Line shape and $D^{(*)}\bar{D}^{(*)}$ probabilities of $\psi(3770)$ from the $e^+e^- \to D\bar{D}$ reaction

Q. X. Yu,[1,2,*] W. H. Liang,[3,4,†] M. Bayar,[5,‡] and E. Oset[2,3,§]

[1]College of Nuclear Science and Technology, Beijing Normal University, Beijing 100875, China
[2]Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC Institutos de Investigación de Paterna, Aptdo.22085, 46071 Valencia, Spain
[3]Department of Physics, Guangxi Normal University, Guilin 541004, China
[4]Guangxi Key Laboratory of Nuclear Physics and Technology, Guangxi Normal University, Guilin 541004, China
[5]Department of Physics, Kocaeli University, Izmit 41380, Turkey

(Received 10 February 2019; published 5 April 2019)

We have performed a calculation of the $D\bar{D}$, $D\bar{D}^*$, $D^*\bar{D}$, $D^*\bar{D}^*$ components in the wave function of the $\psi(3770)$. For this we make use of the $^3P_0$ model to find the coupling of $\psi(3770)$ to these components, that with an elaborate angular momentum algebra can be obtained with only one parameter. Then we use data for the $e^+e^- \to D\bar{D}$ reaction, from where we determine a form factor needed in the theoretical framework, as well as other parameters needed to evaluate the meson-meson self-energy of the $\psi(3770)$. Once this is done we determine the Z probability to still have a vector core and the probability to have the different meson components. We find Z about 80%–85%, and the individual meson-meson components are rather small, providing new empirical information to support the largely $q\bar{q}$ component of vector mesons, and the $\psi(3770)$ in particular. A discussion is done of the meaning of the terms obtained for the case of the open channels where the concept of probability cannot be strictly used.

DOI: 10.1103/PhysRevD.99.076002

I. INTRODUCTION

The nature of hadronic resonances is a field of continuous debate [1–4]. The simple picture of mesons as $q\bar{q}$ objects and baryons as qqq objects gave an impressive boost to hadron physics and large amount of mesons and baryons were described with this picture [5]. Yet, the advent of a new wave of experiments in the charm and bottom sectors has brought new information that clearly challenges this early picture in many cases [2–4]. Even in the light quark sector there are mesonic resonances that clearly cannot be represented as $q\bar{q}$ states, as the low lying scalar mesons $[f_0(500), f_0(980), a_0(980), \cdots]$ [6–9]. On the other hand, the elaborate analysis of meson-meson data by means of QCD and large N_c argument concluded that low lying vector mesons are largely $q\bar{q}$ objects [10].

*qixinyu@ific.uv.es, yuqx@mail.bnu.edu.cn
†liangwh@gxnu.edu.cn
‡melahat.bayar@kocaeli.edu.tr
§oset@ific.uv.es

It is unclear whether in the charm or bottom sector one can come to a similar conclusion. In fact, in Ref. [11] as study was made within the quark model of the meson-meson components of the charmonium vector states, and it was concluded that even the ground state J/ψ had only as survival probability as a vector of about 0.69 when the meson-meson components to which it couples were considered. This makes us think that higher excited vector charmonium states could actually have even smaller $q\bar{q}$ components.

In the present work we retake this issue for the $\psi(3770)$ vector state using data from the $e^+e^- \to D\bar{D}$ reactions. We make an elaborate study of the $D\bar{D}$, $D\bar{D}^*$, $D^*\bar{D}$, $D^*\bar{D}^*$ components of this resonance using the $^3P_0$ model for hadronization of $q\bar{q}$ into meson-meson components which requires only one parameter. By means of this and the data of the $e^+e^- \to D^+D^-, D^0\bar{D}^0$ reactions we can determine the parameters of the theory that allows us to evaluate the meson-meson self-energy of the $\psi(3770)$. The data of the $e^+e^- \to D\bar{D}$ reaction are essential for the reliable calculations of the self-energy, since the unknown couplings and a form factor entering the calculation are extracted from the data. In fact the form factor is relevant to the evaluation of the meson-meson probabilities and we show that it is tied to the fast fall down of the $e^+e^- \to D\bar{D}$ cross section above the $\psi(3770)$ peak.

The asymmetry of the $\psi(3770)$ peak observed in the $e^+e^- \to D\bar{D}$ reactions [12–14] has been the subject of intense discussion (see Ref. [15] for a recent review). In Ref. [15] a work similar to the one we do here, but using only the $D\bar{D}$ components, which are the most relevant, is done, and the shape of the $\psi(3770)$ peak is tied to a form factor that is introduced in an empirical way. We also implement this form factor in the same form and two different forms to estimate uncertainties. What we find is that the $\psi(3770)$ is largely a $q\bar{q}$ state and the meson-meson components are small. The Z probability of having a $q\bar{q}$ vector core for the $\psi(3770)$ is about 80%–85% and the individual meson-meson components are small.

This paper is organized as follows. In Sec. II, we establish the formalism of calculating the cross section for $e^+e^- \to D\bar{D}$ through the dressed propagator of $\psi(3770)$, and the meson-meson probabilities in the $\psi(3770)$ wave function. In Sec. III, we present the results on the line shape of $\psi(3770)$ fitting to the experimental data, and then calculate the Z probabilities using the parameters extracted from the fitting. A summary is presented in Sec. IV. The angular momentum algebra employed in the calculations is done explicitly in the Appendix.

II. FORMALISM

Our starting point is the hadronization in the process $\psi \to D^{(*)}\bar{D}^{(*)}$ shown in Fig. 1, where we introduce a $\bar{q}q$ pair with the quantum numbers of the vacuum, and insert it between the quark constituents of $\psi(3770)$, $c\bar{c}$. The insertion of $\bar{q}q$ is implemented in a $^3P_0$ state [16,17], which indicates that the inserted $\bar{q}q$ has positive parity and zero angular momentum, and since $\bar{q}$ has negative parity we need an orbital angular momentum $L=1$ for $\bar{q}q$ to fix the parity, which makes $\bar{q}q$ couple to spin $S=1$, then $S=1$ and $L=1$ couple to total angular momentum $J=0$. The $\psi(3770)$ according to Ref. [5] corresponds to a D-wave $c\bar{c}$ state with no radial excitation, a 1^3D_1 state with $J^{PC}=1^{--}$.

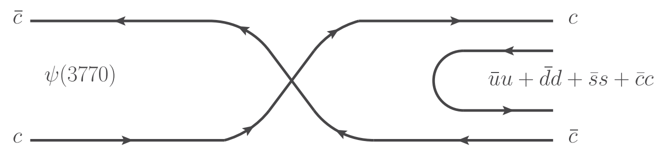

FIG. 1. Hadronization process for $\psi(3770) \to D^{(*)}\bar{D}^{(*)}$.

The hadronization in Fig. 1 proceeds as follows:

$$\psi \to c\bar{c} \to c(\bar{u}u + \bar{d}d + \bar{s}s + \bar{c}c)\bar{c} \to F, \tag{1}$$

with F

$$F = \sum_{i=1}^{4} c\bar{q}_i q_i \bar{c} = \sum_{i=1}^{4} M_{4,i} M_{i,4} = (M^2)_{4,4}, \tag{2}$$

where M corresponds to the following matrix

$$M = (q\bar{q}) = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} & u\bar{c} \\ d\bar{u} & d\bar{d} & d\bar{s} & d\bar{c} \\ s\bar{u} & s\bar{d} & s\bar{s} & s\bar{c} \\ c\bar{u} & c\bar{d} & c\bar{s} & c\bar{c} \end{pmatrix}. \tag{3}$$

Alternatively, we can write $q\bar{q}$ in Eq. (3) in terms of their meson components by means of the ϕ matrix for pseudoscalar mesons with the mixing between η and η' taken into account [18],

$$\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{3}}\eta + \frac{1}{\sqrt{6}}\eta' & \pi^+ & K^+ & \bar{D}^0 \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{3}}\eta + \frac{1}{\sqrt{6}}\eta' & K^0 & D^- \\ K^- & \bar{K}^0 & -\frac{1}{\sqrt{3}}\eta + \sqrt{\frac{2}{3}}\eta' & D_s^- \\ D^0 & D^+ & D_s^+ & \eta_c \end{pmatrix}. \tag{4}$$

Similarly, the vector matrix corresponding to $q\bar{q}$, which is also needed in our calculations, is given by

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & \rho^+ & K^{*+} & \bar{D}^{*0} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & K^{*0} & \bar{D}^{*-} \\ K^{*-} & \bar{K}^{*0} & \phi & D_s^{*-} \\ D^{*0} & D^{*+} & D_s^{*+} & J/\psi \end{pmatrix}. \tag{5}$$

As shown in Eq. (2), where the matrix M could either be the pseudoscalar matrix (which is labeled as P in the following) or the vector matrix (labeled as V), we can have four different types of hadronization of the $\psi(3770)$ leading to PP, PV, VP, and VV. For example, when both M in Eq. (2) are pseudoscalar matrices we have

$$(M^2)_{4,4} \to (\phi\phi)_{4,4} = D^0\bar{D}^0 + D^+D^- + D_s^+D_s^-, \tag{6}$$

where we have neglected η_c^2 which is too heavy to be operative in the meson-meson loop that we shall consider below. It can be noticed that, since the $\psi(3770)$ has isospin zero, the final hadronized combination of $D^0\bar{D}^0 + D^+D^- + D_s^+D_s^-$ has isospin zero. Indeed, recalling the isospin doublets

$$\begin{pmatrix} D^+ \\ -D^0 \end{pmatrix}, \qquad \begin{pmatrix} \bar{D}^0 \\ D^- \end{pmatrix}, \qquad D_s^+, \qquad D_s^-, \tag{7}$$

Eq. (6) can be rewritten in a isospin-zero combination, which is

$$(PP)_{4,4}|I=0\rangle = \sqrt{2}|D\bar{D}, I=0\rangle + |D_s^+D_s^-\rangle. \tag{8}$$

Similarly, we can write the combinations coming from VP, PV, and VV

$$(PV)_{4,4} = D^0\bar{D}^{*0} + D^+D^{*-} + D_s^+D_s^{*-}, \tag{9}$$

$$(VP)_{4,4} = D^{*0}\bar{D}^0 + D^{*+}D^- + D_s^{*+}D_s^-, \tag{10}$$

$$(VV)_{4,4} = D^{*0}\bar{D}^{*0} + D^{*+}D^{*-} + D_s^{*+}D_s^{*-}. \tag{11}$$

Note that the combination $(PV)_{4,4} + (VP)_{4,4}$ that we get has the desired negative C-parity as it corresponds to the $\psi(3770)$ ($CD^* = -\bar{D}^*$ in our formalism).

In order to interpret the line shape of the $\psi(3770)$ we follow the steps of Ref. [15]. We consider the propagator of the vector meson $R \equiv \psi(3770)$

$$G_{\mu\nu}(p) = \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_R^2}\right)G(p), \tag{12}$$

with $G(p) = \frac{1}{p^2 - M_R^2 + i\epsilon}$.

The fact that $\psi(3770)$ couples to PP, PV, VP, VV indicates that $\psi(3770)$ will get a self-energy $\Pi(p)$ that we depict diagrammatically in Fig. 2. One can keep the covariant form of Π, but as shown in Ref. [15] only the transverse part of the propagator is relevant for the discussion here. We argue in a different way, with the same conclusion. In the loop one has $\Pi \sim \int d^4q G(q)G(p-q)$ and the relevant part of it that enters the shape is $Im\Pi$, where the two intermediate mesons are placed on shell. The evaluation of the cross section for $e^+e^- \to D^+D^-$ will place the D, $\bar{D}$ on shell and the D momenta are about 250 MeV. With this small momentum one can neglect the zero component of the ϵ^μ polarization vectors. Indeed, as shown in the Appendix of Ref. [19], the error induced by neglecting the zero component in this case is 0.7%. Hence we need only the spatial component, ϵ^i, and deal with $G_{ij}(p) = \delta_{ij}G(p)(i,j = 1,2,3)$. When we dress the propagator with the self-energy of the diagrams in Fig. 2 we obtain

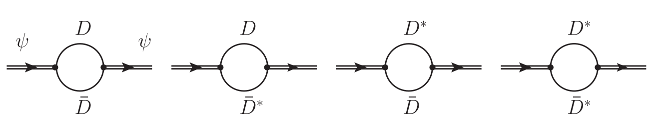

FIG. 2. Contribution to the ψ self-energy for the vector ψ propagator dressed with a meson-meson loop.

$$G(p) = \frac{1}{p^2 - M_R^2 - \Pi(p)}, \tag{13}$$

and we must evaluate $\Pi(p)$. Note that we write M_R rather than M_ψ because M_R is now the bare mass of the resonance. The novelty in the present work with respect to Ref. [15] is that we include the contribution of PV, VP, VV mesons in the self-energy. They only contribute indirectly to the line shape of the $\psi(3770)$ because ImΠ is zero in all these cases. However,

$$\mathrm{Im}G(p) = \frac{\mathrm{Im}\Pi(p)}{(p^2 - M_R^2 - \mathrm{Re}\Pi(p))^2 + (\mathrm{Im}\Pi(p))^2}, \tag{14}$$

and then ImΠ in the numerator comes only from $D\bar{D}$, but Re$\Pi(p)$ in the denominator comes from all the channels. Yet, the most novel thing here is that we will evaluate the probability that the $\psi(3770)$ contains PV, VP, and VV components in its wave function.

The evaluation of Π requires us to relate the strength of the PP, PV, VP, and VV couplings to the $\psi(3770)$. This we can do with the help of the $^3P_0$ model and the details are given in the Appendix. While the evaluation is involved, requiring elaborate sums of many Clebsch-Gordan (CG) coefficients, the results are very simple and we write the $\psi(3770) \to PP, PV, VP, VV$ couplings below

$$V_{\psi,(MM)_i} = g_{\psi,(MM)_i}\boldsymbol{\epsilon q}F(\boldsymbol{q}), \tag{15}$$

with

$$g_{\psi,(MM)_i} = AC_i(i = 1,2,3), \tag{16}$$

and $F(\boldsymbol{q})$ a form factor coming from the integrals of the quark radial wave functions discussed in the Appendix, where A in Eq. (16) is an unknown coefficient to be fitted to the data, and C_i are the coefficients listed in Table I.

The former coefficients are for $\psi(3770)$ assumed a 1^3D_1 state. The terms of the $\Pi(p)$ self-energy are evaluated as follows, see Fig. 3. For D^+D^-, for example, we have

$$-i\Pi(p) = \int \frac{d^4q}{(2\pi)^4}(-i)V_1(-i)V_2\frac{i}{q^2 - m_{D^+}^2 + i\varepsilon} \times \frac{i}{(p-q)^2 - m_{D^-}^2 + i\varepsilon}F(\boldsymbol{q})^2, \tag{17}$$

which gives us

$$\Pi(p) = ig_{\psi,D^+D^-}^2\int\frac{d^4q}{(2\pi)^4}\boldsymbol{q}^2\frac{1}{q^2 - m_{D^+}^2 + i\varepsilon} \times \frac{1}{(p-q)^2 - m_{D^-}^2 + i\varepsilon}F(\boldsymbol{q})^2. \tag{18}$$

The q^0 integration can be done analytically and then we get in the rest frame of the $\psi(3770)$ ($p^0 = \sqrt{s}$)

TABLE I. Coefficients C_i for different components in the loop.

PP	$\lvert C_1\rvert^2 = \frac{1}{12}$	D^+D^-, $D^0\bar{D}^0$, $D_s^+D_s^-$
PV, VP	$\lvert C_2\rvert^2 = \frac{1}{6}\times\frac{1}{4}$	$D^0\bar{D}^{*0}$, $D^{*0}\bar{D}^0$, $D^+\bar{D}^{*-}$, $D^{*+}D^-$, $D_s^+D_s^-$, $D_s^+D_s^{*-}$, $D_s^{*+}D_s^{*-}$
VV	$\lvert C_3\rvert^2 = \frac{1}{12}\times\frac{231}{30}$	$D^{*0}\bar{D}^{*0}$, $D^{*+}D^{*-}$, $D_s^{*+}D_s^{*-}$

$$\Pi(p) = g^2_{\psi,D^+D^-}\tilde{G}(p^0), \tag{19}$$

where $\tilde{G}(p^0)$ has the form

$$\begin{aligned}\tilde{G}(p^0) &= \int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_1(\boldsymbol{q})}\frac{1}{2\omega_2(\boldsymbol{q})}\boldsymbol{q}^2\frac{2\omega_1(\boldsymbol{q})+2\omega_2(\boldsymbol{q})}{(p^0)^2-(\omega_1(\boldsymbol{q})+\omega_2(\boldsymbol{q}))^2+i\varepsilon}F(\boldsymbol{q})^2\\ &= \int \frac{dq}{(2\pi)^2}\frac{\omega_1(\boldsymbol{q})+\omega_2(\boldsymbol{q})}{\omega_1(\boldsymbol{q})\omega_2(\boldsymbol{q})}\frac{\boldsymbol{q}^4}{(p^0)^2-(\omega_1(\boldsymbol{q})+\omega_2(\boldsymbol{q}))^2+i\varepsilon}F(\boldsymbol{q})^2,\end{aligned} \tag{20}$$

with $\omega_1(\boldsymbol{q}) = \sqrt{\boldsymbol{q}^2+m^2_{D^+}}$, $\omega_2(\boldsymbol{q}) = \sqrt{\boldsymbol{q}^2+m^2_{D^-}}$.

Let us note in passing that $\tilde{G}(p^0)$ has a structure similar to the $G(p^0)$ function used in the study of meson-meson interaction [6] except for the extra factor $\boldsymbol{q}^2$ that makes $\tilde{G}(p^0)$ more divergent in the absence of the form factor. However, this form factor makes it convergent and we shall come back to it.

With the former expression for $\tilde{G}(p)$ we can already write the $\psi(3770)$ self-energy as:

$$\begin{aligned}\Pi(p^0) = |A|^2\Big\{&\frac{1}{12}\tilde{G}(p^0)|_{D^0\bar{D}^0}+\frac{1}{12}\tilde{G}(p^0)|_{D^+D^-}+\frac{1}{24}\tilde{G}(p^0)|_{D^0\bar{D}^{*0}}+\frac{1}{24}\tilde{G}(p^0)|_{D^{*0}\bar{D}^0}\\ &+\frac{1}{24}\tilde{G}(p^0)|_{D^+D^{*-}}+\frac{1}{24}\tilde{G}(p^0)|_{D^{*+}\bar{D}^-}+\frac{231}{360}\tilde{G}(p^0)|_{D^{*0}\bar{D}^{*0}}+\frac{231}{360}\tilde{G}(p^0)|_{D^{*+}D^{*-}}\\ &+\frac{1}{12}\tilde{G}(p^0)|_{D_s^+D_s^-}+\frac{1}{24}\tilde{G}(p^0)|_{D_s^+D_s^{*-}}+\frac{1}{24}\tilde{G}(p^0)|_{D_s^{*+}D_s^-}+\frac{231}{360}\tilde{G}(p^0)|_{D_s^{*+}D_s^{*-}}\Big\}.\end{aligned} \tag{21}$$

Rather than evaluating the form factor $F(q)$ with quark wave function we take an empirical attitude as in Ref. [15], and let the data determine this form factor from the shape of the $e^+e^- \to D^+D^-$ cross section. Once again we follow Ref. [15] and write

$$\sigma = -g^2_{\psi e^+e^-}\mathrm{Im}D(M_{\mathrm{inv}}), \tag{22}$$

where M_{inv} is the e^+e^- invariant mass, $\sqrt{s}$, and $g_{\psi e^+e^-}$, as in Ref. [15], will also be determined from the strength of the cross section.

It is also useful to separate σ into the contribution of the different channels $(D^+D^-, D^0\bar{D}^0)$. Then we easily write:

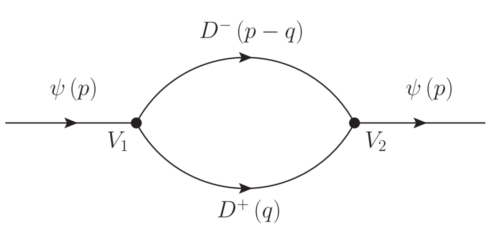

FIG. 3. The ψ propagator dressed with a D^+D^- loop as an example.

$$\sigma_i = -g^2_{\psi e^+e^-}\mathrm{Im}D_i(M_{\mathrm{inv}}), \tag{23}$$

where

$$\mathrm{Im}D_i = \frac{\mathrm{Im}\Pi_i(p)}{(p^2-M_R^2-\mathrm{Re}\Pi(p))^2+(\mathrm{Im}\Pi(p))^2}, \tag{24}$$

where $\Pi_i(p)$ is the contribution to $\mathrm{Im}\Pi(p^2)$ from the D^+D^- or $D^0\bar{D}^0$ channel [see Eq. (21)]. Note that in the denominator we have $\Pi(p)$, meaning that all channels are included here.

A. Meson-meson probabilities in the $\psi(3770)$ wave function

Let us write for convenience, as in Ref. [15],

$$\Pi'(p) = \Pi(p) - \mathrm{Re}(\Pi(M_\psi)), \tag{25}$$

which vanishes at $\sqrt{s} = M_\psi$, and with this choice we can write

$$G(p) = \frac{1}{p^2-M_\psi^2-\Pi'(p)}. \tag{26}$$

We can make an expansion around M_ψ and have

$$G(p) = \frac{1}{p^2 - M_\psi^2 - \mathrm{Re}(\Pi'(p)) - i\mathrm{Im}\Pi(p)} = \frac{1}{p^2 - M_\psi^2 - [\mathrm{Re}(\Pi'(p)) - \mathrm{Re}(\Pi'(M_\psi))] - i\mathrm{Im}\Pi(p)}, \quad (27)$$

since $\mathrm{Re}\Pi'(M_\psi) = 0$ and hence

$$G(p) \simeq \frac{1}{p^2 - M_\psi^2 - \frac{\partial \mathrm{Re}\Pi}{\partial p^2}|_{M_\psi^2}(p^2 - M_\psi^2) - i\mathrm{Im}\Pi(p)} = \frac{1}{(p^2 - M_\psi^2)(1 - \frac{\partial \mathrm{Re}\Pi}{\partial p^2}|_{M_\psi^2}) - i\mathrm{Im}\Pi(p)} = \frac{Z}{p^2 - M_\psi^2 - iZ\mathrm{Im}(p)}, \quad (28)$$

with

$$Z = \frac{1}{1 - \frac{\partial \mathrm{Re}\Pi(p^2)}{\partial p^2}|_{p^2=M_\psi^2}} \simeq 1 + \frac{\partial \mathrm{Re}\Pi}{\partial p^2}|_{p^2=M_\psi^2}. \quad (29)$$

This is the typical wave function renormalization [20] and Z is interpreted as the probability to still have the original vector when it is dressed by the meson-meson components. Conversely $1 - Z$ will be the meson-meson probability of the dressed vector. If $\frac{\partial \mathrm{Re}\Pi}{\partial p^2}$ is reasonably smaller than 1, one can make an expansion as in Eq. (29), and furthermore we have

$$1 - Z = -\frac{\partial \mathrm{Re}\Pi}{\partial p^2}\Big|_{p^2=M_\psi^2}, \quad (30)$$

such that $-\frac{\partial \mathrm{Re}\Pi}{\partial p^2}|_{p^2=M_\psi^2}$ can be interpreted as the meson-meson probability and in particular one can get the contribution of each channel:

$$P_{(MM)i} \simeq -\frac{\partial \mathrm{Re}\Pi_i(p^2)}{\partial p^2}\Big|_{p^2=M_\psi^2}, \quad (31)$$

where Π_i is the contribution of ith channel to Π.

In the case that one has open channels the interpretation of $P_{(MM)i}$ as a probability is not correct [21] and in Sec. III B we shall see the meaning of Eq. (31).

III. RESULTS

In Ref. [15] a form factor was used

$$f_\Lambda(\xi) = e^{-\xi/(4\Lambda^2)} e^{(m_{D^0}^2 + m_{D^+}^2)/(2\Lambda^2)}, \quad (32)$$

with $\xi = 4(\boldsymbol{q}^2 + m^2)$, that is the equivalent to our $F(\boldsymbol{q})^2$, and Λ was fitted to data. We get similar results using this form factor. In addition, we use two other form factors:

$$F(\boldsymbol{q})^2 = \frac{1 + (Rq_{\mathrm{on}})^2}{1 + (Rq)^2}, \quad (33)$$

and

$$F(\boldsymbol{q})^2 = \frac{1 + (Rq_{\mathrm{on}})^4}{1 + (Rq)^4}, \quad (34)$$

with q_{on} the following form for $D\bar{D}$

$$q_{\mathrm{on}} = \frac{\lambda^{1/2}(M_\psi^2, m_D^2, m_{\bar{D}}^2)}{2M_\psi}, \quad (35)$$

where λ is the usual Källén function, and the parameter R is fitted to the data in both cases. We have thus four parameters, as in Ref. [15], which in our case are M_ψ, $g_{\psi e^+e^-}$, A and R. M_ψ is of course very close to the nominal mass of the $\psi(3770)$, $g_{\psi e^+e^-}$ determines the strength of the cross section, A is related to the width of the resonance, and R determines the fall down of the resonance shape above the resonance peak. The parameters are fitted to the data of the cross section for $e^+e^- \to D\bar{D}$ [12–14].

Given the fact that in the Appendix we found that the form factor comes from an integral of the radial wave function of the quarks, and these are the same, independent of the different spin couplings, we assume this form factor to be the same for the PV, VP, and VV cases.

In Fig. 4 we show the results for the $e^+e^- \to D^+D^-$ cross section using the form factor of Eq. (34). The parameters used can be seen in Table II. As we can see, there is a good fit to the data, both above and below the peak, reflecting the asymmetry of the distribution, which does not have a Breit-Wigner form.

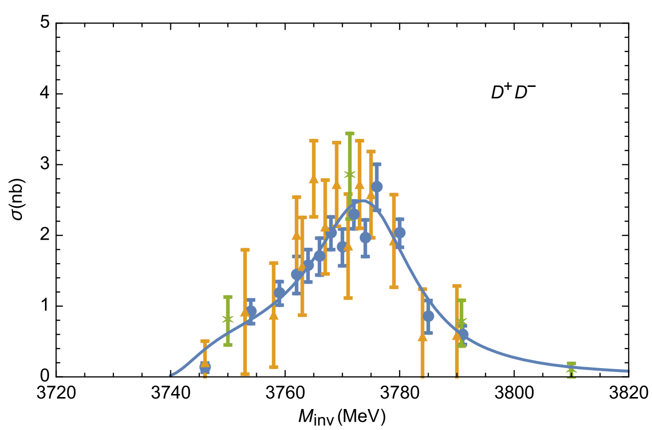

FIG. 4. Cross section of $e^+e^- \to D^+D^-$ fitted to the experimental data (circle [12], triangle [13], star [14]) using the form factor of Eq. (34).

TABLE II. Fitting parameters for Fig. 4.

M_R	3773 MeV
$g^2_{\psi e^+e^-}$	1.40×10^{-6}
R	0.0070 MeV^{-1}
$\lvert A\rvert^2$	1750

TABLE III. Fitting parameters for Fig. 7.

M_R	3773 MeV
$g^2_{\psi e^+e^-}$	1.55×10^{-6}
R	0.0030 MeV^{-1}
$\lvert A\rvert^2$	2756

We should note that the description of the data is a result of the parametrization, and in particular the fall down of the distribution above the peak is related to the parameter R. There is nothing fundamental in this interpretation of the asymmetry. However, the data and particularly the fall down above the threshold determine the range of the form factor, and this is important to make the integral $\tilde{G}(p)$ convergent, such that the probabilities that we obtain are a consequence of the peculiar shape of the $e^+e^- \to D^+D^-$ data. In this sense, the probabilities that we obtain are a prediction based on the $e^+e^- \to D^+D^-$ data, while those in Ref. [11] were based on a particular quark model.

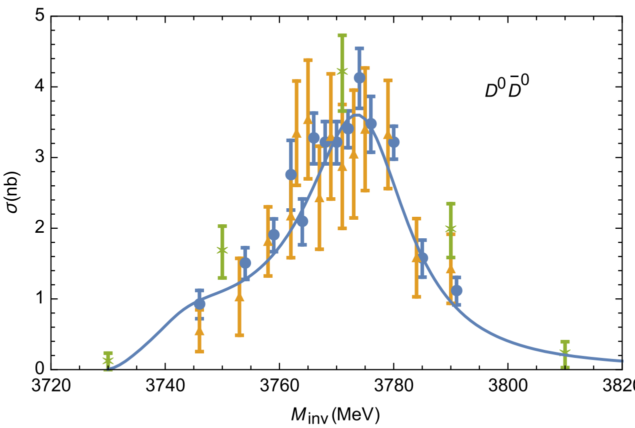

FIG. 5. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^0\bar{D}^0$ reaction, using the form factor of Eq. (34) and the parameters in Table II.

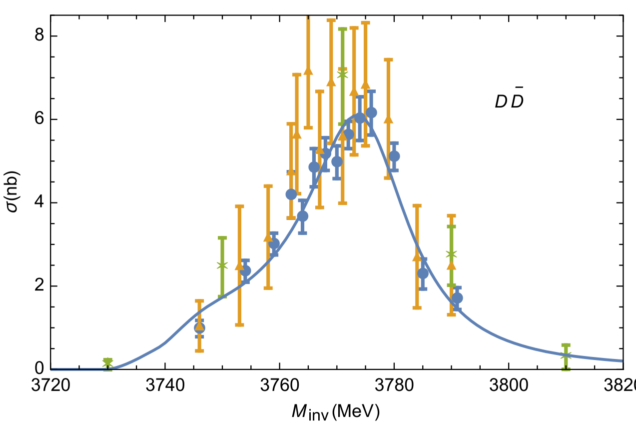

FIG. 6. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^+D^- + D^0\bar{D}^0$ reaction, using the form factor of Eq. (34) and the parameters in Table II.

It is also interesting to evaluate the $e^+e^- \to D^0\bar{D}^0$ cross section and compare with the data; this is done in Fig. 5. We can see that the agreement with the data is also very good, Note that once the $e^+e^- \to D^+D^-$ is fitted, we have no freedom for the $e^+e^- \to D^0\bar{D}^0$, so the latter one is a prediction of the approach.

In Fig. 6 we show the result for the $e^+e^- \to D^+D^- + D^0\bar{D}^0$. Obviously, since the individual cross sections are well produced, so is the sum of the two.

Next we show the result of the calculations using the form factor of Eq. (33). The parameters of the fit are shown in Table III. The result for $e^+e^- \to D^+D^-$, $e^+e^- \to D^0\bar{D}^0$, and $e^+e^- \to D^+D^- + D^0\bar{D}^0$ are shown in Figs. 7–9. We observe a good fit in the region above the peak, but not as good as before below it, although still comparable with the bulk of the data. Concerning our main goal, which is the evaluation of the meson-meson probabilities, the fall down of the cross section above the peak is acceptable. Actually, it is interesting to note that the low part of the spectrum could as well be filled by the contribution of the $\psi(2686)$ as noted in Refs. [22,23]. This resonance is below the $D\bar{D}$ threshold, but it has a very large width that allows it to stretch above it.

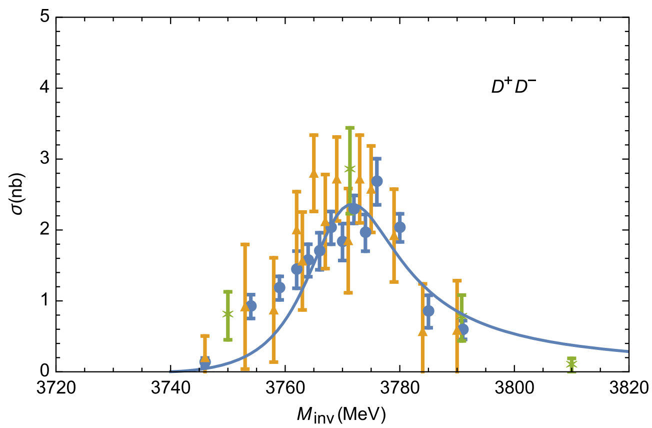

FIG. 7. Cross section of $e^+e^- \to D^+D^-$ fitted to the experimental data (circle [12], triangle [13], star [14]) using the form factor of Eq. (33).

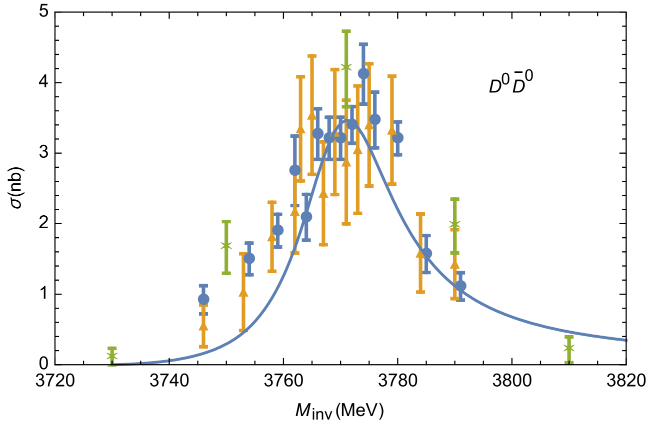

FIG. 8. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^0\bar{D}^0$ reaction, using the form factor of Eq. (33) and the parameters in Table III.

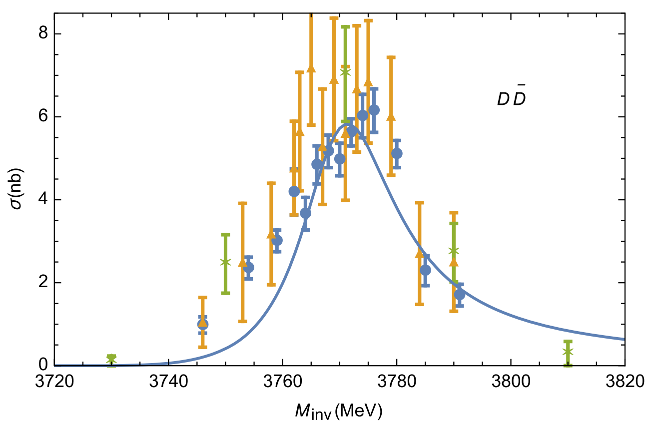

FIG. 9. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^+D^- + D^0\bar{D}^0$ reaction, using the form factor of Eq. (33) and the parameters in Table III.

A. Evaluation of the vector and meson-meson probabilities

In Table IV we show the probability of Eqs. (29) and (31) using the form factor of Eq. (34). What we see is that the probabilities of the D^+D^{*-} + c.c or D^0D^{*0} + c.c are practically zero. However, there is the unpleasant feature that $-\frac{\partial\Pi_{D\bar{D}}}{\partial p^2}|_{p^2=M_\psi^2}$ is complex, and $-\frac{\partial{\rm Re}\Pi_{D\bar{D}}}{\partial p^2}|_{p^2=M_\psi^2}$ $(P_{(MM)})$ is negative. The complex value is unavoidable when one has open channels, but that $-\frac{\partial{\rm Re}\Pi_{D\bar{D}}}{\partial p^2}|_{p^2=M_\psi^2}$, which provides the $D\bar{D}$ probability as we have seen, is negative, is unexpected and unacceptable (see, however, Sec. III B for a more proper interpretation). Fortunately, the value is very small, and could be admitted as an uncertainty related to the approximation implicit in Eq. (28). As a consequence of this negative number, the Z probability of having the original vector in the $\psi(3770)$ wave function is bigger than one, yet, by an amount of 9.3%, which tells us the uncertainties that we have in this approach. It is interesting to note that if we use the form factor of Ref. [15] written in Eq. (32) we get similar results.

TABLE IV. Meson-meson probabilities in the $\psi(3770)$ wave function with the form factor of Eq. (34).

Channels	$-\frac{\partial\Pi}{\partial p^2}\|_{p^2=M_\psi^2}$	$P_{(MM)}$	Z
$D^0\bar{D}^0$	$-0.0555 - 0.0406i$	-0.0555	1.059
D^+D^-	$-0.0879 - 0.0444i$	-0.0879	1.096
$D^0\bar{D}^{*0}$ + c.c	0.0083	0.0083	0.992
$D^+\bar{D}^{*-}$ + c.c	0.0074	0.0074	0.993
$D^{*0}\bar{D}^{*0}$	0.0164	0.0164	0.984
$D^{*+}D^{*-}$	0.0156	0.0156	0.985
$D_s^+D_s^-$	0.0040	0.0040	0.996
$D_s^+D_s^{*-}$ + c.c	0.0014	0.0014	0.999
$D_s^{*+}D_s^{*-}$	0.0054	0.0054	0.995
Total	$-0.0850 - 0.0846i$	-0.0850	1.093

In view of this, we use a form factor more in agreement with phenomenology, which is the one of Eq. (33). This form factor induces a correction to the width

$$\Gamma(s) \to \Gamma_0 \frac{1 + (Rq_{\rm on})^2}{1 + (R\bar{q})^2}, \tag{36}$$

with

$$\bar{q} = \frac{\lambda^{1/2}(s, m_D^2, m_{\bar{D}}^2)}{2\sqrt{s}}, \tag{37}$$

where Γ_0 is the width evaluated at $\sqrt{s} = M_\psi$. This factor is the Blatt-Weisskopf barrier penetration factor [24], commonly used to write the width in usual Breit-Wigner amplitudes. In view of this, we can give more credit to the results that come from this factor. The results can be seen in Table V.

TABLE V. Meson-meson probabilities in the $\psi(3770)$ wave function with the form factor of Eq. (33) [Note that the sum of the total $P_{(MM)}$ and Z is not exactly 1 because of the approximation of Eq. (29)].

Channels	$-\frac{\partial\Pi}{\partial p^2}\|_{p^2=M_\psi^2}$	$P_{(MM)}$	Z
$D^0\bar{D}^0$	$0.0019 + 0.1814i$	0.0019	0.998
D^+D^-	$0.0295 + 0.1862i$	0.0295	0.971
$D^0\bar{D}^{*0}$ + c.c	$0.0264 + 0.0003i$	0.0264	0.974
$D^+\bar{D}^{*-}$ + c.c	$0.0244 + 0.0002i$	0.0244	0.976
$D^{*0}\bar{D}^{*0}$	$0.0708 + 0.0004i$	0.0708	0.934
$D^{*+}D^{*-}$	$0.0681 + 0.0004i$	0.0681	0.936
$D_s^+D_s^-$	$0.0152 + 0.0001i$	0.0152	0.985
$D_s^+D_s^{*-}$ + c.c	0.0065	0.0065	0.994
$D_s^{*+}D_s^{*-}$	0.0268	0.0268	0.974
Total	$0.2696 + 0.3690i$	0.2696	0.787

TABLE VI. Fitting parameters for Fig. 10.

M_R	3775 MeV
$g^2_{\psi e^+e^-}$	1.25×10^{-6}
R	0.0029 MeV^{-1}
$\vert A\vert^2$	1700

Now we can see that all the probabilities are positive and the Z probability is smaller than one. Yet, the results that one obtains indicate small meson-meson probabilities and a total probability for Z to have still a vector component is about 80%.

We can see that in Figs. 7, 8, evaluated with the form factor of Eq. (33) the slope of the cross section above the peak is smaller than in the corresponding Figs. 4 and 5, evaluated with the form factor of Eq. (34). We stated our preference for the form factor of Eq. (33), more in agreement with phenomenology. In view of that we choose a different set of parameters that make the slope above the peak more similar in all cases, paying the price of not having such good agreement at low energies. However, for the meson-meson probabilities that we are concerned about, the slope above the peak is what matters. The parameters of such a set are shown in Table VI, and the results are shown in Figs. 10–12 and Table VII. In this case we find $Z \sim 0.854$. This is a reasonable number, but in view of the results in Table V with the former fit, we can settle the value of Z within 0.80–0.85, which is a reasonable range of uncertainty.

This result is very valuable and we consider it the most important output of the work. There is a continuous debate about the nature of the hadron resonances and it is long since the ideal picture of mesons as pure $q\bar{q}$ and baryons as qqq has been abandoned. With the advent of hadrons in the charm and bottom sectors, the evidence for more complex structures is appalling [2,3]. Yet, in spite of this, an elaborate study combining elements of QCD, large N_c limits and phenomenology concludes that while low lying

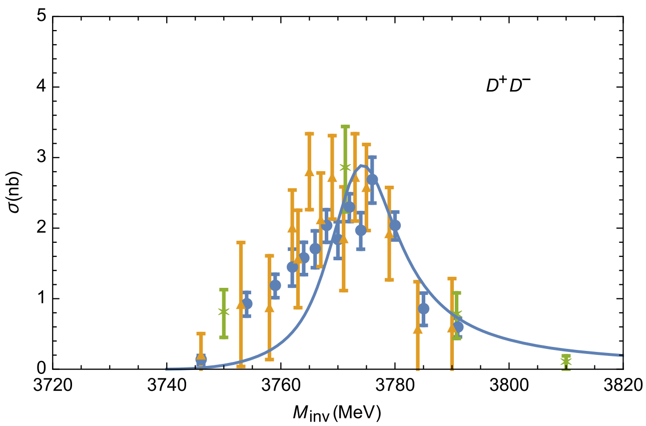

FIG. 10. Cross section of $e^+e^- \to D^+D^-$ fitted to the experimental data (circle [12], triangle [13], star [14]) using the form factor of Eq. (33).

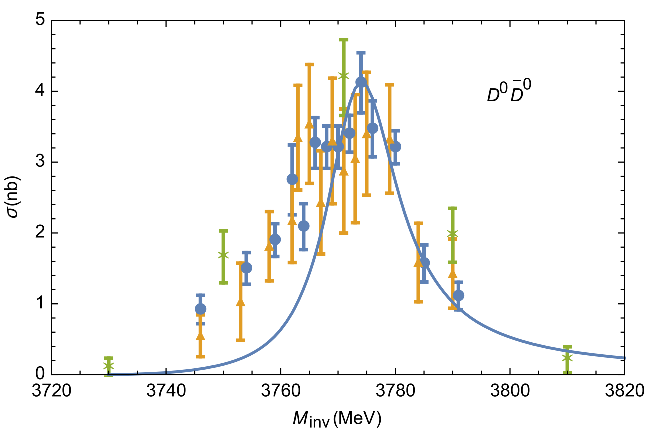

FIG. 11. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^0\bar{D}^0$ reaction, using the form factor of Eq. (33) and the parameters in Table VI.

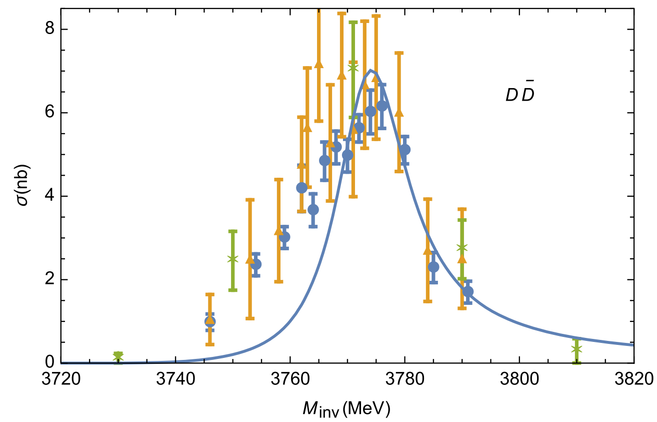

FIG. 12. The comparison of our result with the experimental data (circle [12], triangle [13], star [14]) for the cross section of $e^+e^- \to D^+D^- + D^0\bar{D}^0$ reaction, using the form factor of Eq. (33) and the parameters in Table VI.

TABLE VII. Meson-meson probabilities in the $\psi(3770)$ wave function with the form factor of Eq. (33).

Channels	$-\frac{\partial\Pi}{\partial p^2}\vert_{p^2=M_\psi^2}$	$P_{(MM)}$	Z
$D^0\bar{D}^0$	$0.0001 + 0.1150i$	0.0019	0.998
D^+D^-	$0.0168 + 0.1178i$	0.0295	0.971
$D^0\bar{D}^{*0}$ + c.c	$0.0172 + 0.0002i$	0.0264	0.974
$D^+\bar{D}^{*-}$ + c.c	$0.0158 + 0.0001i$	0.0244	0.976
$D^{*0}\bar{D}^{*0}$	$0.0458 + 0.0003i$	0.0708	0.934
$D^{*+}D^{*-}$	$0.0440 + 0.0002i$	0.0681	0.936
$D_s^+D_s^-$	0.0098	0.0152	0.985
$D_s^+D_s^{*-}$ + c.c	0.0042	0.0065	0.994
$D_s^{*+}D_s^{*-}$	0.0172	0.0268	0.974
Total	$0.1709 + 0.2336i$	0.1709	0.854

scalar mesons, like the σ, $f_0(980)$, $\cdots$ are completely off the $q\bar{q}$ picture, the vector mesons are largely $q\bar{q}$ states [10]. Our result comes in handy when some calculations could make us lose confidence in this picture. Indeed, in Ref. [11], where a calculation within a quark model was done to assess the relevance of the meson-meson components in the vector mesons, even the J/ψ was found to have a Z probability of only 65%, implying that more massive ψ vectors could have an even smaller Z probability. The result of the present paper incorporating the features of the $\psi(3770)$ shape in the $e^+e^- \to D\bar{D}$ reactions, demanded the presence of a form factor that has a consequence the small meson-meson probabilities and the large Z value.

B. Interpretation of $(-)\partial\Pi_i(p^2)/\partial p^2$ for open channels

Let us begin with a clear statement: the interpretation of $(-)\partial\Pi_i(p^2)/\partial p^2$ as a probability when the ith channel is an open channel does not hold, simply because the wave function of the open channel is not normalizable. Asymptotically it goes as e^{ikr}/r and $\int r^2 dr |e^{ikr}/r|^2 = \infty$. Yet, there is a clear meaning to $(-)\partial\Pi_i(p^2)/\partial p^2$ which is derived in Ref. [21] and we take the opportunity to discuss it in the present context.

The first steps on the discussion of molecular probabilities in certain states were given by Weinberg in his celebrated work about the compositeness of the deuteron [25], which has been reviewed and extended recently in relation with the nature of many resonances as composite, or dynamically generated, states [26–30] (see Ref. [31] for a review on the subject). The problem with complex $(-)\partial\Pi_i/\partial p^2$ values to be interpreted as a probability has always been present with no clear answer. Finite and real values are obtained in a finite volume box and a discussion of its meaning is given in Ref. [32] where a more complete reference to works on the subject can be found.

We start with a simple derivation of the Weinberg compositeness condition adapted to our case dealing with relativistic mesons. Assume that we have a potential V in momentum space which generates a bound state at energy E_α. The scattering matrix has a pole at $s_\alpha = E_\alpha^2$, hence

$$T = \frac{V}{1 - VG} = \frac{1}{V^{-1} - G} \simeq \frac{g^2}{s - s_\alpha}, \tag{38}$$

with G the meson-meson loop function, with the last equation valid close to the pole at s_α. g^2 by means of L'Hôpital's rule as

$$g^2 = \lim_{s \to s_\alpha} \frac{s - s_\alpha}{V^{-1} - G} = -\frac{1}{\frac{\partial G}{\partial s}}, \tag{39}$$

where we assume V to be energy independent (see generalization for V energy dependent in [31]). Hence

$$-g^2 \frac{\partial G}{\partial s} = 1, \tag{40}$$

which is the expression for a composite state, stating that $-g^2 \frac{\partial G}{\partial s}$ is the probability to have this meson-meson component in the bound state. The rule is generalized to coupled channels [28,31] for dynamically generated states as

$$\sum_i (-1) g_i^2 \frac{\partial G_i}{\partial s} = 1, \tag{41}$$

and each term represents the probability to have the bound state in the corresponding channel. The derivation of Ref. [28] is done for s-wave interaction, but this is generalized to any partial wave in Ref. [33], and the G function incorporates an extra q^{2l} factor in the integrand, as we have in Eq. (20) for $l = 1$.

A further extension of Eq. (41) for the case of open channels and resonant states is done in Refs. [21,27] and Eq. (41) is proved to hold also in this case, but the couplings and G_i have to be evaluated at the pole in the second Riemann sheet. Since g_i can be complex and so is G_i for the open channels, the corresponding terms in Eq. (41) are complex but the sum is 1 which means there is an extra cancellation of the imaginary parts and then

$$\sum_i (-1) \mathrm{Re}\left(g_i^2 \frac{\partial G_i}{\partial s} \right) = 1. \tag{42}$$

One might be now tempted to associate $\mathrm{Re}(g_i^2 \frac{\partial G}{\partial s})$ to a probability, but this cannot be done for open channels as we showed at the beginning of the subsection. It is then interesting to see the meaning of these terms. The answer to this problem is subtle and is discussed in Ref. [21] (see Section 5 of that work). The open channels can be taken into account by means of a Hamiltonian H which is no longer Hermitian and its eigenstates are not orthogonal. One must introduce a biorthogonal basis of eigenstates of $H, |\lambda_n\rangle$ and of $H^+, |\bar{\lambda}_n\rangle$ and one has

$$\langle \bar{\lambda}_n | \lambda_m \rangle = \delta_{nm}. \tag{43}$$

One also finds there that for the case at work H is not Hermitian but is symmetric and then $|\bar{\lambda}_n\rangle = |\lambda_n^*\rangle$. This has as a consequence that in the derivations of the sum rule in Refs. [28,33] one must substitute

$$\langle \psi_i | \psi_i \rangle \to \langle \bar{\psi}_i | \psi_i \rangle = \int d^3p (\psi_i^*(p))^* \psi_i(p) = \int d^3p \psi_i^2(p), \tag{44}$$

and one also finds that with an appropriate prescription for the global phase of the wave function (which makes the wave function real in the case of a bound state)

$$\langle\bar{\psi}_i|\psi_i\rangle = -g_i^2\frac{\partial G_i}{\partial s}. \quad (45)$$

Hence, the term $\mathrm{Re}(g_i^2\frac{\partial G_i}{\partial s})$ in Eq. (42) has to be interpreted as

$$\mathrm{Re}\left(g_i^2\frac{\partial G_i}{\partial s}\right) \simeq \mathrm{Re}\int d^3p\psi_i^2(p). \quad (46)$$

Note that we get now the integral of $\psi_i^2(p)$ rather than $|\psi_i(p)|^2$ and even for open channels this magnitude is finite since

$$\psi_i(r)^2 \overset{r\to\infty}{\rightarrow} \frac{1}{r}e^{-ik_Rr}e^{k_Ir}, \quad (47)$$

with k_R, k_I the real and imaginary parts of the complex momentum at the pole and now $(e^{-ik_Rr})^2 = e^{-2ik_Rr}$ is a rapidly oscillating function that makes the contribution at large r vanish (yet, the finiteness is better seen in momentum space [21]).

Once the meaning of the terms of the sum rule are clarified, it is clear that Eq. (46) provides a measure of the strength of the wave function in the region of the interaction before the mesons become free propagating particles. In the evaluation of physical processes only the interaction region would be relevant and hence $\mathrm{Re}(g_i^2\frac{\partial G}{\partial s})$ provides us with a measure of the "weight" or "strength" of this component, not a probability in the strict sense which would be infinite.

We have so far assumed that the states we obtain are fully composite states from several coupled channels. In the real world there can be some genuine component, or preexisting component, like in the case we investigate, where the genuine component would be a vector state and the meson-meson components are those we evaluated. In this case the sum rule has to be substituted by [21,27]

$$\sum_i(-1)\mathrm{Re}\left(g_i^2\frac{\partial G}{\partial s}\right) = 1 - Z, \quad (48)$$

where Z is the probability of the genuine component.

Another small point is that since we did not look for poles of the states we evaluated G in the real axis instead of G^{II} in the second Riemann sheet where the sum rule holds. For not so large width as we have here, $G^{II}\sim G^*$ and since g_i are real in our case $\mathrm{Re}(g_i^2G^{II}) \simeq \mathrm{Re}(g_i^2G)$ and we do not have to worry about this detail.

The former discussion has shown the meaning of the $\mathrm{Re}(g_i^2\frac{\partial G}{\partial s})$ terms that we have calculated and this gives full meaning to our calculations in the sense that the weight of the vector component is very large and that of the meson-meson component very small.

IV. SUMMARY AND DISCUSSION

We have performed an evaluation of the meson-meson components in the $\psi(3770)$ wave function, considering PP, PV, VP, and VV components. We found that the determination of such probabilities was much tied to the shape of the $e^+e^-\to D\bar{D}$ reaction, which we described in terms of the $\psi(3770)$ self-energy due to the meson-meson components. Indeed, the shape of the cross section for this reaction determined the range of a form factor that was determined in the evaluation of the meson-meson probabilities of the $\psi(3770)$ wave function. Within uncertainties we found that the Z probability of a vector component in the $\psi(3770)$ is of the order of 80%–85% and the individual meson-meson components are small. This finding is very important, extracting from this phenomenological study the same conclusion obtained from QCD and large N_c behavior, plus meson-meson scattering data, that vector mesons are largely $q\bar{q}$ objects [10]. This is also in line with Z evaluation for the ρ with a different method which gives $Z\sim 0.75$, even with such a large width for the decay to two pions [33]. We also made a discussion that in the case of open channels the concept of probability has to be abandoned but the magnitudes evaluated still provide a measure of the relevant weight of these components such we can still conclude the dominant weight of the vector component in the $\psi(3770)$.

ACKNOWLEDGMENTS

We thank Susana Coito for providing us with the numerical values of the experimental data. Q. X. Y. acknowledges the support from the National Natural Science Foundation of China (Grants No. 11775024 and No. 11575023). W. H. L. acknowledges the support from the National Natural Science Foundation of China (Grants No. 11565007 and No. 11847317). This work is partly supported by the Spanish Ministerio de Economia y Competitividad and European FEDER funds under Contracts No. FIS2017-84038-C2-1-P B and No. FIS2017-84038-C2-2-P B, and the Generalitat Valenciana in the program Prometeo II-2014/068, and the project Severo Ochoa of Instituto de Física Corpuscular, SEV-2014-0398.

APPENDIX: EVALUATION OF THE $\psi(3770)$ COUPLING TO $D\bar{D}$,$D\bar{D}^*$,$D^*\bar{D}$,$D^*\bar{D}^*$

According to Ref. [5] the $\psi(3770)$ is a 1^3D_1 state. This means radial wave function in the ground state, spin 1, and angular momentum of the two quarks $L=2$, coupling later $L=2$ with $S=1$ to give $J=1$. We start with the $c\bar{c}$ spin wave function

$$|S\tilde{M}'\rangle = |1\tilde{M}'\rangle = \sum_{m,m'}\mathcal{C}(s_1,s_2,S;m,m',\tilde{M}')|s_1,m\rangle|s_2,m'\rangle$$
$$= \sum_{m,m'}\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;m,m',\tilde{M}'\right)\left|\frac{1}{2},m\right\rangle\left|\frac{1}{2},m'\right\rangle, \quad (A1)$$

where s_1 and s_2 correspond to the spin of c and $\bar{c}$ in Fig. 1, and m, m' are their third components respectively, while S

and $\tilde{M}'$ are the total spin and third component of $c\bar{c}$. Then after coupling the spin part to the orbital part of $c\bar{c}$, we have

$$|J\tilde{M}\rangle = |1\tilde{M}\rangle = \sum_{M_3',\tilde{M}'} \mathcal{C}(L,S,J;M_3',\tilde{M}',\tilde{M}) Y_{L,M_3'}(\hat{\boldsymbol{r}})|S,\tilde{M}'\rangle$$
$$= \sum_{M_3',\tilde{M}'} \mathcal{C}(2,1,1;M_3',\tilde{M}',\tilde{M}) Y_{2,M_3'}(\hat{\boldsymbol{r}})|1,\tilde{M}'\rangle. \quad \text{(A2)}$$

We do the same to couple the spin and orbital angular momentum of the $q\bar{q}$ vacuum state ${}^3P_0$ in Fig. 1, as done in Refs. [34,35],

$$|1S_3\rangle = \sum_s \mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;s,S_3-s\right)\left|\frac{1}{2},s\right\rangle\left|\frac{1}{2},S_3-s\right\rangle, \quad \text{(A3)}$$

and we combine this state, $|1S_3\rangle$, with the $L=1$ state $Y_{1,M_3}(\hat{\boldsymbol{r}})$ to give $J=0$,

$$|00\rangle = \sum_{M_3} \mathcal{C}(1,1,0;M_3,S_3) Y_{1,M_3}(\hat{\boldsymbol{r}})|1,S_3\rangle, \quad \text{(A4)}$$

implying $M_3+S_3=0$, i.e., $M_3=-S_3$, which allows us to rewrite Eq. (A4) as follows

$$|00\rangle = \sum_{S_3} \mathcal{C}(1,1,0;-S_3,S_3) Y_{1,-S_3}(\hat{\boldsymbol{r}})|1,S_3\rangle$$
$$= \sum_{S_3} (-1)^{1+S_3}\frac{1}{\sqrt{3}} Y_{1,-S_3}(\hat{\boldsymbol{q}})|1,S_3\rangle. \quad \text{(A5)}$$

In addition we have the spatial matrix element, where the c, $\bar{c}$ quark states are in their ground state. Then we have

$$ME(\boldsymbol{q}) = \int d^3\boldsymbol{r}\varphi_c(r)\varphi_q(r)\varphi_{\bar{q}}(r)\varphi_{\bar{c}}(r)e^{i\boldsymbol{q}\cdot\boldsymbol{r}} Y_{1,-S_3}(\hat{\boldsymbol{r}})Y_{2,M_3'}(\hat{\boldsymbol{r}}), \quad \text{(A6)}$$

where $\boldsymbol{q}$ is the exchanged momentum between the two mesons produced after the hadronization, and $e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ can be expanded as

$$e^{i\boldsymbol{q}\cdot\boldsymbol{r}} = 4\pi\sum_l i^l j_l(qr) Y_{l\mu}(\hat{\boldsymbol{q}}) Y^*_{l\mu}(\hat{\boldsymbol{r}}). \quad \text{(A7)}$$

The coupling rule for spherical harmonics permits an easy way of combining three spherical harmonic functions as we show in the following equation, where two of them come from Eq. (A6) and the other one, $Y^*_{l,\mu}(\hat{\boldsymbol{r}})$, from Eq. (A7). After integrating over the full solid angle, we arrive at [36]

$$\int d\Omega Y^*_{l\mu}(\hat{\boldsymbol{r}}) Y_{1,-S_3}(\hat{\boldsymbol{r}}) Y_{2,M_3'}(\hat{\boldsymbol{r}})$$
$$= \left(\frac{15}{4\pi(2l+1)}\right)^{1/2} \mathcal{C}(2,1,l;M_3',-S_3,\mu)\mathcal{C}(2,1,l;0,0,0), \quad \text{(A8)}$$

where for parity reasons $2+1+l$ must be even, hence, $l=1,3$, but $l=1$ is required to have a P-wave coupling of J/ψ to $D\bar{D}$ at the end, such that we obtain (where we use $\mathcal{C}(2,1,1;0,0,0) = -\sqrt{\frac{2}{5}}$)

$$ME(\boldsymbol{q}) = -4\pi i Y_{1,M_3'-S_3}(\hat{\boldsymbol{q}})\sqrt{\frac{2}{4\pi}}\mathcal{C}(2,1,1;M_3',-S_3,M_3'-S_3)$$
$$\times \int r^2 dr \varphi_c(r)\varphi_q(r)\varphi_{\bar{q}}(r)\varphi_{\bar{c}}(r) j_1(qr). \quad \text{(A9)}$$

Since $j_1(qr)$ goes as qr for small values of qr, $ME(\boldsymbol{q})$ grows linearly q for small q, and for that reason we rewrite $ME(\boldsymbol{q})$ as

$$ME(\boldsymbol{q}) = -\frac{4\pi i}{3} q Y_{1,M_3'-S_3}(\hat{\boldsymbol{q}})\sqrt{\frac{2}{4\pi}}\mathcal{C}(2,1,1;M_3',-S_3,M_3'-S_3)$$
$$\times \int r^2 dr \prod_i \varphi_i(r)\frac{3j_1(qr)}{qr} r, \quad \text{(A10)}$$

where the factor $\frac{3j_1(qr)}{qr}$ goes to 1 as qr approaches 0 and is a smooth function, such that the integral in Eq. (A10) is a smooth function of q for small q, the typical form of the form factors and the form that we will take for our empirical form factors. We can write $qY_{1,M_3'-S_3}(\hat{\boldsymbol{q}})$ in Eq. (A10) as $\sqrt{\frac{3}{4\pi}} q_{M_3'-S_3}$ (in spherical basis), which accounts for the vector coupling to two pseudoscalars.

At the same time, by coupling the vacuum state $|00\rangle$ with c, $\bar{c}$ spins we can obtain the final angular momenta of the two mesons produced, $|J_1M_2\rangle$ and $|J_2M_2\rangle$, which is accomplished by means of the Clebsch-Gordan coefficients,

$$|J_1M_1\rangle = \sum_m \mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_1;m,s,M_1\right)\left|\frac{1}{2},m\right\rangle\left|\frac{1}{2},s\right\rangle, \quad \text{(A11)}$$

$$|J_2M_2\rangle = \sum_{m'} \mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_2;S_3-s,m',M_2\right)\left|\frac{1}{2},S_3-s\right\rangle\left|\frac{1}{2},m'\right\rangle, \quad \text{(A12)}$$

where we obtain the constraints: $m+s=M_1$, $S_3-s+m'=M_2$, leading to $m=M_1-s$, $m'=M_2-S_3+s$. Further constraints between S_3 and M_1, M_2 can be derived with the help of Eq. (A1), and S_3 satisfies the relation, $S_3=M_1+M_2-\tilde{M}'$.

Finally, we can write down the matrix element of the transition from $|1\tilde{M}'\rangle$ to $|J_1M_1\rangle|J_2M_2\rangle$ by combining Eqs. (A1), (A2), (A3), (A5), (A11), and (A12),

$$ME = -\frac{4\pi i}{3}\sqrt{\frac{2}{4\pi}}\sum_{\tilde{M}'}\sum_{s}\sum_{S_3}\mathcal{C}(2,1,1;M_3',-S_3,M_3'-S_3)\mathcal{C}(2,1,1;M_3',\tilde{M}',\tilde{M})qY_{1,M_3'-S_3}(\hat{\boldsymbol{q}})$$
$$\times\,\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;M_1-s,M_2-S_3+s,\tilde{M}'\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;s,S_3-s,S_3\right)(-1)^{1+S_3}\frac{1}{\sqrt{3}}$$
$$\times\,\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_1;M_1-s,s,M_1\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_2;S_3-s,M_2-S_3+s,M_2\right). \tag{A13}$$

Now we use $S_3 = M_1 + M_2 - \tilde{M}'$ and the above equation can be rewritten as,

$$ME = -\frac{4\pi i}{3}\sqrt{\frac{2}{4\pi}}\sum_{s}\sum_{\tilde{M}'}\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1-M_2,\tilde{M}-M_1-M_2)$$
$$\times\,\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}',\tilde{M})qY_{1,\tilde{M}-M_1-M_2}(\hat{\boldsymbol{q}})(-1)^{1+M_1+M_2-\tilde{M}'}\frac{1}{\sqrt{3}}$$
$$\times\,\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;M_1-s,\tilde{M}'-M_1+s,\tilde{M}'\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;s,M_1+M_2-\tilde{M}'-s,M_1+M_2-\tilde{M}'\right)$$
$$\times\,\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_1;M_1-s,s,M_1\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_2;M_1+M_2-\tilde{M}'-s,\tilde{M}'-M_1+s,M_2\right), \tag{A14}$$

In Eq. (A14) there are four CG coefficients that depend on s. In order to get an expression with three CG coefficients to be written in terms of Racah coefficients we proceed as follows. First, we need to permute some indices in the fourth CG coefficient in Eq. (A14) as Ref. [36],

$$\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;s,M_1+M_2-\tilde{M}'-s,M_1+M_2-\tilde{M}'\right) = (-1)^{1/2-s}\sqrt{\frac{3}{2}}\mathcal{C}\left(1,\frac{1}{2},\frac{1}{2};M_1+M_2-\tilde{M}',-s,M_1+M_2-\tilde{M}'-s\right), \tag{A15}$$

and together with the last one in Eq. (A14), we can convert them into other two CG coefficients where only one CG coefficient depends on s [36],

$$\mathcal{C}\left(1,\frac{1}{2},\frac{1}{2};M_1+M_2-\tilde{M}',-s,M_1+M_2-\tilde{M}'-s\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_2;M_1+M_2-\tilde{M}'-s,\tilde{M}'-M_1+s,M_2\right)$$
$$= \sum_{j''}\sqrt{2(2j''+1)}\,\mathcal{W}\left(1,\frac{1}{2},J_2,\frac{1}{2};\frac{1}{2},j''\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},j'';-s,-M_1+\tilde{M}'+s,-M_1+\tilde{M}'\right)$$
$$\times\,\mathcal{C}(1,j'',J_2;M_1+M_2-\tilde{M}',-M_1+\tilde{M}',M_2), \tag{A16}$$

where $\mathcal{W}$ is a Racah coefficient [36]. Similarly, we need to permute indices of the third CG coefficient in Eq. (A14) and the first CG coefficient in Eq. (A16) before we move on to the next combination,

$$\mathcal{C}\left(\frac{1}{2},\frac{1}{2},1;M_1-s,\tilde{M}'-M_1+s,\tilde{M}'\right) = (-1)^{1+1/2-M_1+\tilde{M}'+s}\sqrt{\frac{3}{2}}\mathcal{C}\left(1,\frac{1}{2},\frac{1}{2};\tilde{M}',M_1-\tilde{M}'-s,M_1-s\right), \tag{A17}$$

and

$$\mathcal{C}\left(\frac{1}{2},\frac{1}{2},j'';-s,-M_1+\tilde{M}'+s,-M_1+\tilde{M}'\right) = [(-1)^{1/2+1/2-j''}]^2\mathcal{C}\left(\frac{1}{2},\frac{1}{2},j'';M_1-\tilde{M}'-s,s,M_1-\tilde{M}'\right). \tag{A18}$$

We combine now the three CG coefficients from Eqs. (A17) and (A18) and the fifth CG coefficient in Eq. (A14) [36], and since the phase does not depend on s, we can write

$$\sum_s \mathcal{C}\left(1,\frac{1}{2},\frac{1}{2};\tilde{M}',M_1-\tilde{M}'-s,M_1-s\right)\mathcal{C}\left(\frac{1}{2},\frac{1}{2},J_1;M_1-s,s,M_1\right)\times\mathcal{C}\left(\frac{1}{2},\frac{1}{2},j'';M_1-\tilde{M}'-s,s,M_1-\tilde{M}'\right)$$
$$=\sqrt{2(2j''+1)}\mathcal{W}\left(1,\frac{1}{2},J_1,\frac{1}{2};\frac{1}{2},j''\right)\mathcal{C}(1,j'',J_1;\tilde{M}',M_1-\tilde{M}',M_1), \tag{A19}$$

such that Eq. (A14) can be rewritten as

$$ME=-\frac{4\pi i}{3}qY_{1,\tilde{M}-M_1-M_2}(\hat{\boldsymbol{q}})\sqrt{\frac{2}{4\pi}}\sum_{\tilde{M}'}\sum_{j''}[\sqrt{3}(-1)^{1+M_2}(2j''+1)]\prod_{i=1}^{4}\mathcal{C}_i\prod_{j=1}^{2}\mathcal{W}_j, \tag{A20}$$

where $\prod_{i=1}^4\mathcal{C}_i\prod_{j=1}^2\mathcal{W}_j$ can be expressed explicitly as follows,

$$\prod_{i=1}^{4}\mathcal{C}_i\prod_{j=1}^{2}\mathcal{W}_j=\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1-M_2,\tilde{M}-M_1-M_2)\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}',\tilde{M})$$
$$\times\mathcal{C}(1,j'',J_2;M_1+M_2-\tilde{M}',-M_1+\tilde{M}',M_2)\mathcal{C}(1,j'',J_1;\tilde{M}',M_1-\tilde{M}',M_1)$$
$$\times\mathcal{W}\left(1,\frac{1}{2},J_1,\frac{1}{2};\frac{1}{2},j''\right)\mathcal{W}\left(1,\frac{1}{2},J_2,\frac{1}{2};\frac{1}{2},j''\right). \tag{A21}$$

Next we begin evaluating different cases with J_1 and J_2 assigned to particular values, we start with the case where $J_1=0$, $J_2=0$, which corresponds to the PP coupling,

(1) PP: $J_1=0$, $J_2=0$

It implies $M_1=0$, $M_2=0$, and Eq. (A16) leads us to fact that j'' can only be 1 in this case. With these particular quantum numbers we can easily obtain the Racah coefficients,

$$\mathcal{W}\left(1,\frac{1}{2},0,\frac{1}{2};\frac{1}{2},1\right)=\frac{1}{\sqrt{6}}, \tag{A22}$$

and two of the CG coefficients

$$\mathcal{C}(1,1,0;-\tilde{M}',\tilde{M}',0)=(-1)^{1+\tilde{M}'}\sqrt{\frac{1}{3}} \tag{A23}$$

$$\mathcal{C}(1,1,0;\tilde{M}',-\tilde{M}',0)=(-1)^{1-\tilde{M}'}\sqrt{\frac{1}{3}}. \tag{A24}$$

Permuting the first two indices in the first two CG coefficients of Eq. (A21) we obtain the following equation for $|ME|^2$ in this case

$$\left|-\frac{4\pi i}{3}\right|^2 q^2Y_{1,\tilde{M}}(\hat{\boldsymbol{q}})Y^*_{1,\tilde{M}}(\hat{\boldsymbol{q}})\frac{2}{4\pi}[\sqrt{3}\times 3]^2$$
$$\times\left[\sum_{\tilde{M}'}\mathcal{C}(1,2,1;\tilde{M}',\tilde{M}-\tilde{M}',\tilde{M})^2\right]^2\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)^2, \tag{A25}$$

further simplification can be done by replacing $Y_{1,\tilde{M}}(\hat{\boldsymbol{q}})Y^*_{1,\tilde{M}}(\hat{\boldsymbol{q}})$ with

$$\frac{1}{4\pi}\int d\Omega Y_{1,\tilde{M}}(\hat{\boldsymbol{q}})Y^*_{1,\tilde{M}}(\hat{\boldsymbol{q}})=\frac{1}{4\pi}, \tag{A26}$$

as we have to integrate over angles in $\int d^3q$ of the loop. Since

$$\sum_{\tilde{M}'}\mathcal{C}(1,2,1;\tilde{M}',\tilde{M}-\tilde{M}',\tilde{M})^2=1, \tag{A27}$$

we next sum and average $|ME|^2$ over $\tilde{M}$ and we arrive at the final result for $|ME|^2$ summed and averaged over $\tilde{M}$ of Eq. (A25), which is

$$\bar{\sum_{\tilde{M}}}|ME|^2=\frac{1}{12}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi}. \tag{A28}$$

(2) PV: $J_1=1$, $J_2=0$

In this case, we have $M_2=0$, and j'' can be determined with the constraints in Eq. (A21), hence, since $1+j''$ must give $J_2=0$, $j''=1$. Similarly, we can obtain the Racah coefficients in Eq. (A20) with these specific quantum numbers,

$$\mathcal{W}_1\left(1,\frac{1}{2},1,\frac{1}{2};\frac{1}{2},1\right)=\frac{1}{3}, \tag{A29}$$

$$\mathcal{W}_2\left(1,\frac{1}{2},0,\frac{1}{2};\frac{1}{2},1\right)=\frac{1}{\sqrt{6}}, \tag{A30}$$

one of the CG coefficients in Eq. (A20),

$$\mathcal{C}(1,1,0;M_1-\tilde{M}',-M_1+\tilde{M}',0)=(-1)^{1-M_1+\tilde{M}'}\sqrt{\frac{1}{3}}, \tag{A31}$$

and the other three CG coefficients can be combined together to give

$$\sum_{\tilde{M}'}\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1,\tilde{M}-M_1)\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}',\tilde{M})\mathcal{C}(1,1,1;\tilde{M}',M_1-\tilde{M}',M_1)(-1)^{1-M_1+\tilde{M}'}$$
$$=\sum_{\tilde{M}'}(-1)^{1+\tilde{M}'}\sqrt{\frac{3}{5}}\mathcal{C}(1,1,2;\tilde{M},-\tilde{M}',\tilde{M}-\tilde{M}')\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1,\tilde{M}-M_1)$$
$$\times(-1)\mathcal{C}(1,1,1;-\tilde{M}',\tilde{M}'-M_1,-M_1)(-1)^{1-M_1+\tilde{M}'}$$
$$=(-1)^{1-M_1}\sqrt{\frac{3}{5}}\sqrt{15}\mathcal{W}(1,1,1,1;2,1)\mathcal{C}(1,1,1;\tilde{M},-M_1), \tag{A32}$$

where

$$\mathcal{W}(1,1,1,1;2,1)=\frac{1}{6}. \tag{A33}$$

Then, we have a similar equation for $|ME|^2$ in this case after multiplying all terms and squaring, we get for $\overline{\sum}_{\tilde{M}}\sum_{M_1}|ME|^2$

$$\left|-\frac{4\pi i}{3}\right|^2 q^2 Y_{1,\tilde{M}-M_1}(\hat{\boldsymbol{q}})Y^*_{1,\tilde{M}-M_1}(\hat{\boldsymbol{q}})\frac{2}{4\pi}[\sqrt{3}\times 3]^2\frac{1}{3}\left[\sum_{M_1}\sum_{\tilde{M}}\mathcal{C}^2(1,1,1;\tilde{M},M_1-\tilde{M},M_1)\right]^2\left(\frac{3}{5}\right)(15)\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right), \tag{A34}$$

and using the equivalent equation to Eq. (A26) for the spherical harmonics, we have

$$\overline{\sum_{\tilde{M}}}\sum_{M_1}|ME|^2=\frac{1}{24}\left|\frac{4\pi i}{3}\right|^2 q^2\frac{2}{4\pi}\frac{1}{4\pi}. \tag{A35}$$

(3) VP: $J_1=0$, $J_2=1$

We follow closely the previous case (ii) and obtain the same result for $|ME|^2$ in this scenario,

$$\overline{\sum_{\tilde{M}}}\sum_{M_2}|ME|^2=\frac{1}{24}\left|\frac{4\pi i}{3}\right|^2 q^2\frac{2}{4\pi}\frac{1}{4\pi}. \tag{A36}$$

(4) VV: $J_1=1$, $J_2=1$

The calculations in this case is relatively complicated since j'' now can be both 0 and 1 [see CG coefficient in Eq. (A16)]. We thus separate these two situations and present the case with $j''=0$ first.

(a) $j''=0$

As always, first we have the two Racah coefficients of Eq. (A20), which are the same in this case

$$\mathcal{W}\left(1,\frac{1}{2},1,\frac{1}{2};\frac{1}{2},0\right)=-\frac{1}{\sqrt{6}}, \tag{A37}$$

as for the two of the CG coefficients in Eq. (A20) that contain j'', we have

$$\mathcal{C}(1,0,1;M_1+M_2-\tilde{M}',-M_1+\tilde{M}',M_2)=1, \tag{A38}$$

$$\mathcal{C}(1,0,1;\tilde{M}',M_1-\tilde{M}',M_1)=1, \tag{A39}$$

with the condition that $\tilde{M}'=M_1$. Furthermore, the other two CG coefficients in Eq. (A20) can be rewritten as

$$\sum_{\tilde{M}'}\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1-M_2,\tilde{M}-M_1-M_2)$$
$$\times\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}',\tilde{M})\delta_{\tilde{M}',M_1}$$
$$=\mathcal{C}(2,1,1;\tilde{M}-M_1,-M_2,\tilde{M}-M_1-M_2)$$
$$\times\mathcal{C}(2,1,1;\tilde{M}-M_1,M_1,\tilde{M}), \tag{A40}$$

the square of the Eq. (A40) gives us

$$\mathcal{C}(2,1,1;\tilde{M}-M_1,-M_2,\tilde{M}-M_1-M_2)^2$$
$$\times\mathcal{C}(2,1,1;\tilde{M}-M_1,M_1,\tilde{M})^2$$
$$=\frac{3}{5}\mathcal{C}(1,1,2;M_2,\tilde{M}-M_1-M_2,\tilde{M}-M_1)^2$$
$$\times\mathcal{C}(2,1,1;\tilde{M}-M_1,M_1,\tilde{M})^2, \tag{A41}$$

where we write the second term $\mathcal{C}(2,1,1;\tilde{M}-M_1,M_1,\tilde{M})^2$ as $\mathcal{C}(2,1,1;\tilde{M}-M_1,\tilde{M}-(\tilde{M}-M_1),\tilde{M})^2$, and then sum over M_2, $\tilde{M}-M_1$, and $\tilde{M}$.

We obtain the following values with $\tilde{M}-M_1$ and $\tilde{M}$ fixed,

$$\sum_{M_2}\mathcal{C}(1,1,2;M_2,\tilde{M}-M_1-M_2,\tilde{M}-M_1)^2=1, \tag{A42}$$

and with $\tilde{M}$ fixed, we have

$$\sum_{\tilde{M}-M_1}\mathcal{C}(2,1,1;\tilde{M}-M_1,\tilde{M}-(\tilde{M}-M_1),\tilde{M})^2=1, \tag{A43}$$

and the sum over $\tilde{M}$ gives 3. We shall take the factor $\frac{1}{3}$ from the average at the end.

Finally, following the same steps used in the previous cases, we obtain $|ME|^2$ in this case,

$$\sum_{\tilde{M}}\sum_{M_1}\sum_{M_2}|ME|_a^2=\frac{3}{20}\left|\frac{4\pi i}{3}\right|^2 q^2\frac{2}{4\pi}\frac{1}{4\pi}. \tag{A44}$$

(b) $j''=1$

In this case, we have two of the CG coefficients in Eq. (A21) that can be rewritten as

$$\begin{aligned}
&\mathcal{C}(1,j'',J_2;M_1+M_2-\tilde{M}',-M_1+\tilde{M}',M_2)\mathcal{C}(1,j'',J_1;\tilde{M}',M_1-\tilde{M}',M_1)\\
&=\mathcal{C}(1,1,1;M_1+M_2-\tilde{M}',-M_1+\tilde{M}',M_2)\mathcal{C}(1,1,1;\tilde{M}',M_1-\tilde{M}',M_1)\\
&=(-1)^{-M_1-M_2}\mathcal{C}(1,1,1;M_2,\tilde{M}'-M_1-M_2,\tilde{M}'-M_1)\mathcal{C}(1,1,1;M_1,-\tilde{M}',M_1-\tilde{M}')\\
&=(-1)^{1+\tilde{M}'}\mathcal{C}(1,1,1;M_1,-\tilde{M}',M_1-\tilde{M}')\mathcal{C}(1,1,1;-\tilde{M}'+M_1,\tilde{M}'-M_1-M_2,-M_2)\\
&=\sum_{j'''}(-1)^{1+\tilde{M}'}[3(2j'''+1)]^{1/2}\mathcal{W}(1,1,1,1;1,j''')\mathcal{C}(1,1,j''';-\tilde{M}',\tilde{M}'-M_1-M_2)\mathcal{C}(1,j''',1;M_1,-M_1-M_2),
\end{aligned} \tag{A45}$$

where we separate the CG coefficients and only one depends on $\tilde{M}'$, and that one can be combined together with the other two CG coefficients of Eq. (A21) to give

$$\begin{aligned}
&\sum_{\tilde{M}'}(-1)^{1+\tilde{M}'}\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1-M_2,\tilde{M}-M_1-M_2)\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}',\tilde{M})\mathcal{C}(1,1,j''';-\tilde{M}',\tilde{M}'-M_1-M_2)\\
&=\mathcal{C}(1,1,2;\tilde{M},-\tilde{M}',\tilde{M}-\tilde{M}')\mathcal{C}(2,1,1;\tilde{M}-\tilde{M}',\tilde{M}'-M_1-M_2,\tilde{M}-M_1-M_2)\mathcal{C}(1,1,j''';-\tilde{M}',\tilde{M}'-M_1-M_2)\\
&=[5(2j'''+1)]^{1/2}\mathcal{W}(1,1,1,1;2,j''')\mathcal{C}(1,j''',1;\tilde{M},-M_1-M_2).
\end{aligned} \tag{A46}$$

In this way, we now have the following equation for Eq. (A21) in this case

$$\begin{aligned}
&\sum_{j'''}\frac{\sqrt{15}}{9}(2j'''+1)\mathcal{W}(1,1,1,1;1,j''')\mathcal{W}(1,1,1,1;2,j''')\mathcal{C}(1,j''',1;M_1,-M_1-M_2)\mathcal{C}(1,j''',1;\tilde{M},-M_1-M_2)\\
&=\sum_{j'''}(-1)^{-M_1-\tilde{M}}\left(\frac{\sqrt{15}}{3}\right)\mathcal{W}(1,1,1,1;1,j''')\mathcal{W}(1,1,1,1;2,j''')\\
&\times\mathcal{C}(1,1,j''';M_1,M_2,M_1+M_2)\mathcal{C}(1,1,j''';\tilde{M},M_1+M_2-\tilde{M},M_1+M_2),
\end{aligned} \tag{A47}$$

similarly, for these two CG coefficients in Eq. (A47) we will sum over M_1, $\tilde{M}$, M_1+M_2 when we square, which leads us to

$$\sum_{M_1}\mathcal{C}(1,1,j_1''';M_1,M_2,M_1+M_2)\mathcal{C}(1,1,j_2''';M_1,M_2,M_1+M_2)=\delta_{j_1''',j_2'''}, \tag{A48}$$

where we keep $\tilde{M}$ and M_1+M_2 fixed, and a similar thing can be done to the other CG coefficients when we square

$$\sum_{\tilde{M}}\mathcal{C}(1,1,j_1''';\tilde{M},M_1+M_2-\tilde{M},M_1+M_2)\mathcal{C}(1,1,j_2''';\tilde{M},M_1+M_2-\tilde{M},M_1+M_2)=\delta_{j_1''',j_2'''}, \tag{A49}$$

and sum over M_1+M_2 will give us a factor of 3, which is the same as we obtained in the last case. Then we have the following equation when we square Eq. (A47)

$$\sum_{j'''}\left(3\times\frac{15}{9}\right)\mathcal{W}(1,1,1,1;1,j''')^2\mathcal{W}(1,1,1,1;2,j''')^2=\sum_{j'''}5\mathcal{W}(1,1,1,j''';1,1)^2\mathcal{W}(2,1,1,j''';1,1)^2=\frac{1}{81}\frac{426}{80},\tag{A50}$$

where we sum over j''' for $j'''=0$, 1, 2 and all the Racah coefficients used in Eq. (A50) are listed below.

$$\mathcal{W}(1,1,1,0;1,1)=\frac{1}{3},\quad \mathcal{W}(2,1,1,0;1,1)=\frac{1}{3},$$
$$\mathcal{W}(1,1,1,1;1,1)=\frac{1}{6},\quad \mathcal{W}(2,1,1,1;1,1)=-\frac{1}{6},$$
$$\mathcal{W}(1,1,1,2;1,1)=-\frac{1}{6},\quad \mathcal{W}(2,1,1,2;1,1)=\frac{1}{30}.\tag{A51}$$

Consequently, we have $|ME|^2$ in this case as

$$\sum_{\tilde{M}}\sum_{M_1}\sum_{M_2}|ME|_b^2=27\times\frac{1}{81}\frac{426}{80}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi}=\frac{213}{120}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi}.\tag{A52}$$

Crossed terms are calculated to be 0 in this particular case, and we then add up parts (a) and (b) taking into account the factor $(\frac{1}{3})$ from the average over $\tilde{M}$. Then we arrive at

$$\bar{\sum_{\tilde{M}}}\sum_{M_1}\sum_{M_2}(|ME|_a^2+|ME|_b^2)=\frac{231}{360}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi}.\tag{A53}$$

To sum it up, we have obtained all the scattering amplitudes $\bar{\sum}\sum|t|^2$ with different types of interactions: PP, PV, VP, and VV, we present here again for clarity

$$PP:\ \frac{1}{12}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi},$$
$$PV:\ \frac{1}{24}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi},$$
$$VP:\ \frac{1}{24}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi},$$
$$VV:\ \frac{231}{360}\left|\frac{4\pi i}{3}\right|^2q^2\frac{2}{4\pi}\frac{1}{4\pi}.\tag{A54}$$

On top of that, there is a constant common to all the decay modes which would appear in the hadronization process. Then we can omit $|-\frac{4\pi i}{3}|^2\frac{1}{4\pi}\frac{2}{4\pi}$ in Eq. (A54) and replace it with a factor $|A|^2$, which is fitted to the experimental data.

[1] V. Crede and W. Roberts, Rep. Prog. Phys. **76**, 076301 (2013).
[2] H. X. Chen, W. Chen, X. Liu, and S. L. Zhu, Phys. Rep. **639**, 1 (2016).
[3] H. X. Chen, W. Chen, X. Liu, Y. R. Liu, and S. L. Zhu, Rep. Prog. Phys. **80**, 076201 (2017).
[4] F. K. Guo, C. Hanhart, U. G. Meissner, Q. Wang, Q. Zhao, and B. S. Zou, Rev. Mod. Phys. **90**, 015004 (2018).
[5] S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).
[6] J. A. Oller and E. Oset, Nucl. Phys. **A620**, 438 (1997).
[7] N. Kaiser, Eur. Phys. J. A **3**, 307 (1998).
[8] M. P. Locher, V. E. Markushin, and H. Q. Zheng, Eur. Phys. J. C **4**, 317 (1998).
[9] J. Nieves and E. Ruiz Arriola, Nucl. Phys. **A679**, 57 (2000).
[10] J. R. Pelaez, Phys. Rep. **658**, 1 (2016).
[11] T. Barnes and E. S. Swanson, Phys. Rev. C **77**, 055206 (2008).
[12] M. Ablikim *et al.* (BES Collaboration), Phys. Lett. B **668**, 263 (2008).
[13] M. Ablikim *et al.* (BES Collaboration), Phys. Rev. Lett. **97**, 121801 (2006).
[14] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. D **76**, 111105 (2007).
[15] S. Coito and F. Giacosa, Nucl. Phys. **A981**, 38 (2019).
[16] L. Micu, Nucl. Phys. **B10**, 521 (1969).
[17] A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, Phys. Rev. D **8**, 2223 (1973).
[18] A. Bramon, A. Grau, and G. Pancheri, Phys. Lett. B **283**, 416 (1992).
[19] S. Sakai, E. Oset, and A. Ramos, Eur. Phys. J. A **54**, 10 (2018).
[20] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1980), p. 705.
[21] F. Aceti, L. R. Dai, L. S. Geng, E. Oset, and Y. Zhang, Eur. Phys. J. A **50**, 57 (2014).
[22] Y. J. Zhang and Q. Zhao, Phys. Rev. D **81**, 034011 (2010).
[23] G. Y. Chen and Q. Zhao, Phys. Lett. B **718**, 1369 (2013).

[24] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Springer, New York, 1979).
[25] S. Weinberg, Phys. Rev. **137**, B672 (1965).
[26] V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, and A. E. Kudryavtsev, Phys. Lett. B **586**, 53 (2004).
[27] T. Hyodo, D. Jido, and A. Hosaka, Phys. Rev. C **85**, 015201 (2012).
[28] D. Gamermann, J. Nieves, E. Oset, and E. Ruiz Arriola, Phys. Rev. D **81**, 014029 (2010).
[29] T. Sekihara, T. Hyodo, and D. Jido, Prog. Theor. Exp. Phys. **2015**, 063D04 (2015).
[30] Z. H. Guo and J. A. Oller, Phys. Rev. D **93**, 096001 (2016).
[31] T. Hyodo, Int. J. Mod. Phys. A **28**, 1330045 (2013).
[32] Y. Tsuchida and T. Hyodo, Phys. Rev. C **97**, 055213 (2018).
[33] F. Aceti and E. Oset, Phys. Rev. D **86**, 014012 (2012).
[34] L. R. Dai, R. Pavao, S. Sakai, and E. Oset, Eur. Phys. J. A **55**, 20 (2019).
[35] W. H. Liang, M. Bayar, and E. Oset, Eur. Phys. J. C **77**, 39 (2017).
[36] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, New York, 1957).