Polarized backgrounds of relic gravitons

Massimo Giovannini[*]

Department of Physics, CERN, 1211 Geneva 23, Switzerland
and INFN, Section of Milan-Bicocca, 20126 Milan, Italy

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The polarizations of the tensor modes of the geometry evolving in cosmological backgrounds are treated as the components of a bispinor whose dynamics follows from an appropriate gauge-invariant action. This novel framework is closely analog to the (optical) Jones calculus and leads to a compact classification of the various interactions able to polarize the relic gravitons.

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I. POLARIZED RELIC GRAVITONS

The stochastic backgrounds of gravitational radiation may be formed by relic gravitons parametrically amplified in the early Universe, as suggested long ago by Grishchuk [1]. In a general relativistic context, the action of the relic gravitons has been derived, for the first time, by Ford and Parker [2]. In conventional inflationary models, the two polarizations of the gravitational waves do not interact [3], and consequently the low-frequency branch of the spectrum, ranging between the aHz and 100 aHz, is unpolarized [4]. The same conclusion holds at higher frequencies, e.g., in the mHz band and in the audio band. Hereunder, we shall conventionally refer to the audio band as the region between a few Hz and 10 kHz; standard prefixes will be used throughout when needed (e.g.1 aHz $= 10^{-18}$ Hz, 1 mHz $= 10^{-3}$ Hz and so on and so forth).

At late times, the evolution of the relic gravitons is affected by various anisotropic stresses whose transverse and traceless modes could induce a certain degree of polarization. After neutrino decoupling, the corresponding anisotropic stress slightly suppresses the relic graviton background [5], but it is unable to polarize the spectrum either in the audio band or in the mHz range. The polarization of the graviton background induced by the anisotropic stresses typically involves a limited interval of frequencies reflecting the physical properties of the source. For instance, the anisotropic stress of the hypermagnetic knots (i.e., maximally gyrotropic configurations of the hypermagnetic fields) could polarize the stochastic backgrounds of relic gravitons over intermediate frequencies approximately ranging between a few μHz and 10 kHz [6]. According to a complementary perspective, the mutual interaction of the two tensor polarizations might be ultimately responsible for the overall polarization of the cosmic graviton background. This is what happens, for instance, when the logic of the effective field theory is applied, for instance, to single-field inflationary models [7].

Indeed, the large-scale observations can be interpreted in the light of a very simple class of single-field inflationary models whose action can be written in terms of the canonically normalized inflaton φ as

$$S_{\varphi} = \int d^4x\sqrt{-g}\left[-\frac{R}{2\ell_P^2} + \frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\varphi\partial_{\beta}\varphi - V(\varphi)\right], \quad (1.1)$$

where $\ell_P = \sqrt{8\pi G}$, g denotes the determinant of the four-dimensional metric, and $V(\varphi)$ is the inflaton potential. Within the notations employed in this paper, we also have that $\ell_P = 1/\overline{M}_P$; $\overline{M}_P$ is the reduced Planck mass related to $M_P = 1/\sqrt{G}$ as $\overline{M}_P = M_P/\sqrt{8\pi}$; τ will denote throughout the conformal time coordinate. We finally remind the reader that in this paper the latin indices are all Euclidian and, when this is the case, there is no difference between covariant and contravariant components.

Equation (1.1) is just the first term of a generic effective field theory of inflation [7] where the higher derivatives are suppressed by the negative powers of a large mass $M < M_P$ that characterizes the fundamental theory underlying the effective description. Assuming general covariance, as already remarked in Ref. [7], it is possible to write down, for instance, the leading correction containing four derivatives and consisting of ten terms. Among these terms, one has to do with the parity-violating interactions and contains the product of the Riemann tensor with its dual, i.e., $\tilde{R}^{\mu\alpha\nu\beta}R_{\mu\alpha\nu\beta}$; a term of the same kind will also contain the product of the Weyl tensor (which is the traceless part of the

[*]massimo.giovannini@cern.ch

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Riemann tensor) with its dual, i.e., $\tilde{C}^{\mu\alpha\nu\beta}C_{\mu\alpha\nu\beta}$. These terms are able to polarize the background of the relic gravitons, but there could also be different terms coming into play if we consider, for instance, the interactions with the gauge fields.

The main purpose of this paper is to scrutinize and classify the mutual interactions of the tensor polarizations in cosmological backgrounds by expressing their gauge-invariant action in terms of appropriate bispinors whose components coincide with the polarized amplitudes. In this context, the parity-violating interactions mentioned in the previous paragraph as well as other possible contributions will be parametrized in terms of an effective action reducing to the Ford-Parker action in the unpolarized case [2]. In optical applications, the Jones calculus stipulates that the electric fields of the waves are organized in a two-dimensional column vector. In the analyses of optical phenomena, the Jones approach is customarily contrasted with the Mueller calculus where the polarization is described by a four-dimensional (Mueller) column vector of which the components are the four Stokes parameters [8]. The description of the tensor polarizations in terms of bispinors allow for a general classification of the various interaction terms which can be expressed in a much more compact and revealing form. Different Pauli matrices (or, by stretching the language a bit, different directions in "isospace") parametrize the various interactions between the two polarization. The obtained action is invariant under infinitesimal diffeomorphisms; it reduces to the Ford-Parker action when the interaction between the polarizations is absent, and it contains at most two derivatives with respect to the conformal time coordinate. While it seems plausible to suggest that *any* generally covariant model leading to a mutual interaction between the two linear polarizations should fit within the scheme of this paper, examples will be provided with the aim of showing how the arbitrary couplings introduced in the general form of the action can be explicitly computed in concrete models.

The layout of this paper is the following. In Sec. II, we shall outline the main idea and the general form of the effective action. We shall also examine some particular cases with the purpose of illustrating the simplifications introduced by the use of the bispinors. In Sec. III, we shall examine the opposite perspective by considering two classes of generally covariant modes and by showing that they naturally fit in the general scheme proposed in the paper. Section IV discusses some specific applications aimed at illustrating the use if the bispinors for the general solution of the evolution equations of the two polarizations of the graviton. Finally, Sec. V, contains the concluding remarks.

II. GENERAL FORM OF THE ACTION FOR THE BISPINORS

A. Basic considerations

The tensor fluctuations of conformally flat background geometries are defined as $g_{\mu\nu}(\vec{x},\tau) = \bar{g}_{\mu\nu} + \delta_t^{(1)} g_{\mu\nu}$, where $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, $a(\tau)$ is the scale factor and $\eta_{\mu\nu}$ is the Minkowski metric. Overall, the metric fluctuations contain ten independent components; only two describe the cosmic gravitons and are parametrized in terms of a rank-2 tensor in three spatial dimensions, i.e., $\delta_t^{(1)} g_{ij} = -a^2(\tau) h_{ij}(\vec{x},\tau)$, where h_{ij} is divergenceless and traceless (i.e., $h_i^{\ i} = \partial_i h_j^{\ i} = 0$). Since the tensor modes of the geometry are real quantities, they can be expressed in Fourier space as

$$h_{ij}(\vec{x},\tau) = \frac{\sqrt{2}\ell_P}{(2\pi)^{3/2}} \int d^3k h_{ij}(\vec{k},\tau) e^{-i\vec{k}\cdot\vec{x}},$$
$$h_{ij}(\vec{k},\tau) = e_{ij}^{\otimes} h_{\otimes}(\vec{k},\tau) + e_{ij}^{\oplus} h_{\oplus}(\vec{k},\tau), \qquad (2.1)$$

where $h^*_{ij}(\vec{k},\tau) = h_{ij}(-\vec{k},\tau)$ and the factor $\sqrt{2}\ell_P$ appearing in Eq. (2.1) is determined, as we shall see in a moment, by demanding that the action for each of the two polarizations is canonically normalized. The two (orthogonal) polarizations are $e_{ij}^{\otimes} = (\hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j)$ and $e_{ij}^{\oplus} = (\hat{m}_i \hat{m}_j - \hat{n}_i \hat{n}_j)$, where $\hat{m}$, $\hat{n}$, and $\hat{k}$ form a triplet of mutually orthogonal unit vectors in the three spatial dimensions (i.e., $\hat{m}\times\hat{n} = \hat{k}$).

Following the spirit (if not the letter) of the Jones calculus [8], the tensor polarizations can be arranged in a bispinor, be it Ψ, the components of which are given by $h_{\oplus}(\vec{k},\tau)$ and by $h_{\otimes}(\vec{k},\tau)$, respectively. The action describing the dynamics of Ψ must be invariant under infinitesimal diffeomorphisms; it must contain (at most) two derivatives with respect to τ, and it should reduce to the Ford-Parker action [2] when the interactions between the polarizations are absent. Putting together these three requirements,[1] we are led to the following expression,

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \{a^2(\tau)[\partial_\tau \Psi^\dagger \partial_\tau \Psi - k^2 \Psi^\dagger \Psi] + \Psi^\dagger(\vec{v}\cdot\vec{\sigma})\Psi + \partial_\tau\Psi^\dagger(\vec{r}\cdot\vec{\sigma})\partial_\tau\Psi + \Psi^\dagger(\vec{p}\cdot\vec{\sigma})\partial_\tau\Psi + \partial_\tau\Psi^\dagger(\vec{q}\cdot\vec{\sigma})\Psi\}, \quad \Psi = \begin{pmatrix} h_{\oplus} \\ h_{\otimes} \end{pmatrix}, \qquad (2.2)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and σ_i (with $i = 1, 2, , 3$) are the three Pauli matrices. The dagger denotes, as usual, the transposed and complex conjugate of the corresponding spinor or matrix. In the parametrization of Eq. (2.2), the vector $\vec{r}(k,\tau)$ is dimensionless, while the two vectors $\vec{v}(k,\tau)$, $\vec{p}(k,\tau)$, and $\vec{q}(k,\tau)$ are all dimensional and may otherwise contain an arbitrary dependence on k. Finally, since the quantum Hamiltonian associated with the action (2.3) must be Hermitian, we are led to demand that $\vec{p} = \vec{q}$. Thanks to this plausible requirement, the terms containing a

[1] It should be stressed that these requirements are physically complementary but conceptually separate.

single conformal time derivative can be eliminated by dropping a total time derivative, while $\vec{b}$ is redefined as $\vec{v} \to \vec{b} = \vec{v} - \partial_\tau \vec{p}$. Therefore, the canonical form of Eq. (2.3) is always expressible as

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \{a^2(\tau)[\partial_\tau \Psi^\dagger \partial_\tau \Psi - k^2 \Psi^\dagger \Psi] + \Psi^\dagger(\vec{b}\cdot\vec{\sigma})\Psi + \partial_\tau \Psi^\dagger (\vec{r}\cdot\vec{\sigma})\partial_\tau \Psi\}. \quad (2.3)$$

In the noninteracting limit (i.e., when all the vectors are vanishing identically), the actions (2.2)–(2.3) both coincide with the result obtained in Ref. [2], and the two polarizations evolve independently. The physical model and the form of the interaction is specified by the components of the vectors $\vec{b}(k,\tau)$ and $\vec{r}(k,\tau)$.

By construction, the action (2.3) is invariant under infinitesimal diffeomorphisms; it reduces to the Ford-Parker action when the interaction between the polarizations is absent, and it contains at most two derivatives with respect to the conformal time coordinate. Furthermore, the interactions between the two polarizations are invariant under rotations in isospace. Before discussing the advantages and the implications of the description in terms of bispinors, it is useful to examine some potentially different viewpoints. There might be some who would like to introduce terms proportional to $\Psi^\dagger\Psi$. This kind of term does not mix the polarizations, and it would correspond to a massive contribution that does not disappear in the limit where the interaction between the polarizations vanishes. Let us finally remark that the quantum Hamiltonian describing the parametric amplification of the polarizations is necessarily Hermitian since the vectors describing the couplings are all real.

B. Advantages and implications of the bispinor description

A recurrent theme in the present paper will be the advantages of the spinor description in comparison with the case where the polarization and treated independently. This aspect should be already clear; however, to make it even more transparent, we shall unpack some specific cases contained in the general form of the action. These terms will have specific implications in the examples more specifically studied in Sec. III. When Eq. (2.3) involves the only diagonal Pauli matrix (i.e., σ_3) the action follows from Eq. (2.3) by choosing $\vec{b} = (0,0,b_3)$ and $\vec{r} = (0,0,r_3)$,

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \{a^2(\tau)[(\partial_\tau h_\otimes \partial_\tau h_\otimes^* + \partial_\tau h_\oplus \partial_\tau h_\oplus^*) - k^2(h_\otimes h_\otimes^* + h_\oplus h_\oplus^*)] + b_3(k,\tau)(h_\oplus^* h_\oplus - h_\otimes h_\otimes^*) + r_3(k,\tau)(\partial_\tau h_\oplus^* \partial_\tau h_\oplus - \partial_\tau h_\otimes \partial_\tau h_\otimes^*)\}. \quad (2.4)$$

Depending upon the specific forms of r_3 and b_3, the evolution of $h_\oplus$ and $h_\otimes$ can be different, but the corresponding equations for $h_\oplus$ and $h_\otimes$ do not mix. Conversely, whenever the interaction involves either σ_2 or σ_1, the two polarizations are coupled in the linear basis. For instance, when $\vec{b} = (0,b_2,0)$ and $\vec{r} = (0,r_2,0)$, Eq. (2.3) becomes

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \{a^2(\tau)[(\partial_\tau h_\otimes^* \partial_\tau h_\otimes + \partial_\tau h_\oplus^* \partial_\tau h_\oplus) - k^2(h_\otimes^* h_\otimes + h_\oplus^* h_\oplus)] + ib_2(k,\tau)(h_\oplus h_\otimes^* - h_\otimes h_\oplus^*) + ir_2(k,\tau)(\partial_\tau h_\oplus \partial_\tau h_\otimes^* - \partial_\tau h_\otimes \partial_\tau h_\oplus^*)\}. \quad (2.5)$$

The action (2.5) becomes diagonal in the circular basis where the Fourier amplitude of Eq. (2.1) reads now $h_{ij}(\vec{k},\tau) = [e_{ij}^{(R)} h_R(\vec{k},\tau) + e_{ij}^{(L)} h_L(\vec{k},\tau)]$, and $h_L = (h_\oplus + i h_\otimes)/\sqrt{2}$ and $h_R = (h_\oplus - i h_\otimes)/\sqrt{2}$. For the sake of precision, we remind the reader that, in the present paper, the right (i.e., R) and left (i.e., L) polarizations are defined as $e_{ij}^{(R)} = (e_{ij}^{\oplus} + i e_{ij}^{\otimes})/\sqrt{2}$ and $e_{ij}^{(L)} = (e_{ij}^{\oplus} - i e_{ij}^{\otimes})/\sqrt{2}$. The relation between the linear and the circular tensor amplitudes follows easily from Eq. (2.1).

With these specifications, in the circular basis, the action of Eq. (2.5) is

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \{a^2(\tau)[(\partial_\tau h_R^* \partial_\tau h_R + \partial_\tau h_L^* \partial_\tau h_L) - k^2(h_R^* h_R + h_L^* h_L)] + b_2(k,\tau)(h_R^* h_R - h_L^* h_L) + r_2(k,\tau)(\partial_\tau h_R^* \partial_\tau h_R - \partial_\tau h_L^* \partial_\tau h_L)\}. \quad (2.6)$$

Once more, the two circular amplitudes will obey two different equations that are, however, decoupled and will eventually produce a net degree of polarization, as we shall more concretely illustrate in a moment. Needless to say, Eq. (2.6) can be swiftly derived by working directly with bispinors; indeed, the action (2.3) in the case $\vec{\sigma} = (0,\sigma_2,0)$ is

$$S_{\mathrm{pol}} = \frac{1}{2}\int d^3k \int d\tau\{a^2(\tau)[\partial_\tau\Psi^\dagger\partial_\tau\Psi - k^2\Psi^\dagger\Psi] + b_2(k,\tau)\Psi^\dagger\sigma_2\Psi + r_2(k,\tau)\partial_\tau\Psi^\dagger\sigma_2\partial_\tau\Psi\}, \tag{2.7}$$

and it becomes diagonal by performing the following unitary transformation,

$$\Psi = U\Phi, \qquad U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} h_R \\ h_L \end{pmatrix}, \tag{2.8}$$

where $U^\dagger = U^{-1}$. Since $U^\dagger\sigma_2 U = \sigma_3$, we have that Eq. (2.3) becomes

$$S_{\mathrm{pol}} = \frac{1}{2}\int d^3k \int d\tau\{a^2(\tau)[\partial_\tau\Phi^\dagger\partial_\tau\Phi - k^2\Phi^\dagger\Phi] + b_2(k,\tau)\Phi^\dagger\sigma_3\Phi + r_2(k,\tau)\partial_\tau\Phi^\dagger\sigma_3\partial_\tau\Phi\}. \tag{2.9}$$

Equation (2.9) can be expressed in an even more compact form by introducing two appropriate matrices Z and W:

$$S_{\mathrm{pol}} = \frac{1}{2}\int d^3k \int d\tau[\partial_\tau\Phi^\dagger Z\partial_\tau\Phi - \Phi^\dagger W\Phi]. \tag{2.10}$$

The two matrices appearing in Eq. (2.10) are $Z(k,\tau) = \{[a^2(\tau)+r_2(k,\tau)]P_R + [a^2(\tau)-r_2(k,\tau)]P_L\}$ and $W(k,\tau) = \{[k^2a^2(\tau) - b_2(k,\tau)]P_R + [k^2a^2(\tau) + b_2(k,\tau)]P_L\}$, where $P_L = (I-\sigma_3)/2$ and $P_R = (I+\sigma_3)/2$ denote the left and right projectors, while I is the identity matrix. The same steps leading to Eqs. (2.9) and (2.10) can be repeated when the interaction is dictated by σ_1 rather than by σ_2. Since σ_1 has only real entries, the analog of Eq. (2.7) can be easily derived from Eq. (2.3). The resulting action will only contain r_1 and b_1 and can diagonalized by the following unitary transformation,

$$\Psi = V\Xi, V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \qquad \Xi = \begin{pmatrix} h_+ \\ h_- \end{pmatrix}, \tag{2.11}$$

where Ξ is defined in the new basis provided by the sum and by the difference of the two linear polarizations (i.e., $h_\pm = (h_\oplus \pm h_\otimes)/\sqrt{2}$). By plugging Eq. (2.11) into Eq. (2.3) written in the case $\vec{\sigma} = (\sigma_1, 0, 0)$, we are now led to

$$S_{\mathrm{pol}} = \frac{1}{2}\int d^3k \int d\tau[\partial_\tau\Xi^\dagger\tilde{Z}\partial_\tau\Xi - \Xi^\dagger\tilde{W}\Xi], \tag{2.12}$$

where $\tilde{Z}(k,\tau) = \{[a^2(\tau)+r_1(k,\tau)]P_R + [a^2(\tau)-r_1(k,\tau)]P_L\}$ and $\tilde{W}(k,\tau) = \{[k^2a^2(\tau) - b_1(k,\tau)]P_R + [k^2a^2(\tau) + b_1(k,\tau)]P_L\}$ in full analogy with the results of Eq. (2.10). As in the case of the circular basis, the components of Ξ obey two different equations, which are decoupled.

III. TWO CONCRETE MODELS

A. Purely gravitational case

While Eq. (2.3) purportedly describes the most general interaction of the two tensor polarizations evolving in conformally flat backgrounds, the reverse must also be true, and any concrete model will have to correspond to a specific choice of $\vec{b}(k,\tau)$ and $\vec{r}(k,\tau)$. Along this perspective, the action of the relic gravitons may contain a parity-violating term [7,9–11] that involves the dual the Riemann (or Weyl) tensor,

$$S = -\frac{1}{2\ell_P^2}\int d^4x\sqrt{-g}R - \frac{\beta}{8}\int d^4x\sqrt{-g}f(\varphi)\tilde{R}^{\mu\alpha\nu\beta}R_{\mu\alpha\nu\beta},$$
$$\tilde{R}^{\mu\alpha\nu\beta} = \frac{1}{2}E^{\mu\alpha\rho\sigma}R_{\rho\sigma}{}^{\nu\beta}, \tag{3.1}$$

where g is the determinant of the four-dimensional metric, $E^{\mu\alpha\rho\sigma} = \epsilon^{\mu\alpha\rho\sigma}/\sqrt{-g}$, and $\epsilon^{\mu\alpha\rho\sigma}$ is the Levi-Cività symbol; β is just a numerical constant, while $f(\varphi)$ contains the dimensionless coupling to some scalar degree of freedom that can be identified, for instance, with the inflaton or with some other spectator field. Before proceeding, we remark that a complementary class of examples is obtained by replacing the Riemann tensor with the Weyl tensor in Eq. (3.1). Being the traceless part of the Riemann tensor, the Weyl tensor vanishes for a spatially flat Friedmann-Robertson Walker metric; the derivation of the second-order action describing the tensor modes is comparatively easier in the Weyl rather than in the Riemann case that will be specifically studied hereunder.

The action of the tensor modes of the geometry follows, in this example, by perturbing Eq. (3.1) to second order in the amplitude of the tensor modes of the geometry introduced prior to Eq. (2.1). The explicit result is given by

$$\delta_t^{(2)}S = \frac{1}{8\ell_P^2}\int d^4x a^2[\partial_\tau h_{ij}\partial_\tau h_{ij} - \partial_k h_{ij}\partial_k h_{ij}]$$
$$-\frac{\beta}{8}\int d^4x(\partial_\tau f)\epsilon^{ijk}[\partial_\tau h_{qi}\partial_\tau\partial_j h_{kq} - \partial_\ell h_{iq}\partial_\ell\partial_j h_{qk}], \tag{3.2}$$

where ϵ^{ijk} is now the Levi-Cività symbol in three dimensions; we remind the reader that the latin indices are all Euclidian. According to Eq. (2.1), the tensor amplitudes appearing in Eq. (3.2) can be expressed in the linear polarization basis, and the result will be

$$\delta_t^{(2)} S = \frac{1}{2}\int d^3k \int d\tau\{a^2[\partial_\tau h_\oplus \partial_\tau h_\oplus^* + \partial_\tau h_\otimes \partial_\tau h_\otimes^* - k^2(h_\oplus h_\oplus^* + h_\otimes h_\otimes^*)] + ik\beta\ell_P^2 \partial_\tau f[\partial_\tau h_\oplus \partial_\tau h_\otimes^* - \partial_\tau h_\otimes \partial_\tau h_\oplus^* - k^2(h_\oplus h_\otimes^* - h_\otimes h_\oplus^*)]\}. \tag{3.3}$$

If the two linear polarizations appearing in Eq. (3.3) are arranged into the components of the bispinor Ψ, we obtain a particular case of Eqs. (2.3) and (2.7). More specifically, indeed, Eq. (2.7) reproduces exactly Eq. (3.3), provided

$$b_2 = -k^3\beta\ell_P^2(\partial_\tau f), \qquad r_2 = k\beta\ell_P^2(\partial_\tau f). \tag{3.4}$$

As argued in general terms in Eq. (2.6), the resulting action becomes diagonal in the circular polarization basis, and Eq. (3.3) shall then be expressible in the compact form of Eq. (2.10) where now the matrices Z and W are given by $Z(k,\tau) = \{[a^2(\tau) + k\beta\partial_\tau f\ell_P^2]P_R + [a^2(\tau) - k\beta\partial_\tau f\ell_P^2]P_L\}$ and by $W = k^2 Z$.

B. Interactions with the gauge fields

A different class of illustrative models is obtained by considering the case where only $\vec{b}$ does not vanish. While these examples would seem naively difficult to concoct, they may arise from the following generally covariant action,

$$S = -\frac{1}{2\ell_P^2}\int d^4x\sqrt{-g}R - \frac{1}{2\ell_P^2 M^4}\int d^4x f(\varphi) R_{\mu\alpha\nu\beta} Y^{\mu\alpha}\tilde{Y}^{\nu\beta}, \tag{3.5}$$

where $Y_{\mu\nu}$ and $\tilde{Y}^{\mu\nu} = E^{\mu\nu\rho\sigma}Y_{\rho\sigma}/2$ are the gauge field strength and its dual. In Eq. (3.5), M sets the typical scale of the interaction. Note, incidentally, that the explicit powers of M are often omitted in the analysis of effective theories of inflation [7] with the proviso that all constants in the higher derivative terms of the effective action take values that are powers of M indicated by dimensional analysis, with coefficients roughly of order unity. If there exists a family of four-dimensional observers moving with four-velocity u^μ (possibly related with the covariant gradients of a scalar field) in Eq. (3.5), the gauge fields can be covariantly decomposed in their electric and magnetic parts [12] according to $Y_{\mu\alpha} = \mathcal{E}_{[\mu}u_{\alpha]} + E_{\mu\alpha\rho\sigma}u^\rho\mathcal{B}^\sigma$ and to $\tilde{Y}^{\nu\beta} = \mathcal{B}^{[\nu}u^{\beta]} + E^{\nu\beta\rho\sigma}\mathcal{E}_\rho u_\sigma$ (note that $[\ldots]$ denotes an antisymmetric combination of the two corresponding tensorial indices). Inserting this decomposition into Eq. (3.5) and neglecting the electric contributions, we can perturb the action to second order in the amplitude of the tensor modes of the geometry, and the result of this step can be written as

$$\delta_t^{(2)} S = \frac{1}{8\ell_P^2}\int d^4x\{a^2[(\partial_\tau h_{ij})(\partial_\tau h_{ij}) - (\partial_k h_{ij})(\partial_k h_{ij})] + 4a^2 F n_c n_a \epsilon^{bpc}[\partial_\tau h_{pq}(\partial_a h_{qb} - \partial_b h_{qa}) - \partial_\tau h_{aq}(\partial_b h_{qp} - \partial_p h_{bq})]\}, \tag{3.6}$$

where $b_a(\tau) = n_a b(\tau)$ and $F = (b^2 f)/M^4$. To derive Eq. (3.6), we note that, to zeroth order in the tensor amplitude, we have $\tilde{Y}^{0i} = -b^i/a^2$ and $Y^{ij} = -\epsilon^{ijk}b_k/a^2$. To first and second orders in the amplitude of the tensor modes, the previous expressions are corrected as $\delta_t^{(1)}\tilde{Y}^{0i} = h^{ik}b_k/a^2$ and as $\delta_t^{(2)}\tilde{Y}^{0i} = -h^{i\ell}h_\ell{}^k b_k/a^2$.

The action perturbed to second order in the amplitude of the tensor modes of the geometry becomes then

$$S_{\rm pol} = \frac{1}{2}\int d^3k\int d\tau\{a^2[(\partial_\tau h_\oplus \partial_\tau h_\oplus^* + \partial_\tau h_\otimes \partial_\tau h_\otimes^*) - k^2(h_\oplus h_\oplus^* + h_\otimes h_\otimes^*)] + ika^2F[(\partial_\tau h_\oplus)h_\otimes^* - (\partial_\tau h_\otimes)h_\oplus^*] + ika^2F[h_\oplus(\partial_\tau h_\otimes^*) - h_\otimes(\partial_\tau h_\oplus^*)]\}. \tag{3.7}$$

In the linear basis, Eq. (3.7) coincides with Eq. (2.2), provided we choose $\vec{p}(k,\tau) = (0, ka^2F, 0)$. Up to a total derivative, the obtained equation can be brought in the form (2.3) with $\vec{b}(k,\tau) = [0, -k\partial_\tau(a^2F), 0]$ and then diagonalized in the circular basis. If the two previous steps are inverted, the final result does not change, and Eq. (3.7) can be diagonalized[2] before the elimination of the total derivative. In either case, the final form of the action (3.7) is

$$S_{\rm pol} = \frac{1}{2}\int d^3k\int d\tau\{a^2[\partial_\tau\Phi^\dagger\partial_\tau\Phi - k^2\Phi^\dagger\Phi] - k\partial_\tau(a^2F)\Phi^\dagger\sigma_3\Phi\}. \tag{3.8}$$

Equation (3.8) can be finally put in the form (2.10) by choosing $Z = Ia^2$ and $W = \{[k^2a^2 + k\partial_\tau(a^2F)]P_R + [k^2a^2 - k\partial_\tau(a^2F)]P_L\}$.

IV. ILLUSTRATIVE APPLICATIONS OF THE SPINOR ACTION

A. Polarization degree

In the two previous paragraphs, different generally covariant models have been shown to reproduce some particular cases of Eq. (2.3), and this conclusion corroborates the

[2]According to Eq. (2.8), we can go from the linear to the circular basis by positing $\Psi = U\Phi$, where Φ denotes the bispinor in the circular basis. Since $U^\dagger\sigma_2 U = \sigma_3$, the canonical action is easily obtained.

validity of the direct derivation. We now turn to the evaluation of the degree of polarization, and for this purpose, it is interesting to remark that Eq. (2.10) can be simplified even further by defining the rescaled bispinor $\mathcal{M} = (P_R z_R + P_L z_L)\Phi$ of which the components, in the circular basis, are given by $\mu_R = z_R h_R$ and $\mu_L = z_L h_L$. After expressing Eq. (2.10) in terms of $\mathcal{M}$, we obtain

$$S_{\rm pol} = \frac{1}{2}\int d^3k \int d\tau \biggl\{\partial_\tau \mathcal{M}^\dagger \partial_\tau \mathcal{M} + \mathcal{M}^\dagger\biggl[\biggl(\frac{z_R''}{z_R} - \omega_R^2\biggr)P_R + \biggl(\frac{z_L''}{z_L} - \omega_L^2\biggr)P_L\biggr]\mathcal{M}\biggr\}, \tag{4.1}$$

where the prime denotes a derivation with respect to the conformal time coordinate and the same shorthand notation will be employed hereunder. Furthermore, in Eq. (4.1), $z_{R,L}$ and $\omega_{L,R}$ are defined as

$$z_R^2(k,\tau) = a^2(\tau) + r_2(k,\tau), \quad z_L^2(k,\tau) = a^2(\tau) - r_2(k,\tau),$$
$$\omega_R^2(k,\tau) = \frac{k^2 a^2(\tau) - b_2(k,\tau)}{a^2(\tau) + r_2(k,\tau)},$$
$$\omega_L^2(k,\tau) = \frac{k^2 a^2(\tau) + b_2(k,\tau)}{a^2(\tau) - r_2(k,\tau)}. \tag{4.2}$$

For example, the action of Eq. (3.1) implies that $\omega_L = \omega_R = k^2$, while in the case of Eq. (3.5), $r_2 \to 0$ and $z_L = z_R = a(\tau)$. Similarly, the general expressions of Eq. (4.2) may simplify for other specific values of $b_2(k,\tau)$ and $r_2(k,\tau)$. Recalling that Φ denotes the bispinor in the circular basis, the degree of circular polarization can be defined as

$$\Pi_{\rm circ}(k,\tau) = \frac{\Phi^\dagger \sigma_3 \Phi}{\Phi^\dagger\Phi} = \frac{|h_R(k,\tau)|^2 - |h_L(k,\tau)|^2}{|h_R(k,\tau)|^2 + |h_L(k,\tau)|^2}. \tag{4.3}$$

From Eq. (4.1), the evolution of $\mathcal{M}$ reads

$$\partial_\tau^2 \mathcal{M} + \biggl[\biggl(\omega_R^2 - \frac{z_R''}{z_R}\biggr)P_R + \biggl(\omega_L^2 - \frac{z_L''}{z_L}\biggr)P_L\biggr]\mathcal{M} = 0. \tag{4.4}$$

Equation (4.4) reduces to a pair of decoupled equations defined in the circular basis,

$$\mu_R'' + \biggl[\omega_R^2 - \frac{z_R''}{z_R}\biggr]\mu_R = 0, \quad \mu_L'' + \biggl[\omega_L^2 - \frac{z_L''}{z_L}\biggr]\mu_L = 0, \tag{4.5}$$

where we defined $\mu_R = z_R h_R$ and $\mu_L = z_L h_L$. The evolution of the right and of the left movers can be studied within different approximation schemes. The expansion in the conformal coupling parameter, originally explored by Birrell and Davies [13], has been subsequently applied to the case of gravitational waves by Ford [14]. However, since the illustrative goal is to evaluate the degree of circular polarization at high-frequencies, the Wentzel–Kramers–Brillouin (WKB) approximation seems more directly applicable [15]. We remind that the WKB approximation is a method for finding approximate solutions to linear differential equations with varying coefficients. While this method has not been applied so far to the polarized case, this gap will now be bridged, at least partially.

B. Solution for the bispinors in the WKB approximation

The equations for μ_R and μ_L reported in (4.5) closely resemble Schrödinger equations with different k-dependent potentials for the left and right movers [i.e., $V_R(k,\tau) = z_R''/z_R$ and $V_L(k,\tau) = z_L''/z_L$]. Thus, for $\omega_R^2 \gg V_R$ and $\omega_L^2 \gg V_L$, the general solution of Eq. (4.5) is[3]

$$\mu_X(k,\tau) = \frac{1}{\sqrt{2\omega_X}}[\alpha_X e^{-i\int^\tau \omega_X(k,\tau')d\tau'} + \beta_X e^{i\int^\tau \omega_X(k,\tau')d\tau'}],$$
$$\omega_X^2 \gg \biggl|\frac{z_X''}{z_X}\biggr|, \tag{4.6}$$

where α_X and β_X are arbitrary complex numbers and $X = R, L$; Eq. (4.6) holds independently for the left and right movers, provided the variation of ω_X is sufficiently slow (i.e., $|V_X| \ll \omega_X'/\omega_X < |z_X'/z_X|$). In the opposite limit (i.e., $\omega_X^2 \ll V_X$), the general solution of Eq. (4.5) is instead given by

$$\mu_X(k,\tau) = A_X(k) z_X(k,\tau) + B_X(k) z_X(k,\tau)\int^\tau \frac{d\tau'}{z_X^2(k,\tau')},$$
$$\omega_X^2 \ll \biggl|\frac{z_X''}{z_X}\biggr|. \tag{4.7}$$

For a wide class of problems, the potentials $|V_X|$ have a bell-like shape in the conformal time coordinate and vanish in the limit $\tau \to \pm\infty$. The solution (4.6) is valid outside the potential barrier $|V_X|$ (i.e., inside the effective horizon defined by the variation of z_X); the solution (4.7) holds instead when $|V_X|$ dominates against ω_X^2 (or, more precisely, when the corresponding wavelengths are larger than the effective horizon). The turning points are fixed by $\omega_X^2 = V_X$ and will be denoted by τ_{ex} (i.e., the time at which the mode *exits* the effective horizon) and by τ_{re} (i.e., the moment at which the given mode *reenters* the effective horizon). To the left of the barrier, for $\tau < \tau_{ex}$, the solution will be in the form $e^{-i\int^\tau \omega_X(k,\tau')d\tau'}/\sqrt{2\omega_X}$. To the right of the barrier, the solution is instead given by Eq. (4.6). Finally, between the turning points, the solution has the form (4.7). The continuous matching of the three solutions (and of their first derivatives) across the two turning points allows for an explicit determination of α_X and β_X. In the interesting physical case (i.e., $|z_{re}^{(X)}/z_{ex}^{(X)}| \gg 1$), the coefficient β_X is always larger than α_X (i.e., $|\beta_X|^2 \gg |\alpha_X|^2$);

[3]In general terms, since $b_2/r_2 = -k^2$, we shall also have that $\omega_L^2 = \omega_R^2 = k^2$.

thus, we shall only need to mention the results for $|\beta_R|^2$ and $|\beta_L|^2$:

$$|\beta_L|^2 \simeq \frac{1}{4}\left[\frac{z_{re}^{(L)}}{z_{ex}^{(L)}}\right]^2\left[1+\frac{\mathcal{L}_{re}^2}{\omega_L^2}\right]\{1-2\mathcal{L}_{ex}z_{ex}^{(L)2}\mathcal{J}_L + z_{ex}^{(L)4}[\mathcal{L}_{ex}^2+\omega_L^2]\mathcal{J}_L^2\},$$
$$|\beta_R|^2 \simeq \frac{1}{4}\left[\frac{z_{re}^{(R)}}{z_{ex}^{(R)}}\right]^2\left[1+\frac{\mathcal{R}_{re}^2}{\omega_R^2}\right]\{1-2\mathcal{R}_{ex}z_{ex}^{(R)2}\mathcal{J}_R + z_{ex}^{(R)4}[\mathcal{R}_{ex}^2+\omega_R^2]\mathcal{J}_R^2\}. \tag{4.8}$$

In Eq. (4.8), the rates of variation of z_R and z_L (i.e., $\mathcal{R} = z_R'/z_R$ and $\mathcal{L} = z_L'/z_L$) have been introduced, while $\mathcal{J}_L(k,\tau_{ex},\tau_{re})$ and $\mathcal{J}_R(k,\tau_{ex},\tau_{re})$ involve two different integrals between the two turning points:

$$\mathcal{J}_L(k,\tau_{ex},\tau_{re}) = \int_{\tau_{ex}}^{\tau_{re}}\frac{d\tau'}{z_L^2(k,\tau')},$$
$$\mathcal{J}_R(k,\tau_{ex},\tau_{re}) = \int_{\tau_{ex}}^{\tau_{re}}\frac{d\tau'}{z_R^2(k,\tau')}. \tag{4.9}$$

If the interaction between the polarizations ceases after the end of inflation, the high frequencies will cross the barrier the second time when $z_R = z_L = a(\tau)$ and will remain inside the effective horizon thereafter. In this case, the total degree of circular polarization of Eq. (4.3) can be expressed as

$$\Pi_{\rm circ} = \frac{|\beta_R(k)|^2 - |\beta_L(k)|^2}{|\beta_R(k)|^2 + |\beta_L(k)|^2}. \tag{4.10}$$

Equation (4.10) demonstrates that the left and right movers see effectively different potential barriers so that the degree of circular polarization is ultimately determined by the difference of the two dominant mixing coefficients. The explicit evaluation of Eqs. (4.3) and (4.10) is delicate when the potential $V_X \to 0$ in the vicinity of the second turning point (as it happens when the second crossing takes place during radiation where $a'' = 0$ [15]). With these caveats, after inserting Eq. (4.8) into Eq. (4.10), the explicit expression of the polarization degree depends solely on the values of the pump fields and of their first derivatives at the turning points:

$$\Pi_{\rm circ} = \frac{|z_{re}^{(R)}|^2|z_{ex}^{(L)}|^2(k^2+\mathcal{R}_{re}^2) - |z_{ex}^{(R)}|^2|z_{re}^{(L)}|^2(k^2+\mathcal{L}_{re}^2)}{|z_{re}^{(R)}|^2|z_{ex}^{(L)}|^2(k^2+\mathcal{R}_{re}^2) + |z_{ex}^{(R)}|^2|z_{re}^{(L)}|^2(k^2+\mathcal{L}_{re}^2)}. \tag{4.11}$$

If all the modes reenter after the end of inflation (i.e., $z_{re}^{(R)} = z_{re}^{(L)} = a_{re}$), the polarization degree is particularly simple, and it only depends on $z_{ex}^{(X)}$, i.e., $\Pi_{\rm circ} = (|z_{ex}^{(L)}|^2 - |z_{ex}^{(R)}|^2)/(|z_{ex}^{(L)}|^2 + |z_{ex}^{(R)}|^2)$.

C. Explicit evaluations of the polarization degree

While Eqs. (4.10) and (4.11) hold for different functional forms of the pump fields, more explicit expressions also demand further details on z_L and z_R. If we consider, for the sake of illustration, the case $z_R = \sqrt{a^2 + k\beta f'\ell_P^2}$ and $z_L = \sqrt{a^2 - k\beta f'\ell_P^2}$ [discussed in Eq. (3.3) and thereunder] the explicit form of Eq. (4.11) becomes[4]

$$|\Pi_{\rm circ}| = \beta\sqrt{2\epsilon}\left(\frac{H}{\overline{M}_P}\right)^2 = 6\times 10^{-13}\left(\frac{\beta}{0.1}\right)\left(\frac{\epsilon}{0.001}\right)^{3/2} \times\left(\frac{\mathcal{A}_s}{2.41\times 10^{-9}}\right), \tag{4.12}$$

where we assumed, for illustration, that the dependence on the inflaton of $f(\varphi)$ is linear (i.e., $f = \ell_P\varphi$) and that $z_{re}^{(R)} = z_{re}^{(L)} = a_{re}$. In Eq. (4.12), ϵ is the slow-roll parameter that ultimately determines the derivative of φ (i.e., $\varphi' = \sqrt{2\epsilon}aH\overline{M}_P$). The result of Eq. (4.12) becomes a bit different if $z_{re}^{(R)} \neq z_{re}^{(L)} \neq a_{re}$. In this case, Eq. (4.11) implies

$$|\Pi_{\rm circ}| = \beta\left(\frac{H}{\overline{M}_P}\right)^2\left|\sqrt{2\epsilon} - \sqrt{3w_{re}+1}\left(\frac{k}{k_1}\right)^{\frac{3(w_{re}+1)}{3w_{re}+1}}\right|, \tag{4.13}$$

where w_{re} denotes the barotropic index of the plasma when the given mode reenters the effective horizon (i.e., at the second turning point). While Eqs. (4.12) and (4.13) are qualitatively different, they are similar from the quantitative and physical viewpoints. They both hold for all the modes that reentered the effective horizon after the end of inflation but before the onset of the matter epoch (i.e., for frequencies larger than 100 aHz). In practice, however, the estimates apply for frequencies larger than the Hz since we also neglected the presence of the neutrino anisotropic stress [5]. They also suggest that $\Pi_{\rm circ}$ gets larger at high frequencies. More specifically, since k_1 is of the order of GHz (and it corresponds to the Hubble rate at the end of inflation), the degree of polarization is maximal at high frequencies so that the instruments operating in the kHz (or even MHz) regions are potentially more promising than the space-borne interferometers operating below the Hz.

V. CONCLUDING REMARKS

All in all, the polarizations of the tensor modes evolving in cosmological backgrounds have been described in terms

[4]Note that $\mathcal{A}_s$ denotes the amplitude of the scalar power spectrum appearing since $(H/M_P) = \sqrt{\pi\epsilon\mathcal{A}_s}$.

of appropriate bispinors closely analogous to what are commonly referred to as Jones vectors. Unlike the case of polarized optics, the present goal was to obtain a gauge-invariant action containing at most two (conformal) time derivatives and reducing to the standard (Ford-Parker) result when the two polarizations are mutually decoupled and only feel the overall effect of the space-time curvature. After arguing that the interactions potentially leading to polarized relic gravitons can be compactly classified in general terms, we showed that the reverse is also true. For this purpose, the direct derivation has been corroborated by a few examples demonstrating, in a conservative perspective, that different classes of generally covariant models ultimately fit within the scheme of the spinor action proposed here. For an illustrative application, we derived the degree of polarization and its spectral dependence at high frequencies by introducing a suitable WKB approximation where the modes corresponding to each of the two tensor amplitudes obey a different evolution equation and cross their effective horizons at slightly different turning points.

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[1] L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1975)]; Ann. N.Y. Acad. Sci. **302**, 439 (1977).
[2] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).
[3] A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)].
[4] V. A. Rubakov, M. V. Sazhin, and A. V. Veryaskin, Phys. Lett. **115B**, 189 (1982).
[5] S. Weinberg, Phys. Rev. D **69**, 023503 (2004); D. Dicus and W. Repko, Phys. Rev. D **72**, 088302 (2005); W. Zhao and Y. Zhang, Phys. Rev. D **74**, 043503 (2006).
[6] M. Giovannini, Phys. Rev. D **61**, 063502 (2000); **61**, 063004 (2000); Classical Quantum Gravity **34**, 135010 (2017).
[7] S. Weinberg, Phys. Rev. D **77**, 123541 (2008).
[8] B. A. Robson, *The Theory of Polarization Phenomena* (Clarendon, Oxford, 1974).
[9] S.-Y. Pi and R. Jackiw, Phys. Rev. D **68**, 104012 (2003).
[10] A. Lue, L. Wang, and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999).
[11] K. Choi, J. c. Hwang, and K. W. Hwang, Phys. Rev. D **61**, 084026 (2000); J. c. Hwang and H. Noh, Phys. Rev. D **61**, 043511 (2000).
[12] A. Lichnerowicz, *Relativistic Magnetohydrodynamics* (Benjamin, New York, 1967).
[13] N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[14] L. H. Ford, Phys. Rev. D **35**, 2955 (1987); E. D. Schiappacasse and L. H. Ford, Phys. Rev. D **94**, 084030 (2016).
[15] L. P. Grishchuk and M. Solokhin, Phys. Rev. D **43**, 2566 (1991); M. Giovannini, Phys. Rev. D **98**, 103509 (2018).