffmann force [[17], Eq. 6.82] and is not treated as a part of the Nordtvedt effect. The coupling terms depending on high-order multipoles in (538) are fairly small in the Solar System and have never been taken into account so far. Nonetheless, they become large at the latest stage of evolution of coalescing binary systems and can be used for more deep testing of scalar-tensor theory of gravity by gravitational wave detectors.

The other components of the 4-dimensional force standing in the right-hand side of (532) describe gravitational interaction between the internal multipoles of body B and the external multipoles. We have

$$\begin{aligned}F^\mu_{\mathcal{Q}}=&\sum_{l=1}^{\infty}\frac{1}{l!}\bar{g}^{\mu\nu}\mathcal{Q}_{\nu\alpha_1\dots\alpha_l}\mathcal{M}^{\alpha_1\dots\alpha_l}-\sum_{l=2}^{\infty}\frac{l^2+l+4}{(l+1)!}\mathcal{Q}_{\alpha_1\dots\alpha_l}\frac{\mathcal{D}^2_{\rm F}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}}{\mathcal{D}\tau^2}-\sum_{l=2}^{\infty}\frac{2l+1}{l+1}\frac{l^2+2l+5}{(l+1)!}\frac{\mathcal{D}_{\rm F}\mathcal{Q}_{\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\frac{\mathcal{D}_{\rm F}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\\&-\sum_{l=2}^{\infty}\frac{2l+1}{2l+3}\frac{l^2+3l+6}{(l+1)!}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}\frac{\mathcal{D}^2_{\rm F}\mathcal{Q}_{\alpha_1\dots\alpha_l}}{\mathcal{D}\tau^2}+4\sum_{l=1}^{\infty}\frac{l+1}{(l+2)!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{Q}_{\rho\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{S}^{\sigma\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\\&+4\sum_{l=1}^{\infty}\frac{l+1}{l+2}\frac{l+1}{(l+2)!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{S}^{\sigma\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{Q}_{\rho\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}-\frac{2}{\mathcal{M}}\sum_{l=1}^{\infty}\frac{1}{l!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{Q}_{\rho\alpha_1\dots\alpha_l}\mathcal{M}^{\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{S}^{\sigma}}{\mathcal{D}\tau}\\&-\frac{1}{\mathcal{M}}\sum_{l=1}^{\infty}\frac{1}{l!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{S}^{\sigma}\frac{\mathcal{D}_{\rm F}}{\mathcal{D}\tau}(\mathcal{Q}_{\rho\alpha_1\dots\alpha_l}\mathcal{M}^{\alpha_1\dots\alpha_l})\end{aligned}\tag{539}$$

$$F^\mu_{\mathcal{C}}=\sum_{l=1}^{\infty}\frac{l}{(l+1)!}\bar{g}^{\mu\nu}\mathcal{C}_{\nu\alpha_1\dots\alpha_l}\mathcal{S}^{\alpha_1\dots\alpha_l}-\sum_{l=1}^{\infty}\frac{1}{(l+1)!}\varepsilon^{\mu\rho}{}_{\sigma}\left[\mathcal{C}_{\rho\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{M}^{\sigma\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}+\frac{l+1}{l+2}\mathcal{M}^{\sigma\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{C}_{\rho\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\right]\tag{540}$$

$$\begin{aligned}F^\mu_{\mathcal{P}}=2(1-\gamma)\Bigg[&\sum_{l=1}^{\infty}\frac{1}{(l+1)!}\mathcal{P}_{\alpha_1\dots\alpha_l}\frac{\mathcal{D}^2_{\rm F}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}}{\mathcal{D}\tau^2}+\sum_{l=1}^{\infty}\frac{2l+1}{l+1}\frac{1}{(l+1)!}\frac{\mathcal{D}_{\rm F}\mathcal{P}_{\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\frac{\mathcal{D}_{\rm F}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\\&+\sum_{l=1}^{\infty}\frac{2l+1}{2l+3}\frac{1}{(l+1)!}\mathcal{M}^{\mu\alpha_1\dots\alpha_l}\frac{\mathcal{D}^2_{\rm F}\mathcal{P}_{\alpha_1\dots\alpha_l}}{\mathcal{D}\tau^2}-\sum_{l=0}^{\infty}\frac{l+1}{(l+2)!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{P}_{\rho\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{S}^{\sigma\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\\&-\sum_{l=0}^{\infty}\frac{l+1}{l+2}\frac{l+1}{(l+2)!}\varepsilon^{\mu\rho}{}_{\sigma}\mathcal{S}^{\sigma\alpha_1\dots\alpha_l}\frac{\mathcal{D}_{\rm F}\mathcal{P}_{\rho\alpha_1\dots\alpha_l}}{\mathcal{D}\tau}\Bigg].\end{aligned}\tag{541}$$

Time derivatives of the internal and external multipoles of body B in the local coordinates are taken at the fixed value of the spatial coordinates, $w^i=0$, that is at the origin of the local coordinates. The multipoles are STF Cartesian tensors which are orthogonal to 4-velocity of worldline $\mathcal{Z}$ representing the motion of the origin of the local coordinates which coincides with the center of mass of body B. This worldline is not a geodesic on the effective background manifold $\bar{M}$ but is accelerating with the local acceleration Q_α. Therefore, the time derivative of the multipoles corresponds to the Fermi-Walker covariant derivative—denoted as $\mathcal{D}_{\rm F}/\mathcal{D}\tau$—on the background manifold taken along the direction of the 4-velocity vector $\bar{u}^\alpha$ with accounting for the Fermi-Walker transport [[164], Chapter 1, Sec. 4]. For example, the first time derivative taken from 3-dimensional internal multipole $\dot{\mathcal{M}}^L \equiv \dot{\mathcal{M}}^{i_1 i_2 \ldots i_l}$ in the local coordinates is mapped to the 4-dimensional Fermi-Walker covariant derivative as follows:

$$\dot{\mathcal{M}}^L \mapsto \frac{\mathcal{D}_{\rm F}\mathcal{M}^{\alpha_1\alpha_2\ldots\alpha_l}}{\mathcal{D}\tau} \equiv \frac{\mathcal{D}\mathcal{M}^{\alpha_1\alpha_2\ldots\alpha_l}}{\mathcal{D}\tau} + l\mathcal{Q}_\beta u^{\langle\alpha_1}\mathcal{M}^{\alpha_2\ldots\alpha_l\rangle\beta}, \tag{542}$$

where $\mathcal{D}\mathcal{M}^{\langle\alpha_1\alpha_2\ldots\alpha_l\rangle}/\mathcal{D}\tau \equiv \bar{u}^\beta\bar{\nabla}_\beta\mathcal{M}^{\langle\alpha_1\alpha_2\ldots\alpha_l\rangle}$ is a standard covariant derivative of tensor $\mathcal{M}^{\langle\alpha_1\alpha_2\ldots\alpha_l\rangle}$, and $\mathcal{Q}^\alpha$ is 4-acceleration of the origin of the local coordinates. In a similar way, the second time derivative from 3-dimensional internal multipole, $\ddot{\mathcal{M}}^L \equiv \ddot{\mathcal{M}}^{i_1 i_2 \ldots i_l}$, can be mapped to the 4-dimensional Fermi-Walker covariant derivative of the second order by applying the rule (542) two times,

$$\begin{aligned}\ddot{\mathcal{M}}^L \mapsto & \frac{\mathcal{D}^2_{\rm F}\mathcal{M}^{\alpha_1\alpha_2\ldots\alpha_l}}{\mathcal{D}\tau^2} \\ \equiv & \frac{\mathcal{D}^2\mathcal{M}^{\alpha_1\alpha_2\ldots\alpha_l}}{\mathcal{D}\tau^2} + 2l\mathcal{Q}_\beta u^{\langle\alpha_1}\frac{\mathcal{D}\mathcal{M}^{\alpha_2\ldots\alpha_l\rangle\beta}}{\mathcal{D}\tau} \\ & + l\frac{\mathcal{D}\mathcal{Q}_\beta}{\mathcal{D}\tau}u^{\langle\alpha_1}\mathcal{M}^{\alpha_2\ldots\alpha_l\rangle\beta} + l\mathcal{Q}_\beta\mathcal{Q}^{\langle\alpha_1}\mathcal{M}^{\alpha_2\ldots\alpha_l\rangle\beta} \\ & + l^2\mathcal{Q}_\beta\mathcal{Q}_\gamma u^{\langle\alpha_1}u^{\alpha_2}\mathcal{M}^{\alpha_3\ldots\alpha_l\rangle\beta\gamma},\end{aligned} \tag{543}$$

where $\mathcal{D}\mathcal{Q}^\alpha/\mathcal{D}\tau = \bar{u}^\beta\bar{\nabla}_\beta\mathcal{Q}^\alpha$ is the covariant derivative of the 4-acceleration of the origin of the local frame taken along the direction of its 4-velocity.

Comparison of our covariant Eqs. (532)–(541) of translational motion of the center of mass of body B with the corresponding Eq. (467) derived by Dixon [11] will be done in Appendix D 2.

F. Covariant rotational equations of motion

Covariant rotational equations of motion generalize 3-dimensional form (194), (195) of the rotational equations for spin of body B which is a member of an ℕ-body system, to a 4-dimensional, coordinate-independent form. Spin is a vector that is orthogonal to 4-velocity of the worldline $\mathcal{Z}$ of the center of mass of body B and carried out along this worldline according to the Fermi-Walker transportation rule. The covariant form of (194) is based on the Fermi-Walker derivative, and reads

$$\frac{\mathcal{D}_{\rm F}\mathcal{S}^\mu}{\mathcal{D}\tau} = \mathcal{T}^\mu, \tag{544}$$

or more explicitly,

$$\frac{\mathcal{D}\mathcal{S}^\mu}{\mathcal{D}\tau} = \mathcal{T}^\mu - (\mathcal{S}^\beta\mathcal{Q}_\beta)\bar{u}^\mu, \tag{545}$$

where the second term in the right-hand side is due to the fact that the Fermi-Walker transport is executed along the accelerated worldline $\mathcal{Z}$ of the center of mass of body B, the torque $\mathcal{T}^\mu$ is a covariant generalizations of 3-torque (194), and the center-of-mass condition (535) has been implemented. We have

$$\begin{aligned}\mathcal{T}^\mu = & -\varepsilon^{\mu\rho}{}_\sigma\Big[\mathcal{P}_\rho\mathcal{M}^\sigma + 3(\mathcal{P}_\rho - \mathcal{Q}_\rho)\mathcal{Q}_\beta\mathcal{M}^{\sigma\beta} \\ & + (2\beta-\gamma-1)\mathcal{P}\sum_{l=1}^\infty\frac{1}{l!}\mathcal{Q}_{\rho\alpha_1\ldots\alpha_l}\mathcal{M}^{\sigma\alpha_1\ldots\alpha_l}\Big] \\ & - \varepsilon^{\mu\rho}{}_\sigma\sum_{l=1}^\infty\frac{1}{l!}\mathcal{Q}_{\rho\alpha_1\ldots\alpha_l}\mathcal{M}^{\sigma\alpha_1\ldots\alpha_l} \\ & - \varepsilon^{\mu\rho}{}_\sigma\sum_{l=1}^\infty\frac{l+1}{(l+2)l!}\mathcal{C}_{\rho\alpha_1\ldots\alpha_l}\mathcal{S}^{\sigma\alpha_1\ldots\alpha_l},\end{aligned} \tag{546}$$

where the external multipole moments $\mathcal{Q}_{\alpha_1\ldots\alpha_l}$ and $\mathcal{C}_{\alpha_1\ldots\alpha_l}$ are expressed in terms of the Riemann tensor of the background manifold in accordance with Eqs. (529) and (530) respectively. Acceleration $Q^\alpha = -F^\alpha/M$, where the force F^α is taken from (533), and $\mathcal{P}_\alpha$ is defined in (537). It should be noticed that the terms entering the first line of the right-hand side of (546) are present only in the scalar-tensor theory of gravity while the last two terms are the genuine general-relativistic components of the torque caused by the presence of the tidal gravitoelectric and gravitomagnetic fields respectively.

Comparison of our Eq. (545) for evolution of spin of body B with the corresponding Eq. (468) derived by Dixon [11] will be done in Appendix D 3.

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APPENDIX A: AUXILIARY MATHEMATICAL PROPERTIES OF STF TENSORS

The definition of the symmetric trace-free (STF) Cartesian tensor was introduced by Pirani [299] and is given in Eq. (2) of the present paper. Here, we provide the reader with a number of auxiliary algebraic and differential identities involving the STF tensors that were instrumental for doing our computations.

Perhaps one of the most important algebraic identities of the STF tensors is the *index-peeling* formula that helps one to separate a single index from the rest of other STF indices in the STF tensor. Let us demonstrate how this formula is applied in the case of a product of vector with a STF tensor. We denote two STF tensors as $T_L \equiv T_{\langle L\rangle}$ and $R_L \equiv R_{\langle L\rangle}$, and let V_i be an arbitrary covector. The index-peeling formula reads [[80], Eq. (2.14)]

$$V_{\langle i}T_{L\rangle} = \frac{1}{l+1}V_i T_L + \frac{l}{l+1}T_{i\langle L-1}V_{i_l\rangle} - \frac{2l}{(l+1)(2l+1)}V_k T_{k\langle L-1}\delta_{i_l\rangle i}. \qquad \text{(A1)}$$

The index-peeling formula can be applied to two or more indices by successive iterations.

The index-peeling formula (A1) is directly extended from covector V_i to tensors. For example, by replacing $V_i \mapsto \delta_{ij}$ in (A1), and reducing similar terms we can get the following identities [84]:

$$T_{i\langle L}\delta_{j\rangle j} = \frac{2l+3}{2l+1}T_{iL}, \qquad \text{(A2)}$$

$$T_{j\langle L}\delta_{j\rangle i} = \frac{1}{(l+1)(2l+1)}T_{iL}. \qquad \text{(A3)}$$

Replacing $V_i \mapsto R_{Li}$ in (A1) yields [[75], Eq. (4.26)]

$$R_{L\langle i}T_{L\rangle} = \frac{1}{(l+1)(2l+1)}R_{iL}T_L. \qquad \text{(A4)}$$

Two other useful formulas are for a product of the unit vectors $n^i = x^i/r$, where $r = (\delta_{ij}x^i x^j)^{1/2}$. They are [[50], Eqs. (A22a) and (A23)]

$$n^{\langle iL\rangle} = n^i n^{\langle L\rangle} - \frac{l}{2l+1}\delta^{i\langle i_l}n^{L-1\rangle}, \qquad \text{(A5)}$$

$$n^i n^{\langle iL\rangle} = \frac{l+1}{2l+1}n^{\langle L\rangle}. \qquad \text{(A6)}$$

Differential identities for the STF partial derivatives from the radial distance r are [[50], Eqs. (A32) and (A34)]

$$\partial_{\langle L\rangle}r^{-1} = \partial_L r^{-1} = (-1)^l(2l-1)!!\frac{n^{\langle L\rangle}}{r^{l+1}}, \qquad \text{(A7)}$$

$$\partial_{\langle L\rangle}r = (-1)^{l+1}(2l-3)!!\frac{n^{\langle L\rangle}}{r^{l-1}}. \qquad \text{(A8)}$$

A partial spatial derivative from an STF tensor $n^{\langle L\rangle}$ is [[50], Eq. (A24)]

$$r\partial_i n^{\langle L\rangle} = (l+1)n^i n^{\langle L\rangle} - (2l+1)n^{\langle iL\rangle}. \qquad \text{(A9)}$$

Other useful algebraic and differential identities for STF tensors are given in papers [50,75,80,82,84,300].

APPENDIX B: COMPARISON WITH THE RACINE-VINES-FLANAGAN EQUATIONS OF MOTION

Translational equations of motion for arbitrary structured bodies have been derived by Racine and Flanagan [84] with a corrigendum published in [85]. Definitions of the internal multipoles of body B in those papers are the same as in the present paper. The Racine-Vines-Flanagan (RVF) equations of motion are given in [[84], Eqs. (6.11–6.16)] and, besides directly computed terms, contain four terms depending on the STF Cartesian tensor function $\hat{P}^{\mathrm{C}}_K = \hat{P}^{\mathrm{C}}_{\langle K\rangle}$ [[84], Eq. (6.16)] which is[10]

$$\hat{P}^K_{\mathrm{C}} \equiv \ddot{\mathcal{M}}^K_{\mathrm{C}} + 2k\dot{\mathcal{M}}^{\langle K-1}_{\mathrm{C}}v^{i_k\rangle}_{\mathrm{C}} + k(k-1)\mathcal{M}^{\langle K-2}_{\mathrm{C}}v^{i_{k-1}}_{\mathrm{C}}v^{i_k\rangle}_{\mathrm{C}}. \qquad \text{(B1)}$$

Function $\hat{P}^K_{\mathrm{C}}$ enters Eqs. (6.13a), (6.13b), and (6.13g) in [84]. The terms with $\hat{P}^K_{\mathrm{C}}$ must be developed explicitly in order to combine it in similar terms in other parts of the RVF equations of motion.

It is more convenient to develop the products of $\hat{P}^K_{\mathrm{C}}$ with the STF combinations of a unit vector, $n^i_{\mathrm{CB}} = R^i_{\mathrm{CB}}/R_{\mathrm{CB}}$, where $R^i_{\mathrm{CB}} = x^i_{\mathrm{C}} - x^i_{\mathrm{B}}$ is the coordinate distance between centers of mass of bodies B and C. The RVF equations of motion depend on four such combinations which have not been shown in [84,85] so that we present them explicitly. Two of them are products, $n^{\langle iKL\rangle}_{\mathrm{CB}}\mathcal{M}^{jL}_{\mathrm{B}}\hat{P}^{jK}_{\mathrm{C}}$ and $n^{\langle kL\rangle}_{\mathrm{CB}}\mathcal{M}^{L}_{\mathrm{B}}\hat{P}^{iK}_{\mathrm{C}}$, which appear in the first and second terms in the right-hand side of equation (6.12a) in [84]. In order to compute these terms we successively apply the index-peeling formula (A1) two times to separate the index of velocity of body B in $\hat{P}^{iK}_{\mathrm{C}}$ from the STF multi-indices and, then, render contraction of the multi-indices. It yields

[10] Notice that we use indices B and C to label the bodies of an $\mathbb{N}$-body system while Racine and Flanagan [84] use an index B instead of C, and an index A instead of B. We prefer to use our index notations to facilitate the comparison of the equations of motion. Relabeling the RVF equations is achieved with the simple replacements of the body's indices: B → C and A → B.

$$n_{\rm CB}^{\langle iKL\rangle}\mathcal{M}_{\rm B}^{jL}\hat{P}_{\rm C}^{jK}=n_{\rm CB}^{\langle iKL\rangle}\mathcal{M}_{\rm B}^{jL}(\ddot{\mathcal{M}}_{\rm C}^{jK}+2v_{\rm C}^{j}\dot{\mathcal{M}}_{\rm C}^{K})+2kn_{\rm CB}^{\langle ijLK-1\rangle}$$
$$\times\left[v_{\rm C}^{j}\mathcal{M}_{\rm B}^{pL}(\dot{\mathcal{M}}_{\rm C}^{pK-1}+v_{\rm C}^{p}\mathcal{M}_{\rm C}^{K-1})-\frac{1}{2k+1}(2v_{\rm C}^{p}\dot{\mathcal{M}}_{\rm C}^{pK-1}+v_{\rm C}^{2}\mathcal{M}_{\rm C}^{K-1})\mathcal{M}_{\rm B}^{jL}\right]$$
$$+k(k-1)n_{\rm CB}^{\langle ijpLK-2\rangle}v_{\rm C}^{p}\mathcal{M}_{\rm C}^{qK-2}\left(v_{\rm C}^{j}\mathcal{M}_{\rm B}^{qL}-\frac{4}{2k+1}v_{\rm C}^{q}\mathcal{M}_{\rm B}^{jL}\right), \qquad (B2)$$

$$n_{\rm CB}^{\langle kL\rangle}\mathcal{M}_{\rm B}^{L}\hat{P}_{\rm C}^{iK}=n_{\rm CB}^{\langle kL\rangle}\mathcal{M}_{\rm B}^{L}(\ddot{\mathcal{M}}_{\rm C}^{iK}+2v_{\rm C}^{i}\dot{\mathcal{M}}_{\rm C}^{K})+2k\mathcal{M}_{\rm B}^{L}$$
$$\times\left[n_{\rm CB}^{\langle jLK-1\rangle}v_{\rm C}^{j}(\dot{\mathcal{M}}_{\rm C}^{pK-1}+v_{\rm C}^{i}\mathcal{M}_{\rm C}^{K-1})-\frac{1}{2k+1}n_{\rm CB}^{\langle iLK-1\rangle}(2v_{\rm C}^{p}\dot{\mathcal{M}}_{\rm C}^{pK-1}+v_{\rm C}^{2}\mathcal{M}_{\rm C}^{K-1})\right]$$
$$+k(k-1)v_{\rm C}^{j}v_{\rm C}^{p}\mathcal{M}_{\rm B}^{L}\left(n_{\rm CB}^{\langle jpLK-2\rangle}\mathcal{M}_{\rm C}^{iK-2}-\frac{4}{2k+1}n_{\rm CB}^{\langle ijLK-2\rangle}\mathcal{M}_{\rm C}^{pK-2}\right). \qquad (B3)$$

There are two other terms in the RVF equations of motion which contain combinations, $n_{\rm CB}^{\langle kL\rangle}\mathcal{M}_{\rm B}^{iL}\hat{P}_{\rm C}^{K}$ and $n_{\rm CB}^{\langle iKL\rangle}\mathcal{M}_{\rm B}^{L}\hat{P}_{\rm C}^{K}$, in the second and seventh terms of the right-hand side of Eq. (6.12a) in [84]. These terms are easy to deal with. Straightforward application of (B1) and contraction of multi-indices yield

$$n_{\rm CB}^{\langle kL\rangle}\mathcal{M}_{\rm B}^{iL}\hat{P}_{\rm C}^{K}=\mathcal{M}_{\rm B}^{iL}[n_{\rm CB}^{\langle kL\rangle}\ddot{\mathcal{M}}_{\rm C}^{K}+2kn_{\rm CB}^{\langle jLK-1\rangle}\dot{\mathcal{M}}_{\rm C}^{K-1}v_{\rm C}^{j}+k(k-1)n_{\rm CB}^{\langle jpLK-2\rangle}\mathcal{M}_{\rm C}^{K-2}v_{\rm C}^{j}v_{\rm C}^{p}], \qquad (B4)$$

$$n_{\rm CB}^{\langle iKL\rangle}\mathcal{M}_{\rm B}^{L}\hat{P}_{\rm C}^{K}=\mathcal{M}_{\rm B}^{L}[n_{\rm CB}^{\langle iKL\rangle}\ddot{\mathcal{M}}_{\rm C}^{K}+2kn_{\rm CB}^{\langle ijLK-1\rangle}\dot{\mathcal{M}}_{\rm C}^{K-1}v_{\rm C}^{j}+k(k-1)n_{\rm CB}^{\langle ijpLK-2\rangle}\mathcal{M}_{\rm C}^{K-2}v_{\rm C}^{j}v_{\rm C}^{p}]. \qquad (B5)$$

Substituting (B2)–(B5) to Eqs. (6.13a), (6.13b), (6.13g) of the paper [84], and making use of (293), (294) from the present paper in the inverse order, allow us to write down the RVF equations of motion given in [[84], Eq. (6.11)] with typos fixed in [85], as follows:

$$M_{B}a_{B}^{i}=\mathfrak{F}_{\rm N}^{i}+\mathfrak{F}_{\rm pN}^{i}, \qquad (B6)$$

where $M_{\rm B}$ is the inertial (relativistic mass) of body B, $a_{\rm B}^{i}=d^{2}x_{\rm B}^{i}/dt^{2}$ is acceleration of the center of mass of body B, $\mathfrak{F}_{\rm N}^{i}$ is the Newtonian force, and $\mathfrak{F}_{\rm pN}^{i}$ is the post-Newtonian force. After taking into account our Eqs. (B2)–(B5) the RVF forces can be written down similar to our equations (295) and (301) in the form of the partial derivative operator,

$$\mathfrak{F}_{\rm N}^{i}=\sum_{C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{l!n!}\mathcal{M}_{\rm B}^{L}(\tau_{\rm B})\mathcal{M}_{\rm C}^{N}(\tau_{\rm C})\partial_{iLN}R_{\rm BC}^{-1}, \qquad (B7)$$

$$\mathfrak{F}_{\rm pN}^{i}=\frac{1}{2}\sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{l!n!}\mathcal{M}_{\rm B}^{L}[\ddot{\mathcal{M}}_{\rm C}^{N}\partial_{\langle iLN\rangle}-(2\dot{\mathcal{M}}_{\rm C}^{N}v_{\rm C}^{p}+\mathcal{M}_{\rm C}^{N}a_{\rm C}^{p})\partial_{\langle ipLN\rangle}+\mathcal{M}_{\rm C}^{N}v_{\rm C}^{p}v_{\rm C}^{q}\partial_{\langle ipqLN\rangle}]R_{\rm BC}$$
$$+\sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{l!n!}[(\alpha_{\rm RVF}^{iLN}+\beta_{\rm RVF}^{iLN})\partial_{\langle LN\rangle}+(\alpha_{\rm RVF}^{ipLN}+\beta_{\rm RVF}^{ipLN})\partial_{\langle pLN\rangle}+\alpha_{\rm RVF}^{ipqLN}\partial_{\langle pqLN\rangle}$$
$$+(\alpha_{\rm RVF}^{LN}+\beta_{\rm RVF}^{LN}+\gamma_{\rm RVF}^{LN})\partial_{\langle iLN\rangle}+(\mu_{\rm RVF}^{pLN}+\nu_{\rm RVF}^{pLN}+\rho_{\rm RVF}^{pLN})\partial_{\langle ipLN\rangle}+\sigma_{\rm RVF}^{pqLN}\partial_{\langle ipqLN\rangle}]R_{\rm BC}^{-1}$$
$$+3(a_{\rm B}^{k}\ddot{\mathcal{M}}_{\rm B}^{ik}+2\dot{a}_{\rm B}^{k}\dot{\mathcal{M}}_{\rm B}^{ik}+\ddot{a}_{\rm B}^{k}\mathcal{M}_{\rm B}^{ik}), \qquad (B8)$$

where all partial derivatives are understood in the sense of Eqs. (293), (294). We have explicitly indicated the time arguments of the multipoles in the expression for the Newtonian force (B7) which, according to [[84], Eq. (5.9)], are the proper times of the bodies taken on their worldlines at the points of intersection with hypersurface $\mathcal{H}_{t}$ of constant coordinate time t of the global coordinate chart [cf. (297)–(298)],

$$\tau_{\rm B} = u_{\rm B}|_{\boldsymbol{x}=\boldsymbol{x}_{\rm B}} = t + \frac{1}{c^2}\mathcal{A}_{\rm B}(t) + \mathcal{O}\left(\frac{1}{c^4}\right), \qquad \text{(B9)}$$

$$\tau_{\rm C} = u_{\rm C}|_{\boldsymbol{x}=\boldsymbol{x}_{\rm C}} = t + \frac{1}{c^2}\mathcal{A}_{\rm C}(t) + \mathcal{O}\left(\frac{1}{c^4}\right), \qquad \text{(B10)}$$

where time dilation functions $\mathcal{A}_{\rm B}$ and $\mathcal{A}_{\rm C}$ are defined by solutions of the ordinary differential equations (299) and (300).

The coefficients in the RVF post-Newtonian force (B8) can be directly compared to those in our Eq. (301) where we have to take $\beta = \gamma = 1$ in order to bring it to general-relativistic form. The comparison is tedious but rather straightforward. It results in

$$\alpha^{iLN}_{\rm RVF} = \alpha^{iLN}_{\rm F} - \frac{2}{2l+2n+3} v^i_{\rm C}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C}, \qquad \text{(B11)}$$

$$\alpha^{ipLN}_{\rm RVF} = \alpha^{ipLN}_{\rm F} + \frac{2}{2l+2n+5} v^p_{\rm C}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^{iN}_{\rm C} + \left(\frac{2}{2l+3} - \frac{2}{2l+2n+5}\right) v^p_{\rm C}\mathcal{M}^{iL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C}, \qquad \text{(B12)}$$

$$\alpha^{LN}_{\rm RVF} = \alpha^{LN}_{\rm F} - \frac{2}{2l+2n+5} v^k_{\rm C}\mathcal{M}^{kL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C} - \frac{2l+2n+3}{2l+2n+5} v^k_{\rm C}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^{kN}_{\rm C}, \qquad \text{(B13)}$$

$$\mu^{pLN}_{\rm RVF} = \mu^{pLN}_{\rm F} + \frac{2}{2l+2n+7} v^p_{\rm C}\mathcal{M}^{kL}_{\rm B}\dot{\mathcal{M}}^{kN}_{\rm C}, \qquad \text{(B14)}$$

and all other remaining coefficients in (B8) and (301) are identical for $\beta = \gamma = 1$, except for $\rho^{pLN}_{\rm RVF} = 0$. The reason for vanishing $\rho^{pLN}_{\rm RVF}$ is that the local coordinate system adapted to body B has been chosen by Racine and Flanagan [84] as *kinematically* nonrotating with respect to the spatial axes of the global coordinates while we operate with *dynamically* nonrotating local frame of body B. A kinematically nonrotating local frame is not carried out along the worldline of the body's center of mass in accordance with the Fermi-Walker transportation rule. It means that particles of matter moving with respect to the body must experience the centrifugal and Coriolis forces in this frame. These forces become sufficiently large at the latest stages of evolution of inspiralling compact binaries and affect computation of templates of gravitational waveforms. This effect is, however, purely coordinate dependent and can be removed by choosing a dynamically nonrotating local frame adapted to body B which is our choice.

Now, we notice a useful formula

$$\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C} R^p_{\rm C}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\left[\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C}\partial_{\langle iL\rangle pN}R_{\rm C} + v^p_{\rm C}\dot{\mathcal{M}}^{pN}_{\rm C}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\right], \qquad \text{(B15)}$$

whose expansion in terms of the STF derivatives is as follows:

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}&\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C} R^p_{\rm C}\partial_{iLN}\left(\frac{1}{R_{\rm C}}\right)\\
= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}&\frac{(-1)^n}{l!n!}\left\{\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C}\partial_{\langle ipLN\rangle}R_{\rm C} + \frac{2}{2l+2n+3} v^i_{\rm C}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C}\partial_{\langle LN\rangle}\left(\frac{1}{R_{\rm C}}\right)\right.\\
&+\left(\frac{2}{2l+2n+5} - \frac{2}{2l+3}\right)\mathcal{M}^{iL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C}\partial_{\langle pLN\rangle}\left(\frac{1}{R_{\rm C}}\right) - \frac{2}{2l+2n+5}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^{iN}_{\rm C} v^p_{\rm C}\partial_{\langle pLN\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&+\left(\frac{2}{2l+2n+5} v^p_{\rm C}\mathcal{M}^{pL}_{\rm B}\dot{\mathcal{M}}^N_{\rm C} + v^p_{\rm C}\dot{\mathcal{M}}^{pN}_{\rm C}\right)\partial_{\langle iLN\rangle}\left(\frac{1}{R_{\rm C}}\right) - \frac{2}{2l+2n+5} v^p_{\rm C}\dot{\mathcal{M}}^{pN}_{\rm C}\mathcal{M}^L_{\rm B}\partial_{\langle iLN\rangle}\left(\frac{1}{R_{\rm C}}\right)\\
&\left.-\frac{2}{2l+2n+7}\mathcal{M}^{qL}_{\rm B}\dot{\mathcal{M}}^{qN}_{\rm C} v^p_{\rm C}\partial_{\langle ipLN\rangle}\left(\frac{1}{R_{\rm C}}\right)\right\}. \qquad \text{(B16)}\end{aligned}$$

Derivation of (B16) is based on application of (259) and transformation (273) where replacements, $a^i_{\rm C} \to v^i_{\rm C}$ and $\mathcal{M}^L_{\rm C} \to \dot{\mathcal{M}}^L_{\rm C}$ must be done in all terms. Employing (B16) in (B8) we find out that the RVF post-Newtonian force $\mathfrak{F}^i_{\rm pN}$ relates to our post-Newtonian force (301) in a fairly simple way,

$$\begin{aligned}\mathfrak{F}^i_{\rm pN} = F^i_{\rm pN} &+ 3(a^k_{\rm B}\ddot{\mathcal{M}}^{ik}_{\rm B} + 2\dot{a}^k_{\rm B}\dot{\mathcal{M}}^{ik}_{\rm B} + \ddot{a}^k_{\rm B}\mathcal{M}^{ik}_{\rm B}) - \sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\rho^{pLN}_{\rm F}\partial_{ipLN}\left(\frac{1}{R_{\rm BC}}\right)\\
&-\sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C} R^p_{\rm BC}\partial_{iLN}\left(\frac{1}{R_{\rm BC}}\right). \qquad \text{(B17)}\end{aligned}$$

The first three acceleration-dependent terms in the right-hand side of (B17) following $F^i_{\rm pN}$ are identical to those in our Eq. (285). Hence, these terms are due to the different choice of the center of mass of body B in [84] corresponding to the complementary function, $\mathcal{I}^i_{\rm c} = 0$, in the definition of the center of mass of body B as compared to the choice adopted for this function in Eq. (289) of the present paper. The next term in the right-hand side of (B17) depends on coefficient $\rho^{pLN}_{\rm F}$ given in (312). This coefficient defines the relativistic transport of the multipoles adapted to body B, along the worldline of the body's center of mass. Our convention is that the local frame is carried out in accordance with the Fermi-Walker transportation law while Racine and Flanagan [[84], Sec. 5F] decided to make the local frame nonrotating with respect to the spatial axes of the global coordinates. This difference is a matter of choosing either kinematical or dynamical definition of the rotation of the body-adapted local frame and is easy to reconcile.

The very last term in the right-hand side of (B17) is due to the different time arguments of the multipoles $\mathcal{M}_C$ taken at slightly different points on the worldline of body C. Indeed, by comparing (297) with (B9) and (298) with (B10), we conclude that the time arguments of the multipoles $\mathcal{M}_{\rm B}$ of body B are identical, $\tau_{\rm B} = u^*_{\rm B}$, while the time arguments of multipoles of body C are shifted one with respect to another, $\tau_{\rm C} = u^*_{\rm C} + v^k_{\rm C}(t)R^k_{\rm BC}$. Looking back to Fig. 1 we can say that the multipoles $\mathcal{M}_{\rm B}$ of body B are taken at point P while the multipoles $\mathcal{M}_C$ of body C are taken at point R in our approach and at the point Q in the paper by Racine and Flanagan [84]. This observation allows us to connect the RVF Newtonian force (B7) with our Newtonian force (295) by taking the Taylor expansion of the multipoles $\mathcal{M}_{\rm C}$. It yields

$$\mathfrak{F}^i_{\rm N} = F^i_{\rm N} + \sum_{\rm C\neq B}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!n!}\mathcal{M}^L_{\rm B}\dot{\mathcal{M}}^N_{\rm C} v^p_{\rm C} R^p_{\rm BC}\partial_{iLN}\left(\frac{1}{R_{\rm BC}}\right). \tag{B18}$$

The last term in the right-hand side of (B18) exactly cancels the very last term in (B17) after substituting (B17) and (B18) to the total force in the right-hand side of (B6). This makes it clear that our translational equations of motion are essentially the same as those derived by Racine and Flanagan [84] and Racine *et al.* [85] except of several terms which are a matter of slightly different conventions adopted to define the center of mass of the bodies and rotation of the spatial axes of the body-adapted local frame. It is remarkable that the agreement is achieved in spite of using a different mathematical technique based on the Fock-Papapetrou-Chandrasekhar approach [126,134,209,249,301] to the derivation of equations of motion of extended bodies in an ℕ-body system made of matter with continuous stress-energy tensor. Finally, we bring to the attention of the reader the fact that our equations of translational motion are more economic than that given in [84,85] in the sense that the post-Newtonian force $F^i_{\rm pN}$ in our approach has been reduced to the form (314) containing lesser number of terms than the corresponding force $\mathfrak{F}^i_{\rm pN}$ in [84,85]. It might be more effective to implement our form of the equations of motion with quadrupole and higher-order multipoles to the numerical integration of the orbital evolution of tidally deformed neutron star binaries and prediction of gravitational wave signals from the mergers; see, for example, [33–35].

APPENDIX C: THE DIXON MULTIPOLE MOMENTS

Dixon [11] has defined internal multipoles of an extended body B in the normal Riemann coordinates, X^α, by means of a tensor integral (458)

$$I^{\alpha_1\ldots\alpha_l\mu\nu}(z) = \int X^{\alpha_1}\ldots X^{\alpha_l}\hat{T}^{\mu\nu}(z,X)\sqrt{-\bar{g}(z)}DX \quad (l\geq 2) \tag{C1}$$

where $\hat{T}^{\mu\nu}$ is the stress-energy *skeleton* of the body, the integration is performed over the tangent 4-dimensional space to background manifold $\bar{M}$ at point z taken on a reference worldline $\mathcal{Z}$, and the volume element of integration $DX = dX^0\wedge dX^1\wedge dX^2\wedge dX^3$. The reason for the appearance of the skeleton $\hat{T}^{\mu\nu}$ in (C1) instead of the regular stress-energy tensor $T^{\mu\nu}$ was to incorporate the self-field effects of a gravitational field of the body to the definition of the higher-order multipoles.[11] According to [11], the skeleton $\hat{T}^{\mu\nu}(z,x)$ is a distribution [212] defined on the worldline $\mathcal{Z}$ in such a way that it contains complete information about the body but is entirely independent of the geometry of the surrounding spacetime to which the body is embedded. The skeleton is lying on the hyperplane made out of vectors X^α which are orthogonal to the vector of dynamic velocity $\mathfrak{n}^\alpha$. It gives the following constraint:

$$(\mathfrak{n}_\alpha X^\alpha)X^{[\lambda}\hat{T}^{\mu][\nu}X^{\sigma]} = 0, \tag{C2}$$

which points out that the skeleton distribution is concentrated on the hyperplane $\mathfrak{n}_\alpha X^\alpha = 0$.

Definition (C1) suggests that the Dixon multipole moments have the following symmetries:

$$I^{\alpha_1\ldots\alpha_l\mu\nu} = I^{(\alpha_1\ldots\alpha_l)(\mu\nu)}, \tag{C3}$$

where the round parentheses around the tensor indices denote a full symmetrization. In addition to (C3) there are more symmetries of the Dixon multipoles due to

[11]The influence of the self-field effects on multipoles was studied by Thorne [82], Blanchet and Damour [78], and Damour and Iyer [79] with different techniques.

the one-to-one mapping of the microscopic equation of motion (449) to a similar equation for the stress-energy skeleton [11]

$$\bar{\nabla}_\nu \hat{T}^{\mu\nu}(z, X) = 0. \tag{C4}$$

Multiplying (C4) with $X^{\alpha_1}\ldots X^{\alpha_l}X^{\alpha_{l+1}}$, integrating over 4-dimensional volume and taking into account that $\hat{T}^{\mu\nu}$ vanishes outside hyperplane $\mathfrak{n}_\alpha X^\alpha = 0$, yields [[11], Eq. (143)],

$$I^{(\alpha_1\ldots\alpha_l\mu)\nu} = 0, \tag{C5}$$

and a similar relation holds after exchanging indices μ and ν due to symmetry (C3). The number of algebraically independent components of $I^{\alpha_1\ldots\alpha_l\mu\nu}$ obeying (C3) is $N_1(l) = C_3^{l+3} \times C_3^5$ where $C_q^p = \frac{p!}{q!(p-q)!}$ is a binomial coefficient. Constraints (C5) reduce the number of the algebraically independent components of the multipoles $I^{\alpha_1\ldots\alpha_l\mu\nu}$ by $N_2(l) = C_3^{l+4} \times C_3^4$ making the number of linearly independent components of $I^{\alpha_1\ldots\alpha_l\mu\nu}$ equal to $N_3(l) = N_1(l) - N_2(l) = (l+3)(l+2)(l-1)$.

The multipoles $I^{\alpha_1\ldots\alpha_l\mu\nu}$ are coupled to the Riemann tensor $\bar{R}^\alpha{}_{\mu\beta\nu}$ characterizing the curvature of the effective background spacetime. Therefore, they can be replaced with a more suitable set of *reduced* moments $J^{\alpha_1\ldots\alpha_l\lambda\mu\nu\rho}$ which are defined by the following formulas [9,11]:

$$J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu} \equiv I^{\alpha_1\ldots\alpha_p[\lambda[\sigma\mu]\nu]}, \tag{C6}$$

where the square parentheses around the tensor indices denote a full antisymmetrization, and the nested square brackets in (C6) denote the antisimmetrization on pairs of indices $[\lambda, \mu]$ and $[\nu, \rho]$ independently. Definition (C6) tells us that tensor $J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu}$ is fully symmetric with respect to the first p indices and is skew symmetric with respect to the pairs of indices λ, μ and σ, ν,

$$J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu} = J^{(\alpha_1\ldots\alpha_p)[\lambda\mu][\sigma\nu]}. \tag{C7}$$

Among other properties of $J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu}$ we have

$$J^{\alpha_1\ldots\alpha_p\lambda[\mu\sigma\nu]} = 0, \qquad J^{\alpha_1\ldots[\alpha_p\lambda\mu]\sigma\nu} = 0, \tag{C8}$$

which are consequences of the definition (C6), and

$$\mathfrak{n}_{\alpha_1} J^{\alpha_1\ldots\alpha_p\lambda\mu\sigma\nu} = 0, \tag{C9}$$

that is the condition of orthogonality following from the constraint (C2).

Equation (C6) can be transformed to another form. For this we write down the antisymmetric part of (C6) explicitly as a combination of four terms, change notations of indices $\{\alpha_1\ldots\alpha_p\mu\nu\} \to \{\alpha_1\ldots\alpha_{l-2}\alpha_{l-1}\alpha_l\}$, and make a full symmetrization with respect to the set of indices $\{\alpha_1\ldots\alpha_l\}$. It gives

$$J^{(\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l)\nu} = \frac{1}{4}\left[I^{(\alpha_1\ldots\alpha_{l-1}\alpha_l)\mu\nu} - I^{(\alpha_1\ldots\alpha_{l-2}|\mu|\alpha_{l-1}\alpha_l)\nu} - I^{(\alpha_1\ldots\alpha_{l-2}\alpha_{l-1}|\nu\mu|\alpha_l)} + I^{(\alpha_1\ldots\alpha_{l-2}|\mu\nu|\alpha_{l-1}\alpha_l)}\right], \tag{C10}$$

where the indices enclosed in vertical bars are excluded from symmetrization. Remembering that each of the I moments is separately symmetric with respect to the first l and the last two indices we can recast (C10) to the following form:

$$J^{(\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l)\nu} = \frac{1}{4}\left[I^{(\alpha_1\ldots\alpha_{l-1}\alpha_l)\mu\nu} - I^{(\mu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\nu} - I^{(\nu(\alpha_1\ldots\alpha_{l-1}\alpha_l)\mu} + I^{(\mu\nu(\alpha_1\ldots\alpha_{l-2})\alpha_{l-1}\alpha_l)}\right]. \tag{C11}$$

We now use the constraint (C5) and notice that

$$I^{(\alpha_1\ldots\alpha_{l-1}\alpha_l\mu)\nu} = \frac{1}{l+1}\left[I^{\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu} + lI^{(\mu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\nu}\right] = 0, \tag{C12}$$

which gives

$$I^{(\mu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\nu} = -\frac{1}{l} I^{\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu}, \tag{C13}$$

and, because of the symmetry with respect to indices μ and ν,

$$I^{(\nu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\mu} = -\frac{1}{l} I^{\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu}. \tag{C14}$$

We also have

$$I^{(\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu)} = \frac{2!l!}{(l+2)!}\left[I^{\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu} + lI^{(\mu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\nu} + lI^{(\nu(\alpha_1\ldots\alpha_{l-1})\alpha_l)\mu} + \frac{l(l-1)}{2} I^{(\mu\nu(\alpha_1\ldots\alpha_{l-2})\alpha_{l-1}\alpha_l)}\right] = 0, \tag{C15}$$

which yields

$$I^{(\mu\nu(\alpha_1\ldots\alpha_{l-2})\alpha_{l-1}\alpha_l)} = \frac{2}{l(l-1)} I^{\alpha_1\ldots\alpha_{l-1}\alpha_l\mu\nu}. \tag{C16}$$

Replacing (C13), (C14), and (C16) to (C10) yields

$$J^{(\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l)\nu} = \frac{1}{4}\frac{l+1}{l-1} I^{\alpha_1\ldots\alpha_l\mu\nu}, \tag{C17}$$

that shows the algebraic equivalence between the symmetrized $J^{(\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l)\nu}$ and $I^{\alpha_1\ldots\alpha_l\mu\nu}$ multipole moments for

$l \geq 2$. Due to the orthogonality condition (C9) we conclude that

$$\mathfrak{n}_{\alpha_1} I^{\alpha_1 \ldots \alpha_l \mu\nu} = 0 \tag{C18}$$

for the first l indices of $I^{\alpha_1 \ldots \alpha_l \mu\nu}$. The number of these conditions is the same as the number of components of tensor $I^{\alpha_1 \ldots \alpha_{l-1} \mu\nu}$ that is $N_3(l-1) = (l+2)(l+1)(l-2)$. It reduces the number of linearly independent components of $I^{\alpha_1 \ldots \alpha_l \mu\nu}$ to $N = N_3(l) - N_3(l-1) = (l+2)(3l-1)$ [11,136].

APPENDIX D: COMPARISON WITH MATHISSON-PAPAPETROU-DIXON EQUATIONS OF MOTION

1. Comparison of Dixon's and Blanchet-Damour multipole moments

Before comparing our covariant equations of motion (532) and (545) with analogous equations (467) and (468) derived by Dixon [11] in the MPD formalism, we need to establish the correspondence between the Dixon multipole moments $I^{\alpha_1 \ldots \alpha_l \mu\nu}$ and the STF mass and spin multipoles $\mathcal{M}^{\alpha_1 \ldots \alpha_l}$ and $\mathcal{S}^{\alpha_1 \ldots \alpha_l}$ that are used in the present paper. To this end we notice that the original definition (C1) of multipoles $I^{\alpha_1 \ldots \alpha_l \mu\nu}$ contains the time components, X^0, of vector X^α which are nonphysical as they cannot be measured by a local observer with dynamic velocity $\mathfrak{n}^\alpha$ at point z on the reference worldline $\mathcal{Z}$. Only those components of $I^{\alpha_1 \ldots \alpha_l \mu\nu}$ which are orthogonal to $\mathfrak{n}^\alpha$ can be measured. This explains the physical meaning of the orthogonality condition (C18).

It is reasonable to introduce a new notation for the physically meaningful components of Dixon's multipoles,

$$\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu} = P^{\alpha_1}_{\beta_1} \ldots P^{\alpha_l}_{\beta_l} \int_\Sigma X^{\beta_1} \ldots X^{\beta_l} \hat{T}^{\mu\nu}(z, X) \sqrt{-\bar{g}(z)} d\Sigma \quad (l \geq 2), \tag{D1}$$

where the integration is performed in 4-dimensional spacetime over the hypersurface Σ passing through the point z with the element of integration $d\Sigma = \mathfrak{n}^\alpha d\Sigma_\alpha$, and

$$P^\alpha_\beta = \delta^\alpha_\beta + \mathfrak{n}^\alpha \mathfrak{n}_\beta \tag{D2}$$

is the operator of projection on the hypersurface Σ making all vectors X^α in (D1) orthogonal to $\mathfrak{n}^\alpha$. The multipoles $\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu}$ have the same symmetries (C3), (C5) as $I^{\alpha_1 \ldots \alpha_l \mu\nu}$,

$$\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu} = \mathcal{J}^{(\alpha_1 \ldots \alpha_l)(\mu\nu)}, \tag{D3}$$

$$\mathcal{J}^{(\alpha_1 \ldots \alpha_l \mu)\nu} = 0, \tag{D4}$$

while the orthogonality condition (C18) is identically satisfied and is no longer considered as an additional constraint. The projection operator is idempotent [302] that is

$$P^\alpha_\gamma P^\gamma_\beta = P^\alpha_\beta, \tag{D5}$$

which makes only 3 out of 4 components of X^α linearly independent in (D1). On the other hand, the indices μ and ν in $\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu}$ still take values from the set $\{0, 1, 2, 3\}$. Thus, Eq. (D3) tells us that the number of components of $\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu}$ is $C_2^{l+2} \times C_3^5 = 5(l+2)(l+1)$ while the number of constraints (D4) is $C_2^{l+3} \times C_3^4 = 2(l+3)(l+2)$. It gives the number of the algebraically independent components of $\mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu}$ equal to $N = (l+2)(3l-1)$ which exactly coincides with the number of algebraically independent components of Dixon's multipoles $I^{\alpha_1 \ldots \alpha_l \mu\nu}$.

Picking up the local Riemann coordinates in such a way that the X^0 component of vector X^α is directed along the dynamic velocity $\mathfrak{n}^\alpha$ and three other components $X^i = \{X^1, X^2, X^3\}$ are lying in the hypersurface Σ yields the skeleton's structure,

$$\hat{T}^{\mu\nu}(z, X) = \int_{-\infty}^{+\infty} \delta(X^0) \hat{T}^{\mu\nu}_\perp(X^i) dX^0, \tag{D6}$$

where $\delta(X^0)$ is Dirac's delta function and the distribution $\hat{T}^{\mu\nu}_\perp \in \Sigma$. Substituting (D6) to (D1) and taking into account that in these coordinates $DX = dX^0 d\Sigma$, we obtain that Dixon's multipoles $I^{\alpha_1 \ldots \alpha_l \mu\nu} = \mathcal{J}^{\alpha_1 \ldots \alpha_l \mu\nu}$ and, due to the tensor nature of the multipoles, this equality is retained in arbitrary coordinates.

The exact nature of the distribution $\hat{T}^{\mu\nu}_\perp(X^i)$ in full general relativity is not yet known due to the nonlinearity of the Einstein equations. Nonetheless, the Dirac delta function is a reasonable candidate being sufficient to work in the post-Newtonian approximation with corresponding regularization techniques [51]. For the purpose of the present paper it is sufficient to assume that in arbitrary coordinates the stress-energy skeleton (D6) has the following structure [12,13,247]:

$$\hat{T}^{\mu\nu}(z, x) = \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \bar{\nabla}_{\alpha_1 \ldots \alpha_l} \left[\mathfrak{t}^{\alpha_1 \ldots \alpha_l \mu\nu}(z) \frac{\delta_4(x - z)}{\sqrt{-\bar{g}(z)}} \right] \times \frac{ds}{\sqrt{-\bar{g}_{\mu\nu}(z) \mathfrak{n}^\mu \mathfrak{n}^\nu}}, \tag{D7}$$

where s is an affine parameter along the geodesic in direction of the dynamic velocity $\mathfrak{n}^\alpha$, $\delta_4(x - z) \equiv \delta_4[x^\alpha - z^\alpha(s)]$ is 4-dimensional Dirac's delta function, $\mathfrak{t}^{\alpha_1 \ldots \alpha_l \mu\nu}$ are generalized multipole moments defined on the worldline $\mathcal{Z}$ that are orthogonal to $\mathfrak{n}^\alpha$ in the first l indices ($\mathfrak{n}_{\alpha_1} \mathfrak{t}^{\alpha_1 \ldots \alpha_l \mu\nu} = 0$), and $\bar{\nabla}_{\alpha_1 \ldots \alpha_l} \equiv \bar{\nabla}_{\alpha_1} \ldots \bar{\nabla}_{\alpha_l}$ is a covariant derivative of the order l taken with respect to the argument $x \equiv x^\alpha$ of the Dirac delta function on the background manifold. Notice that expression (D7) is a simplification of the original Mathisson theory [4] proposed by Tulczyjew [207]. Dixon [11] did not specify the nature of the singularity entering definition (D7) assuming that Dirac's delta function is solely valid in the pole-dipole approximation while a more general type of distribution is required in the definition of the stress-energy skeleton for higher-order multipoles. The Dirac delta function is widely adopted in computations of equations of motion of relativistic binary systems [29,31,65] amended with corresponding regularization techniques to deal with the singularities in the nonlinear approximations of general relativity [52,53,154,300].

The generalized multipoles $\mathfrak{t}^{\alpha_1\ldots\alpha_l\mu\nu}$ are used to derive the explicit form of the MPD equations of motion in terms of the linear momentum $\mathfrak{p}^\alpha$, angular momentum $S^{\alpha\beta}$, and Dixon's multipole moments $I^{\alpha_1\ldots\alpha_l\mu\nu}$ as demonstrated by Mathisson [4,5], Papapetrou [134,209], Dixon [11], and other researchers [12,13,145,210,258]. It turns out that the generalized multipoles $\mathfrak{t}^{\alpha_1\ldots\alpha_l\mu\nu}$ are effectively equivalent to the body multipoles, $\mathcal{J}^{\alpha_1\ldots\alpha_l\mu\nu}$. Indeed, replacing the stress-energy skeleton (D7) to (C1), transforming the most general coordinates x^α in (D7) to the local Riemannian coordinates X^α, and taking the covariant derivatives yield

$$\mathcal{J}^{\alpha_1\ldots\alpha_l\mu\nu} = P^{\alpha_1}_{\beta_1}\ldots P^{\alpha_l}_{\beta_l}\sum_{n=0}^{\infty}\mathfrak{t}^{\gamma_1\ldots\gamma_p\mu\nu} \times\int X^{\beta_1}\ldots X^{\beta_l}\frac{\partial^n\delta_4(X)}{\partial X^{\gamma_1}\ldots\partial X^{\gamma_n}}DX. \tag{D8}$$

Integrating by parts, taking the partial derivatives from X^α, and accounting for the integral properties of the delta function [212], we conclude

$$\mathcal{J}^{\alpha_1\ldots\alpha_l\mu\nu} = (-1)^l l!\mathfrak{t}^{\alpha_1\ldots\alpha_l\mu\nu}. \tag{D9}$$

To proceed further on, we shall assume that the dynamic velocity $\mathfrak{n}^\alpha$ is equal to the kinematic velocity $\bar{u}^\alpha$. This assumption is consistent with Dixon's mathematical development and agrees with our covariant definition (493) of the linear momentum of an extended body moving on the background spacetime manifold. It also allows us to employ the results obtained previously by Ohashi [12], to retrieve a covariant expression for the generalized multipoles $\mathfrak{t}^{\alpha_1\ldots\alpha_l\mu\nu}$ of the gravitational skeleton $\hat{T}^{\mu\nu}$ from the multipolar expansion of the metric tensor of a single body. We have derived the generalized multipoles of the stress-energy skeleton from [[12], Eq. (3.1)] after reconciling the sign conventions of the metric tensor perturbation and the normalization coefficients of multipoles adopted in [12] with those adopted by Blanchet and Damour [[50], Eq. (2.32)] which we also use in the present paper. The generalized moments of the stress-energy skeleton read

$$\mathfrak{t}^{\alpha_1\ldots\alpha_l\mu\nu} = \frac{(-1)^l}{l!}\left[\bar{u}^\mu\bar{u}^\nu\mathcal{M}^{\alpha_1\ldots\alpha_l} + \frac{2}{l+1}\bar{u}^{(\mu}\dot{\mathcal{M}}^{\nu)\alpha_1\ldots\alpha_l} + \frac{1}{(l+1)(l+2)}\ddot{\mathcal{M}}^{\mu\nu\alpha_1\ldots\alpha_l}\right] - \frac{(-1)^l}{l!}\left[\frac{2l}{l+1}\bar{u}^{(\mu}\varepsilon_\beta{}^{\nu)\langle\alpha_1}\mathcal{S}^{\alpha_2\ldots\alpha_l\rangle\beta} + \frac{2}{l+2}\varepsilon_\beta{}^{\langle\alpha_1(\mu}\dot{\mathcal{S}}^{\nu)\alpha_2\ldots\alpha_l\rangle\beta}\right], \tag{D10}$$

where the dot above functions denotes the Fermi-Walker covariant derivative (542) and (543). Comparing (D10) with (D9) we obtain the relationship between the Dixon internal multipoles and the mass and spin multipoles used in the present paper,

$$\mathcal{J}^{\alpha_1\ldots\alpha_l\mu\nu} = \bar{u}^\mu\bar{u}^\nu\mathcal{M}^{\alpha_1\ldots\alpha_l} + \frac{2}{l+1}\bar{u}^{(\mu}\dot{\mathcal{M}}^{\nu)\alpha_1\ldots\alpha_l} + \frac{1}{(l+1)(l+2)}\ddot{\mathcal{M}}^{\mu\nu\alpha_1\ldots\alpha_l} - \frac{2l}{l+1}\bar{u}^{(\mu}\varepsilon_\beta{}^{\nu)\langle\alpha_1}\mathcal{S}^{\alpha_2\ldots\alpha_l\rangle\beta} - \frac{2}{l+2}\varepsilon_\beta{}^{\langle\alpha_1(\mu}\dot{\mathcal{S}}^{\nu)\alpha_2\ldots\alpha_l\rangle\beta}. \tag{D11}$$

We still have to take into account the identity (D4) in order to eliminate linearly dependent components of $\mathcal{J}^{\alpha_1\ldots\alpha_l\mu\nu}$. The most easy way is to take the double skew-symmetric part with respect to the last four indices as shown in Eq. (C6). It yields

$$I^{\alpha_1\ldots\alpha_l\mu\nu} \equiv \mathcal{J}^{\alpha_1\ldots[a_{l-1}[\alpha_l\mu]\nu]} = 4\left\{\mathcal{M}^{\langle\alpha_1\ldots[\alpha_{l-1}[\alpha_l\rangle}u^{\mu]}u^{\nu]} + \frac{l}{l+1}\mathcal{S}^{\beta\langle\alpha_1\ldots[\alpha_{l-1}}u^{(\mu]}\varepsilon^{\alpha_l\rangle\nu)}{}_\beta\right\}, \tag{D12}$$

where we have taken into account that in calculating the skew-symmetric part of 4-velocity u^μ with a purely spatial tensor we have, for example,

$$\mathcal{M}^{\alpha_1\ldots\alpha_{l-1}[\alpha_l}\bar{u}^{\mu]} = \pi^{\alpha_l}_{\beta_l}\mathcal{M}^{\alpha_1\ldots\alpha_{l-1}[\beta_l}\bar{u}^{\mu]} = \frac{1}{2}\mathcal{M}^{\alpha_1\ldots\alpha_{l-1}\alpha_l}\bar{u}^\mu, \tag{D13}$$

and so on. Relation between Dixon's J and I multipole moments has been defined in (C17). Substituting expression (D12) for the Dixon multipoles I in the right-hand side of (C17) provides a correspondence between the symmetrized Dixon multipoles J and the Blanchet-Damour mass and spin multipoles in the following form:

$$J^{(\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l)\nu} = \frac{l+1}{l-1}\left[\mathcal{M}^{\langle\alpha_1\ldots[\alpha_{l-1}[\alpha_l\rangle}\bar{u}^{\mu]}\bar{u}^{\nu]} + \frac{l}{l+1}\mathcal{S}^{\beta\langle\alpha_1\ldots[\alpha_{l-1}}\bar{u}^{(\mu]}\varepsilon^{\alpha_l\rangle\nu)}{}_{\beta}\right], \quad \text{(D14)}$$

where the antisymmetrization goes over the pair of indices $[\alpha_{l-1}\mu]$ and $[\alpha_l\nu]$. Contracting both sides of (D14) with 4-velocity allows us to express the Blanchet-Damour mass and spin multipoles in terms of projections of the Dixon multipoles onto 4-velocity of the center of mass of the body. More specifically, we have

$$\mathcal{M}^{\alpha_1\ldots\alpha_l} = 4\frac{l-1}{l+1}J^{\langle\alpha_1\ldots\alpha_{l-1}|\mu|\alpha_l\rangle\nu}\bar{u}_\mu\bar{u}_\nu \quad (l\geq 2) \quad \text{(D15)}$$

$$\mathcal{S}^{\alpha_1\ldots\alpha_l} = 2\frac{l-1}{l}J^{\langle\alpha_1\ldots\alpha_{l-1}|\mu\nu\sigma|}\varepsilon^{\alpha_l\rangle}{}_{\mu\nu}\bar{u}_\sigma \quad (l\geq 2). \quad \text{(D16)}$$

It is worth emphasizing that in this section we work in the framework of general relativity. Therefore, all internal mass and spin multipoles, $\mathcal{M}^{\alpha_1\ldots\alpha_l}$ and $\mathcal{S}^{\alpha_1\ldots\alpha_l}$, have only general-relativistic value with vanishing scalar field contribution. In particular, the mass dipole, $\mathcal{M}^i = 0$, due to the choice of the origin of the local coordinates at the center of mass of the body.

2. Comparison of translational equations of motion

In order to compare our translational equations of motion (532) with Dixon's equation (467) we need to symmetrize the covariant derivatives in the right-hand side of (467). It is achieved with the help of the following algebraic transformation:

$$\begin{aligned}&\bar{\nabla}_{\alpha(\beta_1\ldots\beta_{l-2}}R_{|\mu|\beta_{l-1}\beta_l)\nu}J^{\beta_1\ldots\beta_{l-1}\mu\beta_l\nu}\\ &=\bar{\nabla}_{(\alpha\beta_1\ldots\beta_{l-2}}R_{|\mu|\beta_{l-1}\beta_l)\nu}J^{\beta_1\ldots\beta_{l-1}\mu\beta_l\nu}\\ &+\frac{2}{l+1}\bar{\nabla}_{\nu(\beta_1\ldots\beta_{l-2}}R_{|\mu|\beta_{l-1}\beta_l)\alpha}J^{\beta_1\ldots\beta_{l-1}\mu\beta_l\nu}+\mathcal{O}(R^2), \quad \text{(D17)}\end{aligned}$$

where the residual terms are proportional to the square of the Riemann tensor, and have been discarded. These quadratic-in-curvature terms are important for the post-Newtonian equations of motion but complicate the equations which follow and, hence, will be omitted every time when they appear. Substituting (D14) to the right-hand side of (D17) yields

$$\begin{aligned}&\bar{\nabla}_{\alpha(\beta_1\ldots\beta_{l-2}}R_{|\mu|\beta_{l-1}\beta_l)\nu}J^{\beta_1\ldots\beta_{l-1}\mu\beta_l\nu}\\ &=\frac{l+1}{l-1}\left[\mathcal{E}_{\alpha\beta_1\ldots\beta_l}\mathcal{M}^{\beta_1\ldots\beta_l}+\frac{l}{l+1}\mathcal{C}_{\alpha\beta_1\ldots\beta_l}\mathcal{S}^{\beta_1\ldots\beta_l}\right]\\ &+\mathcal{O}(R^2), \quad \text{(D18)}\end{aligned}$$

where the external multipole moments $\mathcal{E}_{\alpha_1\ldots\alpha_l}$ and $\mathcal{C}_{\alpha_1\ldots\alpha_l}$ have been defined in (528) and (530) respectively. Substituting (D18) to the right-hand side of (467) recasts it to

$$\begin{aligned}\frac{\mathcal{D}\mathfrak{p}_\alpha}{\mathcal{D}\tau} &= \frac{1}{2}\bar{u}^\beta S^{\mu\nu}\bar{R}_{\mu\nu\beta\alpha}\\ &+\sum_{l=2}^{\infty}\frac{1}{l!}\left[\mathcal{E}_{\alpha\beta_1\ldots\beta_l}\mathcal{M}^{\beta_1\ldots\beta_l}+\frac{l}{l+1}\mathcal{C}_{\alpha\beta_1\ldots\beta_l}\mathcal{S}^{\beta_1\ldots\beta_l}\right]\\ &+\mathcal{O}(R^2). \quad \text{(D19)}\end{aligned}$$

The very first term in the right-hand side depending on $S^{\alpha\beta}$ can be incorporated to the sum over the spin moments by making use of the duality relation between the body's intrinsic spin $\mathcal{S}^\alpha$ and spin-tensor[12] $S^{\alpha\beta}$

$$S^{\mu\nu} = -\varepsilon^{\mu\nu}{}_\alpha\mathcal{S}^\alpha, \quad \text{(D20)}$$

where the Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma}$ has been defined above in (531). It yields

$$\bar{u}^\beta S^{\mu\nu}\bar{R}_{\mu\nu\beta\alpha} = \mathcal{C}_{\alpha\beta}\mathcal{S}^\beta, \quad \text{(D21)}$$

where $\mathcal{C}_{\alpha\beta}$ is given by (530) for $l=2$. Making use of (D20) allows us to rewrite (D19) in the final form

$$\begin{aligned}\frac{\mathcal{D}\mathfrak{p}_\alpha}{\mathcal{D}\tau} &= \sum_{l=2}^{\infty}\frac{1}{l!}\mathcal{E}_{\alpha\beta_1\ldots\beta_l}\mathcal{M}^{\beta_1\ldots\beta_l}\\ &+\sum_{l=1}^{\infty}\frac{l}{(l+1)!}\mathcal{C}_{\alpha\beta_1\ldots\beta_l}\mathcal{S}^{\beta_1\ldots\beta_l}+\mathcal{O}(R^2). \quad \text{(D22)}\end{aligned}$$

Thus, Dixon's equation of translational motion (467) given in terms of Dixon's internal multipoles and Veblen's tensor extensions of the Riemann tensor are brought to the form (D22) given in terms of the gravitoelectric, $\mathcal{E}_{\alpha\beta_1\ldots\beta_l}$, and gravitomagnetic, $\mathcal{C}_{\alpha\beta_1\ldots\beta_l}$, external multipoles as well as mass, $\mathcal{M}^{\beta_1\ldots\beta_l}$ and spin, $\mathcal{S}^{\beta_1\ldots\beta_l}$ internal multipoles. Comparing with the complete covariant form of the translational equations of motion (532)–(541) taken for the case of general relativity one can see that Dixon's equation reproduces only two terms in the complete expression for the post-Newtonian force, more specifically, the very first term of the post-Newtonian force $F^\alpha_{\mathcal{Q}}$ in (539) and that of $F^\alpha_{\mathcal{C}}$ in (540). The terms which are missed in the Dixon's translational equations of motion but are present in our Eqs. (532)–(541) include the quadratic-in-curvature terms through (529) and the terms which depend on the time derivatives of multipoles, both external and internal ones. The terms with the time derivatives of the multipoles must be present in the equations of motion but they have been

[12]The minus sign in (D20) appears because Dixon's definition (451) of $S^{\alpha\beta}$ has an opposite sign as compared to our definition (182) of spin $\mathcal{S}^\alpha$.

omitted by Dixon as he has taken into account only his J multipoles while, in fact, all components of the Dixon's I multipoles must be taken into account. Independent derivation of the translational equations of motion by Racine and Flanagan [84] and Racine *et al.* [85] with different mathematical technique corroborates our conclusions about the missing terms in Dixon's translational equations of motion (467). It does not mean that the MPD formalism is erroneous. It merely indicates that much more work is required to take into account all the missing contributions to the post-Newtonian translational equations of motion derived in the framework of the Mathisson variational dynamics.

3. Comparison of rotational equations of motion

Dixon's equations of rotational motion are given by Eq. (468). The first term in the right-hand side of this equation vanishes in our approach because the linear momentum of the body $\mathfrak{p}^\alpha$ is chosen to be parallel to 4-velocity $\bar{u}^\alpha$ of the center of mass of body B. We express the spin of the body $\mathcal{S}^\alpha$ in terms of the spin tensor $S^{\lambda\sigma}$ by inverting (D20),

$$\mathcal{S}^\alpha = -\frac{1}{2}\varepsilon^\alpha{}_{\lambda\sigma}S^{\lambda\sigma}. \tag{D23}$$

Taking a covariant derivative from both sides of (D23) and replacing the covariant derivative from $S^{\beta\gamma}$ with the terms from the right side of (468) yields

$$\frac{\mathcal{D}\mathcal{S}^\alpha}{\mathcal{D}\tau} = -\varepsilon^\alpha{}_{\lambda\sigma}\sum_{l=1}^{\infty}\frac{1}{l!}\nabla_{(\beta_1\ldots\beta_{l-1}}\bar{R}_{|\mu|\rho\beta_l)\nu}g^{\rho\lambda}\Big[\mathcal{M}^{\sigma\beta_1\ldots\beta_{l-1}\beta_l}\bar{u}^\mu\bar{u}^\nu + \frac{l+1}{l+2}\mathcal{S}^{\sigma\gamma\beta_1\ldots\beta_{l-1}}\bar{u}^\mu\varepsilon^{\beta_l\nu}{}_\gamma\Big], \tag{D24}$$

where we have also used (D14) to replace the Dixon internal multipole moments with the Blanchet-Damour mass and spin multipoles. Now, we employ the covariant definitions (528) and (530) of the gravitoelectric and gravitomagnetic external multipoles in (D24) that takes on the following form:

$$\frac{\mathcal{D}\mathcal{S}^\alpha}{\mathcal{D}\tau} = -\varepsilon^{\alpha\lambda}{}_\sigma\sum_{l=1}^{\infty}\frac{1}{l!}\Big[\mathcal{E}_{\lambda\beta_1\ldots\beta_l}\mathcal{M}^{\sigma\beta_1\ldots\beta_l} + \frac{l+1}{l+2}\mathcal{C}_{\lambda\beta_1\ldots\beta_l}\mathcal{S}^{\sigma\beta_1\ldots\beta_l}\Big]. \tag{D25}$$

Now, we can compare Dixon's equation of rotational motion (D25) with our Eq. (544) where only general-relativistic terms in the torque (546) must be retained. These terms are making up the third and fourth lines in (546) and they are in a perfect agreement with Dixon's torque in the right-hand side of (D25). The difference between (D25) and (545) is in the presence of the very last term in the right-hand side of (545) as compared with (D25). This term is associated with the Fermi-Walker transport of spin along an accelerated worldline of the body center of mass. The absence of this term in Dixon's rotational equation of motion (D25) tells us that the reference worldline $\mathcal{W}$ of the origin of the normal Riemann coordinates used by Dixon [11,136] for computation of his own results is a timelike geodesic which, in the most general case, does not coincide with the worldline $\mathcal{Z}$ of the body center of mass because of the gravitational interaction of the internal moments of the body with the external gravitoelectric and gravitomagnetic multipoles.

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