Lepton mixing predictions from S_4 in the tridirect CP approach to two right-handed neutrino models

Gui-Jun Ding,[1,*] Stephen F. King,[2,†] and Cai-Chang Li[1,‡]

[1]*Interdisciplinary Center for Theoretical Study and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*
[2]*Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom*

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We perform an exhaustive analysis of all possible breaking patterns arising from $S_4 \rtimes H_{CP}$ in a new *tridirect CP approach* to the minimal seesaw model with two right-handed neutrinos, and construct a realistic flavor model along these lines. According to this approach, separate residual flavor and CP symmetries persist in the charged lepton, "atmospheric" and "solar" right-handed neutrino sectors, i.e., we have *three* symmetry sectors rather than the usual two of the *semidirect CP approach* (charged leptons and neutrinos). Following the *tridirect CP approach*, we find 26 kinds of independent phenomenologically interesting mixing patterns. Eight of them predict a normal ordering (NO) neutrino mass spectrum and the other 18 predict an inverted ordering (IO) neutrino mass spectrum. For each phenomenologically interesting mixing pattern, the corresponding predictions for the Pontecorvo-Maki-Nakagawa-Sakata matrix, the lepton mixing parameters, the neutrino masses and the effective mass in neutrinoless double beta decay are given in a model-independent way. One breaking pattern with an NO spectrum and two breaking patterns with IO spectra correspond to form dominance. We find that the lepton mixing matrices of three kinds of breaking patterns with NO spectra and one form dominance breaking pattern with an IO spectrum preserve the first column of the tribimaximal mixing matrix, i.e., yield a TM1 mixing matrix.

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I. INTRODUCTION

The discovery of neutrino oscillations implied that neutrinos have masses and there is mixing in the lepton sector. According to the neutrino oscillation experimental data, the 3σ ranges of the leptonic mixing angles and neutrino mass-squared differences are [1]

$$
\begin{gathered}
0.272 \leq \sin^2\theta_{12} \leq 0.346, \qquad 6.80\times 10^{-5}\ \mathrm{eV}^2 \leq \Delta m^2_{21} \leq 8.02\times 10^{-5}\ \mathrm{eV}^2,\\
\begin{cases} 0.01981 \leq \sin^2\theta_{13} \leq 0.02436, & 0.418 \leq \sin^2\theta_{23} \leq 0.613, \quad (\mathrm{NO}),\\ 0.02006 \leq \sin^2\theta_{13} \leq 0.02452, & 0.435 \leq \sin^2\theta_{23} \leq 0.616, \quad (\mathrm{IO}),\end{cases}\\
\begin{cases} 2.399\times 10^{-3}\ \mathrm{eV}^2 \leq \Delta m^2_{31} \leq 2.593\times 10^{-3}\ \mathrm{eV}^2, & (\mathrm{NO}),\\ -2.562\times 10^{-3}\ \mathrm{eV}^2 \leq \Delta m^2_{32} \leq -2.369\times 10^{-3}\ \mathrm{eV}^2, & (\mathrm{IO}),\end{cases}
\end{gathered}
\tag{1.1}
$$

where the symbols "NO" and "IO" denote normal ordering and inverted ordering neutrino mass spectra, respectively. We do not know the origin of neutrino mass and lepton mixing so far, although these results are consistent with some theories. The leading candidate for a framework of neutrino mass and lepton mixing is the type I seesaw mechanism which involves additional heavy right-handed Majorana neutrinos [2–4]. However the seesaw mechanism is very difficult to be tested experimentally, because it introduces many additional parameters in the neutrino Yukawa couplings and the right-handed

*dinggj@ustc.edu.cn
†king@soton.ac.uk
‡lcc0915@mail.ustc.edu.cn

neutrino masses are typically of the grand unified theory scale such that they are generally far beyond the reach of the LHC. In order to obtain testable predictions, it is natural to follow the idea of minimality (as discussed in e.g., Ref. [5]), i.e., focusing on the seesaw theories with smaller numbers of parameters.

The most minimal version of the seesaw mechanism involves two additional right-handed neutrinos [6,7]. In order to increase the predictive power of the two right-handed neutrino seesaw model, various schemes to reduce the number of free parameters have been suggested, such as postulating one [8] or two [7] texture zeros; however the latter models with two texture zeros are now phenomenologically excluded for NO [9–11]. In the charged lepton diagonal basis, together with a diagonal right-handed neutrino mass matrix, the idea of constrained sequential dominance (CSD) has been proposed, involving a Dirac mass matrix with one texture zero and a restricted form of the Yukawa couplings [12]. The CSD(n) scheme [12–19] assumes that the coupling of one right-handed neutrino (called "atmospheric") with ν_L is proportional to (0,1,1), while the second right-handed neutrino (called "solar") has couplings to ν_L proportional to $(1, n, n-2)$ with positive integer n, where $\nu_L \equiv (\nu_e, \nu_\mu, \nu_\tau)_L^T$ denote the left-handed neutrino fields. The CSD(n) models generally [12–19] predict a TM1 mixing matrix and normal mass hierarchy with a massless neutrino $m_1 = 0$ [20]. Predictions for lepton mixing parameters and neutrino masses have been made for the cases of $n=1$ [12], $n=2$ [13], $n=3$ [14–16], $n=4$ [17,18] and $n \geq 5$ [19]. It turns out that the CSD(3) model also called the littlest seesaw (LS) model can successfully accommodate the experimental data on neutrino masses and mixing angles [14–16]. The LS model can yield the baryon asymmetry of the Universe via leptogenesis [21–23]. The LS structure can also be incorporated into grand unified models [21,24,25]. In practice the LS model can be achieved by introducing S_4 family symmetry, which is spontaneously broken by flavon fields with particular vacuum alignments governed by remnant subgroups of S_4 [15,16]. Furthermore, from the breaking of A_5 flavor symmetry to different residual subgroups in the charged lepton, atmospheric neutrino and solar neutrino sectors, we can obtain the viable golden LS model which predicts the GR1 lepton mixing pattern [26]. Here the GR1 mixing matrix preserves the first column of the golden ratio mixing matrix.

The leptonic CP violation is one of the most urgent questions in neutrino oscillation physics. The indication of CP violation in the neutrino sector has been reported by the T2K [27] and NOνA collaborations [28], and the Dirac CP phase δ_{CP} will be intensively probed experimentally in the forthcoming years. In order to address this question theoretically, non-Abelian discrete flavor symmetry combined with generalized CP symmetry has been widely exploited to explain the lepton mixing angles and to predict CP-violating phases [29–60]. Both flavor symmetry G_f and CP symmetry H_{CP} are imposed at high energy scales, and the full symmetry is $G_f \rtimes H_{CP}$. In the successful *semidirect CP approach*, the original symmetry $G_f \rtimes H_{CP}$ is spontaneously broken down to G_l and $G_\nu \rtimes H^\nu_{CP}$ in the charged lepton sector and the neutrino sector at lower energies, respectively.

Recently we extended the above *semidirect CP approach* to propose a so-called *tridirect CP approach* [61] based on the two right-handed neutrino seesaw mechanism, and a new variant of the LS model was found. In the *tridirect CP approach*, the common residual symmetry of the neutrino sector is split into two branches: the residual symmetries $G_{\rm atm} \rtimes H^{\rm atm}_{CP}$ and $G_{\rm sol} \rtimes H^{\rm sol}_{CP}$ associated with the "atmospheric" and "solar" right-handed neutrino sectors respectively. An Abelian subgroup G_l is assumed to be preserved by the charged lepton mass matrix and it allows for the distinction of three generations of charged leptons. It is the combination of these *three* residual symmetries that provides a new way of fixing the lepton mixing parameters and neutrino masses in the *tridirect CP approach*.

In the present work, we shall extend the analysis of the *tridirect CP approach* for two right-handed neutrino models considerably, beyond the few examples studied in Ref. [61], to an exhaustive model-independent analysis of *all* possible phenomenologically viable lepton flavor mixing patterns which arise from the breaking of the parent symmetry $S_4 \rtimes H_{CP}$. The lepton mixing matrix is not restricted to TM1 mixing anymore and the mass ordering of the neutrino masses can be either NO or IO. We shall find eight independent phenomenologically interesting mixing patterns for the case of NO neutrino masses and 18 independent phenomenologically interesting mixing patterns for the case of IO. The eight breaking patterns for NO are labeled as $\mathcal{N}_1 \sim \mathcal{N}_8$ and the other 18 for IO are labeled as $\mathcal{I}_1 \sim \mathcal{I}_{18}$. For each possible breaking pattern, we numerically analyze the predictions of the mixing parameters, the three neutrino masses and the effective mass in neutrinoless double beta decay. We find that all four breaking patterns $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{I}_5$ give rise to TM1 mixing. For the cases of $\mathcal{N}_5$, $\mathcal{I}_4$ and $\mathcal{I}_5$, the two columns of the Dirac neutrino mass matrix are orthogonal to each other and consequently have the texture of form dominance [62–64] is reproduced. Furthermore, we implement the case of $\mathcal{N}_4$ with $x=-4$ and $\eta = \pm 3\pi/4$ in an explicit model based on $S_4 \rtimes H_{CP}$, and the required vacuum alignment needed to achieve the remnant symmetries is dynamically realized. In this model, the absolute value of the first column of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is fixed to be $\left(2\sqrt{\frac{6}{37}}, \sqrt{\frac{13}{74}}, \sqrt{\frac{13}{74}}\right)^T$.

The paper is organized as follows. In Sec. II, we recall the framework of the *tridirect CP approach* to two right-handed neutrino models, and we present the generic procedures of how to derive the lepton flavor mixing and neutrino masses from remnant symmetries in the *tridirect CP approach* in a model-independent way. In Sec. III, we perform a model-independent analysis of five kinds of phenomenologically viable breaking patterns achievable from the underlying symmetry $S_4 \rtimes H_{CP}$ in the *tridirect CP approach* with NO neutrino masses. In Sec. IV, a general analysis of five kinds of breaking patterns with IO neutrino masses is presented. In Sec. V, we present a new version of the LS model based on $S_4 \rtimes H_{CP}$ from the *tridirect CP approach*. The vacuum alignment, the LO structure and the next-to-leading-order (NLO) corrections of the model are discussed. Section VI is devoted to our conclusion. The group theory of S_4 and all of its Abelian subgroups are presented in Appendix A. In Appendix B, we study the breaking patterns $\mathcal{N}_6 \sim \mathcal{N}_8$ in a model-independent way. The analysis of the remaining 13 kinds of breaking patterns with IO is given in Appendix C.

II. THE TRIDIRECT *CP* APPROACH

In the scenario with a discrete flavor group G_f and generalized CP symmetry H_{CP}, G_f and H_{CP} should be compatible with each other, and they fulfill the following consistency condition [29,30,65,66]:

$$X_{\mathbf{r}}\rho^*_{\mathbf{r}}(g)X^\dagger_{\mathbf{r}} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f, \quad X_{\mathbf{r}} \in H_{CP}, \tag{2.1}$$

where $\rho_{\mathbf{r}}(g)$ is the representation matrix of the element g in the irreducible representation $\mathbf{r}$ of G_f, and $X_{\mathbf{r}}$ is the generalized CP transformation matrix of H_{CP}. Moreover, the physically well-defined generalized CP transformations should be class-inverting automorphisms of G_f [65]. It requires that the elements g^{-1} and g' in Eq. (2.1) belong to the same conjugacy class of G_f. The automorphism in Eq. (2.1) thus implies that the mathematical structure of the group comprising G_f and CP is in general a semidirect product $G_f \rtimes H_{CP}$ [29].

In the present work, we shall perform a comprehensive study of lepton mixing patterns which can be obtained from the flavor group S_4 and CP symmetry in the tridirect CP approach [61]. In the following, we shall first review how the tridirect CP approach allows us to predict the lepton mixing and neutrino masses in terms of few parameters. In the tridirect CP approach, the assumed family and CP symmetry $G_f \rtimes H_{CP}$ at high energy scales is spontaneously broken down to an Abelian subgroup G_l which is capable of distinguishing the three generations in the charged lepton sector, and it is broken to $G_{\rm atm} \rtimes H^{\rm atm}_{CP}$ and $G_{\rm sol} \rtimes H^{\rm sol}_{CP}$ in the atmospheric and solar neutrino sectors respectively. A sketch of the tridirect CP approach for two right-handed neutrino models is illustrated in Fig. 1. In the right-handed neutrino diagonal basis, the effective Lagrangian is given by

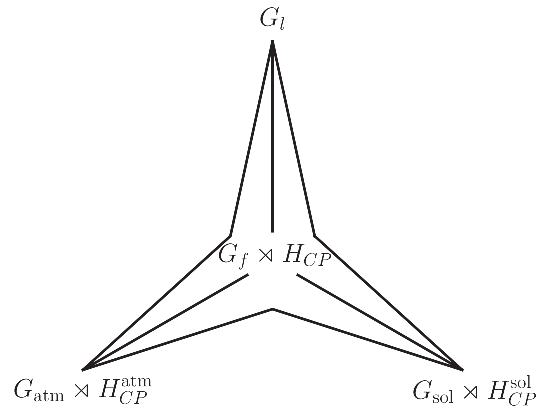

FIG. 1. A sketch of the tridirect CP approach for two right-handed neutrino models, where the high-energy family and CP symmetry $G_f \rtimes H_{CP}$ is spontaneously broken down to $G_{\rm atm} \rtimes H^{\rm atm}_{CP}$ in the sector of one of the right-handed neutrinos, and $G_{\rm sol} \rtimes H^{\rm sol}_{CP}$ in the sector of the other right-handed neutrino, with the charged lepton sector having a different residual flavor symmetry G_l.

$$\begin{aligned}\mathcal{L} = &-y_l L\phi_l E^c - y_{\rm atm} L\phi_{\rm atm} N^c_{\rm atm} - y_{\rm sol} L\phi_{\rm sol} N^c_{\rm sol} \\ &-\frac{1}{2}x_{\rm atm}\xi_{\rm atm} N^c_{\rm atm} N^c_{\rm atm} - \frac{1}{2}x_{\rm sol}\xi_{\rm sol} N^c_{\rm sol} N^c_{\rm sol} + \text{H.c.},\end{aligned} \tag{2.2}$$

where we use the two-component notation for the fermion fields. The notation L stands for the left-handed lepton doublets and $E^c \equiv (e^c, \mu^c, \tau^c)^T$ are the right-handed charged leptons, the flavons $\xi_{\rm atm}$ and $\xi_{\rm sol}$ are standard model singlets, and the flavons ϕ_l, $\phi_{\rm sol}$ and $\phi_{\rm atm}$ can be either Higgs fields or combinations of the electroweak Higgs doublet together with flavons. All four coupling constants $y_{\rm atm}$, $y_{\rm sol}$, $x_{\rm atm}$ and $x_{\rm sol}$ would be constrained to be real if we impose CP symmetry.

Without loss of generality, we assume that the three generations of left-handed leptons doublets transform as a faithful three-dimensional representation $\mathbf{3}$ under G_f. The residual symmetry G_l in the charged lepton sector requires that the Hermitian combination $m^\dagger_l m_l$ must be invariant under the action of G_l, i.e.,

$$\rho^\dagger_{\mathbf{3}}(g_l) m^\dagger_l m_l \rho_{\mathbf{3}}(g_l) = m^\dagger_l m_l, \quad g_l \in G_l, \tag{2.3}$$

where the charged lepton mass matrix m_l is defined using the convention $l^c m_l l$. The diagonalization matrix of the Hermitian combination $m^\dagger_l m_l$ is defined as U_l with $U^\dagger_l m^\dagger_l m_l U_l = \text{diag}(m^2_e, m^2_\mu, m^2_\tau)$. From Eq. (2.3), we find that the unitary matrix U_l can be derived from

$$U^\dagger_l \rho_{\mathbf{3}}(g_l) U_l = \rho^{\rm diag}_{\mathbf{3}}(g_l), \tag{2.4}$$

where $\rho^{\rm diag}_{\mathbf{3}}(g_l)$ is a diagonal matrix whose entries are three eigenvalues of $\rho_{\mathbf{3}}(g_l)$. In the atmospheric neutrino sector and the solar neutrino sector, as the residual symmetries contain both flavor symmetry and CP symmetry, the following restricted consistency conditions should be satisfied:

$$X^{\rm atm}_{\mathbf{r}}\rho^*_{\mathbf{r}}(g^{\rm atm}_i)(X^{\rm atm}_{\mathbf{r}})^{-1}=\rho_{\mathbf{r}}(g^{\rm atm}_j),\quad g^{\rm atm}_i, g^{\rm atm}_j\in G_{\rm atm},\qquad X^{\rm atm}_{\mathbf{r}}\in H^{\rm atm}_{CP}, \tag{2.5a}$$

$$X^{\rm sol}_{\mathbf{r}}\rho^*_{\mathbf{r}}(g^{\rm sol}_i)(X^{\rm sol}_{\mathbf{r}})^{-1}=\rho_{\mathbf{r}}(g^{\rm sol}_j),\quad g^{\rm sol}_i, g^{\rm sol}_j\in G_{\rm sol},\qquad X^{\rm sol}_{\mathbf{r}}\in H^{\rm sol}_{CP}. \tag{2.5b}$$

The consistency conditions indicate that the mathematical structure of the residual flavor and CP symmetries is a semidirect product for $i\neq j$ and it reduces to a direct product for the case of $i=j$. The consistency equations in Eqs. (2.5a) and (2.5b) can be used to find the residual CP symmetry consistent with the residual flavor symmetries of the atmospheric neutrino and solar neutrino sectors, respectively. In the atmospheric and solar neutrino sectors, the residual symmetries imply that the vacuum alignments of flavons $\phi_{\rm atm}$ and $\phi_{\rm sol}$ should be invariant under the symmetries $G_{\rm atm}\rtimes H^{\rm atm}_{CP}$ and $G_{\rm sol}\rtimes H^{\rm sol}_{CP}$ respectively, i.e.,

$$\rho_{\mathbf{r}}(g^{\rm atm})\langle\phi_{\rm atm}\rangle=\langle\phi_{\rm atm}\rangle,\quad X^{\rm atm}_{\mathbf{r}}\langle\phi_{\rm atm}\rangle^*=\langle\phi_{\rm atm}\rangle, \tag{2.6a}$$

$$\rho_{\mathbf{r}}(g^{\rm sol})\langle\phi_{\rm sol}\rangle=\langle\phi_{\rm sol}\rangle,\qquad X^{\rm sol}_{\mathbf{r}}\langle\phi_{\rm sol}\rangle^*=\langle\phi_{\rm sol}\rangle, \tag{2.6b}$$

where $\langle\phi_{\rm atm}\rangle$ and $\langle\phi_{\rm sol}\rangle$ denote the vacuum alignments of flavons $\phi_{\rm atm}$ and $\phi_{\rm sol}$, respectively. After electroweak and flavor symmetry breaking, the flavons ϕ_l, $\phi_{\rm atm}$, $\phi_{\rm sol}$, $\xi_{\rm atm}$ and $\xi_{\rm sol}$ acquire nonvanishing vacuum expectation values (VEVs). From the Lagrangian in Eq. (2.2), one can read out the neutrino Dirac mass matrix and the heavy Majorana mass matrix,

$$m_D=(\,y_{\rm atm}U_a\langle\phi_{\rm atm}\rangle,\ y_{\rm sol}U_s\langle\phi_{\rm sol}\rangle\,),\qquad m_N=\begin{pmatrix} x_{\rm atm}\langle\xi_{\rm atm}\rangle & 0\\ 0 & x_{\rm sol}\langle\xi_{\rm sol}\rangle\end{pmatrix}, \tag{2.7}$$

where U_a and U_s are two constant matrices and they are constituted by the Clebsch-Gordan (CG) coefficients which appear in the contractions $y_{\rm atm}L\phi_{\rm atm}N^c_{\rm atm}$ and $y_{\rm sol}L\phi_{\rm sol}N^c_{\rm sol}$, respectively. For the sake of convenience in the following, we shall parametrize the combinations $U_a\langle\phi_{\rm atm}\rangle\equiv\boldsymbol{v}_{\rm atm}v_{\phi_a}$ and $U_s\langle\phi_{\rm sol}\rangle\equiv\boldsymbol{v}_{\rm sol}v_{\phi_s}$, where $\boldsymbol{v}_{\rm atm}$ and $\boldsymbol{v}_{\rm sol}$ are three-dimensional column vectors and they denote the directions of the vacuum alignment, and v_{ϕ_a} and v_{ϕ_s} are the overall scales of corresponding flavons. The light effective Majorana neutrino mass matrix is given by the seesaw formula $m_\nu=-m_Dm_N^{-1}m_D^T$; then we find that m_ν takes the form

$$m_\nu=-\frac{y^2_{\rm atm}}{x_{\rm atm}}\frac{U_a\langle\phi_{\rm atm}\rangle\langle\phi_{\rm atm}\rangle^TU_a^T}{\langle\xi_{\rm atm}\rangle}-\frac{y^2_{\rm sol}}{x_{\rm sol}}\frac{U_s\langle\phi_{\rm sol}\rangle\langle\phi_{\rm sol}\rangle^TU_s^T}{\langle\xi_{\rm sol}\rangle},\quad \equiv e^{i\varphi_a}[m_a\boldsymbol{v}_{\rm atm}\boldsymbol{v}^T_{\rm atm}+m_se^{i\eta}\boldsymbol{v}_{\rm sol}\boldsymbol{v}^T_{\rm sol}], \tag{2.8}$$

where the overall phase φ_a is given by $\varphi_a=\arg(-y^2_{\rm atm}v^2_{\phi_a}/(x_{\rm atm}\langle\xi_{\rm atm}\rangle))$, $m_a=|y^2_{\rm atm}v^2_{\phi_a}/(x_{\rm atm}\langle\xi_{\rm atm}\rangle)|$, $m_s=|y^2_{\rm sol}v^2_{\phi_s}/(x_{\rm sol}\langle\xi_{\rm sol}\rangle)|$ and $\eta=\arg(-y^2_{\rm sol}v^2_{\phi_s}/(x_{\rm sol}\langle\xi_{\rm sol}\rangle))-\varphi_a$. The overall phase φ_a can be absorbed into the lepton field and it will always be omitted in the following. For convenience the notation $r\equiv m_s/m_a$ will be used throughout this paper. If the roles of $G_{\rm atm}\rtimes H^{\rm atm}_{CP}$ and $G_{\rm sol}\rtimes H^{\rm sol}_{CP}$ are switched, the two columns of the Dirac mass matrix m_D will be exchanged. Thus the same neutrino mass matrix would be obtained if one interchanges $y_{\rm atm}$ with $y_{\rm sol}$, $x_{\rm atm}$ with $x_{\rm sol}$, and $\langle\xi_{\rm atm}\rangle$ with $\langle\xi_{\rm sol}\rangle$.

In the following we shall give the detailed procedures for analyzing the phenomenological predictions of the tridirect CP approach in a model-independent way, and we shall present the generic expressions of the lepton mixing matrix and neutrino masses. One can easily check that the neutrino mass matrix m_ν of Eq. (2.8) satisfies

$$m_\nu\boldsymbol{v}_{\rm fix}=(0,0,0)^T, \tag{2.9}$$

with

$$\boldsymbol{v}_{\rm fix}\equiv\boldsymbol{v}_{\rm atm}\times\boldsymbol{v}_{\rm sol}, \tag{2.10}$$

where $\boldsymbol{v}_{\rm atm}\times\boldsymbol{v}_{\rm sol}$ denotes the cross product of $\boldsymbol{v}_{\rm atm}$ and $\boldsymbol{v}_{\rm sol}$. The normalized vector of $\boldsymbol{v}_{\rm fix}$ is defined as $\hat{\boldsymbol{v}}_{\rm fix}\equiv\boldsymbol{v}_{\rm fix}/\sqrt{\boldsymbol{v}^\dagger_{\rm fix}\boldsymbol{v}_{\rm fix}}$. Equation (2.9) implies that $\hat{\boldsymbol{v}}_{\rm fix}$ is an eigenvector of m_ν with zero eigenvalue. As a result, the first (third) column of U_ν is determined to be $\hat{\boldsymbol{v}}_{\rm fix}$ for an NO (IO) mass spectrum, where U_ν is the diagonalization matrix of m_ν with $U^T_\nu m_\nu U_\nu=\mathrm{diag}(0,m_2,m_3)$ for the NO case and $U^T_\nu m_\nu U_\nu=\mathrm{diag}(m_1,m_2,0)$ for the IO case. In order to diagonalize the above neutrino mass matrix, we first perform a unitary transformation $U_{\nu1}$, where the unitary matrix $U_{\nu1}$ can take the following form[1]:

$$U_{\nu1}=\begin{cases}(\hat{\boldsymbol{v}}_{\rm fix},\hat{\boldsymbol{v}}^*_{\rm atm},\hat{\boldsymbol{v}}'_{\rm sol}) & \text{for NO},\\ (\hat{\boldsymbol{v}}^*_{\rm atm},\hat{\boldsymbol{v}}'_{\rm sol},\hat{\boldsymbol{v}}_{\rm fix}) & \text{for IO},\end{cases} \tag{2.11}$$

with

$$\hat{\boldsymbol{v}}_{\rm atm}=\frac{\boldsymbol{v}_{\rm atm}}{\sqrt{\boldsymbol{v}^\dagger_{\rm atm}\boldsymbol{v}_{\rm atm}}},\qquad \hat{\boldsymbol{v}}'_{\rm sol}=\hat{\boldsymbol{v}}^*_{\rm fix}\times\hat{\boldsymbol{v}}_{\rm atm}. \tag{2.12}$$

[1]In general the unitary matrix $U_{\nu1}$ is not unique. In the case $\boldsymbol{v}^\dagger_{\rm sol}\boldsymbol{v}_{\rm atm}\neq0$, $U_{\nu1}$ can multiply a unitary rotation from the right-hand side in the (23) and (12) planes for NO and IO respectively.

Then the neutrino mass matrix becomes

$$m'_\nu = U_{\nu1}^T m_\nu U_{\nu1} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & z & w \end{pmatrix} & \text{for NO}, \\ \begin{pmatrix} y & z & 0 \\ z & w & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for IO}, \end{cases} \tag{2.13}$$

where the expressions of the parameters y, z and w are

$$\begin{aligned} y &\equiv |y|e^{i\phi_y} = m_a \boldsymbol{v}_{\text{atm}}^\dagger \boldsymbol{v}_{\text{atm}} + e^{i\eta} m_s (\hat{\boldsymbol{v}}_{\text{atm}}^\dagger \boldsymbol{v}_{\text{sol}})^2, \\ z &\equiv |z|e^{i\phi_z} = e^{i\eta} m_s \sqrt{(\hat{\boldsymbol{v}}_{\text{atm}} \times \boldsymbol{v}_{\text{sol}})^\dagger (\hat{\boldsymbol{v}}_{\text{atm}} \times \boldsymbol{v}_{\text{sol}})} (\hat{\boldsymbol{v}}_{\text{atm}}^\dagger \boldsymbol{v}_{\text{sol}}), \\ w &\equiv |w|e^{i\phi_w} = e^{i\eta} m_s (\hat{\boldsymbol{v}}_{\text{atm}} \times \boldsymbol{v}_{\text{sol}})^\dagger (\hat{\boldsymbol{v}}_{\text{atm}} \times \boldsymbol{v}_{\text{sol}}). \end{aligned} \tag{2.14}$$

The neutrino mass matrix m'_ν in Eq. (2.13) can be diagonalized through the standard procedure, as shown in Refs. [32,61],

$$U_{\nu2}^T m'_\nu U_{\nu2} = \begin{cases} \text{diag}(0, m_2, m_3) & \text{for NO}, \\ \text{diag}(m_1, m_2, 0) & \text{for IO}, \end{cases} \tag{2.15}$$

where the unitary matrix $U_{\nu2}$ can be written as

$$U_{\nu2} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta e^{i(\psi+\rho)/2} & \sin\theta e^{i(\psi+\sigma)/2} \\ 0 & -\sin\theta e^{i(-\psi+\rho)/2} & \cos\theta e^{i(-\psi+\sigma)/2} \end{pmatrix} & \text{for NO}, \\ \begin{pmatrix} \cos\theta e^{i(\psi+\rho)/2} & \sin\theta e^{i(\psi+\sigma)/2} & 0 \\ -\sin\theta e^{i(-\psi+\rho)/2} & \cos\theta e^{i(-\psi+\sigma)/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for IO}. \end{cases} \tag{2.16}$$

We find that the light neutrino masses are

$$\begin{aligned} m_l^2 &= \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 - \frac{|w|^2 - |y|^2}{\cos 2\theta}\right], \\ m_h^2 &= \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 + \frac{|w|^2 - |y|^2}{\cos 2\theta}\right], \end{aligned} \tag{2.17}$$

with $m_1 = 0$, $m_2 = m_l$, $m_3 = m_h$ for the NO case and $m_1 = m_l$, $m_2 = m_h$, $m_3 = 0$ for the IO case. The rotation angle θ is determined by

$$\begin{aligned} \sin 2\theta &= \frac{2|z|\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}{\sqrt{(|w|^2 - |y|^2)^2 + 4|z|^2[|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)]}}, \\ \cos 2\theta &= \frac{|w|^2 - |y|^2}{\sqrt{(|w|^2 - |y|^2)^2 + 4|z|^2[|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)]}}. \end{aligned} \tag{2.18}$$

It is obvious that $\sin 2\theta$ is always non-negative. The expressions of the phases ψ, ρ and σ are given by

$$\begin{aligned} \sin\psi &= \frac{-|y|\sin(\phi_y - \phi_z) + |w|\sin(\phi_w - \phi_z)}{\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\psi &= \frac{|y|\cos(\phi_y - \phi_z) + |w|\cos(\phi_w - \phi_z)}{\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \sin\rho &= -\frac{(m_2^2 - |z|^2)\sin\phi_z + |y||w|\sin(\phi_y + \phi_w - \phi_z)}{m_2\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\rho &= \frac{(m_2^2 - |z|^2)\cos\phi_z + |y||w|\cos(\phi_y + \phi_w - \phi_z)}{m_2\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \sin\sigma &= -\frac{(m_3^2 - |z|^2)\sin\phi_z + |y||w|\sin(\phi_y + \phi_w - \phi_z)}{m_3\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\sigma &= \frac{(m_3^2 - |z|^2)\cos\phi_z + |y||w|\cos(\phi_y + \phi_w - \phi_z)}{m_3\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}. \end{aligned} \tag{2.19}$$

Thus the lepton mixing matrix is determined to be

$$U_{\rm PMNS} = P_l U_l^\dagger U_{\nu 1} U_{\nu 2}, \tag{2.20}$$

where P_l is a generic permutation matrix since the charged lepton masses are not constrained in this approach, and it can take the following six possible forms:

$$P_{123}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad P_{132}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad P_{213}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$
$$P_{231}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad P_{312}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad P_{321}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}. \tag{2.21}$$

If two mixing matrices are related by the exchange of the second and third rows, we shall only consider one of them. The reason is that the atmospheric mixing angle θ_{23} becomes $\pi/2-\theta_{23}$, the Dirac CP phase δ_{CP} becomes $\pi+\delta_{CP}$ and the other mixing parameters are unchanged after the second and third rows of a PMNS matrix are permuted. We notice that if both NO and IO neutrino mass spectra can be achieved for a residual symmetry, the lepton mixing matrix of IO can be obtained from the corresponding one of NO by multiplying P_{312} from the right side, and the expressions of the parameters y, z and w in m'_ν are identical in the NO and IO cases.

In the present work, we will adopt the standard parametrization of the lepton mixing matrix [67],

$$U=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23}\end{pmatrix}\mathrm{diag}(1,e^{i\frac{\beta}{2}},1), \tag{2.22}$$

where $c_{ij}\equiv\cos\theta_{ij}$, $s_{ij}\equiv\sin\theta_{ij}$, δ_{CP} is the Dirac CP violation phase and β is the Majorana CP phase. There is a second Majorana phase if the lightest neutrino is not massless. As regards the CP violation, two weak basis invariants J_{CP} [68] and I_1 [69–73] associated with the CP phases δ_{CP} and β respectively can be defined,

$$J_{CP}=\Im(U_{11}U_{33}U_{13}^*U_{31}^*)=\frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\cos\theta_{13}\sin\delta_{CP},$$
$$I_1=\begin{cases}\Im(U_{12}^2U_{13}^{*2})=\frac{1}{4}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin(\beta+2\delta_{CP}) & \text{for NO},\\ \Im(U_{12}^2U_{11}^{*2})=\frac{1}{4}\cos^4\theta_{13}\sin^2 2\theta_{12}\sin\beta & \text{for IO}.\end{cases} \tag{2.23}$$

Given a set of residual symmetries $\{G_l, G_{\rm atm}\rtimes H_{CP}^{\rm atm}, G_{\rm sol}\rtimes H_{CP}^{\rm sol}\}$, the explicit forms of U_l, $\langle\phi_{\rm atm}\rangle$ and $\langle\phi_{\rm sol}\rangle$ can be straightforwardly determined. Using the general formulas of Eqs. (2.4), (2.11), (2.16), (2.17), and (2.20), we can extract the predictions for the lepton mixing matrix and neutrino masses.

III. MIXING PATTERNS DERIVED FROM S_4 WITH NO NEUTRINO MASSES

In this section, we shall consider all possible residual subgroups arising from the breaking of S_4 flavor symmetry and CP, and the resulting predictions for lepton mixing parameters and neutrino masses are studied. The group theory of S_4 and all the CG coefficients in our basis are reported in Appendix A. S_4 has 20 nontrivial Abelian subgroups which contain nine Z_2 subgroups, four Z_3 subgroups, three Z_4 subgroups, and four $K_4\cong Z_2\times Z_2$ subgroups. In our basis given in Appendix A, the generalized CP transformation compatible with the S_4 flavor symmetry is of the same form as the flavor symmetry transformation [31], i.e.,

$$X_{\mathbf r}=\rho_{\mathbf r}(g),\qquad g\in S_4, \tag{3.1}$$

where g can be any of the 24 group elements of S_4.

As discussed in Sec. II, the flavor symmetry S_4 is broken to the Abelian subgroup G_l which is capable of distinguishing the three generations in the charged lepton sector. Then G_l can be taken to be any one of the 11 subgroups Z_3, Z_4 and K_4 of S_4. The vacuum alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ preserve different residual symmetries $G_{\rm atm}\rtimes H_{CP}^{\rm atm}$ and $G_{\rm sol}\rtimes H_{CP}^{\rm sol}$, respectively. The residual flavor symmetries $G_{\rm atm}$ and $G_{\rm sol}$ can be any one of the 20 Abelian subgroups of S_4. After including residual CP symmetry, we find that there are altogether 4400 kinds of possible breaking patterns. But these breaking patterns are not all independent from each other. If a pair of residual flavor symmetries $\{G'_l, G'_{\rm atm}, G'_{\rm sol}\}$ is conjugated to the pair of groups $\{G_l, G_{\rm atm}, G_{\rm sol}\}$ under an element of S_4, i.e.,

$$G'_l=hG_lh^{-1},\qquad G'_{\rm atm}=hG_{\rm atm}h^{-1},$$
$$G'_{\rm sol}=hG_{\rm sol}h^{-1},\qquad h\in S_4, \tag{3.2}$$

then these two breaking patterns will lead to the same predictions for mixing parameters [32,35,37]. As a result, it is sufficient to analyze the independent residual flavor symmetries not related by group conjugation and the compatible remnant CP. From Appendix A, we find that all the Z_3 subgroups of S_4 are conjugate to each other, all the Z_4 subgroups are related to each other under group conjugation, $K_4^{(S,TST^2)}$ is a normal subgroup of S_4, and the other three K_4 subgroups are conjugate to each other. As a consequence, it is sufficient to only consider four types of residual symmetries in the charged lepton sector, i.e., $G_l = Z_3^T,\ Z_4^{TSU},\ K_4^{(S,TST^2)}$ and $K_4^{(S,U)}$, while both $G_{\rm atm}$ and $G_{\rm sol}$ can be any one of these 20 subgroups of S_4. In the present work, we assume that the three generations of left-handed lepton doublets are assigned to transform as an S_4 triplet **3**. For G_l being the above four kinds of subgroups, up to permutations and phases of the column vectors, the diagonalization matrix of the Hermitian combination $m_l^\dagger m_l$ can be fixed to be

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } G_l = Z_3^T,$$
$$U_l = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2\omega & -2\omega & 2\omega \\ -(\sqrt{3}+1)\omega^2 & (1-\sqrt{3})\omega^2 & 2\omega^2 \\ \sqrt{3}-1 & \sqrt{3}+1 & 2 \end{pmatrix}, \quad \text{for } G_l = Z_4^{TSU},$$
$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{for } G_l = K_4^{(S,TST^2)},$$
$$U_l = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{for } G_l = K_4^{(S,U)}. \tag{3.3}$$

The residual CP symmetries in the atmospheric neutrino sector and the solar neutrino sector have to be compatible with the residual flavor symmetries, and the restricted consistency conditions in Eqs. (2.5a) and (2.5b) must be fulfilled. For the residual flavor symmetries $G_{\rm atm}$ and $G_{\rm sol}$ being the 20 subgroups of S_4, the corresponding residual CP transformations consistent with these subgroups are listed in Table I. In this work, we assume that the flavon fields $\phi_{\rm atm}$ and $\phi_{\rm sol}$ are assigned to transform as S_4 triplets **3** and **3**′, respectively. In our working basis, the S_4 singlet contraction rules for $\mathbf{3}\otimes\mathbf{3}\to\mathbf{1}$ and $\mathbf{3}\otimes\mathbf{3}'\to\mathbf{1}'$ imply $(L\phi_{\rm atm})_{\mathbf{1}} = L_1\phi_{{\rm atm}_1} + L_2\phi_{{\rm atm}_3} + L_3\phi_{{\rm atm}_2}$ and $(L\phi_{\rm sol})_{\mathbf{1}'} = L_1\phi_{{\rm sol}_1} + L_2\phi_{{\rm sol}_3} + L_3\phi_{{\rm sol}_2}$. As a consequence, we can read out the matrices U_a and U_s as follows:

$$U_a = U_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

In other words, the column vectors $\boldsymbol{v}_{\rm atm}$ and $\boldsymbol{v}_{\rm sol}$ defined above Eq. (2.8) are $\boldsymbol{v}_{\rm atm} = P_{132}\langle\phi_{\rm atm}\rangle/v_{\phi_a}$ and $\boldsymbol{v}_{\rm sol} = P_{132}\langle\phi_{\rm sol}\rangle/v_{\phi_s}$. Hence the column vectors $\boldsymbol{v}_{\rm atm}$ and $\boldsymbol{v}_{\rm sol}$ can be obtained by exchanging the second and third elements of the columns $\langle\phi_{\rm atm}\rangle/v_{\phi_a}$ and $\langle\phi_{\rm sol}\rangle/v_{\phi_s}$, respectively. The most general VEVs of the flavons $\phi_{\rm atm}$ and $\phi_{\rm sol}$ which preserve the possible residual symmetries in Table I are summarized in Table II. For some residual flavor groups, not all the compatible residual CP transformations in Table I are explicitly listed in Table II; this is because the invariant vacuum alignments for the shown residual CP symmetry and those not shown only differ by an overall factor of i. The contribution of the overall factor of i can be absorbed into the couplings $x_{\rm atm}$ and $x_{\rm sol}$. Following the procedures presented in Sec. II, we can straightforwardly obtain the expressions of the mixing parameters (three mixing angles, one Dirac CP phase and one Majorana CP phase) and the neutrino masses for each possible residual symmetry $\{G_l, G_{\rm atm}\rtimes H_{CP}^{\rm atm}, G_{\rm sol}\rtimes H_{CP}^{\rm sol}\}$.

In order to single out all independent viable breaking patterns from all possible breaking patterns in the tridirect CP approach, we will first find all possible independent pairs of $\{G_l, G_{\rm atm}\rtimes H_{CP}^{\rm atm}, G_{\rm sol}\rtimes H_{CP}^{\rm sol}\}$ which are not related by group conjugation given in Eq. (3.2). In order to quantitatively assess how well a residual symmetry can describe the experimental data on mixing parameters and neutrino masses [1], we define a χ^2 function to estimate the goodness of fit of a chosen set of values of the input parameters,

$$\chi^2 = \sum_{i=1}^{5}\left(\frac{P_i(x,\eta,m_a,r) - O_i}{\sigma_i}\right)^2, \tag{3.5}$$

where the input parameters m_a, $r = m_s/m_a$ and η are defined in Eq. (2.8), the parameter x parametrizes the vacuum of the flavon $\phi_{\rm sol}$, O_i denote the global best-fit values of the observable quantities including the mixing angles $\sin^2\theta_{ij}$ and the mass splittings Δm_{21}^2 and Δm_{3l}^2 ($\Delta m_{3l}^2 = \Delta m_{31}^2$ for NO and $\Delta m_{3l}^2 = \Delta m_{32}^2$ for IO), and σ_i refer to the 1σ deviations of the corresponding quantities. The values of O_i and σ_i are taken from the global data analysis [1]. $P_i \in \{\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, \Delta m_{21}^2, \Delta m_{3l}^2\}$ are the theoretical predictions for the five physical observable quantities as functions of x, η, m_a, r. Here the contribution of the Dirac phase δ_{CP} is not included in the χ^2 function. The reason is that the value of δ_{CP} is less constrained at present. For each set of the input parameters x, η, m_a and r, we can extract the predictions for P_i and the corresponding χ^2. We have carried out the χ^2 minimization.

TABLE I. The possible residual flavor subgroups and the compatible residual CP transformations.

$G_{\rm atm}$ ($G_{\rm sol}$)	$X_{\rm atm}$ ($X_{\rm sol}$)	$G_{\rm atm}$ ($G_{\rm sol}$)	$X_{\rm atm}$ ($X_{\rm sol}$)
Z_4^{TSU}	$\{SU, T, STS, TST^2U, U, ST, TS, T^2STU\}$	$Z_4^{ST^2U}$	$\{SU, T^2, ST^2S, T^2STU, U, ST^2, T^2S, TST^2U\}$
$Z_4^{TST^2U}$	$\{1, S, TST^2U, T^2STU, TST^2, T^2ST, U, SU\}$	Z_3^T	$\{1, T, T^2, U, TU, T^2U\}$
Z_3^{ST}	$\{S, STS, T^2, U, STU, T^2SU\}$	Z_3^{TS}	$\{S, T, ST^2S, U, TSU, ST^2U\}$
Z_3^{STS}	$\{1, STS, ST^2S, U, STSU, ST^2SU\}$	Z_2^S	$\{1, S, TST^2U, T^2STU, TST^2, T^2ST, U, SU\}$
$Z_2^{TST^2}$	$\{SU, T^2STU, T^2, ST^2S, U, TST^2U, ST^2, T^2S\}$	$Z_2^{T^2ST}$	$\{SU, TST^2U, T, STS, U, T^2STU, ST, TS\}$
Z_2^U	$\{1, U, S, SU\}$	Z_2^{TU}	$\{U, T, STS, T^2STU\}$
Z_2^{SU}	$\{1, SU, S, U\}$	$Z_2^{T^2U}$	$\{U, T^2, ST^2S, TST^2U\}$
Z_2^{STSU}	$\{U, STS, T, T^2STU\}$	$Z_2^{ST^2SU}$	$\{U, ST^2S, T^2, TST^2U\}$
$K_4^{(S,TST^2)}$	all elements of S_4	$K_4^{(S,U)}$	$\{1, S, U, SU, TST^2, T^2ST, TST^2U, T^2STU\}$
$K_4^{(TST^2,T^2U)}$	$\{U, T^2, ST^2S, TST^2U, SU, ST^2, T^2S, T^2STU\}$	$K_4^{(T^2ST,TU)}$	$\{U, T, STS, T^2STU, SU, ST, TS, TST^2U\}$

TABLE II. The possible residual symmetries and the corresponding constraints on the vacuum configurations of the flavon fields $\phi_{\rm atm}$ and $\phi_{\rm sol}$ which transform as $\mathbf{3}$ and $\mathbf{3}'$ respectively. The parameter x is a generic real number. The VEVs of $\phi_{\rm atm}$ invariant under the actions of the four K_4 subgroups are $(0,0,0)^T$. The VEVs of $\phi_{\rm sol}$ invariant under the three Z_4 subgroups and the normal subgroup $K_4^{(S,TST^2)}$ are also $(0,0,0)^T$. Comparing with Table I, for some residual flavor subgroups we only show the invariant vacuum alignments for part of the CP transformations consistent with them. The reason is that the invariant VEVs for the remaining compatible CP transformations can be obtained by multiplying the above given VEVs by an overall factor of i, and the contribution of the overall factor of i can be compensated by shifting the signs of the couplings $x_{\rm atm}$ and $x_{\rm sol}$.

VEVs of $\phi_{\rm atm}$

$G_{\rm atm}$	$X_{\rm atm}$	$\langle\phi_{\rm atm}\rangle/v_{\phi_a}$	$G_{\rm atm}$	$X_{\rm atm}$	$\langle\phi_{\rm atm}\rangle/v_{\phi_a}$
Z_2^S	$\{1, TST^2U, S, T^2STU\}$	$(1,1,1)^T$	$Z_2^{TST^2}$	$\{SU, T^2, ST^2S, T^2STU\}$	$(1,\omega^2,\omega)^T$
$Z_2^{T^2ST}$	$\{SU, T, TST^2U, STS\}$	$(1,\omega,\omega^2)^T$	Z_2^U	$\{1, U\}$	$(0,1,-1)^T$
Z_2^{TU}	$\{U, T\}$	$(0,-\omega,\omega^2)^T$	Z_2^{SU}	$\{1, SU\}$	$(2,-1,-1)^T$
$Z_2^{T^2U}$	$\{U, T^2\}$	$(0,-\omega^2,\omega)^T$	Z_2^{STSU}	$\{T, T^2STU\}$	$(2,-\omega,-\omega^2)^T$
$Z_2^{ST^2SU}$	$\{T^2, TST^2U\}$	$(2,-\omega^2,-\omega)^T$	Z_3^T	$\{1, T, T^2\}$	$(1,0,0)^T$
Z_3^{ST}	$\{S, STS, T^2\}$	$(1,-2\omega^2,-2\omega)^T$	Z_3^{TS}	$\{S, T, ST^2S\}$	$(1,-2\omega,-2\omega^2)^T$
Z_3^{STS}	$\{1, STS, ST^2S\}$	$(1,-2,-2)^T$	Z_4^{TSU}	$\{SU, T, STS, TST^2U\}$	$(1,\omega,\omega^2)^T$
$Z_4^{ST^2U}$	$\{SU, T^2, ST^2S, T^2STU\}$	$(1,\omega^2,\omega)^T$	$Z_4^{TST^2U}$	$\{1, S, TST^2U, T^2STU\}$	$(1,1,1)^T$

VEVs of $\phi_{\rm sol}$

$G_{\rm sol}$	$X_{\rm sol}$	$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	$G_{\rm sol}$	$X_{\rm sol}$	$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$
Z_2^U	$\{1, U\}$ $\{S, SU\}$	$(1,x,x)^T$ $(1+2ix, 1-ix, 1-ix)^T$	Z_2^{TU}	$\{U, T\}$ $\{STS, T^2STU\}$	$(1,x\omega,x\omega^2)^T$ $(1+2ix, \omega(1-ix), \omega^2(1-ix))^T$
Z_2^{SU}	$\{1, SU\}$ $\{S, U\}$	$(1,x,2-x)^T$ $(1,1+ix,1-ix)^T$	$Z_2^{T^2U}$	$\{U, T^2\}$ $\{ST^2S, TST^2U\}$	$(1,x\omega^2,x\omega)^T$ $(1+2ix, \omega^2(1-ix), \omega(1-ix))^T$
Z_2^{STSU}	$\{U, STS\}$ $\{T, T^2STU\}$	$(\frac{\sqrt{3}x-1}{2}, 1+ix, 1-ix)^T$ $(1,(2x+1)\omega,(1-2x)\omega^2)^T$	$Z_2^{ST^2SU}$	$\{U, ST^2S\}$ $\{T^2, TST^2U\}$	$(\frac{-1-\sqrt{3}x}{2}, 1+ix, 1-ix)^T$ $(1,(2x+1)\omega^2,(1-2x)\omega)^T$
Z_2^S	$\{1, U, S, SU\}$	$(1,1,1)^T$	$Z_2^{TST^2}$	$\{U, T^2, TST^2U, ST^2S\}$	$(1,\omega^2,\omega)^T$
$Z_2^{T^2ST}$	$\{U, T, T^2STU, STS\}$	$(1,\omega,\omega^2)^T$	Z_3^T	$\{1, U, T, TU, T^2, T^2U\}$	$(1,0,0)^T$
Z_3^{ST}	$\{U, S, STU,$ $STS, T^2SU, T^2\}$	$(1,-2\omega^2,-2\omega)^T$	Z_3^{TS}	$\{U, S, TSU, T,$ $ST^2U, ST^2S\}$	$(1,-2\omega,-2\omega^2)^T$
Z_3^{STS}	$\{1, U, STS, STSU,$ $ST^2S, ST^2SU\}$	$(1,-2,-2)^T$	$K_4^{(S,U)}$	$\{1, U, S, SU\}$	$(1,1,1)^T$
$K_4^{(TST^2,T^2U)}$	$\{U, T^2, TST^2U, ST^2S\}$	$(1,\omega^2,\omega)^T$	$K_4^{(T^2ST,TU)}$	$\{U, T, T^2STU, STS\}$	$(1,\omega,\omega^2)^T$

TABLE III. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} in neutrinoless double beta decay for all viable residual symmetries, where the parameters x, η, m_a and $r \equiv m_s/m_a$ are treated as free parameters. The residual CP transformation associated with atmospheric neutrinos can be read out from Table II. We only show one representative residual CP transformation of the solar neutrino sector since the other residual CP transformations can be obtained by multiplying the residual flavor symmetry $G_{\rm sol}$ with the given CP transformation from the left-hand side.

NO for x, η, m_a and $r \equiv m_s/m_a$ being free parameters											
	$(G_l, G_{\rm atm}, G_{\rm sol})$	$X_{\rm sol}$	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$\mathcal{N}_1$	(Z_3^T, Z_2^U, Z_2^{SU})	1	0.383	0.0224	0.318	0.580	−0.386	0.335	8.597	50.249	3.100
		U	0.383	0.0224	0.318	0.580	−0.386	0.910	8.597	50.249	3.725
$\mathcal{N}_2$	$(Z_3^T, Z_3^{ST}, Z_2^{SU})$	1	0.383	0.0224	0.318	0.580	−0.386	0.754	8.596	50.249	3.798
		U	0.383	0.0224	0.318	0.580	−0.386	0.996	8.596	50.249	3.604
$\mathcal{N}_3$	(Z_3^T, Z_2^S, Z_2^{SU})	U	4.321	0.0225	0.318	0.538	−0.447	0.444	8.603	50.242	3.064
$\mathcal{N}_4$	$(Z_3^T, Z_2^{TST^2}, Z_2^U)$	1	5.081	0.0225	0.337	0.563	−0.407	0.284	8.601	50.244	2.950
$\mathcal{N}_5$	$(K_4^{(S,U)}, Z_2^{TU}, Z_2^{TU})$	U	20.461	0.0225	0.256	0.582	0	−0.265	8.597	50.249	3.026
$\mathcal{N}_6$	$(Z_4^{TSU}, Z_3^T, Z_2^{SU})$	U	8.698	0.0226	0.345	0.554	−0.419	0.202	8.605	50.239	2.638
$\mathcal{N}_7$	$(K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$	1	12.254	0.0224	0.328	0.513	−0.482	0.502	8.600	50.245	3.099
		U	11.621	0.0224	0.327	0.514	0	0	8.601	50.244	3.877
$\mathcal{N}_8$	$(K_4^{(S,TST^2)}, Z_2^U, Z_2^{TU})$	U	5.768	0.0228	0.298	0.537	−0.451	0.365	8.539	50.326	2.615

IO for x, η, m_a and $r \equiv m_s/m_a$ being free parameters											
	$(G_l, G_{\rm atm}, G_{\rm sol})$	$X_{\rm sol}$	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_1 (meV)	m_2 (meV)	m_{ee} (meV)
$\mathcal{I}_1$	(Z_3^T, Z_3^{ST}, Z_2^U)	1	17.640	0.0226	0.310	0.5	−0.928	0.306	49.377	50.120	43.792
$\mathcal{I}_2$	$(Z_3^T, Z_2^{SU}, Z_2^{TU})$	U	17.640	0.0226	0.310	0.5	−0.682	0.843	49.377	50.120	21.168
$\mathcal{I}_3$	$(K_4^{(S,U)}, Z_2^{TST^2}, Z_2^U)$	1	17.640	0.0226	0.310	0.5	−0.495	0.102	49.377	50.120	47.946
		S	17.640	0.0226	0.310	0.5	−0.495	0.102	49.377	50.120	47.946
$\mathcal{I}_4$	$(K_4^{(S,U)}, Z_2^{TU}, Z_2^{TU})$	U	20.419	0.0227	0.256	0.582	0	1	49.377	50.120	23.384
$\mathcal{I}_5$	$(Z_3^T, Z_2^{SU}, Z_2^{SU})$	U	18.008	0.0227	0.318	0.5	−0.5	0.743	49.377	50.120	24.840
$\mathcal{I}_6$	$(Z_3^T, Z_2^{TST^2}, Z_2^U)$	1	17.640	0.0226	0.310	0.5	0.913	−0.389	49.377	50.120	41.048
$\mathcal{I}_7$	(Z_3^T, Z_2^U, Z_2^{TU})	U	17.640	0.0226	0.310	0.5	0.975	−0.175	49.377	50.120	46.918
$\mathcal{I}_8$	$(Z_3^T, Z_2^U, Z_2^{STSU})$	U	17.640	0.0226	0.310	0.5	−0.761	0.759	49.377	50.119	24.569
$\mathcal{I}_9$	$(Z_3^T, Z_2^{SU}, Z_2^{STSU})$	U	17.640	0.0226	0.310	0.5	−0.954	0.249	49.377	50.120	45.347
$\mathcal{I}_{10}$	$(Z_4^{TSU}, Z_2^S, Z_2^{TU})$	U	17.640	0.0226	0.310	0.5	−0.00465	−0.102	49.377	50.120	47.946
		STS	17.640	0.0226	0.310	0.5	−0.00465	−0.102	49.377	50.120	47.946
$\mathcal{I}_{11}$	$(Z_4^{TSU}, Z_2^S, Z_2^{T^2U})$	U	17.640	0.0226	0.310	0.5	−0.128	−0.548	49.377	50.120	34.480
		ST^2S	17.640	0.0226	0.310	0.5	−0.372	0.548	49.377	50.120	34.480
$\mathcal{I}_{12}$	$(Z_4^{TSU}, Z_2^U, Z_2^{TU})$	U	17.640	0.0226	0.310	0.5	−0.772	0.729	49.377	50.120	25.920
$\mathcal{I}_{13}$	$(Z_4^{TSU}, Z_2^{TU}, Z_2^U)$	1	17.640	0.0226	0.310	0.5	0.834	−0.636	49.377	50.120	30.323
$\mathcal{I}_{14}$	$(K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$	1	17.640	0.0226	0.310	0.5	−0.104	−0.448	49.377	50.120	38.772
		U	2.046	0.0225	0.310	0.607	−0.604	−0.448	49.377	50.120	38.778
$\mathcal{I}_{15}$	$(K_4^{(S,TST^2)}, Z_2^U, Z_2^{TU})$	U	17.640	0.0226	0.310	0.5	−0.666	−0.636	49.377	50.120	30.323
$\mathcal{I}_{16}$	$(K_4^{(S,U)}, Z_2^{TST^2}, Z_2^{TU})$	STS	17.640	0.0226	0.310	0.5	−0.872	0.548	49.377	50.120	34.480
$\mathcal{I}_{17}$	$(K_4^{(S,U)}, Z_2^{TU}, Z_2^U)$	S	28.676	0.0225	0.310	0.477	0.915	−0.548	49.377	50.120	34.486
$\mathcal{I}_{18}$	$(K_4^{(S,U)}, Z_2^{TU}, Z_2^{T^2U})$	ST^2S	9.241	0.0227	0.310	0.523	−0.743	0.510	49.377	50.120	36.178

After performing the χ^2 analysis for all possible breaking patterns in the tridirect CP approach, we find eight independent interesting mixing patterns with NO and 18 interesting mixing patterns with IO. All the viable cases and the corresponding predictions for mixing parameters and neutrino masses are summarized in Table III. Then we proceed to study the eight NO viable cases (five cases in this section and three cases in Appendix B) one by one.

($\mathcal{N}_1$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^U, Z_2^{SU})$, $X_{\rm atm} = \{1, U\}$

(i) $X_{\rm sol} = \{1, SU\}$

For this breaking pattern, the charged lepton mass matrix $m_l^\dagger m_l$ is diagonal such that the unitary transformation U_l is the identity matrix, as shown in Eq. (3.3). From Table I, we find that there are four possible residual CP transformations which are compatible with the residual family symmetry Z_2^U in the atmospheric neutrino sector. For the residual CP transformations $X_{\rm atm} = \{1, U\}$, the VEV alignment of the flavon $\phi_{\rm atm}$ is

$$\langle\phi_{\rm atm}\rangle = v_{\phi_a}(0, 1, -1)^T, \tag{3.6}$$

where v_{ϕ_a} is a real parameter with dimensions of mass. For the other two residual CP transformations $X_{\rm atm} = \{S, SU\}$, the alignment of the flavon $\phi_{\rm atm}$ is

$$\langle\phi_{\rm atm}\rangle = i v_{\phi_a}(0, 1, -1)^T, \tag{3.7}$$

which differs from the vacuum configuration of Eq. (3.6) by an overall factor of i. We see from Eq. (2.8) that this overall factor of i can be absorbed into the sign of the coupling constant $x_{\rm atm}$. Hence the two alignments in Eqs. (3.6) and (3.7) will give rise to the same light neutrino mass matrix, and it is sufficient to consider one of them. For other residual symmetries discussed in the following, if two alignments of $\phi_{\rm atm}$ differ by an overall factor of i only one of them will be studied as well. Without loss of generality, here we shall choose the atmospheric vacuum in Eq. (3.6), i.e., the residual CP is $X_{\rm atm} = \{1, U\}$ in the atmospheric neutrino sector.

First we consider the solar residual CP transformations $X_{\rm sol} = \{1, SU\}$; then the VEV of the flavon field $\phi_{\rm sol}$ reads as

$$\langle\phi_{\rm sol}\rangle = v_{\phi_s}(1, x, 2-x)^T, \tag{3.8}$$

where x is a dimensionless real number and v_{ϕ_s} is a real parameter with dimensions of mass. Consequently the Dirac neutrino mass matrix m_D and the heavy right-handed neutrino Majorana mass matrix m_N take the following forms:

$$m_D = \begin{pmatrix} 0 & y_{\rm sol} v_{\phi_s} \\ -y_{\rm atm} v_{\phi_a} & (2-x) y_{\rm sol} v_{\phi_s} \\ y_{\rm atm} v_{\phi_a} & x y_{\rm sol} v_{\phi_s} \end{pmatrix},$$
$$m_N = \begin{pmatrix} x_{\rm atm}\langle\xi_{\rm atm}\rangle & 0 \\ 0 & x_{\rm sol}\langle\xi_{\rm sol}\rangle \end{pmatrix}, \tag{3.9}$$

where the couplings $y_{\rm atm}$ and $y_{\rm sol}$ are real since the theory is invariant under CP. Using the seesaw formula, we can obtain the low-energy effective light neutrino mass matrix

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & 2-x & x \\ 2-x & (x-2)^2 & (2-x)x \\ x & (2-x)x & x^2 \end{pmatrix}, \tag{3.10}$$

where an overall unphysical phase has been omitted and it will be neglected for the other cases in the following, and the parameters m_a, m_s and η are defined in Eq. (2.8). We find that the above neutrino mass matrix m_ν fulfills

$$m_\nu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

It implies that the column vector $(2, -1, -1)^T$ is an eigenvector of m_ν with zero eigenvalue. Subsequently we follow the procedure given in Sec. II to perform a unitary transformation $U_{\nu 1}$,

$$m'_\nu = U_{\nu 1}^T m_\nu U_{\nu 1}, \tag{3.12}$$

with

$$U_{\nu 1} = U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.13}$$

where $U_{\rm TB}$ is the well-known tribimaximal (TB) mixing matrix. The neutrino mass matrix m'_ν is block diagonal, and its entries are given by

$$y = 3 m_s e^{i\eta}, \qquad z = \sqrt{6}(x-1) m_s e^{i\eta},$$
$$w = 2(m_a + (x-1)^2 m_s e^{i\eta}). \tag{3.14}$$

Furthermore, m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$ given in Eq. (2.16), i.e.,

$$U_{\nu 2}^T m'_\nu U_{\nu 2} = {\rm diag}(0, m_2, m_3). \tag{3.15}$$

The expressions of m_2, m_3 and $U_{\nu 2}$ can be straightforwardly obtained by inserting the parameters y, z and w into Eqs. (2.17)–(2.19). As a result, we can read out the lepton mixing matrix as

$$U_{\mathrm{PMNS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{\cos\theta}{\sqrt{3}} & \frac{e^{i\psi}\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} + \frac{e^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{e^{i\psi}\sin\theta}{\sqrt{3}} - \frac{\cos\theta}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} - \frac{e^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} + \frac{e^{i\psi}\sin\theta}{\sqrt{3}} \end{pmatrix} P_\nu, \tag{3.16}$$

with

$$P_\nu = \mathrm{diag}(1, e^{i(\psi+\rho)/2}, e^{i(-\psi+\sigma)/2}). \tag{3.17}$$

In the following, the Majorana phase matrix P_ν will be omitted for simplicity. We see that the neutrino mixing matrix is the so-called TM1 mixing pattern in which the first column of the tribimaximal mixing is preserved. The three lepton mixing angles for this mixing matrix are

$$\sin^2\theta_{13} = \frac{\sin^2\theta}{3}, \qquad \sin^2\theta_{12} = \frac{2\cos^2\theta}{5+\cos 2\theta},$$
$$\sin^2\theta_{23} = \frac{1}{2} - \frac{\sqrt{6}\sin 2\theta\cos\psi}{5+\cos 2\theta}. \tag{3.18}$$

Eliminating the free parameter θ, a sum rule between the solar mixing angle θ_{12} and the reactor mixing angle θ_{13} is found,

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{2}{3}. \tag{3.19}$$

Plugging in the best-fit value of $\sin^2\theta_{13} = 0.02241$ [1], we find that the solar mixing angle is

$$\sin^2\theta_{12} \simeq 0.318, \tag{3.20}$$

which is within the 3σ region [1]. For the lepton mixing matrix in Eq. (3.16), the two CP invariants are given by

$$J_{CP} = \frac{\sin 2\theta \sin\psi}{6\sqrt{6}}, \qquad I_1 = \frac{1}{36}\sin^2 2\theta \sin(\rho-\sigma). \tag{3.21}$$

The so-called TM1 mixing matrix indicates the following sum rule among the Dirac CP phase δ_{CP} and mixing angles:

$$\cos\delta_{CP} = \frac{(3-5\cos 2\theta_{13})\cot 2\theta_{23}}{4\sin\theta_{13}\sqrt{3\cos 2\theta_{13}-1}}. \tag{3.22}$$

It is easy to check that $\theta_{23} = \pi/4$ leads to $\cos\delta_{CP} = 0$ which corresponds to maximal CP violation $\delta_{CP} = \pm\pi/2$. The neutrino masses m_2 and m_3 depend on all four input parameters x, η, m_a and m_s while the mixing parameters and mass ratio m_2^2/m_3^2 only depend on x, η and $r \equiv m_s/m_a$. In the case where η, m_a and r are free parameters, we find that the experimental data on the mixing angles and the neutrino masses can be achieved for some special x.

In order to show concrete examples, some benchmark values of the parameters x and η are considered and the numerical results for the mixing parameters and neutrino masses are listed in Table IV. The solar flavon alignment ϕ_{sol} for these representative values of x takes a relatively simple form; consequently we expect that it should not be difficult to be realized dynamically in an explicit model. We show the predictions for the effective Majorana mass m_{ee} in neutrinoless double beta decay in the last column of Table IV, where the effective mass m_{ee} is defined as [67],

$$m_{ee} = |m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2|. \tag{3.23}$$

From Table IV, we can see that the measured values of the lepton mixing angles and the mass splittings Δm_{21}^2 and Δm_{31}^2 can be accommodated for certain choices of x, η, m_a and r. For the benchmark value $x = -1$, the solar flavon alignment is $\langle\phi_{\mathrm{sol}}\rangle = (1,-1,3)^T v_{\phi_s}$, which is exactly the littlest seesaw model with CSD(3) originally proposed in Ref. [15]. The solar vacuum $\langle\phi_{\mathrm{sol}}\rangle = (1,-3,1)^T v_{\phi_s}$ for $x = 3$ corresponds to another version of the littlest seesaw model [16]. Moreover, the value $x = 4$ leads to the vacuum $\langle\phi_{\mathrm{sol}}\rangle = (1,4,-2)^T v_{\phi_s}$, and the CSD(4) scenario [17] is reproduced. From Table IV, we see that a smaller χ^2 than the original LS model [15–17,74] can be achieved for the values $x = -1/2$ and $\eta = \pm\pi/2$, and the corresponding vacuum alignment $\langle\phi_{\mathrm{sol}}\rangle \propto (2,-1,5)$ seems simple and it should be easy to realize in a concrete model.

Furthermore, we perform a comprehensive numerical analysis. The three input parameters x, r and η are randomly scanned over $x \in [-20, 20]$, $r \in [0, 20]$ and $\eta \in [-\pi, \pi]$. We only keep the points for which the resulting mixing angles $\sin^2\theta_{ij}$ and the mass ratio m_2^2/m_3^2 are in the experimentally preferred 3σ regions [1]. The parameter m_a can be fixed by requiring that the individual squared mass differences Δm_{21}^2 and Δm_{31}^2 are reproduced. Then the predictions for the CP-violating phases δ_{CP} and β and the neutrino masses as well as m_{ee} can be extracted. In the end we find that the allowed regions of the parameters x, $|\eta|$, and r are $[-2.072, -0.287] \cup [2.463, 4.683]$, $[0.414\pi, 0.861\pi]$, and $[0.0400, 0.166]$, respectively. As regards the predictions for the mixing angles, we find that any values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ in their 3σ ranges can be achieved. The the solar mixing angle is in a narrow region $0.317 \le \sin^2\theta_{12} \le 0.319$, which arises from the TM1 sum rule in Eq. (3.19).

TABLE IV. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_1$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^U, Z_2^{SU})$ and $X_{\rm sol} = \{1, SU\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1,3,-1)^T$	3	$\pm\frac{2\pi}{3}$	26.843	0.0998	19.625	0.0222	0.318	0.488	∓0.516	∓0.403	8.586	50.263	2.680
$(1,-1,3)^T$	-1	$\pm\frac{2\pi}{3}$	26.798	0.101	10.716	0.0225	0.318	0.513	±0.482	∓0.401	8.628	50.212	2.694
$(1,4,-2)^T$	4	$\pm\frac{4\pi}{5}$	35.249	0.0565	14.196	0.0241	0.317	0.575	∓0.398	∓0.474	8.316	50.609	1.990
		$\pm\frac{5\pi}{6}$	36.720	0.0532	3.841	0.0227	0.318	0.610	∓0.338	∓0.554	8.560	50.297	1.954
$(1,-2,4)^T$	-2	$\pm\frac{4\pi}{5}$	35.242	0.0566	68.409	0.0243	0.317	0.425	±0.601	∓0.473	8.339	50.581	1.995
$(1,\frac{7}{2},-\frac{3}{2})^T$	$\frac{7}{2}$	$\pm\frac{3\pi}{4}$	31.121	0.0734	6.567	0.0231	0.318	0.541	∓0.444	∓0.447	8.462	50.425	2.285
		$\pm\frac{4\pi}{5}$	33.006	0.0674	9.388	0.0210	0.319	0.589	∓0.366	∓0.544	8.806	49.994	2.223
$(1,\frac{8}{3},-\frac{2}{3})^T$	$\frac{8}{3}$	$\pm\frac{3\pi}{5}$	24.618	0.121	45.788	0.0209	0.319	0.456	∓0.564	∓0.385	8.841	49.949	2.990
$(1,\frac{10}{3},-\frac{4}{3})^T$	$\frac{10}{3}$	$\pm\frac{3\pi}{4}$	30.566	0.0777	4.332	0.0218	0.318	0.548	∓0.432	∓0.474	8.689	50.139	2.375
$(1,-\frac{1}{2},\frac{5}{2})^T$	$-\frac{1}{2}$	$\pm\frac{\pi}{2}$	22.359	0.145	2.475	0.0220	0.318	0.599	±0.354	∓0.316	8.672	50.158	3.242
$(1,-\frac{3}{2},\frac{7}{2})^T$	$-\frac{3}{2}$	$\pm\frac{3\pi}{4}$	31.101	0.0737	35.893	0.0233	0.317	0.460	±0.555	∓0.445	8.493	50.386	2.293
$(1,-\frac{2}{3},\frac{8}{3})^T$	$-\frac{2}{3}$	$\pm\frac{3\pi}{5}$	24.560	0.123	13.654	0.0212	0.319	0.545	±0.435	∓0.383	8.888	49.890	3.009
$(1,-\frac{4}{3},\frac{10}{3})^T$	$-\frac{4}{3}$	$\pm\frac{3\pi}{4}$	30.550	0.0780	38.724	0.0220	0.318	0.453	±0.567	∓0.472	8.716	50.105	2.383
$(1,-\frac{3}{4},\frac{11}{4})^T$	$-\frac{3}{4}$	$\pm\frac{3\pi}{5}$	24.578	0.120	2.837	0.0222	0.318	0.551	±0.429	∓0.367	8.668	50.164	2.948
$(1,-\frac{5}{4},\frac{13}{4})^T$	$-\frac{5}{4}$	$\pm\frac{3\pi}{4}$	30.265	0.0802	46.446	0.0213	0.319	0.450	±0.573	∓0.486	8.824	49.971	2.428
$(1,-\frac{3}{5},\frac{13}{5})^T$	$-\frac{3}{5}$	$\pm\frac{\pi}{2}$	22.215	0.142	11.399	0.0232	0.317	0.606	±0.347	∓0.297	8.312	50.614	3.156
$(1,-\frac{4}{5},\frac{14}{5})^T$	$-\frac{4}{5}$	$\pm\frac{3\pi}{5}$	24.587	0.118	2.595	0.0228	0.318	0.554	±0.425	∓0.357	8.532	50.335	2.911
$(1,-\frac{6}{5},\frac{16}{5})^T$	$-\frac{6}{5}$	$\pm\frac{3\pi}{4}$	30.090	0.0816	53.229	0.0208	0.319	0.448	±0.577	∓0.494	8.887	49.891	2.456
$(1,-\frac{7}{5},\frac{17}{5})^T$	$-\frac{7}{5}$	$\pm\frac{3\pi}{4}$	30.772	0.0762	35.632	0.0225	0.318	0.455	±0.562	∓0.461	8.627	50.213	2.346
$(1,-\frac{5}{6},\frac{17}{6})^T$	$-\frac{5}{6}$	$\pm\frac{3\pi}{5}$	24.592	0.117	4.998	0.0231	0.318	0.556	±0.422	∓0.351	8.441	50.452	2.886

The predicted ranges of the Dirac CP phase $|\delta_{CP}|$ and Majorana CP phase $|\beta|$ are $[0.299\pi, 0.624\pi]$ and $[0.273\pi, 0.608\pi]$, respectively. These predictions may be tested at future long-baseline experiments, as discussed in Ref. [74]. The allowed ranges of the mixing parameters for other breaking patterns are also obtained by randomly varying the parameters x, r and η in the ranges $x \in [-20, 20]$, $r \in [0, 20]$ and $\eta \in [-\pi, \pi]$. We will not explicitly mention this point in the following.

(ii) $X_{\rm sol} = \{S, U\}$

From Table II, we find that the VEV of $\phi_{\rm sol}$ is proportional to $(1, 1+ix, 1-ix)^T$. Inserting the vacuum configuration of the flavons $\phi_{\rm atm}$ and $\phi_{\rm sol}$ into Eq. (2.8), we can obtain the light effective Majorana neutrino mass matrix as follows:

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & 1-ix & 1+ix \\ 1-ix & (1-ix)^2 & 1+x^2 \\ 1+ix & 1+x^2 & (1+ix)^2 \end{pmatrix}. \tag{3.24}$$

In the following, we will not give the expression of m_ν for each possible case, since it is just Eq. (2.8) with the appropriate VEVs replaced. It is easy to check that the column vector $(2,-1,-1)^T$ is an eigenvector of m_ν with zero eigenvalue. In order to diagonalize the light neutrino mass matrix m_ν in the above equation, we first perform a unitary transformation $U_{\nu 1}$, where $U_{\nu 1}$ is taken to be the TB mixing matrix $U_{\rm TB}$. Then the neutrino mass matrix m'_ν is of block diagonal form with nonzero elements y, z and w,

$$y = 3m_s e^{i\eta}, \qquad z = i\sqrt{6}x m_s e^{i\eta}, \qquad w = 2(m_a - x^2 m_s e^{i\eta}). \tag{3.25}$$

The neutrino mass matrix m'_ν can be exactly diagonalized by $U_{\nu 2}$ shown in Eq. (2.16). It is easy to check that the PMNS matrix takes the same form as Eq. (3.16), and it is also a TM1 mixing matrix. Therefore the expressions of the mixing angles and CP invariants are still given by Eqs. (3.18) and (3.21), respectively. However, the explicit

TABLE V. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_1$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^U, Z_2^{SU})$ and $X_{\rm sol} = \{S, U\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 1\pm 4i, 1\mp 4i)^T$	±4	0	47.378	0.0320	15.257	0.0228	0.318	0.5	∓0.5	0	8.562	50.295	1.515
$(1, 1\pm\frac{7i}{2}, 1\mp\frac{7i}{2})^T$	$\pm\frac{7}{2}$	0	43.643	0.0383	20.623	0.0211	0.319	0.5	∓0.5	0	8.734	50.083	1.670
$(1, 1\pm\frac{5i}{4}, 1\mp\frac{5i}{4})^T$	$\pm\frac{5}{4}$	$\pm\frac{3\pi}{4}$	19.427	0.187	7.914	0.0230	0.318	0.603	±0.352	∓0.888	8.361	50.554	3.626
$(1, 1\pm\frac{6i}{5}, 1\mp\frac{6i}{5})^T$	$\pm\frac{6}{5}$	$\pm\frac{3\pi}{4}$	19.634	0.187	2.177	0.0223	0.318	0.602	±0.350	∓0.881	8.619	50.222	3.675
		$\pm\frac{4\pi}{5}$	19.242	0.192	2.540	0.0228	0.318	0.583	±0.382	∓0.909	8.455	50.434	3.694
		$\pm\frac{5\pi}{6}$	19.030	0.195	6.400	0.023	0.318	0.570	±0.403	∓0.926	8.362	50.552	3.702
		$\mp\frac{5\pi}{6}$	19.049	0.194	56.898	0.0229	0.318	0.430	±0.597	±0.926	8.355	50.561	3.696
$(1, 1\pm\frac{7i}{6}, 1\mp\frac{7i}{6})^T$	$\pm\frac{7}{6}$	π	18.720	0.201	21.977	0.0231	0.318	0.5	±0.5	1	8.335	50.586	3.754
		$\pm\frac{3\pi}{4}$	19.777	0.187	6.184	0.0218	0.318	0.602	±0.348	∓0.876	8.792	50.011	3.705
		$\pm\frac{4\pi}{5}$	19.391	0.192	0.557	0.0223	0.318	0.583	±0.381	∓0.905	8.634	50.204	3.726
		$\pm\frac{5\pi}{6}$	19.184	0.195	0.917	0.0226	0.318	0.570	±0.402	∓0.923	8.546	50.317	3.736
		$\mp\frac{5\pi}{6}$	19.204	0.194	51.371	0.0225	0.318	0.431	±0.598	±0.923	8.537	50.328	3.729

dependence of the parameters y, z and w on m_a, m_s, η, x differs from that of the above case with $X_{\rm sol} = \{1, SU\}$. Hence distinct predictions for mixing parameters are reached. We can check that the neutrino mass matrix m_ν in Eq. (3.24) has the following symmetry properties:

$$\begin{aligned} m_\nu(-x, r, \eta) &= P_{132}^T m_\nu(x, r, \eta) P_{132}, \\ m_\nu(-x, r, -\eta) &= m_\nu^*(x, r, \eta). \end{aligned} \tag{3.26}$$

The former implies that the reactor and solar mixing angles are invariant, the atmospheric angle changes from θ_{23} to $\pi/2 - \theta_{23}$ and the Dirac phase changes from δ_{CP} to $\pi + \delta_{CP}$ under the transformation $x \to -x$. The latter implies that all the lepton mixing angles are kept intact and the signs of all CP violation phases are reversed by changing $x \to -x$ and $\eta \to -\eta$. Once the values of x and η are fixed, the light neutrino mass matrix m_ν only depends on two free parameters m_a and m_s whose values can be determined by the neutrino mass squared differences Δm^2_{21} and Δm^2_{31}. Then we can extract the predictions for the three lepton mixing angles and CP violation phases δ_{CP} and β. The best-fit values of the mixing parameters and neutrino masses for some benchmark values of x and η are shown in Table V. The most interesting points are $\eta = 0$ and π which predict a maximal atmospheric mixing angle, maximal Dirac phase and trivial Majorana phase. The reason is because the general neutrino mass m_ν shown in Eq. (3.24) has an accidental $\mu\tau$ reflection symmetry in the cases of $\eta = 0$ and π [75]. Realistic values of the mixing angles and mass ratio m_2^2/m_3^2 can be obtained for $x = \pm 4$, $\pm 7/2$, $\pm 7/6$ in the case of $\eta = 0$ or π. In order to describe the experimental data at the 3σ level [1], the three input parameters are constrained to be $|x| \in [1.045, 1.346] \cup [2.952, 4.754]$, $|\eta| \in [0, 0.112\pi] \cup [0.674\pi, \pi]$ and $r \in [0.0250, 0.0519] \cup [0.169, 0.214]$. For this mixing pattern, the solar angle θ_{12} is predicted to be in the range of [0.317, 0.319] and the other two mixing angles θ_{13} and θ_{23} can take any values within their 3σ ranges. Furthermore, the absolute values of the two CP phases δ_{CP} and β are predicted to lie in the regions $[0.298\pi, 0.623\pi]$ and $[0, 0.251\pi] \cup [0.830\pi, \pi]$, respectively.

($\mathcal{N}_2$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_3^{ST}, Z_2^{SU})$, $X_{\rm atm} = \{S, STS, T^2\}$

(i) $X_{\rm sol} = \{1, SU\}$

The unitary transformation U_l is an identity matrix up to a permutation of columns because the residual symmetry $G_l = Z_3^T$ is diagonal in our working basis. The possible residual CP transformations $X_{\rm sol}$ and the corresponding VEVs of the flavon field $\phi_{\rm sol}$ are the same as those of case $\mathcal{N}_1$. Here the VEVs of the flavon $\phi_{\rm atm}$ are proportional to $(1, -2\omega^2, -2\omega)^T$, i.e.,

$$\langle\phi_{\rm atm}\rangle = v_{\phi_a}(1, -2\omega^2, -2\omega)^T, \tag{3.27}$$

where v_{ϕ_a} is a real number with dimensions of mass. For $X_{\rm sol} = \{1, SU\}$, the alignment of $\phi_{\rm sol}$ is given in Eq. (3.8). The light neutrino mass matrix m_ν can be easily obtained from Eq. (2.8). After performing a TB transformation $U_{\rm TB}$, the light neutrino mass matrix m_ν will have the block-diagonal form m'_ν in Eq. (2.13) for the NO case, and the nonzero elements y, z and w in m'_ν take the following forms:

$$\begin{aligned} y &= 3(m_a + m_s e^{i\eta}), \\ z &= 3\sqrt{2} i m_a + \sqrt{6}(x-1) m_s e^{i\eta}, \\ w &= -6 m_a + 2(x-1)^2 m_s e^{i\eta}. \end{aligned} \tag{3.28}$$

TABLE VI. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_2$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_3^{ST}, Z_2^{SU})$ and $X_{\rm sol} = \{1, SU\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1,4,-2)^T$	4	$-\frac{3\pi}{4}$	4.030	0.369	8.810	0.0222	0.318	0.519	−0.473	−0.640	8.581	50.270	3.158
$(1,5,-3)^T$	5	$-\frac{2\pi}{3}$	4.120	0.226	38.945	0.0223	0.318	0.452	−0.567	−0.715	8.726	50.093	3.742
$(1,-2,4)^T$	−2	$\frac{3\pi}{4}$	4.050	0.365	22.417	0.0227	0.318	0.482	−0.525	0.642	8.570	50.285	3.182
$(1,-3,5)^T$	−3	$\frac{2\pi}{3}$	4.132	0.224	3.930	0.0226	0.318	0.549	−0.432	0.717	8.720	50.100	3.755
$(1,-5,7)^T$	−5	$\frac{3\pi}{5}$	3.955	0.120	4.755	0.0224	0.318	0.601	−0.352	0.765	8.781	50.025	3.835
$(1,-6,8)^T$	−6	$\frac{3\pi}{5}$	3.832	0.0935	3.674	0.0224	0.318	0.607	−0.342	0.778	8.502	50.374	3.737
$(1,\frac{7}{2},-\frac{3}{2})^T$	$\frac{7}{2}$	$-\frac{4\pi}{5}$	3.862	0.531	3.408	0.0225	0.318	0.572	−0.397	−0.577	8.783	50.022	2.512
		$-\frac{5\pi}{6}$	3.574	0.600	5.012	0.0228	0.318	0.603	−0.350	−0.564	8.430	50.466	2.024
$(1,\frac{11}{2},-\frac{7}{2})^T$	$\frac{11}{2}$	$-\frac{2\pi}{3}$	4.011	0.190	43.619	0.0221	0.318	0.446	−0.576	−0.733	8.434	50.462	3.690
$(1,-\frac{7}{2},\frac{11}{2})^T$	$-\frac{7}{2}$	$\frac{2\pi}{3}$	4.022	0.189	4.322	0.0224	0.318	0.554	−0.424	0.734	8.430	50.466	3.702
$(1,-\frac{11}{2},\frac{17}{2})^T$	$-\frac{11}{2}$	$\frac{3\pi}{5}$	3.889	0.106	2.511	0.0224	0.318	0.604	−0.347	0.772	8.630	50.210	3.782
$(1,-\frac{5}{3},\frac{11}{3})^T$	$-\frac{5}{3}$	$\frac{4\pi}{5}$	3.858	0.474	48.139	0.0230	0.318	0.437	−0.587	0.596	8.499	50.377	2.609
$(1,-\frac{10}{3},\frac{16}{3})^T$	$-\frac{10}{3}$	$\frac{2\pi}{3}$	4.056	0.200	2.635	0.0224	0.318	0.553	−0.426	0.729	8.519	50.352	3.718
$(1,-\frac{7}{4},\frac{15}{4})^T$	$-\frac{7}{4}$	$\frac{3\pi}{4}$	4.137	0.409	32.646	0.0231	0.318	0.476	−0.532	0.623	8.884	49.895	3.175
		$\frac{4\pi}{5}$	3.835	0.454	49.278	0.0228	0.318	0.440	−0.583	0.604	8.380	50.529	2.633
$(1,-\frac{9}{4},\frac{17}{4})^T$	$-\frac{9}{4}$	$\frac{3\pi}{4}$	3.979	0.326	27.092	0.0224	0.318	0.488	−0.517	0.660	8.309	50.618	3.195
$(1,-\frac{11}{4},\frac{19}{4})^T$	$-\frac{11}{4}$	$\frac{2\pi}{3}$	4.196	0.246	10.766	0.0227	0.318	0.545	−0.437	0.707	8.896	49.879	3.787
$(1,-\frac{13}{4},\frac{21}{4})^T$	$-\frac{13}{4}$	$\frac{2\pi}{3}$	4.074	0.205	2.306	0.0225	0.318	0.552	−0.428	0.726	8.565	50.291	3.727
$(1,-\frac{8}{5},\frac{18}{5})^T$	$-\frac{8}{5}$	$\frac{4\pi}{5}$	3.877	0.491	49.170	0.0231	0.318	0.435	−0.591	0.590	8.600	50.245	2.591
$(1,-\frac{9}{5},\frac{19}{5})^T$	$-\frac{9}{5}$	$\frac{3\pi}{4}$	4.118	0.399	29.007	0.0230	0.318	0.477	−0.531	0.627	8.817	49.980	3.176
		$\frac{4\pi}{5}$	3.823	0.443	51.042	0.0228	0.318	0.442	−0.581	0.609	8.313	50.614	2.648
$(1,-\frac{11}{5},\frac{21}{5})^T$	$-\frac{11}{5}$	$\frac{3\pi}{4}$	3.992	0.334	25.304	0.0225	0.318	0.486	−0.519	0.656	8.358	50.558	3.192

This implies that the first column of $U_{\rm TB}$ is an eigenvector of m_ν with zero eigenvalue. Hence the lepton mixing matrix is the TM1 mixing pattern. In order to achieve the Dirac CP phase δ_{CP} around $-\pi/2$ which is preferred by the present data [1], we take $U_l = P_{132}$. Then the PMNS matrix can be obtained by exchanging the second and third rows of the mixing matrix in Eq. (3.16). Comparing with the expressions of the three mixing angles in Eq. (3.18), $\sin^2\theta_{23}$ becomes $1 - \sin^2\theta_{23}$ and the other two mixing angles are kept intact. The overall sign of the Jarlskog invariant J_{CP} in Eq. (3.21) is reversed while the Majorana invariant I_1 is invariant. The sum rules among mixing angles and Dirac CP phase in Eqs. (3.19) and (3.22) are fulfilled as well.

Similar to previous cases, we perform a χ^2 analysis for the neutrino mass matrix, and the numerical results for some benchmark values of x and η are reported in Table VI. We can see that the measured values of the mixing angles and the neutrino masses can be well accommodated and the Dirac CP phase δ_{CP} is approximately maximal for all the typical values of x and η. Furthermore, we notice that the vacuum $\langle\phi_{\rm sol}\rangle \propto (2, 7, -3)^T$ for $x = 7/2$, $\eta = -4\pi/5$ is relatively simple and it can describe the experimental data quite well. Similar to the LS model [15–17,74], we expect that this alignment might provide an interesting opportunity for model building. Requiring all three mixing angles and the mass ratio m_2^2/m_3^2 to lie in their 3σ ranges, we find that the allowed regions of the parameters x, η, and r are $[-9.433, -1.314] \cup [3.189, 7.120]$, $[-0.854\pi, -0.614\pi] \cup [0.555\pi, 0.831\pi]$, and $[0.0485, 0.756]$, respectively. The possible values of δ_{CP} lie in the interval $[-0.623\pi, -0.296\pi]$, and the allowed range of the Majorana phase is $[-0.773\pi, -0.527\pi] \cup [0.558\pi, 0.810\pi]$. We see that this breaking pattern can accommodate a nearly maximal Dirac CP phase. The other mixing angles except θ_{12} can take any values within their 3σ ranges. The solar mixing angle $\sin^2\theta_{12}$ is close to 0.318 and this is generally true for TM1 mixing, as shown in Eqs. (3.19) and (3.20).

(ii) $X_{\rm sol} = \{S, U\}$

The explicit forms of the vacua of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ invariant under the assumed residual symmetries

TABLE VII. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_2$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_3^{ST}, Z_2^{SU})$ and $X_{\rm sol} = \{S, U\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 1+\frac{3i}{2}, 1-\frac{3i}{2})^T$	$\frac{3}{2}$	π	35.677	1.044	16.323	0.0218	0.318	0.5	−0.5	0	8.526	50.343	1.560
$(1, 1-\frac{7i}{2}, 1+\frac{7i}{2})^T$	$-\frac{7}{2}$	$-\frac{\pi}{3}$	1.534	1.111	2.981	0.0215	0.319	0.590	0.367	0.745	8.535	50.330	2.806
$(1, 1-3i, 1+3i)^T$	-3	$\frac{\pi}{5}$	1.474	1.491	44.990	0.0234	0.317	0.444	0.576	−0.848	8.674	50.157	3.498
		$-\frac{\pi}{5}$	1.479	1.484	4.378	0.0233	0.317	0.556	0.423	0.849	8.699	50.126	3.501
		$\frac{\pi}{6}$	1.461	1.503	36.885	0.0224	0.318	0.453	0.565	−0.874	8.568	50.288	3.537
		$-\frac{\pi}{6}$	1.464	1.498	2.928	0.0223	0.318	0.547	0.434	0.874	8.587	50.262	3.538
$(1, 1-4i, 1+4i)^T$	-4	$-\frac{3\pi}{7}$	1.600	0.853	4.993	0.0217	0.319	0.61	0.333	0.667	8.557	50.303	2.322
		$-\frac{4\pi}{9}$	1.596	0.859	4.271	0.0228	0.318	0.612	0.336	0.651	8.585	50.265	2.278

can be found from Table II, i.e., $\langle\phi_{\rm atm}\rangle \propto (1, -2\omega^2, -2\omega)^T$ and $\langle\phi_{\rm sol}\rangle \propto (1, 1+ix, 1-ix)^T$. Similar to previous cases, we can perform a TB transformation to obtain the block-diagonal neutrino mass matrix m'_ν. The nonvanishing elements y, z and w of m'_ν are given by

$$\begin{aligned} y &= 3(m_a + m_s e^{i\eta}), \\ z &= \sqrt{6} i(\sqrt{3} m_a + x m_s e^{i\eta}), \\ w &= -2(3m_a + x^2 m_s e^{i\eta}). \end{aligned} \tag{3.29}$$

We can further introduce the unitary transformation $U_{\nu2}$ to diagonalize the neutrino mass matrix m'_ν, as generally shown in Eqs. (2.15) and (2.16). As a consequence, the lepton mixing matrix is also the TM1 pattern, and the sum rules in Eqs. (3.19) and (3.22) are satisfied as well. However, the dependence of the mixing parameters on the input parameters m_a, m_s, η and x are different, consequently the above two mixing patterns of $\mathcal{N}_2$ with $X_{\rm sol} = \{1, SU\}$ and $X_{\rm sol} = \{S, U\}$ lead to different predictions. In Table VII, we present the results of our χ^2 analysis for some simple values of x and η. We find that accordance with experimental data can be achieved for certain values of m_a and r. In the case of $\eta = \pi$, both the atmospheric mixing θ_{23} and the Dirac CP phase δ_{CP} are maximal, while the Majorana CP phase β is trivial. We notice that realistic values of mixing angles and m_2^2/m_3^2 can be obtained for $x = 3/2, -3$ in the case of $\eta = 0$ or π. If we require that $\sin^2\theta_{ij}$ and m_2^2/m_3^2 lie in their 3σ regions [1], we find that the allowed regions of the parameters x, η, and r are $[-12.192, -2.744] \cup [1.350, 1.534]$, $[-\pi, 0.326\pi] \cup [0.804\pi, \pi]$, and $[0.0330, 1.750]$, respectively. The predictions for the CP phases are $|\delta_{CP}| \in [0.295\pi, 0.624\pi]$ and $\beta \in [-\pi, -0.749\pi] \cup [-0.248\pi, \pi]$.

$(\mathcal{N}_3)$ $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^S, Z_2^{SU})$, $X_{\rm atm} = \{1, S, TST^2U, T^2STU\}$, $X_{\rm sol} = \{S, U\}$

For the concerned residual flavor symmetry $G_{\rm sol} = Z_2^{SU}$ in the solar neutrino sector, the residual CP transformation $X_{\rm sol}$ can only be $X_{\rm sol} = \{S, U\}$ in order to achieve agreement with experimental data. In this case the charged lepton diagonalization matrix U_l is also the identity matrix, and the vacuum expectation values of the flavon fields $\phi_{\rm atm}$ and $\phi_{\rm sol}$ read as

$$\langle\phi_{\rm atm}\rangle = v_{\phi_a}(1, 1, 1)^T, \qquad \langle\phi_{\rm sol}\rangle = v_{\phi_s}(1, 1+ix, 1-ix)^T. \tag{3.30}$$

We find that the column vector $(2, -1, -1)^T$ is an eigenvector of m_ν with zero eigenvalue. This neutrino mass matrix m_ν can be simplified to a block-diagonal form by performing a $U_{\rm TB}$ transformation. Then we can obtain the three nonzero elements y, z and w:

$$y = 3(m_a + m_s e^{i\eta}), \qquad z = i\sqrt{6} x m_s e^{i\eta}, \qquad w = -2x^2 m_s e^{i\eta}. \tag{3.31}$$

The neutrino mass matrix m'_ν can be diagonalized by performing the unitary transformation $U_{\nu2}$. Thus the lepton mixing matrix is the TM1 pattern shown in Eq. (3.16). The expressions of the lepton mixing angles are the same as those in Eq. (3.18) and the two CP invariants J_{CP} and I_1 are still given by Eq. (3.21). The sum rules in Eqs. (3.19) and (3.22) are satisfied as well. Furthermore, we find that the neutrino mass matrix m_ν has the following symmetry properties:

$$\begin{aligned} m_\nu(-x, r, \eta) &= P_{132}^T m_\nu(x, r, \eta) P_{132}, \\ m_\nu(-x, r, -\eta) &= m_\nu^*(x, r, \eta). \end{aligned} \tag{3.32}$$

These identities indicate that the mixing angles θ_{12} and θ_{13} remain invariant, θ_{23} becomes $\pi/2 - \theta_{23}$ and the Dirac phase changes from δ_{CP} to $\pi + \delta_{CP}$ under the

TABLE VIII. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_3$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^S, Z_2^{SU})$ and $X_{\rm sol} = \{S, U\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 1\pm 4i, 1\mp 4i)^T$	± 4	π	3.080	0.473	20.478	0.0209	0.319	0.5	± 0.5	0	8.562	50.296	1.624
		$\pm\frac{3\pi}{4}$	3.115	0.465	10.932	0.0239	0.317	0.532	± 0.457	∓ 0.252	8.618	50.223	2.328
		$\mp\frac{3\pi}{4}$	3.107	0.466	34.132	0.0239	0.317	0.468	± 0.543	± 0.252	8.598	50.248	2.321
		$\pm\frac{4\pi}{5}$	3.106	0.467	7.318	0.0229	0.318	0.527	± 0.464	∓ 0.201	8.609	50.235	2.113
		$\mp\frac{4\pi}{5}$	3.100	0.468	26.561	0.0229	0.318	0.474	± 0.536	± 0.201	8.592	50.255	2.107
		$\pm\frac{5\pi}{6}$	3.100	0.468	7.928	0.0223	0.318	0.523	± 0.469	∓ 0.168	8.599	50.246	1.980
		$\mp\frac{5\pi}{6}$	3.095	0.469	24.266	0.0223	0.318	0.478	± 0.531	± 0.168	8.585	50.264	1.975
$(1, 1\pm 5i, 1\mp 5i)^T$	± 5	$\pm\frac{\pi}{2}$	3.055	0.310	12.152	0.0207	0.319	0.536	± 0.447	∓ 0.503	8.630	50.210	3.198
		$\mp\frac{\pi}{2}$	3.047	0.311	38.352	0.0207	0.319	0.464	± 0.553	± 0.503	8.606	50.238	3.190
		$\pm\frac{\pi}{3}$	3.045	0.309	12.182	0.024	0.317	0.531	± 0.459	∓ 0.670	8.556	50.303	3.609
		$\mp\frac{\pi}{3}$	3.038	0.310	34.705	0.024	0.317	0.469	± 0.541	± 0.670	8.537	50.328	3.603
		$\pm\frac{2\pi}{5}$	3.052	0.309	5.409	0.0228	0.318	0.534	± 0.453	∓ 0.603	8.594	50.252	3.462
		$\mp\frac{2\pi}{5}$	3.045	0.310	30.253	0.0228	0.318	0.466	± 0.547	± 0.603	8.574	50.280	3.455
$(1, 1\pm\frac{9}{2}i, 1\mp\frac{9}{2}i)^T$	$\pm\frac{9}{2}$	$\pm\frac{2\pi}{3}$	3.067	0.378	10.006	0.021	0.319	0.535	± 0.450	∓ 0.335	8.602	50.243	2.682
		$\mp\frac{2\pi}{3}$	3.059	0.379	35.123	0.021	0.319	0.465	± 0.550	± 0.335	8.580	50.271	2.675
		$\pm\frac{3\pi}{5}$	3.074	0.376	4.416	0.0226	0.318	0.538	± 0.447	∓ 0.403	8.602	50.243	2.931
		$\mp\frac{3\pi}{5}$	3.066	0.378	32.040	0.0226	0.318	0.462	± 0.553	± 0.403	8.578	50.274	2.923
$(1, 1\pm\frac{11}{2}i, 1\mp\frac{11}{2}i)^T$	$\pm\frac{11}{2}$	0	3.032	0.258	15.565	0.0229	0.318	0.5	± 0.5	1	8.578	50.274	3.816
		$\pm\frac{\pi}{4}$	3.045	0.258	10.643	0.0213	0.319	0.523	± 0.467	∓ 0.752	8.636	50.202	3.642
		$\mp\frac{\pi}{4}$	3.040	0.258	27.523	0.0213	0.319	0.477	± 0.533	± 0.752	8.621	50.220	3.637
		$\pm\frac{\pi}{5}$	3.042	0.258	9.429	0.0219	0.318	0.519	± 0.473	∓ 0.802	8.619	50.222	3.705
		$\mp\frac{\pi}{5}$	3.038	0.258	23.435	0.0219	0.318	0.481	± 0.527	± 0.802	8.607	50.237	3.701
		$\pm\frac{\pi}{6}$	3.040	0.258	9.667	0.0222	0.318	0.516	± 0.477	∓ 0.835	8.608	50.235	3.739
		$\mp\frac{\pi}{6}$	3.036	0.258	21.569	0.0222	0.318	0.484	± 0.523	± 0.835	8.598	50.248	3.736

transformation $x \to -x$. Moreover, by changing x to $-x$ and η to $-\eta$ simultaneously, all the lepton mixing angles are unchanged and the signs of all CP violation phases are reversed. Detailed numerical analyses show that accordance with experimental data can be achieved for certain values of x, m_a, r and η, and the corresponding benchmark numerical results are listed in Table VIII. We find that acceptable values of the mixing angles and m_2^2/m_3^2 can be obtained for $x = \pm 4$, $\eta = \pi$ and $x = \pm 11/2$, $\eta = 0$. If all three mixing angles and m_2^2/m_3^2 are restricted to their 3σ regions [1], the viable ranges of the input parameters $|x|$ and r are [3.641, 5.911] and [0.213, 0.568] respectively while any value of $\eta \in [-\pi, \pi]$ is viable. Then the atmospheric mixing angle $\sin^2\theta_{23}$ and the Dirac CP phase δ_{CP} are predicted to be $0.458 \leq \sin^2\theta_{23} \leq 0.542$ and $|\delta_{CP}| \in [0.443\pi, 0.557\pi]$, respectively. The Majorana CP phase β can take any value between $-\pi$ and π.

In summary, we find that all three of the above breaking patterns $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ predict a TM1 lepton mixing matrix and the experimental data [1] can be described very well. All three breaking patterns predict a normal mass hierarchy with $m_1 = 0$ and the sum rules in Eqs. (3.19) and (3.22). In fact these two sum rules are common to all TM1 mixing matrices. The prospects for testing the two sum rules in future neutrino facilities have been discussed [76]. Under the assumption of TM1 mixing, the structure of the Dirac mass matrix has been analyzed in Refs. [77,78] in the framework of a two right-handed neutrino seesaw model, generally with more parameters are involved than in the tridirect CP models.

$(\mathcal{N}_4)$ $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^{TST^2}, Z_2^U)$, $X_{\rm atm} = \{SU, ST^2S, T^2, T^2STU\}$, $X_{\rm sol} = \{1, U\}$

This breaking pattern has been studied in great detail by us [61]. Hence we shall not repeat the analysis here. When all three lepton mixing angles and the neutrino mass ratio m_2^2/m_3^2 are restricted to their 3σ regions [1], we find that the parameters x, $|\eta|$ and r should be in the ranges of $[-6.238, -3.365]$, $[0.347\pi, \pi]$ and [0.154, 0.607], respectively. Moreover, we show the results of the χ^2 analysis for some benchmark values of x and η in Table IX. The case of $x = -7/2$, $\eta = \pi$ in Table IX has been realized in a concrete model [61]. In Sec. V of the present work, we

TABLE IX. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_4$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^{TST^2}, Z_2^U)$ and $X_{\rm sol} = \{1, U\}$. Here we choose many benchmark values for the parameters x and η. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
-4	$\boldsymbol{\frac{3\pi}{4}}$	**3.708**	**0.423**	**48.401**	**0.0227**	**0.336**	**0.441**	**−0.587**	**−0.264**	**8.566**	**50.289**	**2.825**
	$\boldsymbol{-\frac{3\pi}{4}}$	**3.723**	**0.421**	**5.168**	**0.0226**	**0.336**	**0.560**	**−0.412**	**0.264**	**8.603**	**50.242**	**2.840**
	$\frac{4\pi}{5}$	3.674	0.430	51.698	0.0205	0.338	0.451	−0.576	−0.211	8.535	50.331	2.569
	$-\frac{4\pi}{5}$	3.686	0.428	16.028	0.0204	0.338	0.549	−0.424	0.211	8.565	50.291	2.581
-5	$\frac{3\pi}{5}$	3.723	0.266	67.144	0.021	0.345	0.424	−0.621	−0.422	8.594	50.252	3.545
	$-\frac{3\pi}{5}$	3.745	0.264	11.786	0.0211	0.345	0.577	−0.379	0.422	8.643	50.193	3.565
-6	$\frac{2\pi}{5}$	3.720	0.181	69.680	0.024	0.349	0.428	−0.606	−0.627	8.441	50.452	3.980
	$-\frac{2\pi}{5}$	3.738	0.180	17.443	0.0241	0.349	0.572	−0.394	0.627	8.479	50.404	3.997
$\boldsymbol{-\frac{7}{2}}$	$\boldsymbol{\pi}$	**3.716**	**0.557**	**17.524**	**0.0227**	**0.331**	**0.5**	**−0.5**	**0**	**8.611**	**50.232**	**1.647**
$-\frac{9}{2}$	$\frac{2\pi}{3}$	3.708	0.332	58.776	0.0216	0.341	0.429	−0.609	−0.352	8.567	50.289	3.271
	$-\frac{2\pi}{3}$	3.727	0.330	7.352	0.0216	0.341	0.571	−0.391	0.352	8.611	50.232	3.289

shall construct a model to realize the breaking pattern with $x = -4$ and $\eta = \pm 3\pi/4$. The corresponding best-fit values of the input parameters, mixing angles, CP phases and neutrino masses are showed in bold font in Table IX.

$(\mathcal{N}_5)(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,U)}, Z_2^{TU}, Z_2^{TU})$, $X_{\rm atm} = \{U, T\}$, $X_{\rm sol} = \{U, T\}$

In this case, the unitary transformation U_l is the TB mixing matrix $U_{\rm TB}$. From Table II, we find that the vacuum alignments of the flavons $\phi_{\rm atm} \sim \mathbf{3}$ and $\phi_{\rm sol} \sim \mathbf{3}'$ are dictated by the residual symmetry to be

$$\langle \phi_{\rm atm} \rangle = v_{\phi_a}(0, -\omega, \omega^2)^T, \qquad \langle \phi_{\rm sol} \rangle = v_{\phi_s}(1, \omega x, \omega^2 x)^T. \tag{3.33}$$

It is easy to check that the two column vectors $\langle \phi_{\rm atm} \rangle$ and $\langle \phi_{\rm sol} \rangle$ are orthogonal to each other, i.e., $\langle \phi_{\rm atm} \rangle^\dagger \langle \phi_{\rm sol} \rangle = 0$. This scenario is referred to as form dominance in the literature [62–64]. We can straightforwardly obtain the light neutrino mass matrix by using Eq. (2.8). In this case, $\langle \phi_{\rm atm} \rangle$ and $\langle \phi_{\rm sol} \rangle$ are proportional to two columns of U_ν which is the diagonalization matrix of m_ν,

$$U_\nu = \begin{pmatrix} -\frac{2x}{\sqrt{2+4x^2}} & 0 & \frac{1}{\sqrt{1+2x^2}} \\ \frac{\omega}{\sqrt{2(1+2x^2)}} & \frac{\omega}{\sqrt{2}} & \frac{\omega x}{\sqrt{1+2x^2}} \\ \frac{\omega^2}{\sqrt{2(1+2x^2)}} & -\frac{\omega^2}{\sqrt{2}} & \frac{\omega^2 x}{\sqrt{1+2x^2}} \end{pmatrix} \mathrm{diag}(1, 1, e^{-\frac{i\eta}{2}}), \tag{3.34}$$

with

$$U_\nu^T m_\nu U_\nu = \mathrm{diag}(0, 2m_a, (1+2x^2)m_s). \tag{3.35}$$

It implies that the three neutrino masses are 0, $2m_a$ and $(1+2x^2)m_s$ which are independent of the phase η. Including the contribution $U_l = U_{\rm TB}$ from the charged lepton sector, we find that the lepton mixing matrix is given by

$$U = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{1+2x^2}} & \frac{i}{2} & \frac{3x}{\sqrt{6(1+2x^2)}} \\ \frac{1-4x}{2\sqrt{3(1+2x^2)}} & -\frac{i}{2} & \frac{2+x}{\sqrt{6(1+2x^2)}} \\ \frac{1+2x}{\sqrt{6(1+2x^2)}} & -\frac{i}{\sqrt{2}} & \frac{x-1}{\sqrt{3(1+2x^2)}} \end{pmatrix} \mathrm{diag}(1, 1, e^{-\frac{i\eta}{2}}). \tag{3.36}$$

The three lepton mixing angles read as

$$\sin^2\theta_{13} = \frac{3x^2}{4x^2+2}, \qquad \sin^2\theta_{12} = \frac{2x^2+1}{2x^2+4},$$
$$\sin^2\theta_{23} = \frac{(x+2)^2}{3(x^2+2)}, \tag{3.37}$$

which are expressed in terms of one real parameter x. Furthermore, we can derive the following sum rules among the mixing angles:

$$\sin^2\theta_{12}\cos^2\theta_{13} = \frac{1}{4},$$
$$\sin^2\theta_{23} = \frac{6 - 7\sin^2\theta_{13} \pm 2\sin\theta_{13}\sqrt{2(3-4\sin^2\theta_{13})}}{9\cos^2\theta_{13}}$$
$$\simeq \frac{6 - \sin^2\theta_{13} \pm 2\sqrt{6}\sin\theta_{13}}{9}, \tag{3.38}$$

where the first sum rule in Eq. (3.38) has already appeared in the literature [37], and the sign "±" in the second sum rule depends on the value of x. For the best-fitting value of the reactor angle $\sin^2\theta_{13} = 0.02241$ [1], the solar mixing angle is determined to be $\sin^2\theta_{12} = 0.256$ and the atmospheric mixing angle is $\sin^2\theta_{23} = 0.746$ or $\sin^2\theta_{23} = 0.582$. We see that the latter value of $\sin^2\theta_{23}$ is compatible with the preferred values from the global data analysis [1]. The value of $\sin^2\theta_{12}$ is rather close to its 3σ lower limit 0.275. As a result, we suggest that this mixing pattern is a good leading-order approximation since accordance with experimental data should be easily achieved after subleading contributions are taken into account in a concrete model. Furthermore, we find that the two CP rephasing invariants J_{CP} and I_1 are

$$J_{CP} = 0, \qquad I_1 = -\frac{(2+x)^2 \sin\eta}{24(1+2x^2)}. \tag{3.39}$$

Hence the Dirac CP phase is trivial for any value of x. From the expressions of mixing angles in Eq. (3.37) and the Majorana invariant in Eq. (3.39), we find that the Majorana CP phase β is determined to be

$$\beta = \eta + \pi. \tag{3.40}$$

It implies that a trivial Majorana CP phase is obtained for $\eta = 0$ or π. For $\eta = \pm\pi/2$, the Majorana CP phase is maximal. As an example, we take the representative value $x = -1/8$. Thus the VEV of the flavon field $\phi_{\rm sol}$ is proportional to $(1, -\omega/8, -\omega^2/8)^T$ and the PMNS matrix is

$$U = \frac{1}{2\sqrt{22}} \begin{pmatrix} 8 & \sqrt{22}i & -\sqrt{2} \\ 4 & -\sqrt{22}i & 5\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{11}i & -6 \end{pmatrix} \mathrm{diag}(1, 1, e^{-\frac{i\eta}{2}}). \tag{3.41}$$

The three mixing angles are determined to be

$$\sin^2\theta_{13} = \frac{1}{44} \simeq 0.0227,$$
$$\sin^2\theta_{12} = \frac{11}{43} \simeq 0.256,$$
$$\sin^2\theta_{23} = \frac{25}{43} \simeq 0.581. \tag{3.42}$$

Wee see that the reactor and atmospheric mixing angles are compatible with the preferred values from the global fit [1] at the 3σ level. Furthermore, in the case of $x = -1/8$ the two neutrino mass squared differences only depend on the values of m_a and r, as shown in Fig. 2. We find that the best-fit values of the two neutrino mass squared differences Δm^2_{21} and Δm^2_{32} can be reproduced.

In order to increase the readability of the paper, the remaining three viable cases $\mathcal{N}_6$, $\mathcal{N}_7$ and $\mathcal{N}_8$ are moved to Appendix B. The reason is that the

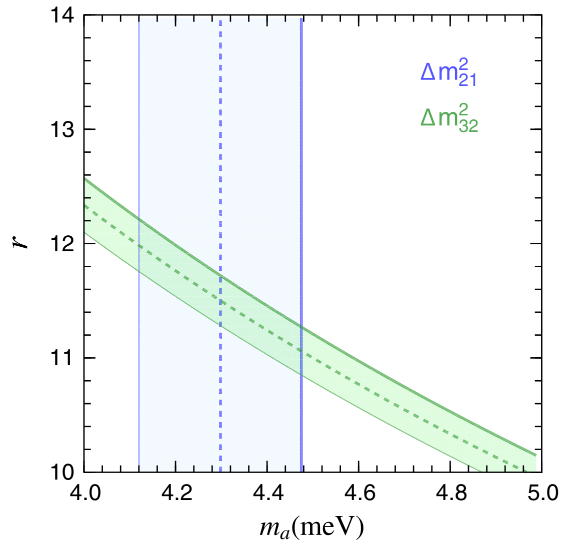

FIG. 2. Contour plot of Δm^2_{21} and Δm^2_{32} in the $r - m_a$ plane for the residual symmetry $\mathcal{N}_5$ with the benchmark value $x = -1/8$. The 3σ lower (upper) bounds of the neutrino mass squared differences are labeled with thin (thick) solid curves, and the dashed contour lines represent the corresponding best-fit values.

diagonalization matrix of the charged lepton mass matrix and the phenomenologically interesting alignments of the flavon $\phi_{\rm sol}$ may be not simple enough to be realized in a concrete model.

IV. MIXING PATTERNS DERIVED FROM S_4 WITH IO NEUTRINO MASSES

From Table III, we see that the breaking of S_4 and CP symmetries in the tridirect CP approach can lead to 18 viable mixing patterns with IO neutrino masses. In the following, we proceed to study five viable cases among them and present their predictions for the lepton mixing angles, CP-violating phases and neutrino masses. The other viable breaking patterns are shown in Appendix C. The last two of the five breaking patterns in this section will lead to the form dominance texture.

$(\mathcal{I}_1)$ $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_3^{ST}, Z_2^U)$, $X_{\rm atm} = \{S, STS, T^2\}$, $X_{\rm sol} = \{1, U\}$

The diagonalization matrix of the charged lepton mass matrix $m_l^\dagger m_l$ is a unity matrix because of the residual symmetry $G_l = Z_3^T$. The given residual symmetries fix the vacuum of the flavon fields $\phi_{\rm atm}$ and $\phi_{\rm sol}$ to be

$$\langle\phi_{\rm atm}\rangle = v_{\phi_a}(1, -2\omega^2, -2\omega)^T, \qquad \langle\phi_{\rm sol}\rangle = v_{\phi_s}(1, x, x)^T, \tag{4.1}$$

where x is real. The light neutrino mass matrix can be simplified to a block-diagonal form m'_ν by performing a unitary transformation $U_{\nu 1}$, where the unitary matrix $U_{\nu 1}$ is

$$U_{\nu 1}=\begin{pmatrix} 0 & -i\sqrt{\frac{x^2-2x+4}{7x^2-2x+4}} & \frac{i\sqrt{6}x}{\sqrt{7x^2-2x+4}} \\ \frac{-2i\omega^2-ix}{\sqrt{2(x^2-2x+4)}} & \frac{\sqrt{3}x(2\omega^2+x)}{\sqrt{(x^2-2x+4)(7x^2-2x+4)}} & \frac{2\omega^2+x}{\sqrt{2(7x^2-2x+4)}} \\ \frac{-ix-2i\omega}{\sqrt{2(x^2-2x+4)}} & -\frac{\sqrt{3}x(x+2\omega)}{\sqrt{(x^2-2x+4)(7x^2-2x+4)}} & \frac{-x-2\omega}{\sqrt{2(7x^2-2x+4)}} \end{pmatrix}. \tag{4.2}$$

As shown in Eq. (2.13), m'_ν can be parametrized by three parameters y, z and w with

$$\begin{aligned} y &= -\frac{2((x-4)^2 m_a + x^2(x-1)^2 m_s e^{i\eta})}{x^2-2x+4}, \\ z &= -\frac{\sqrt{2(7x^2-2x+4)}((x-4)m_a + x(x-1)m_s e^{i\eta})}{x^2-2x+4}, \\ w &= -\frac{(7x^2-2x+4)(m_a+m_s e^{i\eta})}{x^2-2x+4}. \end{aligned} \tag{4.3}$$

The neutrino mass matrix m'_ν can be exactly diagonalized by a second unitary transformation $U_{\nu 2}$ in Eq. (2.16). Then the lightest neutrino mass m_3 is vanishing and the other two neutrino masses $m_{1,2}$ can be obtained from Eq. (2.17). The lepton mixing matrix is determined to be

$$U=\begin{pmatrix} e^{-i\psi}\sqrt{\frac{x^2-2x+4}{7x^2-2x+4}}\sin\theta & -\sqrt{\frac{x^2-2x+4}{7x^2-2x+4}}\cos\theta & \frac{\sqrt{6}x}{\sqrt{7x^2-2x+4}} \\ -\frac{i\cos\theta}{\sqrt{2}}-\frac{\sqrt{3}xe^{-i\psi}\sin\theta}{\sqrt{7x^2-2x+4}} & \frac{\sqrt{3}x\cos\theta}{\sqrt{7x^2-2x+4}}-\frac{ie^{i\psi}\sin\theta}{\sqrt{2}} & \sqrt{\frac{x^2-2x+4}{2(7x^2-2x+4)}} \\ \frac{i\cos\theta}{\sqrt{2}}-\frac{\sqrt{3}xe^{-i\psi}\sin\theta}{\sqrt{7x^2-2x+4}} & \frac{ie^{i\psi}\sin\theta}{\sqrt{2}}+\frac{\sqrt{3}x\cos\theta}{\sqrt{7x^2-2x+4}} & \sqrt{\frac{x^2-2x+4}{2(7x^2-2x+4)}} \end{pmatrix}P_\nu, \tag{4.4}$$

with

$$P_\nu=\begin{pmatrix} e^{i(\rho+\psi)/2} & 0 & 0 \\ 0 & e^{i(\sigma-\psi)/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

For the IO neutrino masses, P_ν will always be defined as in Eq. (4.5) and it will be omitted for simplicity in the following. We see that the residual symmetry fixes the third column of the mixing matrix to be $(\sqrt{12}x, \sqrt{x^2-2x+4}, \sqrt{x^2-2x+4})^T$ up to normalization. From the lepton mixing matrix given in Eq. (4.4), we can extract the expressions of the mixing angles and CP invariants as follows:

$$\begin{aligned} &\sin^2\theta_{13}=\frac{6x^2}{4-2x+7x^2}, \qquad \sin^2\theta_{12}=\cos^2\theta, \\ &\sin^2\theta_{23}=\frac{1}{2}, \qquad J_{CP}=-\frac{\sqrt{3}x(x^2-2x+4)\sin 2\theta\cos\psi}{2\sqrt{2}(7x^2-2x+4)^{3/2}}, \\ &I_1=-\frac{(x^2-2x+4)^2\sin^2 2\theta\sin(\rho-\sigma)}{4(7x^2-2x+4)^2}. \end{aligned} \tag{4.6}$$

Notice that the atmospheric mixing angle θ_{23} is exactly maximal and the reactor mixing angle θ_{13} only depends on the parameter x. Imposing the experimentally favored 3σ region of the reactor mixing angle $0.02068 \leq \sin^2\theta_{13} \leq 0.02463$ [1], we find that the parameter x is constrained to be in the narrow ranges $[-0.134, -0.122] \cup [0.115, 0.126]$.

Subsequently we shall perform a comprehensive numerical analysis. The input parameters x, r and η are treated as random numbers in the intervals $[-20, 20]$, $[0, 20]$ and $[-\pi, \pi]$ respectively; then we calculate the values of the mixing parameters and mass ratio m_1^2/m_2^2. Imposing the experimentally preferred 3σ regions of the three mixing angles and $\Delta m^2_{21}/|\Delta m^2_{32}|$ [1], we find that the allowed regions of the parameters x, $|\eta|$ and r are $[-0.134, -0.122] \cup [0.115, 0.126]$, $[0.9796\pi, 0.9916\pi]$ and $[8.630, 8.713]$, respectively. The mixing angles θ_{13} and θ_{12} can take any values in their 3σ intervals [1]. The Dirac CP phase δ_{CP} and the absolute value of the Majorana CP phase β are predicted to be in the ranges $[-0.944\pi, -0.913\pi] \cup [-0.0873\pi, -0.0563\pi] \cup [0.0326\pi, 0.0503\pi] \cup [0.950\pi, 0.967\pi]$ and $[0.178\pi, 0.193\pi] \cup [0.294\pi, 0.321\pi]$, respectively. Notice that the Dirac CP phase is around 0 or π;

consequently this breaking pattern would be ruled out if the signal of maximal δ_{CP} is confirmed by future neutrino facilities.

As an illustrative example, we choose the solar alignment parameter $x=-1/8$ which gives a relatively simple vacuum of $\phi_{\rm sol}$. Accordingly the third column of the PMNS mixing matrix is $\left(\sqrt{\frac{2}{93}},\sqrt{\frac{91}{186}},\sqrt{\frac{91}{186}}\right)^T\simeq(0.147,0.699,0.699)^T$ which is compatible with experimental data. The results of the χ^2 analysis are

$$\begin{gathered}\eta=-0.983\pi,\qquad m_a=5.721\ \text{meV},\qquad r=8.673,\qquad \chi^2_{\rm min}=20.595,\qquad \sin^2\theta_{13}=0.0215,\\ \sin^2\theta_{12}=0.310,\qquad \sin^2\theta_{23}=0.5,\qquad \delta_{CP}=-0.0715\pi,\qquad \beta=-0.304\pi,\\ m_1=49.377\ \text{meV},\qquad m_2=50.120\ \text{meV},\qquad m_3=0\ \text{meV},\qquad m_{ee}=43.904\ \text{meV}.\end{gathered}\tag{4.7}$$

($\mathcal{I}_2$) $(G_l,G_{\rm atm},G_{\rm sol})=(Z_3^T,Z_2^{SU},Z_2^{TU})$, $X_{\rm atm}=\{1,SU\}$, $X_{\rm sol}=\{U,T\}$

The combination $m^\dagger l m_l$ is enforced to be diagonal by the residual symmetry $G_l=Z_3^T$. The vacuum of the flavon $\phi_{\rm atm}$ is proportional to $(2,-1,-1)^T$ for this breaking pattern. Here only the residual CP symmetry $X_{\rm sol}=\{U,T\}$ can give a phenomenologically viable mixing pattern, and the VEV of $\phi_{\rm sol}$ is $\langle\phi_{\rm sol}\rangle=v_{\phi_s}(1,x\omega,x\omega^2)^T$. From the alignments $\langle\phi_{\rm atm}\rangle$, $\langle\phi_{\rm sol}\rangle$ and Eq. (2.8), we can obtain the general form of the light neutrino mass matrix m_ν. Similar to the previous case, we first perform a unitary transformation $U_{\nu1}$ on m_ν with

$$U_{\nu1}=\begin{pmatrix}0 & i\sqrt{\frac{8x^2-4x+2}{11x^2-4x+2}} & \frac{i\sqrt{3}x}{\sqrt{11x^2-4x+2}}\\ \frac{-2x\omega-1}{\sqrt{2(4x^2-2x+1)}} & -\frac{\sqrt{3}x(2x\omega+1)}{\sqrt{2(4x^2-2x+1)(11x^2-4x+2)}} & \frac{2x\omega+1}{\sqrt{11x^2-4x+2}}\\ \frac{-2x\omega^2-1}{\sqrt{2(4x^2-2x+1)}} & \frac{\sqrt{3}x(2x\omega^2+1)}{\sqrt{2(4x^2-2x+1)(11x^2-4x+2)}} & \frac{-2x\omega^2-1}{\sqrt{11x^2-4x+2}}\end{pmatrix}.\tag{4.8}$$

Then the resulting neutrino mass matrix m'_ν is block diagonal and the nonzero elements y, z and w are

$$\begin{aligned}y&=\frac{4(x-1)^2m_a+x^2(1-4x)^2m_se^{i\eta}}{8x^2-4x+2},\\ z&=\frac{i\sqrt{11x^2-4x+2}(4(x-1)m_a+x(4x-1)m_se^{i\eta})}{8x^2-4x+2},\\ w&=-\frac{(11x^2-4x+2)(4m_a+m_se^{i\eta})}{8x^2-4x+2}.\end{aligned}\tag{4.9}$$

In general m'_ν can be diagonalized by the unitary matrix $U_{\nu2}$ given in Eq. (2.16) for the IO case, and the lightest neutrino is massless $m_3=0$. Thus we can read out the PMNS mixing matrix as

$$U=\begin{pmatrix}-\sqrt{\frac{8x^2-4x+2}{11x^2-4x+2}}e^{-i\psi}\sin\theta & \sqrt{\frac{8x^2-4x+2}{11x^2-4x+2}}\cos\theta & \frac{\sqrt{3}x}{\sqrt{11x^2-4x+2}}\\ -\frac{\cos\theta}{\sqrt{2}}+\frac{\sqrt{3}xe^{-i\psi}\sin\theta}{\sqrt{2(11x^2-4x+2)}} & \frac{-e^{i\psi}\sin\theta}{\sqrt{2}}-\frac{\sqrt{3}x\cos\theta}{\sqrt{2(11x^2-4x+2)}} & \sqrt{\frac{4x^2-2x+1}{11x^2-4x+2}}\\ \frac{\cos\theta}{\sqrt{2}}+\frac{\sqrt{3}xe^{-i\psi}\sin\theta}{\sqrt{2(11x^2-4x+2)}} & \frac{e^{i\psi}\sin\theta}{\sqrt{2}}-\frac{\sqrt{3}x\cos\theta}{\sqrt{2(11x^2-4x+2)}} & \sqrt{\frac{4x^2-2x+1}{11x^2-4x+2}}\end{pmatrix},\tag{4.10}$$

which leads to the following expressions for the mixing angles and the CP invariants:

$$\begin{gathered}\sin^2\theta_{13}=\frac{3x^2}{11x^2-4x+2},\qquad \sin^2\theta_{12}=\cos^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},\\ J_{CP}=\frac{\sqrt{3}x(4x^2-2x+1)\sin2\theta\sin\psi}{2(11x^2-4x+2)^{3/2}},\qquad I_1=-\frac{(4x^2-2x+1)^2\sin^22\theta\sin(\rho-\sigma)}{(11x^2-4x+2)^2}.\end{gathered}\tag{4.11}$$

Note that the atmospheric mixing angle θ_{23} is exactly 45°. The precisely measured value of the reactor mixing angle θ_{13} at the 3σ level leads to the admissible intervals $x\in[-0.153,-0.138]\cup[0.108,0.118]$. Furthermore, a comprehensive numerical analysis is performed. In order to be compatible with experimental data, we find that the allowed regions of the parameter x, $|\eta|$ and r are $[-0.153, -0.138]\cup[0.108,0.118]$, $[0.9965\pi,\pi]$ and $[5.666, 5.817]$, respectively. The predictions for the two CP phases are $\delta_{CP}\in[-0.777\pi,-0.223\pi]\cup[0.145\pi,0.239\pi]\cup[0.761\pi, 0.855\pi]$ and $|\beta|\in[0.636\pi,\pi]$. The mixing angles θ_{13} and θ_{12} can take any values in their 3σ ranges. For the benchmark values $x=-1/7$ and $\eta=\pi$, the third column of the mixing matrix is $\left(\sqrt{\frac{3}{137}},\sqrt{\frac{67}{137}},\sqrt{\frac{67}{137}}\right)^T$ in the experimentally favored range. The χ^2 analysis gives the following best-fitting values:

$$\begin{aligned}
&m_a=12.453\ \text{meV},\qquad r=5.707,\qquad \chi^2_{\text{min}}=26.114,\\
&\sin^2\theta_{13}=0.0219,\qquad \sin^2\theta_{12}=0.278,\\
&\sin^2\theta_{23}=0.5,\qquad \delta_{CP}=-0.5\pi,\qquad \beta=\pi,\\
&m_1=49.374\ \text{meV},\qquad m_2=50.117\ \text{meV},\\
&m_3=0\ \text{meV},\qquad m_{ee}=21.261\ \text{meV}.
\end{aligned}\tag{4.12}$$

The Dirac phase δ_{CP} is predicted to be exactly maximal while the Majorana phase β takes a CP-conserving value. The reason is that the neutrino mass matrix in this case fulfills the $\mu\tau$ reflection symmetry $m_\nu|_{\eta=\pi}=P^T_{132}m^*_\nu|_{\eta=\pi}P_{132}$ for $\eta=\pi$. Note that the phase $\eta=\pi$ is easy to dynamically realize in an explicit model; see Ref. [61] for an example.

$(\mathcal{I}_3)$ $(G_l,G_{\text{atm}},G_{\text{sol}})=(K_4^{(S,U)},Z_2^{TST^2},Z_2^U)$, $X_{\text{atm}}=\{SU, T^2, ST^2S, T^2STU\}$

(i) $X_{\text{sol}}=\{1,U\}$

In this case, the atmospheric and solar alignments invariant under the actions of the residual symmetries are

$$\langle\phi_{\text{atm}}\rangle=v_{\phi_a}(1,\omega^2,\omega)^T,\qquad \langle\phi_{\text{sol}}\rangle=v_{\phi_s}(1,x,x)^T,\tag{4.13}$$

where x is a real parameter. The diagonalization matrix of the above neutrino mass matrix can be written as $U_\nu=U_{\nu1}U_{\nu2}$ with

$$U_{\nu1}=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{2+x}{\sqrt{3(2+2x+5x^2)}} & -\frac{i\sqrt{3}x}{\sqrt{2+2x+5x^2}}\\ \frac{\omega^2}{\sqrt{3}} & \frac{\omega^2-(2-\omega)x}{\sqrt{3(2+2x+5x^2)}} & \frac{x-\omega^2}{\sqrt{2+2x+5x^2}}\\ \frac{\omega}{\sqrt{3}} & \frac{\omega-(2-\omega^2)x}{\sqrt{3(2+2x+5x^2)}} & \frac{\omega-x}{\sqrt{2+2x+5x^2}}\end{pmatrix},\tag{4.14}$$

and $U_{\nu2}$ is shown in Eq. (2.16) for the IO case. After performing the unitary transformation $U_{\nu1}$, we obtain a block-diagonal neutrino mass matrix m'_ν with nonzero elements

$$\begin{aligned}
y&=3m_a+\frac{1}{3}(1-x)^2m_se^{i\eta},\\
z&=\frac{1}{3}(x-1)\sqrt{2+2x+5x^2}m_se^{i\eta},\\
w&=\frac{1}{3}(2+2x+5x^2)m_se^{i\eta}.
\end{aligned}\tag{4.15}$$

Including the contribution of the charged lepton sector $U_l=U_{\text{TB}}$ determined by $G_{\text{sol}}=K_4^{(S,U)}$, we find that the lepton mixing matrix takes the following form:

$$U=\begin{pmatrix}\frac{(1+2x)e^{-i\psi}\sin\theta}{\sqrt{2+2x+5x^2}} & \frac{(1+2x)\cos\theta}{\sqrt{2+2x+5x^2}} & \frac{i(1-x)}{\sqrt{2+2x+5x^2}}\\ -\frac{\cos\theta}{\sqrt{2}}+\frac{(x-1)e^{-i\psi}\sin\theta}{\sqrt{2(2+2x+5x^2)}} & \frac{(x-1)\cos\theta}{\sqrt{2(2+2x+5x^2)}}+\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{i(1+2x)}{\sqrt{2(2+2x+5x^2)}}\\ \frac{\cos\theta}{\sqrt{2}}+\frac{(x-1)e^{-i\psi}\sin\theta}{\sqrt{2(2+2x+5x^2)}} & \frac{(x-1)\cos\theta}{\sqrt{2(2+2x+5x^2)}}-\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{i(1+2x)}{\sqrt{2(2+2x+5x^2)}}\end{pmatrix}.\tag{4.16}$$

Then we can read out the expressions of the mixing angles and CP invariants as

$$\begin{aligned}
&\sin^2\theta_{13}=\frac{(x-1)^2}{2+2x+5x^2},\qquad \sin^2\theta_{12}=\cos^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},\\
&J_{CP}=\frac{(x-1)(2x+1)^2\sin2\theta\sin\psi}{4(2+2x+5x^2)^{3/2}},\qquad I_1=-\frac{(2x+1)^4\sin^22\theta\sin(\rho-\sigma)}{4(2+2x+5x^2)^2}.
\end{aligned}\tag{4.17}$$

The atmospheric mixing angle θ_{23} is exactly maximal. Requiring that the three mixing angles lie in their 3σ ranges [1], we find that the admissible ranges of the input parameters x, $|\eta|$ and r are $[0.638,0.662]\cup[1.615,1.699]$, $[0.965\pi,0.975\pi]$ and $[0.441,0.480]\cup[1.594,1.630]$, respectively. The two CP phases are predicted to be $0.487\pi\le|\delta_{CP}|\le0.503\pi$ and

$0.0962\pi \le |\beta| \le 0.108\pi$. Similar to previous cases, we also give a benchmark example which could be easily achieved in a model. The alignment parameter is $x = 2/3$, and thus the vacuum of $\phi_{\rm sol}$ is $(1, \frac{2}{3}, \frac{2}{3})^T v_{\phi_s}$. The fixed column of the PMNS matrix takes the form $(\frac{1}{5\sqrt{2}}, \frac{7}{10}, \frac{7}{10})^T \simeq (0.141, 0.7, 0.7)^T$. The best-fit values of the mixing parameters and neutrino masses for this example are

$$\begin{aligned}
&\eta = -0.969\pi, \qquad m_a = 16.794\ \text{meV},\\
&r = 1.579, \qquad \chi^2_{\rm min} = 33.640, \qquad \sin^2\theta_{13} = 0.02,\\
&\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5,\\
&\delta_{CP} = -0.497\pi, \qquad \beta = 0.0964\pi,\\
&m_1 = 49.377\ \text{meV}, \qquad m_2 = 50.120\ \text{meV},\\
&m_3 = 0\ \text{meV}, \qquad m_{ee} = 48.137\ \text{meV}.
\end{aligned} \tag{4.18}$$

We see that θ_{13} is rather close to its 3σ lower limit 0.2068 [1]. Hence this mixing pattern could be considered as a good leading-order approximation since accordance with experimental data can be easily achieved after subleading contributions are taken into account in a model.

(ii) $X_{\rm sol} = \{S, SU\}$

For this kind of residual symmetry, the solar flavon alignment is $\langle\phi_{\rm sol}\rangle \propto (1+2ix, 1-ix, 1-ix)$, and the neutrino mass matrix can be easily read off from Eq. (2.8). We choose the first unitary transformation $U_{\nu1}$ as

$$U_{\nu1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{ix-1}{\sqrt{3(1+x^2)}} & \frac{-i-x}{\sqrt{3(1+x^2)}} \\ \frac{\omega^2}{\sqrt{3}} & \frac{i\omega x-1}{\sqrt{3(1+x^2)}} & \frac{-i\omega-x}{\sqrt{3(1+x^2)}} \\ \frac{\omega}{\sqrt{3}} & \frac{i\omega^2 x-1}{\sqrt{3(1+x^2)}} & \frac{-i\omega^2-x}{\sqrt{3(1+x^2)}} \end{pmatrix}. \tag{4.19}$$

Then the nonvanishing elements of the neutrino mass matrix m'_ν are given by

$$\begin{aligned}
&y = 3m_a - 3x^2 m_s e^{i\eta}, \qquad z = -3ix\sqrt{1+x^2}\, m_s e^{i\eta},\\
&w = 3(1+x^2) m_s e^{i\eta}.
\end{aligned} \tag{4.20}$$

We can further diagonalize m'_ν by the unitary matrix $U_{\nu2}$. The lepton mixing matrix is determined to be of the form

$$U = \begin{pmatrix} \frac{e^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & \frac{\cos\theta}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} \\ \frac{i\cos\theta}{\sqrt{2}} + \frac{x e^{-i\psi}\sin\theta}{\sqrt{2(1+x^2)}} & \frac{x\cos\theta}{\sqrt{2(1+x^2)}} - \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{2(1+x^2)}} \\ \frac{i\cos\theta}{\sqrt{2}} - \frac{x e^{-i\psi}\sin\theta}{\sqrt{2(1+x^2)}} & -\frac{x\cos\theta}{\sqrt{2(1+x^2)}} - \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2(1+x^2)}} \end{pmatrix}. \tag{4.21}$$

Then the lepton mixing angles and CP invariants are

$$\begin{aligned}
&\sin^2\theta_{13} = \frac{x^2}{1+x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},\\
&J_{CP} = -\frac{x\sin 2\theta\cos\psi}{4(1+x^2)^{3/2}}, \qquad I_1 = -\frac{\sin^2 2\theta\sin(\rho-\sigma)}{4(1+x^2)^2}.
\end{aligned} \tag{4.22}$$

The atmospheric mixing angle is maximal as well. We find $|x| \in [0.145, 0.159]$, $|\eta| \in [0.0245\pi, 0.0345\pi]$ and $r \in [0.945, 0.954]$ in order to accommodate the experimental data at the 3σ level [1]. The absolute values of the CP-violating phases δ_{CP} and β are predicted to lie in the regions $[0.488\pi, 0.503\pi]$ and $[0.0971\pi, 0.110\pi]$ respectively. For the benchmark value of $x = 1/4\sqrt{3}$, the third column of the mixing matrix is $(1/7, 2\sqrt{6}/7, 2\sqrt{6}/7)^T \simeq (0.143, 0.700, 0.700)^T$, and the results of the χ^2 analysis are

$$\begin{aligned}
&\eta = 0.0307\pi, \qquad m_a = 16.798\ \text{meV}, \qquad r = 0.955, \qquad \chi^2_{\rm min} = 29.075, \qquad \sin^2\theta_{13} = 0.0204,\\
&\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.497\pi, \qquad \beta = 0.0973\pi,\\
&m_1 = 49.377\ \text{meV}, \qquad m_2 = 50.120\ \text{meV}, \qquad m_3 = 0\ \text{meV}, \qquad m_{ee} = 48.108\ \text{meV}.
\end{aligned} \tag{4.23}$$

($\mathcal{I}_4$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,U)}, Z_2^{TU}, Z_2^{TU})$, $X_{\rm atm} = \{U, T\}$, $X_{\rm sol} = \{U, T\}$

In this case, the residual CP symmetries are the same as those of the case $\mathcal{N}_5$. Therefore we can straightforwardly read out the lepton mixing matrix from Eq. (3.36) by exchanging the first and the third columns,

$$U = \begin{pmatrix} \frac{3x}{\sqrt{6(1+2x^2)}} & \frac{i}{2} & \frac{\sqrt{3}}{2\sqrt{1+2x^2}} \\ \frac{2+x}{\sqrt{6(1+2x^2)}} & -\frac{i}{2} & \frac{1-4x}{2\sqrt{3(1+2x^2)}} \\ \frac{x-1}{\sqrt{3(1+2x^2)}} & -\frac{i}{\sqrt{2}} & \frac{1+2x}{\sqrt{6(1+2x^2)}} \end{pmatrix} \mathrm{diag}(e^{-\frac{i\eta}{2}}, 1, 1), \tag{4.24}$$

which leads to mixing angles expressed by

$$\sin^2\theta_{13} = \frac{3}{4+8x^2}, \qquad \sin^2\theta_{12} = \frac{1+2x^2}{1+8x^2},$$
$$\sin^2\theta_{23} = \frac{(1-4x)^2}{3(1+8x^2)}. \tag{4.25}$$

The three mixing angles only depend on the parameter x. Then the two sum rules among the three mixing angles can be obtained. We find that the two sum rules here are the same as the sum rules in Eq. (3.38). Moreover, the CP-violating phases δ_{CP} and β are determined to be

$$\sin\delta_{CP} = 0, \qquad \beta = \eta + \pi. \tag{4.26}$$

This implies that the Dirac CP phase takes CP-conserving values. The Majorana CP phase β differs from η by π. For the benchmark value $x = 4$, the alignment of the solar flavon ϕ_{sol} is proportional to $(1, 4\omega, 4\omega^2)^T$ and the PMNS matrix is

$$U = \frac{1}{2\sqrt{22}} \begin{pmatrix} 8 & \sqrt{22}i & \sqrt{2} \\ 4 & -\sqrt{22}i & -5\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{11}i & 6 \end{pmatrix} \mathrm{diag}(e^{-\frac{i\eta}{2}}, 1, 1). \tag{4.27}$$

The three mixing angles are then

$$\sin^2\theta_{13} = \frac{1}{44} \simeq 0.0227, \qquad \sin^2\theta_{12} = \frac{11}{43} \simeq 0.256,$$
$$\sin^2\theta_{23} = \frac{25}{43} \simeq 0.581. \tag{4.28}$$

We find that the solar mixing angle θ_{12} is rather close to its 3σ lower bound [1]. Hence this mixing pattern can be regarded as a good leading-order approximation.

($\mathcal{I}_5$) $(G_l, G_{\mathrm{atm}}, G_{\mathrm{sol}}) = (Z_3^T, Z_2^{SU}, Z_2^{SU})$, $X_{\mathrm{atm}} = \{1, SU\}$, $X_{\mathrm{sol}} = \{U, S\}$

The residual symmetry fixes the vacuum alignments of ϕ_{atm} and ϕ_{sol} to be

$$\langle\phi_{\mathrm{atm}}\rangle = v_{\phi_a}(2, -1, -1)^T,$$
$$\langle\phi_{\mathrm{sol}}\rangle = v_{\phi_s}(1, 1+ix, 1-ix)^T, \tag{4.29}$$

which are orthogonal to each other. Hence the neutrino Yukawa coupling is has the texture of form dominance [62–64]. The column vectors $\langle\phi_{\mathrm{atm}}\rangle$ and $\langle\phi_{\mathrm{sol}}\rangle$ in Eq. (4.29) constitute two columns of the neutrino diagonalization matrix U_ν,

$$U_\nu = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3+2x^2}} & \frac{i\sqrt{2}x}{\sqrt{3(3+2x^2)}} \\ -\frac{1}{\sqrt{6}} & \frac{1+ix}{\sqrt{3+2x^2}} & \frac{3+2ix}{\sqrt{6(3+2x^2)}} \\ -\frac{1}{\sqrt{6}} & \frac{1-ix}{\sqrt{3+2x^2}} & \frac{2ix-3}{\sqrt{6(3+2x^2)}} \end{pmatrix} \mathrm{diag}(1, e^{-\frac{i\eta}{2}}, 1), \tag{4.30}$$

with

$$U_\nu^T m_\nu U_\nu = \mathrm{diag}(6m_a, m_s(3+2x^2), 0). \tag{4.31}$$

The residual flavor symmetry $G_l = Z_3^T$ enforces the unitary transformation, and U_l is an identity matrix up to the permutations of columns. Consequently the lepton mixing matrix coincides with U_ν, and it is the TM1 mixing pattern. Then the lepton mixing angles and CP invariants are determined to be

$$\sin^2\theta_{13} = \frac{2x^2}{9+6x^2}, \qquad \sin^2\theta_{12} = \frac{3}{9+4x^2}, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = -\frac{x}{9+6x^2}, \qquad I_1 = -\frac{2\sin\eta}{9+6x^2}. \tag{4.32}$$

We note that the atmospheric mixing angle θ_{23} is maximal and the other two mixing angles depend on a single real parameter x. The following sum rule between the reactor mixing angle and the solar mixing angle is satisfied:

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{2}{3}. \tag{4.33}$$

Form the weak-basis invariants J_{CP} and I_1 in Eq. (4.32), we find that the CP phases δ_{CP} and β are

$$\sin\delta_{CP} = -\mathrm{sign}(x), \qquad \beta = -\eta. \tag{4.34}$$

Hence the Dirac phase is maximally CP violating with $\delta_{CP} = \pm\pi/2$, and the Majorana phase β is equal to the opposite of η. We take the alignment parameter $x = 1/3$ as an example. Then the solar vacuum $\langle\phi_{\mathrm{sol}}\rangle$ is proportional to the column vector $(1, 1+i/3, 1-i/3)^T$, and the PMNS matrix is of constant value,

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{3}{\sqrt{29}} & i\sqrt{\frac{2}{87}} \\ -\frac{1}{\sqrt{6}} & \frac{3+i}{\sqrt{29}} & \frac{9+2i}{\sqrt{174}} \\ -\frac{1}{\sqrt{6}} & \frac{3-i}{\sqrt{29}} & -\frac{9-2i}{\sqrt{174}} \end{pmatrix} \mathrm{diag}(1, 1, e^{-\frac{i\eta}{2}}), \tag{4.35}$$

which leads to the following expressions for the mixing angles and δ_{CP}:

$$\sin^2\theta_{13} = \frac{2}{87} \simeq 0.0230, \qquad \sin^2\theta_{12} = \frac{27}{87} \simeq 0.318,$$
$$\sin^2\theta_{23} = \frac{1}{2}, \qquad \delta_{CP} = -\frac{\pi}{2}. \tag{4.36}$$

This mixing pattern can accommodate the experimental results very well [1]. From Table III, we see that there are still 13 other kinds of breaking patterns with IO and their phenomenological predictions for the lepton mixing parameters and neutrino masses are discussed in Appendix C.

V. MODEL CONSTRUCTION

In Secs. III and IV, Appendixes B and C, we have performed a model-independent analysis for the lepton mixing patterns which can be derived from $S_4 \rtimes H_{CP}$ in the tridirect CP approach. Inspired by the above model-independent analysis, in this section we shall construct a supersymmetric model with the flavor symmetry S_4 and a CP symmetry, and the symmetry-breaking pattern $\mathcal{N}_4$ is realized due to the nonvanishing vacuum expectation values of some flavons. The phenomenological predictions of $\mathcal{N}_4$ have been studied in detail in Ref. [61], and some numerical benchmark examples are listed in Table IX. The reasons why we construct a model to realize the breaking pattern $\mathcal{N}_4$ are as follows. First the TM1 mixing matrix predicted in the cases of $\mathcal{N}_{1,2,3}$ has been widely discussed in the literature. Second the vacuum alignments of ϕ_{atm} and ϕ_{sol} are quite simple, i.e., $\langle\phi_{\text{atm}}\rangle \propto (1,\omega^2,\omega)^T$ and $\phi_{\text{sol}} \propto (1,x,x)^T$. Third the minimum of χ^2 in $\mathcal{N}_4$ is rather small for certain values of x and η, as shown in Table IX. In particular the experimentally measured lepton mixing angles and neutrino masses can be described very well for the case of $x=-4$ and $\eta = \pm 3\pi/4$. We formulate our model in the framework of the seesaw mechanism with two right-handed neutrinos. The three generations of left-handed lepton doublets L are embedded into an S_4 triplet representation **3**, while the right-handed charged leptons e^c, μ^c and τ^c are all singlets **1** under the family symmetry S_4. The two right-handed neutrinos ν^c_{atm} and ν^c_{sol} are assumed to transform as **1** and **1′** respectively. In order to ensure the needed vacuum alignment, to forbid unwanted couplings and to reproduce the observed charged lepton mass hierarchies, the auxiliary symmetry $Z_5 \times Z_8 \times Z'_8$ is imposed. The shaping symmetry Z_8 disentangles the charged lepton sector from the neutrino sector, $Z_5 \times Z'_8$ further distinguishes the atmospheric neutrino sector from the solar neutrino sector, and the entire auxiliary symmetry imposes different powers of flavon fields for the electron, muon and tauon terms such that the observed charged lepton mass hierarchies are reproduced. In this model, the original symmetry $S_4 \rtimes H_{CP}$ is spontaneously broken to Z_3^T, $Z_2^{TST^2} \times X_{\text{atm}}$ and $Z_2^U \times X_{\text{sol}}$ in the charged lepton, atmospheric neutrino and solar neutrino sectors respectively, where the residual CP transformations are $X_{\text{atm}} = SU$ and $X_{\text{sol}} = U$. As a consequence, the desired vacua $\langle\phi_{\text{atm}}\rangle \propto (1,\omega^2,\omega)^T$ and $\langle\phi_{\text{sol}}\rangle \propto (1,-4,-4)^T$ can be achieved. Furthermore, we note that other flavon fields besides ϕ_{atm}, ϕ_{sol}, ξ_{atm} and ξ_{sol} are usually needed in order to realize the desired remnant symmetry. The relevant flavon fields, driving fields and their transform properties under the imposed flavor symmetry $S_4 \times Z_5 \times Z_8 \times Z'_8$ are collected in Table X.

A. Vacuum alignment

We adopt the now standard F-term alignment mechanism [79] to generate the appropriate vacuum alignments of the flavor-symmetry-breaking flavons. The leading-order (LO) driving superpotential w_d which is invariant under the imposed $S_4 \times Z_5 \times Z_8 \times Z'_8$ takes the following form:

$$w_d = w_d^l + w_d^{\text{atm}} + w_d^{\text{sol}}, \tag{5.1}$$

where w_d^l, w_d^{atm} and w_d^{sol} are used to realize the LO vacuum alignments of the flavons in the charged lepton sector, the atmospheric neutrino sector and the solar neutrino sector, respectively. They can be expressed as

TABLE X. The lepton, Higgs and flavon superfields and their transformation properties under the flavor symmetry $S_4 \times Z_5 \times Z_8 \times Z'_8$, where $\omega_5 \equiv e^{2\pi i/5}$ and $\omega_8 \equiv e^{\pi i/4}$. In addition, we assume a standard $U(1)_R$ symmetry under which all lepton fields carry a unit charge and the driving fields indicated with the superscript "0" have charge $+2$ while the Higgs and flavon fields are uncharged.

	L	e^c	μ^c	τ^c	ν^c_{atm}	ν^c_{sol}	$H_{u,d}$	η_l	ϕ_l	ξ_a	ϕ_a	ξ_s	η_s	χ_s	φ_s	Δ_s	ϕ_s	ψ_s	ξ_l^0	ϕ_l^0	ϕ_a^0	σ^0	ρ^0	η^0	χ^0	φ^0	Δ^0	κ^0
S_4	**3**	**1**	**1**	**1**	**1**	**1′**	**1**	**2**	**3**	**1**	**3**	**1**	**2**	**3′**	**3′**	**3′**	**3′**	**3′**	**1**	**3′**	**3′**	**2**	**2**	**2**	**3′**	**3′**	**3′**	**1**
Z_5	ω_5^4	ω_5^3	ω_5^4	1	1	1	1	ω_5	ω_5	1	ω_5	1	ω_5^3	ω_5^2	1	ω_5^3	ω_5	ω_5^4	ω_5^3	ω_5^3	ω_5^3	ω_5^2	ω_5	ω_5^4	1	1	ω_5^2	ω_5^3
Z_8	ω_8^7	ω_8^6	ω_8^7	1	ω_8^5	ω_8	1	ω_8	ω_8	ω_8^6	-1	ω_8^6	1	1	1	1	1	1	ω_8^6	ω_8^6	1	1	1	1	1	1	1	1
Z'_8	ω_8	ω_8^7	ω_8^7	ω_8^7	ω_8^5	1	1	1	1	ω_8^6	ω_8^2	1	ω_8^5	ω_8^5	ω_8	ω_8^4	ω_8^7	ω_8^5	1	1	ω_8^4	ω_8^6	ω_8^6	ω_8^6	ω_8^6	ω_8^7	ω_8^4	ω_8^2

$$\begin{aligned}
w_d^l &= g_1\xi_l^0(\eta_l\eta_l)_{\mathbf{1}} + g_2\xi_l^0(\phi_l\phi_l)_{\mathbf{1}} + g_3(\phi_l^0(\eta_l\phi_l)_{\mathbf{3}'})_{\mathbf{1}} + g_4(\phi_l^0(\phi_l\phi_l)_{\mathbf{3}'})_{\mathbf{1}},\\
w_d^{\rm atm} &= h_1(\phi_a^0(\phi_a\phi_a)_{\mathbf{3}'})_{\mathbf{1}},\\
w_d^{\rm sol} &= f_1(\rho^0(\chi_s\chi_s)_{\mathbf{2}})_{\mathbf{1}} + f_2(\sigma^0(\psi_s\psi_s)_{\mathbf{2}})_{\mathbf{1}} + f_3(\eta^0(\eta_s\eta_s)_{\mathbf{2}})_{\mathbf{1}} + f_4(\eta^0(\chi_s\psi_s)_{\mathbf{2}})_{\mathbf{1}} + f_5(\chi^0(\eta_s\chi_s)_{\mathbf{3}'})_{\mathbf{1}} + f_6(\chi^0(\varphi_s\varphi_s)_{\mathbf{3}'})_{\mathbf{1}}\\
&\quad + M_\varphi(\varphi^0\varphi_s)_{\mathbf{1}} + f_7(\varphi^0(\chi_s\psi_s)_{\mathbf{3}'})_{\mathbf{1}} + M_\Delta(\Delta_s^0\Delta_s)_{\mathbf{1}} + f_8(\Delta_s^0(\chi_s\phi_s)_{\mathbf{3}'})_{\mathbf{1}} + f_9\kappa_s^0(\chi_s\varphi_s)_{\mathbf{1}} + f_{10}\kappa_s^0(\phi_s\phi_s)_{\mathbf{1}},
\end{aligned} \tag{5.2}$$

where the subscript $(\cdots)_{\mathbf{r}}$ denotes a contraction of the S_4 indices into the irreducible representation $\mathbf{r}$. All of the couplings g_i $(i = 1, \ldots, 4)$, h_1, f_i $(i = 1, \ldots, 10)$ and mass parameters M_φ, M_Δ in Eq. (5.2) are real because the theory is invariant under the generalized CP symmetry. In the supersymmetry limit, the vacuum alignment is achieved through the vanishing F terms of the driving fields. In the charged lepton sector, the F-term conditions obtained from the driving fields ξ_l^0 and ϕ_l^0 are given by

$$\begin{aligned}
\frac{\partial w_d^l}{\partial\xi_l^0} &= 2g_1\eta_{l_1}\eta_{l_2} + g_2(\phi_{l_1}^2 + 2\phi_{l_2}\phi_{l_3}) = 0,\\
\frac{\partial w_d^l}{\partial\phi_{l_1}^0} &= g_3(\eta_{l_1}\phi_{l_2} - \eta_{l_2}\phi_{l_3}) + 2g_4(\phi_{l_1}^2 - \phi_{l_2}\phi_{l_3}) = 0,\\
\frac{\partial w_d^l}{\partial\phi_{l_2}^0} &= g_3(\eta_{l_1}\phi_{l_1} - \eta_{l_2}\phi_{l_2}) + 2g_4(\phi_{l_2}^2 - \phi_{l_1}\phi_{l_3}) = 0,\\
\frac{\partial w_d^l}{\partial\phi_{l_3}^0} &= g_3(\eta_{l_1}\phi_{l_3} - \eta_{l_2}\phi_{l_1}) + 2g_4(\phi_{l_3}^2 - \phi_{l_1}\phi_{l_2}) = 0.
\end{aligned} \tag{5.3}$$

We find that these equations are satisfied by the alignment

$$\langle\eta_l\rangle = (0,1)^T v_{\eta_l}, \qquad \langle\phi_l\rangle = (0,1,0)^T v_{\phi_l}, \qquad \text{with} \quad v_{\phi_l} = \frac{g_3}{2g_4}v_{\eta_l}, \tag{5.4}$$

where v_{η_l} is undetermined. In the atmospheric neutrino sector, the vacuum is determined by the F-term conditions associated with the driving field ϕ_a^0

$$\begin{aligned}
\frac{\partial w_d^{\rm atm}}{\partial\phi_{a_1}^0} &= 2h_1(\phi_{a_1}^2 - \phi_{a_2}\phi_{a_3}) = 0,\\
\frac{\partial w_d^{\rm atm}}{\partial\phi_{a_2}^0} &= 2h_1(\phi_{a_2}^2 - \phi_{a_1}\phi_{a_3}) = 0,\\
\frac{\partial w_d^{\rm atm}}{\partial\phi_{a_3}^0} &= 2h_1(\phi_{a_3}^2 - \phi_{a_1}\phi_{a_2}) = 0,
\end{aligned} \tag{5.5}$$

which lead to the following vacuum alignment of ϕ_a:

$$\langle\phi_a\rangle = (1,\omega^2,\omega)^T v_{\phi_a}. \tag{5.6}$$

Then we turn to the vacuum of the solar neutrino sector. The F-flatness conditions of the driving fields ρ^0 and σ^0 are given by

$$\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial\rho_1^0} &= f_1(2\chi_{s_1}\chi_{s_2} + \chi_{s_3}^2) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\rho_2^0} &= f_1(2\chi_{s_1}\chi_{s_3} + \chi_{s_2}^2) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\sigma_1^0} &= f_2(2\psi_{s_1}\psi_{s_2} + \psi_{s_3}^2) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\sigma_2^0} &= f_2(2\psi_{s_1}\psi_{s_3} + \psi_{s_2}^2) = 0.
\end{aligned} \tag{5.7}$$

A solution to the above equations is

$$\langle\chi_s\rangle = v_{\chi_s}(1,0,0)^T, \qquad \langle\psi_s\rangle = v_{\psi_s}(1,-2,-2)^T. \tag{5.8}$$

The vacuum of the doublet flavon η originates from the F term of η^0,

$$\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial\eta_1^0} &= f_3\eta_{s_1}^2 + f_4(\chi_{s_1}\psi_{s_2} + \chi_{s_2}\psi_{s_1} + \chi_{s_3}\psi_{s_3}) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\eta_2^0} &= f_3\eta_{s_2}^2 + f_4(\chi_{s_1}\psi_{s_3} + \chi_{s_2}\psi_{s_2} + \chi_{s_3}\psi_{s_1}) = 0.
\end{aligned} \tag{5.9}$$

An extremum solution is given by,

$$\langle\eta_s\rangle = v_{\eta_s}(1,1)^T, \quad \text{with} \quad v_{\eta_s}^2 = \frac{2f_4}{f_3}v_{\chi_s}v_{\psi_s}. \tag{5.10}$$

Furthermore, the F terms of the driving field χ^0 are of the form

$$\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial\chi_1^0} &= f_5(\eta_{s_1}\chi_{s_2} + \eta_{s_2}\chi_{s_3}) + 2f_6(\varphi_{s_1}^2 - \varphi_{s_2}\varphi_{s_3}) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\chi_2^0} &= f_5(\eta_{s_1}\chi_{s_1} + \eta_{s_2}\chi_{s_2}) + 2f_6(\varphi_{s_2}^2 - \varphi_{s_1}\varphi_{s_3}) = 0,\\
\frac{\partial w_d^{\rm sol}}{\partial\chi_3^0} &= f_5(\eta_{s_1}\chi_{s_3} + \eta_{s_2}\chi_{s_1}) + 2f_6(\varphi_{s_3}^2 - \varphi_{s_1}\varphi_{s_2}) = 0,
\end{aligned} \tag{5.11}$$

which generate the alignment

$$\langle\varphi_s\rangle = v_{\varphi_s}(1,-1,-1)^T, \quad \text{with} \quad v_{\varphi_s}^2 = -\frac{f_5}{4f_6}v_{\eta_s}v_{\chi_s}. \tag{5.12}$$

Similarly we can read out the F-flatness condition of the driving field φ^0,

$$\frac{\partial w_d^{\rm sol}}{\partial \varphi_1^0} = M_\varphi \varphi_{s_1} + f_7(2\chi_{s_1}\Delta_{s_1} - \chi_{s_2}\Delta_{s_3} - \chi_{s_3}\Delta_{s_2}) = 0,$$
$$\frac{\partial w_d^{\rm sol}}{\partial \varphi_2^0} = M_\varphi \varphi_{s_3} + f_7(2\chi_{s_2}\Delta_{s_2} - \chi_{s_1}\Delta_{s_3} - \chi_{s_3}\Delta_{s_1}) = 0,$$
$$\frac{\partial w_d^{\rm sol}}{\partial \varphi_3^0} = M_\varphi \varphi_{s_2} + f_7(2\chi_{s_3}\Delta_{s_3} - \chi_{s_1}\Delta_{s_2} - \chi_{s_2}\Delta_{s_1}) = 0. \tag{5.13}$$

Considering the vacuum configurations of χ_s and φ_s given in Eqs. (5.8) and (5.12), we see that the vacuum expectation values of Δ_s are

$$\langle \Delta_s \rangle = v_{\Delta_s}(1, 2, 2)^T, \quad \text{with} \quad v_{\Delta_s} = -\frac{M_\varphi v_{\varphi_s}}{2 f_7 v_{\chi_s}}. \tag{5.14}$$

In order to realize the desired solar alignment $\langle \phi_s \rangle \propto (1, -4, -4)^T$, we consider the F terms of the driving field Δ^0,

$$\frac{\partial w_d^{\rm sol}}{\partial \Delta_1^0} = M_\Delta \Delta_{s_1} + f_8(2\chi_{s_1}\phi_{s_1} - \chi_{s_2}\phi_{s_3} - \chi_{s_3}\phi_{s_2}) = 0,$$
$$\frac{\partial w_d^{\rm sol}}{\partial \Delta_2^0} = M_\Delta \Delta_{s_3} + f_8(2\chi_{s_2}\phi_{s_2} - \chi_{s_3}\phi_{s_1} - \chi_{s_1}\phi_{s_3}) = 0,$$
$$\frac{\partial w_d^{\rm sol}}{\partial \Delta_3^0} = M_\Delta \Delta_{s_2} + f_8(2\chi_{s_3}\phi_{s_3} - \chi_{s_1}\phi_{s_2} - \chi_{s_2}\phi_{s_1}) = 0, \tag{5.15}$$

which uniquely determine the vacuum of the solar flavon ϕ_s to be

$$\langle \phi_s \rangle = v_{\phi_s}(1, -4, -4)^T, \quad \text{with} \quad v_{\phi_s} = -\frac{M_\Delta v_{\Delta_s}}{2 f_8 v_{\chi_s}}. \tag{5.16}$$

Analogously the flatness condition of the driving field κ^0 gives rise to

$$\frac{\partial w_d^{\rm sol}}{\partial \kappa^0} = f_9(\varphi_{s_1}\chi_{s_1} + \varphi_{s_2}\chi_{s_3} + \varphi_{s_3}\chi_{s_2}) + f_{10}(\phi_{s_1}^2 + 2\phi_{s_2}\phi_{s_3}) = 0. \tag{5.17}$$

It is easy to solve this equation and obtain

$$v_{\phi_s}^2 = -\frac{f_9 v_{\chi_s} v_{\varphi_s}}{33 f_{10}}. \tag{5.18}$$

Now we have obtained the vacuum alignments of all flavons $\eta_s, \psi_s, \chi_s, \varphi_s, \Delta_s$ and ϕ_s in the solar neutrino sector by adopting the standard F-term alignment mechanism. In other words, the needed vacuum alignment $\langle \phi_s \rangle \propto (1, -4, -4)^T$ is realized. Next we shall fix the overall phases of all VEVs of flavons in the atmospheric and solar neutrino sectors. From the alignments of flavons ξ_s, $\eta_s, \chi_s, \psi_s, \varphi_s$ and ϕ_s shown above, we find that the VEVs of these fields are invariant under the subgroup Z_2^U. In order to obtain the phase with $\eta = \pm\frac{3\pi}{4}$, we introduce the S_4 singlet fields in Table XI. Then the driving superpotential which is used to obtain the phases of all the VEVs of flavons in the neutrino sector is

$$\begin{aligned} w_d^{\rm phase} = {} & M_1^2\zeta_1^0 + x_1\zeta_1^0\Omega_1^2 + M_2^2\zeta_2^0 + x_2\zeta_2^0\Omega_2^2 + M_3^2\zeta_3^0 + x_3\zeta_3^0\Omega_3\Omega_4 + M_4^2\zeta_4^0 + x_4\zeta_4^0\Omega_5\Omega_6 + M_{\Omega_1}\Omega_1^0\Omega_1 \\ & + x_5\Omega_1^0\xi_s^2 + M_{\Omega_2}\Omega_2^0\Omega_2 + x_6\Omega_2^0\xi_a^2 + M_{\Omega_3}\Omega_3^0\Omega_3 + x_7\Omega_3^0(\eta_s\eta_s)_{\mathbf{1}} + x_8\Omega_3^0(\chi_s\psi_s)_{\mathbf{1}} + M_{\Omega_4}\Omega_4^0\Omega_4 \\ & + x_9\Omega_4^0(\varphi_s\psi_s)_{\mathbf{1}} + M_{\Omega_5}\Omega_5^0\Omega_5 + x_{10}\Omega_5^0(\phi_a\chi_s)_{\mathbf{1}'} + M_{\Omega_6}\Omega_6^0\Omega_6 + x_{11}\Omega_6^0(\phi_a\phi_s)_{\mathbf{1}'}, \end{aligned} \tag{5.19}$$

where the couplings x_i and mass parameters M_i^2 and M_{Ω_i} are all real. The F-term conditions from the above superpotential are

TABLE XI. The transformation rules of the singlet flavon and driving superfields which are used to determine the phases of the flavon VEVs.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1^0	Ω_2^0	Ω_3^0	Ω_4^0	Ω_5^0	Ω_6^0	$\zeta_{1,2,3,4}^0$
S_4	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{1}$
Z_5	1	1	ω_5	ω_5^4	ω_5^3	ω_5^2	1	1	ω_5^4	ω_5	ω_5^2	ω_5^3	1
Z_8	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1
Z_8'	1	−1	ω_8^2	ω_8^6	ω_8^7	ω_8	1	−1	ω_8^6	ω_8^2	ω_8	ω_8^7	1

$$\frac{\partial w_d^{\rm phase}}{\partial \zeta_1^0} = M_1^2 + x_1\Omega_1^2 = 0, \qquad \frac{\partial w_d^{\rm phase}}{\partial \zeta_2^0} = M_2^2 + x_2\Omega_2^2 = 0,$$
$$\frac{\partial w_d^{\rm phase}}{\partial \zeta_3^0} = M_3^2 + x_3\Omega_3\Omega_4 = 0, \qquad \frac{\partial w_d^{\rm phase}}{\partial \zeta_4^0} = M_4^2 + x_4\Omega_5\Omega_6 = 0,$$
$$\frac{\partial w_d^{\rm phase}}{\partial \Omega_1^0} = M_{\Omega_1}\Omega_1 + x_5\xi_s^2 = 0, \qquad \frac{\partial w_d^{\rm phase}}{\partial \Omega_2^0} = M_{\Omega_2}\Omega_2 + x_6\xi_a^2 = 0,$$
$$\frac{\partial w_d^{\rm phase}}{\partial \Omega_3^0} = M_{\Omega_3}\Omega_3 + 2x_7 v_{\eta_s}^2 + x_8 v_{\chi_s} v_{\psi_s} = 0, \qquad \frac{\partial w_d^{\rm phase}}{\partial \Omega_4^0} = M_{\Omega_4}\Omega_4 + 5x_9 v_{\varphi_s} v_{\psi_s} = 0,$$
$$\frac{\partial w_d^{\rm phase}}{\partial \Omega_5^0} = M_{\Omega_5}\Omega_5 + x_{10} v_{\chi_s} v_{\phi_a} = 0, \qquad \frac{\partial w_d^{\rm phase}}{\partial \Omega_6^0} = M_{\Omega_6}\Omega_6 + 5x_{11} v_{\phi_s} v_{\phi_a} = 0. \tag{5.20}$$

From the above equations and the relations among the VEVs of flavons in the solar neutrino sector, we can achieve the phases of the VEVs of all the flavons in the solar and atmospheric neutrino sectors. Since the expressions of the flavon VEVs are a little redundant and they are not used in the following discussion, we will not show them here. In order to obtain the phase $\eta = \pm\frac{3\pi}{4}$, the only valid ratio is $\frac{v_{\phi_s}^2 v_{\xi_a}}{v_{\phi_a}^2 v_{\xi_s}}$, where v_{ξ_s} and v_{ξ_a} are the VEVs of the flavon fields ξ_s and ξ_a, respectively. The expression of this ratio is

$$\frac{v_{\phi_s}^2 v_{\xi_a}}{v_{\phi_a}^2 v_{\xi_s}} = \frac{(\frac{5}{33})^{3/4} x_4 x_{10} x_{11} f_5 M_{\Delta_s} M_{\varphi_s}}{16\sqrt{2} f_6 f_7 f_8 M_{\Omega_5} M_{\Omega_6} M_4^2} \times \left(\frac{x_1 x_5^2 f_4^2 f_9^3 M_2^2 M_3^2 M_{\Delta_s}^2 M_{\varphi_s}^2 M_{\Omega_2}^2 M_{\Omega_3} M_{\Omega_4}}{(f_3 x_8 + 4 f_4 x_7) x_2 x_3 x_6^2 x_9 f_3 f_7^2 f_8^2 f_{10}^3 M_1^2 M_{\Omega_1}^2}\right)^{\frac{1}{4}}. \tag{5.21}$$

Since all the couplings x_i, f_i and mass parameters M_i in Eq. (5.21) are real, we see that the phases of the ratio $\frac{v_{\phi_s}^2 v_{\xi_a}}{v_{\phi_a}^2 v_{\xi_s}}$ are $e^{\frac{ik\pi}{4}}$ $(i = 0, 1, \ldots, 7)$. In the present work we shall concentrate on the solution with

$$\arg\left(\frac{v_{\phi_s}^2 v_{\xi_a}}{v_{\phi_a}^2 v_{\xi_s}}\right) = \pm\frac{3\pi}{4}, \tag{5.22}$$

which would happen one in four times by chance. In order to obtain the observed hierarchy among the charged lepton masses, we assume

$$\frac{v_{\eta_l}}{\Lambda} \sim \frac{v_{\phi_l}}{\Lambda} \sim \lambda^2, \tag{5.23}$$

where λ is the Cabibbo angle with $\lambda \simeq 0.23$. Moreover, the VEVs of flavons in the neutrino sector are expected to be of the same order of magnitude and we will take them to be of the same order as the VEVs of flavons in the charged lepton sector, i.e.,

$$\frac{v_{\xi_a}}{\Lambda} \sim \frac{v_{\phi_a}}{\Lambda} \sim \frac{v_{\xi_s}}{\Lambda} \sim \frac{v_{\phi_s}}{\Lambda} \sim \frac{v_{\eta_s}}{\Lambda} \sim \frac{v_{\chi_s}}{\Lambda} \sim \frac{v_{\psi_s}}{\Lambda} \sim \frac{v_{\varphi_s}}{\Lambda} \sim \frac{v_{\Delta_s}}{\Lambda} \sim \frac{v_{\Omega_i}}{\Lambda} \sim \lambda^2, \tag{5.24}$$

where v_{Ω_i} $(i = 1, \ldots, 6)$ are the VEVs of flavons Ω_i. Now we will briefly touch on the subleading corrections to the driving superpotential given above. We first start with the corrections to the driving superpotential w_d^l which contains the driving fields ξ_l^0 and ϕ_l^0. We find that its NLO corrections are suppressed by $1/\Lambda^2$ with respect to the renormalizable terms in Eq. (5.2). The subleading contributions to the driving superpotential $w_d^{\rm atm}$ and $w_d^{\rm sol}$ involve three flavon fields. The corresponding corrections to the leading-order terms in $w_d^{\rm atm}$ and $w_d^{\rm sol}$ are of relative order λ^2.

B. The structure of the model

The lowest-dimensional Yukawa operators of the charged lepton mass terms, which are invariant under the imposed flavor symmetry $S_4 \times Z_5 \times Z_8 \times Z_8'$, can be written as

$$\begin{aligned} w_l = {} & \frac{y_\tau}{\Lambda}(L\phi_l)_{\mathbf{1}}\tau^c H_d + \frac{y_{\mu_1}}{\Lambda^2}(L(\eta_l\phi_l)_{\mathbf{3}})_{\mathbf{1}}\mu^c H_d + \frac{y_{\mu_2}}{\Lambda^2}(L(\phi_l\phi_l)_{\mathbf{3}})_{\mathbf{1}}\mu^c H_d + \frac{y_{e_1}}{\Lambda^3}(L\phi_l)_{\mathbf{1}}(\eta_l\eta_l)_{\mathbf{1}} e^c H_d + \frac{y_{e_2}}{\Lambda^3}((L\phi_l)_{\mathbf{2}}(\eta_l\eta_l)_{\mathbf{2}})_{\mathbf{1}} e^c H_d \\ & + \frac{y_{e_3}}{\Lambda^3}((L\eta_l)_{\mathbf{3}}(\phi_l\phi_l)_{\mathbf{3}})_{\mathbf{1}} e^c H_d + \frac{y_{e_4}}{\Lambda^3}((L\eta_l)_{\mathbf{3'}}(\phi_l\phi_l)_{\mathbf{3'}})_{\mathbf{1}} e^c H_d + \frac{y_{e_5}}{\Lambda^3}(L\phi_l)_{\mathbf{1}}(\phi_l\phi_l)_{\mathbf{1}} e^c H_d + \frac{y_{e_6}}{\Lambda^3}((L\phi_l)_{\mathbf{2}}(\phi_l\phi_l)_{\mathbf{2}})_{\mathbf{1}} e^c H_d \\ & + \frac{y_{e_7}}{\Lambda^3}((L\phi_l)_{\mathbf{3}}(\phi_l\phi_l)_{\mathbf{3}})_{\mathbf{1}} e^c H_d + \frac{y_{e8}}{\Lambda^3}((L\phi_l)_{\mathbf{3'}}(\phi_l\phi_l)_{\mathbf{3'}})_{\mathbf{1}} e^c H_d, \end{aligned} \tag{5.25}$$

where all couplings are real due to the generalized CP symmetry. With the VEVs of η_l and ϕ_l in Eq. (5.4), we find that the charged lepton mass matrix is diagonal with the three charged lepton masses being

$$m_e = \left|(y_{e_6} - 2y_{e_8} - 2y_{e_4} v_{\eta_l}/v_{\phi_l} + y_{e_2} v_{\eta_l}^2/v_{\phi_l}^2)\frac{v_{\phi_l}^3}{\Lambda^3}\right| v_d,$$
$$m_\mu = \left|y_{\mu_1}\frac{v_{\eta_l} v_{\phi_l}}{\Lambda^2}\right| v_d, \qquad m_\tau = \left|y_\tau \frac{v_{\phi_l}}{\Lambda}\right| v_d, \tag{5.26}$$

where $v_d = \langle H_d\rangle$. Note that in order to obtain the mass hierarchies of the charged leptons $m_e : m_\mu : m_\tau \simeq \lambda^4 : \lambda^2 : 1$, the auxiliary symmetry Z_8 is imposed, where $\lambda \simeq 0.23$ is the Cabibbo angle. The auxiliary symmetry Z_8 imposes different powers of η_l and ϕ_l to couple with the electron, muon and tau lepton mass terms. From Eq. (5.25), we find that the electron, muon and tau masses arise at order $(\langle\Phi_l\rangle/\Lambda)^3$, $(\langle\Phi_l\rangle/\Lambda)^2$ and $\langle\Phi_l\rangle/\Lambda$ respectively, where Φ_l refers to either η_l or ϕ_l. If we assume that $\langle\Phi_l\rangle/\Lambda$ is of order λ^2, then the mass hierarchy of the charged leptons can be reproduced. Moreover, the subleading operators related to e^c, μ^c and τ^c comprise four flavons and consequently are suppressed by $1/\Lambda^4$. Such corrections for the charged lepton masses and lepton mixing parameters can be neglected.

Now we come to the neutrino sector. The light neutrino masses are given by the famous type-I seesaw mechanism with two right-handed neutrinos. The most general LO superpotential for the neutrino masses is

$$w_\nu = \frac{y_a}{\Lambda}(L\phi_a)_{\mathbf{1}} H_u \nu^c_{\rm atm} + \frac{y_s}{\Lambda}(L\phi_s)_{\mathbf{1}'} H_u \nu^c_{\rm sol} + x_a \nu^c_{\rm atm}\nu^c_{\rm atm}\xi_a + x_s \nu^c_{\rm sol}\nu^c_{\rm sol}\xi_s, \tag{5.27}$$

where the four coupling constants y_a, y_s, x_a and x_s are real because the theory is required to be invariant under the generalized CP transformation. From the vacuum alignments of the flavons ϕ_a and ϕ_s, we can read out the Dirac and Majorana mass matrices as follows:

$$M_D = \begin{pmatrix} y_a v_{\phi_a} & y_s v_{\phi_s} \\ \omega y_a v_{\phi_a} & -4 y_s v_{\phi_s} \\ \omega^2 y_a v_{\phi_a} & -4 y_s v_{\phi_s} \end{pmatrix}\frac{v_u}{\Lambda}, \quad M_N = \begin{pmatrix} x_a v_{\xi_a} & 0 \\ 0 & x_s v_{\xi_s}\end{pmatrix}, \tag{5.28}$$

where $v_u = \langle H_u\rangle$ and the expressions of the VEVs v_{ξ_a}, v_{ξ_s}, v_{ϕ_a}, v_{ϕ_s} are shown in Sec. V A. After applying the seesaw formula, the effective light neutrino mass matrix can be written as

$$m_\nu = -\frac{v_u^2}{\Lambda^2}\left[\frac{y_a^2 v_{\phi_a}^2}{x_a v_{\xi_a}}\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega\end{pmatrix} + \frac{y_s^2 v_{\phi_s}^2}{x_s v_{\xi_s}}\begin{pmatrix} 1 & -4 & -4 \\ -4 & 16 & 16 \\ -4 & 16 & 16\end{pmatrix}\right]. \tag{5.29}$$

In Sec. V A, we have taken the solution where the phase of the ratio $\frac{v_{\phi_s}^2 v_{\xi_a}}{v_{\phi_a}^2 v_{\xi_s}}$ is $\pm\frac{3\pi}{4}$. Up to the overall phase of the neutrino mass matrix in Eq. (5.29), we see that this neutrino mass matrix is of the same form as the general mass matrix in breaking pattern $\mathcal{N}_4$ [61] but with

$$x = -4, \qquad m_a = \left|\frac{y_a^2 v_{\phi_a}^2}{x_a v_{\xi_a}}\frac{v_u^2}{\Lambda^2}\right|, \qquad m_s e^{i\eta} = \left|\frac{y_s^2 v_{\phi_s}^2}{x_s v_{\xi_s}}\frac{v_u^2}{\Lambda^2}\right| e^{\mp\frac{3\pi i}{4}}, \tag{5.30}$$

for the case of $x_a x_s > 0$. In the following, we will briefly touch on the subleading corrections to the superpotential given in Secs. V A and V B. Furthermore,we find that the next-to-leading-order operators of w_ν are suppressed by $1/\Lambda^2$ with respect to the LO contributions and therefore can be neglected.

From the standard procedure shown in Sec. II, we find that the above model predicts the following LO lepton mixing matrix:

$$U_{\rm PMNS} = \frac{1}{\sqrt{74}}\begin{pmatrix} 4\sqrt{3} & -i\sqrt{26}\cos\theta & -i\sqrt{26}e^{i\psi}\sin\theta \\ \sqrt{13} & 2\sqrt{6}i\cos\theta - \sqrt{37}e^{-i\psi}\sin\theta & 2\sqrt{6}ie^{i\psi}\sin\theta + \sqrt{37}\cos\theta \\ \sqrt{13} & 2\sqrt{6}i\cos\theta + \sqrt{37}e^{-i\psi}\sin\theta & 2\sqrt{6}ie^{i\psi}\sin\theta - \sqrt{37}\cos\theta \end{pmatrix} P_\nu, \tag{5.31}$$

where the diagonal phase P_ν is given in Eq. (3.17). All the parameters θ, ψ, σ and ρ only depend on one input parameter $r = m_s/m_a$. In the case of $\eta = -\frac{3\pi}{4}$, the three mixing angles and the two CP invariants can be expressed in terms of r as

$$\sin^2\theta_{13} = \frac{13}{74}\left(1 - \frac{15(781r^2 + 3\sqrt{2}r - 7)}{13C_r}\right), \quad \sin^2\theta_{12} = 1 - \frac{48C_r}{15(781r^2 + 3\sqrt{2}r - 7) + 61C_r},$$
$$\sin^2\theta_{23} = \frac{1}{2} + \frac{740\sqrt{6}r}{15(781r^2 + 3\sqrt{2}r - 7) + 61C_r}, \quad J_{CP} = \frac{3\sqrt{3}(-308r^2 + 25\sqrt{2}r - 2)}{37C_r}, \quad I_1 = \frac{\sqrt{2}(1 - 196r^2)}{37C_r}, \tag{5.32}$$

where the parameter C_r is defined as

$$C_r = \sqrt{(1089r^2 - 25\sqrt{2}r + 9)^2 - 21904r^2}. \quad (5.33)$$

A sum rule between the reactor mixing angle and the solar mixing angle is easy to obtain

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{48}{74}. \quad (5.34)$$

Inserting the experimentally preferred 3σ range $0.01981 \leq \sin^2\theta_{13} \leq 0.02436$ [1], we obtain the prediction for the solar mixing angle

$$0.3352 \leq \sin^2\theta_{12} \leq 0.3365. \quad (5.35)$$

Furthermore, we find that the two nonzero neutrino masses are determined to be

$$m_2^2 = \frac{m_a^2}{2}(1089r^2 - 25\sqrt{2}r + 9 - C_r),$$
$$m_3^2 = \frac{m_a^2}{2}(1089r^2 - 25\sqrt{2}r + 9 + C_r). \quad (5.36)$$

The neutrino masses m_2^2 and m_3^2 are dependent on the free parameters m_a and r. As described above, all lepton mixing parameters and the mass ratio m_2/m_3 depend only on the ratio $r = m_s/m_a$. We plot the results of the mixing parameters and mass ratio m_2/m_3 with respect to the input parameter r in Fig. 3. At the best-fit point $r = 0.421$, we see that the mixing angles θ_{13}, θ_{23} and the mass ratio m_2/m_3 lie within their 1σ ranges, while θ_{12} lies just outside its 1σ range and within its 3σ range. The Dirac CP phase δ_{CP} and the Majorana CP phase β are predicted to be $\delta_{CP} \simeq -0.412\pi$ and $\beta \simeq 0.264\pi$, respectively. If we require that all three lepton mixing angles and two mass squared differences lie in their corresponding experimentally preferred 3σ intervals [1], then we find that the lepton mixing parameters and the neutrino masses are predicted to be in rather narrow regions,

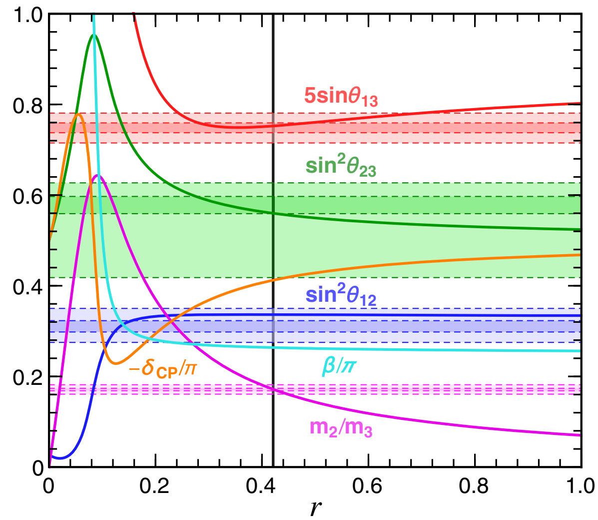

FIG. 3. The predicted values of our model with $\eta = -3\pi/4$ for the mixing parameters and mass ratio m_2/m_3 as a function of r. Horizontal bands show the experimentally determined 1σ and 3σ ranges [1] for each parameter. The black vertical line denotes the best-fit value $r_{\rm bf} = 0.421$ for r for which the χ^2 function reaches a global minimum.

$$0.3362 \leq \sin^2\theta_{12} \leq 0.3364, \quad 0.02254 \leq \sin^2\theta_{13} \leq 0.02280, \quad 0.556 \leq \sin^2\theta_{23} \leq 0.564,$$
$$-0.418 \leq \delta_{CP}/\pi \leq -0.406, \quad 0.263 \leq \beta/\pi \leq 0.264, \quad 2.690~\text{meV} \leq m_{ee} \leq 2.985~\text{meV},$$
$$8.240~\text{meV} \leq m_2 \leq 8.950~\text{meV}, \quad 49.265~\text{meV} \leq m_3 \leq 51.235~\text{meV}. \quad (5.37)$$

Therefore this model is very predictive and it should be easily falsified. The next-generation reactor neutrino oscillation experiments JUNO [80] and RENO-50 [81] are expected to reduce the error of θ_{12} to about 0.1° or around 0.3%. The oscillation parameters θ_{12}, θ_{23} and δ_{CP} will be precisely measured by the future long-baseline experiments DUNE [82–84], T2HK [85], T2HKK [86]. Hence this breaking pattern can be checked by future neutrino facilities. Furthermore, we expect that a more ambitious facility such as the Neutrino Factory [87–89] could provide more stringent tests of our approach. We see that the light neutrino mass matrix in Eq. (5.29) has the following symmetry property:

$$m_\nu(m_a, r, -\eta) = P_{132}^T m_\nu^*(m_a, r, \eta) P_{132}. \quad (5.38)$$

Therefore the atmospheric mixing angle changes from θ_{23} to $\pi/2 - \theta_{23}$, the Dirac CP phase changes from δ_{CP} to $\pi - \delta_{CP}$, the Majorana CP phase will become the opposite and the other observable quantities remain unchanged under the transformation $\eta \to -\eta$. The predictions for $\eta = \frac{3\pi}{4}$ can be easily obtained from the results of $\eta = -\frac{3\pi}{4}$. Hence we shall not show the predictions for $\eta = \frac{3\pi}{4}$.

VI. CONCLUSION

In the present paper, guided by the principles of symmetry and minimality, we have analyzed the possible symmetry-breaking patterns of $S_4 \rtimes H_{CP}$ in the tridirect CP approach [61] based on the two right-handed neutrino seesaw mechanism. In the tridirect CP approach, the high-energy flavor and generalized CP symmetry

$S_4 \rtimes H_{CP}$ is spontaneously broken down to an Abelian subgroup G_l (non Z_2 subgroups) in the charged lepton sector, to $G_{\rm atm} \rtimes H_{CP}^{\rm atm}$ in one right-handed neutrino sector and to $G_{\rm sol} \rtimes H_{CP}^{\rm sol}$ in the other right-handed neutrino sector, as illustrated in Fig. 1. In this work, we assumed that the flavon field $\phi_{\rm atm}$ which couples to the right-handed $N_{\rm atm}$ and the left-handed lepton doublets L is assigned to transform as an S_4 triplet **3**, and the flavon $\phi_{\rm sol}$ which couples to the right-handed $N_{\rm sol}$ and the left-handed lepton doublets L transforms as the three-dimensional representation $\mathbf{3}'$ under S_4. Then the two columns of the neutrino Dirac mass matrix are determined by the vacuum alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$, respectively. Furthermore, we have given the basic procedure of predicting lepton flavor mixing and neutrino mass from residual symmetries in the tridirect CP approach in a model-independent way and we found that the first (third) column of the PMNS matrix is fixed by the diagonalization matrix U_l of the charged lepton mass matrix and the vacuum alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ for the NO (IO) spectrum. Notice that the alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ are enforced by the residual symmetries $G_{\rm atm} \rtimes H_{CP}^{\rm atm}$ and $G_{\rm sol} \rtimes H_{CP}^{\rm sol}$ respectively in the tridirect CP approach. The results of this paper only depend on the structure of the flavor symmetry and the assumed symmetry-breaking patterns, and they are independent of how the required residual symmetries are dynamically realized. In concrete models, the desired vacuum of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ could be achieved in the context of supersymmetry or extra dimensions.

After considering all possible breaking patterns arising from S_4 flavor symmetry combined with the corresponding generalized CP symmetry in a model-independent way, we found eight phenomenologically interesting mixing patterns with NO spectra labeled as $\mathcal{N}_1 \sim \mathcal{N}_8$ and 18 phenomenologically interesting mixing patterns with IO spectra labeled as $\mathcal{I}_1 \sim \mathcal{I}_{18}$; please see Table III. For each phenomenologically interesting mixing pattern, we have analyzed the corresponding predictions for the PMNS matrix, the lepton mixing parameters, the neutrino masses and the effective mass in neutrinoless double beta decay in a model-independent way in the tridirect CP approach. There is one form dominance breaking pattern with an NO spectrum ($\mathcal{N}_5$) and two form dominance breaking patterns with IO spectra ($\mathcal{I}_4$ and $\mathcal{I}_5$). We found that three kinds of breaking patterns with NO spectra ($\mathcal{N}_1 \sim \mathcal{N}_3$) and one form dominance breaking pattern with an IO spectrum ($\mathcal{I}_5$) yield the TM1 mixing matrix. For each of these four kinds of breaking patterns with the TM1 mixing matrix, two sum rules among the mixing angles and Dirac CP phase corresponding to TM1 mixing were obtained. Furthermore, we performed a numerical analysis for each breaking pattern that is able to give a successful description of the lepton mixing parameters and the neutrino masses in terms of four real input parameters x, η, m_a and r. In the breaking patterns with NO spectra, we also gave the χ^2 results for some benchmark values of x and η, where the parameter x comes from the VEV of the flavon $\phi_{\rm sol}$. The simple values of x and η are very useful in model building. Once the values of x and η were fixed, we obtained a highly predictive theory of neutrino mass and lepton mixing, in which all lepton mixing parameters and the neutrino masses are determined by only two real input parameters m_a and r. In the breaking pattern $\mathcal{N}_1$, for the benchmark value $x = -1$ which leads to $\langle\phi_{\rm sol}\rangle = (1,-1,3)^T v_{\phi_s}$, it is exactly the littlest seesaw model with CSD(3) which was originally proposed in Ref. [15]. The solar vacuum $\langle\phi_{\rm sol}\rangle = (1,-3,1)^T v_{\phi_s}$ for $x = 3$ corresponds to another version of the littlest seesaw model [16]. Moreover, for the vacuum $\langle\phi_{\rm sol}\rangle = (1,4,-2)^T v_{\phi_s}$ with $x = 4$, the CSD(4) scenario [17] is reproduced. Furthermore, we showed the best-fit values of the neutrino masses and the mixing parameters for a simple value of x for each of the 18 breaking patterns with IO spectra.

Guided by the above model-independent analysis, we constructed a successful flavor model involving two right-handed neutrinos based on S_4 and generalized CP symmetry to realize the breaking pattern $\mathcal{N}_4$ with $x = -4$ and $\eta = \pm\frac{3\pi}{4}$, in which the original symmetry $S_4 \rtimes H_{CP}$ is spontaneously broken down to Z_3^T in the charged lepton sector, to $Z_2^{TST^2} \times X_{\rm atm}$ in the atmospheric neutrino sector and to $Z_2^U \times X_{\rm sol}$ in the solar neutrino sector, where the residual CP transformations $X_{\rm atm} = SU$ and $X_{\rm sol} = U$. In this model, the first column of the PMNS matrix is fixed to be $\left(2\sqrt{\frac{6}{37}}, \sqrt{\frac{13}{74}}, \sqrt{\frac{13}{74}}\right)^T$. This model has not appeared so far in the literature. We found that this model is a powerful model for predicting lepton mixing parameters and neutrino masses. In particular, all the lepton mixing parameters and the neutrino masses are restricted to rather narrow regions in this model as in Eq. (5.37).

In summary, we have performed an exhaustive analysis of all possible breaking patterns arising from $S_4 \rtimes H_{CP}$ in a new tridirect CP approach to the minimal seesaw model with two right-handed neutrinos and have constructed a realistic flavor model along these lines. According to this approach, separate residual flavor and CP symmetries persist in the charged lepton, "atmospheric" and "solar" right-handed neutrino sectors, resulting in *three* symmetry sectors rather than the usual two of the semidirect CP approach. Following the tridirect CP approach, we have found 26 kinds of independent phenomenologically interesting mixing patterns. Eight of them predict an NO neutrino mass spectrum and the other 18 predict an IO neutrino mass spectrum. For each phenomenologically interesting mixing pattern, the corresponding predictions for the PMNS matrix, the lepton mixing parameters, the neutrino masses and the effective mass in neutrinoless double beta decay were given in a model-independent way. One breaking pattern with an NO spectrum and two breaking patterns with IO spectra correspond to form dominance. We have found that the lepton mixing matrices of three kinds of breaking patterns with NO spectra and one form of the dominance breaking pattern with an IO spectrum preserve the first column of the TB mixing matrix, corresponding to the TM1 mixing matrix.

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APPENDIX A: GROUP THEORY OF S_4

S_4 is the permutation group of four objects, and it has 24 elements. In the present work, we shall adopt the same convention as in Ref. [31]. The S_4 group can be generated by three generators S, T and U with the multiplication rules

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1. \tag{A1}$$

S_4 group has 20 Abelian subgroups which contain nine Z_2 subgroups, four Z_3 subgroups, three Z_4 subgroups and four $K_4 \cong Z_2 \times Z_2$ subgroups. These Abelian subgroups can be expressed in terms of the generators S, T and U as follows.

(1) Z_2 subgroups:

$$\begin{aligned} &Z_2^{ST^2SU} = \{1, ST^2SU\}, \qquad Z_2^{TU} = \{1, TU\}, \qquad Z_2^{STSU} = \{1, STSU\}, \\ &Z_2^{T^2U} = \{1, T^2U\}, \qquad Z_2^{U} = \{1, U\}, \qquad Z_2^{SU} = \{1, SU\}, \\ &Z_2^{S} = \{1, S\}, \qquad Z_2^{T^2ST} = \{1, T^2ST\}, \qquad Z_2^{TST^2} = \{1, TST^2\}. \end{aligned} \tag{A2}$$

The former six Z_2 subgroups are conjugate to each other, and the latter three subgroups are related to each other by group conjugation as well.

(2) Z_3 subgroups:

$$\begin{aligned} &Z_3^{ST} = \{1, ST, T^2S\}, \qquad Z_3^{T} = \{1, T, T^2\}, \\ &Z_3^{STS} = \{1, STS, ST^2S\}, \qquad Z_3^{TS} = \{1, TS, ST^2\}, \end{aligned} \tag{A3}$$

which are related to each other under group conjugation.

(3) Z_4 subgroups:

$$Z_4^{TST^2U} = \{1, TST^2U, S, T^2STU\}, \quad Z_4^{ST^2U} = \{1, ST^2U, TST^2, T^2SU\}, \quad Z_4^{TSU} = \{1, TSU, T^2ST, STU\}. \tag{A4}$$

All the above Z_4 subgroups are conjugate to each other.

(4) K_4 subgroups:

$$\begin{aligned} K_4^{(S,TST^2)} &\equiv Z_2^{S} \times Z_2^{TST^2} = \{1, S, TST^2, T^2ST\}, \\ K_4^{(S,U)} &\equiv Z_2^{S} \times Z_2^{U} = \{1, S, U, SU\}, \\ K_4^{(TST^2,T^2U)} &\equiv Z_2^{TST^2} \times Z_2^{T^2U} \equiv \{1, TST^2, T^2U, ST^2SU\}, \\ K_4^{(T^2ST,TU)} &\equiv Z_2^{T^2ST} \times Z_2^{TU} = \{1, T^2ST, TU, STSU\}, \end{aligned} \tag{A5}$$

where $K_4^{(S,TST^2)}$ is a normal subgroup of S_4, and the other three K_4 subgroups are conjugate to each other.

S_4 has five irreducible representations which contain two singlet irreducible representations $\mathbf{1}$ and $\mathbf{1}'$, one two-dimensional representation $\mathbf{2}$ and two three-dimensional irreducible representations $\mathbf{3}$ and $\mathbf{3}'$. In this work, we choose the same basis as that of Ref. [31], i.e., the representation matrix of the generator T is diagonal. The representation matrices for the three generators are listed in Table XII. Moreover, the Kronecker products of two irreducible representations of the S_4 group are

$$\begin{aligned}
&\mathbf{1}\otimes\mathbf{R}=\mathbf{R}, \quad \mathbf{1}'\otimes\mathbf{1}'=\mathbf{1}, \quad \mathbf{1}'\otimes\mathbf{2}=\mathbf{2}, \quad \mathbf{1}'\otimes\mathbf{3}=\mathbf{3}', \quad \mathbf{1}'\otimes\mathbf{3}'=\mathbf{3},\\
&\mathbf{2}\otimes\mathbf{2}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}, \quad \mathbf{2}\otimes\mathbf{3}=\mathbf{2}\otimes\mathbf{3}'=\mathbf{3}\otimes\mathbf{3}',\\
&\mathbf{3}\otimes\mathbf{3}=\mathbf{3}'\otimes\mathbf{3}'=\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{3}', \quad \mathbf{3}\otimes\mathbf{3}'=\mathbf{1}'\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{3}',
\end{aligned} \tag{A6}$$

where $\mathbf{R}$ stands for any irreducible representation of S_4.

We now list the CG coefficients for our basis. All the CG coefficients can be written in the form $\mathbf{R_1}\otimes\mathbf{R_2}$, where $\mathbf{R_1}$ and $\mathbf{R_2}$ are two irreducible representations of S_4. We shall use α_i to denote the elements of the first representation and β_i stands for the elements of the second representation of the tensor product. For the product of the singlet $\mathbf{1}'$ with a doublet or a triplet, we have

$\mathbf{1}'\otimes\mathbf{2}$	$\mathbf{1}'\otimes\mathbf{3}=\mathbf{3}'$	$\mathbf{1}'\otimes\mathbf{3}'=\mathbf{3}$
$\mathbf{2}\sim\begin{pmatrix}\alpha\beta_1\\-\alpha\beta_2\end{pmatrix}$	$\mathbf{3}'\sim\begin{pmatrix}\alpha\beta_1\\\alpha\beta_2\\\alpha\beta_3\end{pmatrix}$	$\mathbf{3}\sim\begin{pmatrix}\alpha\beta_1\\\alpha\beta_2\\\alpha\beta_3\end{pmatrix}$

The CG coefficients for the products involving the doublet representation $\mathbf{2}$ are found to be

$\mathbf{2}\otimes\mathbf{2}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}$	$\mathbf{2}\otimes\mathbf{3}=\mathbf{3}\oplus\mathbf{3}'$	$\mathbf{2}\otimes\mathbf{3}'=\mathbf{3}\oplus\mathbf{3}'$
$\mathbf{1}\sim\alpha_1\beta_2+\alpha_2\beta_1$ $\mathbf{1}'\sim\alpha_1\beta_2-\alpha_2\beta_1$	$\mathbf{3}\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3\\\alpha_1\beta_3+\alpha_2\beta_1\\\alpha_1\beta_1+\alpha_2\beta_2\end{pmatrix}$	$\mathbf{3}\sim\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_3\\\alpha_1\beta_3-\alpha_2\beta_1\\\alpha_1\beta_1-\alpha_2\beta_2\end{pmatrix}$
$\mathbf{2}\sim\begin{pmatrix}\alpha_2\beta_2\\\alpha_1\beta_1\end{pmatrix}$	$\mathbf{3}'\sim\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_3\\\alpha_1\beta_3-\alpha_2\beta_1\\\alpha_1\beta_1-\alpha_2\beta_2\end{pmatrix}$	$\mathbf{3}'\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3\\\alpha_1\beta_3+\alpha_2\beta_1\\\alpha_1\beta_1+\alpha_2\beta_2\end{pmatrix}$

Finally, for the products among the triplet representations $\mathbf{3}$ and $\mathbf{3}'$, we have

$\mathbf{3}\otimes\mathbf{3}=\mathbf{3}'\otimes\mathbf{3}'=\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{3}'$	$\mathbf{3}\otimes\mathbf{3}'=\mathbf{1}'\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{3}'$
$\mathbf{1}\sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2$	$\mathbf{1}'\sim\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2$
$\mathbf{2}\sim\begin{pmatrix}\alpha_2\beta_2+\alpha_1\beta_3+\alpha_3\beta_1\\\alpha_3\beta_3+\alpha_1\beta_2+\alpha_2\beta_1\end{pmatrix}$	$\mathbf{2}\sim\begin{pmatrix}\alpha_2\beta_2+\alpha_1\beta_3+\alpha_3\beta_1\\-(\alpha_3\beta_3+\alpha_1\beta_2+\alpha_2\beta_1)\end{pmatrix}$
$\mathbf{3}\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\\alpha_1\beta_2-\alpha_2\beta_1\\\alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}$	$\mathbf{3}\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\\2\alpha_2\beta_2-\alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}$
$\mathbf{3}'\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\\2\alpha_2\beta_2-\alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}$	$\mathbf{3}'\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\\alpha_1\beta_2-\alpha_2\beta_1\\\alpha_3\beta_1-\alpha_1\beta_3\end{pmatrix}$

TABLE XII. The representation matrices of the generators S, T and U for the five irreducible representations of S_4 in the chosen basis, where $\omega = e^{2\pi i/3}$.

	S	T	U
1, **1′**	1	1	± 1
2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3, **3′**	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

APPENDIX B: OTHER MIXING PATTERNS WITH NO

$(\mathcal{N}_6)$ $(G_l, G_{\mathrm{atm}}, G_{\mathrm{sol}}) = (Z_4^{TSU}, Z_3^T, Z_2^{SU})$, $X_{\mathrm{atm}} = \{1, T, T^2\}$, $X_{\mathrm{sol}} = \{U, S\}$

Here the diagonalization matrix of the charged lepton mass matrix U_l is given in Eq. (3.3). Here only residual CP transformations $X_{\mathrm{sol}} = \{U, S\}$ can accommodate the present experimental data on lepton mixing. In this kind of breaking pattern the VEVs of the flavon fields ϕ_{atm} and ϕ_{sol} are

$$\langle \phi_{\mathrm{atm}} \rangle = v_{\phi_a}(1,0,0)^T, \qquad \langle \phi_{\mathrm{sol}} \rangle = v_{\phi_s}(1, 1+ix, 1-ix)^T. \tag{B1}$$

We can straightforwardly read out the resulting neutrino matrix from Eq. (2.8). The neutrino mass matrix m_ν has an eigenvalue of 0, and the corresponding eigenvector is $(0, 1+ix, -1+ix)^T$. It is convenient to first perform a unitary transformation $U_{\nu 1}$ with

$$U_{\nu 1} = \begin{pmatrix} 0 & i\sqrt{\frac{2(x^2+1)}{5x^2-2\sqrt{3}x+3}} & \frac{1-\sqrt{3}x}{\sqrt{5x^2-2\sqrt{3}x+3}} \\ \frac{ix+1}{\sqrt{2(x^2+1)}} & \frac{(i-x)(\sqrt{3}x-1)}{\sqrt{2(x^2+1)(5x^2-2\sqrt{3}x+3)}} & \frac{ix+1}{\sqrt{5x^2-2\sqrt{3}x+3}} \\ \frac{-1+ix}{\sqrt{2(x^2+1)}} & \frac{(i+x)(\sqrt{3}x-1)}{\sqrt{2(x^2+1)(5x^2-2\sqrt{3}x+3)}} & \frac{1-ix}{\sqrt{5x^2-2\sqrt{3}x+3}} \end{pmatrix}. \tag{B2}$$

Then the neutrino mass matrix m'_ν is a block-diagonal matrix and it can be diagonalized by a unitary matrix $U_{\nu 2}$ in the (2,3) sector. The nonzero parameters y, z and w in the m'_ν are given by

$$\begin{aligned} y &= \frac{2(x^2+1)(m_a + 3x^2 m_s e^{i\eta})}{5x^2-2\sqrt{3}x+3}, \\ z &= i\frac{\sqrt{2(x^2+1)}((1-\sqrt{3}x)m_a + \sqrt{3}x(2x^2-\sqrt{3}x+3)m_s e^{i\eta})}{5x^2-2\sqrt{3}x+3}, \\ w &= \frac{(\sqrt{3}x-1)^2 m_a + (4x^4 - 4\sqrt{3}x^3 + 15x^2 - 6\sqrt{3}x + 9)m_s e^{i\eta}}{5x^2-2\sqrt{3}x+3}. \end{aligned} \tag{B3}$$

Then the lepton mixing matrix is determined to be

$$U = \frac{1}{2\sqrt{3}} \begin{pmatrix} \frac{\sqrt{2}(x+\sqrt{3})}{\sqrt{x^2+1}} & -\sqrt{\frac{10x^2-4\sqrt{3}x+6}{x^2+1}}\cos\theta & -\sqrt{\frac{10x^2-4\sqrt{3}x+6}{x^2+1}}e^{i\psi}\sin\theta \\ \sqrt{\frac{5x^2-2\sqrt{3}x+3}{x^2+1}} & \frac{(x+\sqrt{3})\cos\theta}{\sqrt{x^2+1}} - \sqrt{6}e^{-i\psi}\sin\theta & \sqrt{6}\cos\theta + \frac{(\sqrt{3}+x)e^{i\psi}\sin\theta}{\sqrt{x^2+1}} \\ \sqrt{\frac{5x^2-2\sqrt{3}x+3}{x^2+1}} & \frac{(x+\sqrt{3})\cos\theta}{\sqrt{x^2+1}} + \sqrt{6}e^{-i\psi}\sin\theta & -\sqrt{6}\cos\theta + \frac{(\sqrt{3}+x)e^{i\psi}\sin\theta}{\sqrt{x^2+1}} \end{pmatrix}. \tag{B4}$$

One can straightforwardly extract the lepton mixing angles and CP phases as follows:

$$\begin{aligned}
\sin^2\theta_{13} &= \frac{(5x^2-2\sqrt{3}x+3)\sin^2\theta}{6(x^2+1)},\\
\sin^2\theta_{12} &= \frac{(5x^2-2\sqrt{3}x+3)\cos^2\theta}{6(x^2+1)-(5x^2-2\sqrt{3}x+3)\sin^2\theta},\\
\sin^2\theta_{23} &= \frac{1}{2}+\frac{(\sqrt{3}x+3)\sqrt{x^2+1}\sin 2\theta\cos\psi}{\sqrt{2}(6(x^2+1)-(5x^2-2\sqrt{3}x+3)\sin^2\theta)},\\
J_{CP} &= -\frac{(5\sqrt{3}x^3+9x^2-3\sqrt{3}x+9)\sin 2\theta\sin\psi}{72\sqrt{2}(x^2+1)^{3/2}},\\
I_1 &= \frac{(5x^2-2\sqrt{3}x+3)^2\sin^2 2\theta\sin(\rho-\sigma)}{144(x^2+1)^2}.
\end{aligned} \tag{B5}$$

A sum rule between the solar mixing angle θ_{12} and the reactor mixing angle θ_{13} is satisfied,

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{1}{6}+\frac{1+\sqrt{3}x}{3(1+x^2)}. \tag{B6}$$

For a fixed value of x, the mixing angles θ_{13} and θ_{12} are strongly correlated with each other. On the other hand, the 3σ ranges of θ_{13} and θ_{12} [1] will restrict the possible value of the input parameter x ($0.310 \leq x \leq 0.925$). Furthermore, we can derive the following sum rule among the Dirac CP phase δ_{CP} and mixing angles:

$$\cos\delta_{CP} = \frac{\cos 2\theta_{23}((9+2\sqrt{3}x+7x^2)\sin^2\theta_{13}-3+2\sqrt{3}x-5x^2)}{2\sin\theta_{13}\sin 2\theta_{23}\sqrt{(\sqrt{3}+x)^2(3(1+x^2)\cos 2\theta_{13}+2x(x-\sqrt{3}))}}. \tag{B7}$$

We note that a maximal θ_{23} leads to a maximal Dirac CP phase δ_{CP}. It is easy to check that the neutrino mass matrix m_ν has the symmetry property

$$m_\nu(x,r,-\eta) = P_{132}^T m_\nu^*(x,r,\eta)P_{132}. \tag{B8}$$

It implies that the atmospheric angle changes from θ_{23} to $\pi/2-\theta_{23}$, the Dirac phase δ_{CP} becomes $\pi-\delta_{CP}$, the Majorana CP phase β will become negative and the other output parameters are kept intact under a change of the sign of the parameter η. For fixed values of x and η, all the mixing angles, CP phases and neutrino masses are fully determined by m_a and r. As an example, we shall give the predictions for $x=\frac{1}{3}$ and $\eta=-\frac{4\pi}{5}$. For this case, the fixed column of the PMNS matrix is $\frac{1}{2\sqrt{15}}(3\sqrt{3}+1, \sqrt{16-3\sqrt{3}}, \sqrt{16-3\sqrt{3}})^T$. Furthermore, we shall perform a conventional χ^2 analysis, and the best-fit values of the input and output parameters are

$$\begin{aligned}
&m_a = 11.910~\text{meV}, \quad r=1.372, \quad \chi^2_{\text{min}}=8.753, \quad \sin^2\theta_{13}=0.0227,\\
&\sin^2\theta_{12}=0.345, \quad \sin^2\theta_{23}=0.557, \quad \delta_{CP}/\pi=-0.415, \quad \beta/\pi=0.215,\\
&m_1=0, \quad m_2=8.606~\text{meV}, \quad m_3=50.238~\text{meV}, \quad m_{ee}=2.720~\text{meV}.
\end{aligned} \tag{B9}$$

By comprehensively scanning over the parameter space of x, η and r, if we require the three mixing angles and mass ratio m_2^2/m_3^2 to be in their 3σ regions [1], we find that all three input parameters are restricted to the narrow intervals $0.311 \leq x \leq 0.381$, $0.730\pi \leq |\eta| \leq \pi$ and $1.270 \leq r \leq 1.487$. Furthermore, θ_{12} is found to lie in a narrow interval around its 3σ upper bound $0.334 \leq \sin^2\theta_{12} \leq 0.350$, and θ_{23} can only take values in the range [0.425, 0.575]. The predictions for the two CP phases δ_{CP} and β are $-0.611\pi \leq \delta_{CP} \leq -0.389\pi$ and $-0.281\pi \leq \beta \leq 0.281\pi$, respectively.

($\mathcal{N}_7$) $(G_l, G_{\text{atm}}, G_{\text{sol}}) = (K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$, $X_{\text{atm}} = \{1, T, T^2\}$

(i) $X_{\text{sol}} = \{1, SU\}$

The original symmetry $S_4 \rtimes H_{CP}$ is broken into $K_4^{(S,TST^2)}$ in the charged lepton sector; consequently the diagonalization matrix of the Hermitian combination of the charged lepton mass matrix $m_l^\dagger m_l$ is given in Eq. (3.3). The atmospheric alignment is determined to be along the direction $(1,0,0)^T$. Two types of residual CP

transformations $X_{\rm sol}=\{1,SU\}$ and $X_{\rm sol}=\{S,U\}$ are compatible with the residual flavor $G_{\rm sol}=Z_2^{SU}$, and the vacuum alignments $\langle\phi_{\rm sol}\rangle$ invariant under the residual flavor and CP symmetries are proportional to $(1,x,2-x)^T$ and $(1,1+ix,1-ix)^T$, respectively. The neutrino mass matrix can be obtained by applying the general formula Eq. (2.8). It is easy to check that the column vector $(0,x,x-2)^T$ is an eigenvector of the neutrino mass matrix and the corresponding eigenvalue is 0. Before diagonalizing the neutrino mass matrix m_ν, it is useful to perform a unitary transformation $U_{\nu1}$, and this unitary transformation will make m'_ν a block-diagonal matrix. Here the unitary matrix $U_{\nu1}$ takes the following form:

$$U_{\nu1}=\begin{pmatrix} 0 & -\sqrt{\frac{x^2-2x+2}{x^2-2x+4}} & -\frac{\sqrt{2}}{\sqrt{x^2-2x+4}} \\ \frac{x}{\sqrt{2(x^2-2x+2)}} & \frac{x-2}{\sqrt{(x^2-2x+2)(x^2-2x+4)}} & \frac{2-x}{\sqrt{2(x^2-2x+4)}} \\ \frac{x-2}{\sqrt{2(x^2-2x+2)}} & -\frac{x}{\sqrt{(x^2-2x+2)(x^2-2x+4)}} & \frac{x}{\sqrt{2(x^2-2x+4)}} \end{pmatrix}. \tag{B10}$$

Then the nonzero parameters y, z and w of m'_ν are given by

$$\begin{aligned} y&=\frac{(x^2-2x+2)(m_a+9m_se^{i\eta})}{x^2-2x+4},\\ z&=\frac{\sqrt{2(x^2-2x+2)}(m_a-3(x-1)^2m_se^{i\eta})}{x^2-2x+4},\\ w&=\frac{2(m_a+(x-1)^4m_se^{i\eta})}{x^2-2x+4}. \end{aligned} \tag{B11}$$

Then we need to put m'_ν into diagonal form with real non-negative masses, which can be done exactly by using the standard procedure shown in Sec. II, i.e., $U_{\nu2}^Tm'_\nu U_{\nu2}=\mathrm{diag}(0,m_2,m_3)$. Hence the lepton mixing matrix is determined to be

$$U=\begin{pmatrix} \frac{\sqrt{2}(x-1)}{\sqrt{3(x^2-2x+2)}} & -\sqrt{\frac{x^2-2x+4}{3(x^2-2x+2)}}\cos\theta & -\sqrt{\frac{x^2-2x+4}{3(x^2-2x+2)}}e^{i\psi}\sin\theta \\ \sqrt{\frac{x^2-2x+4}{6(x^2-2x+2)}} & \frac{(x-1)\cos\theta}{\sqrt{3(x^2-2x+2)}}+\frac{ie^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{(x-1)e^{i\psi}\sin\theta}{\sqrt{3(x^2-2x+2)}}-\frac{i\cos\theta}{\sqrt{2}} \\ \sqrt{\frac{x^2-2x+4}{6(x^2-2x+2)}} & \frac{(x-1)\cos\theta}{\sqrt{3(x^2-2x+2)}}-\frac{ie^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{(x-1)e^{i\psi}\sin\theta}{\sqrt{3(x^2-2x+2)}}+\frac{i\cos\theta}{\sqrt{2}} \end{pmatrix}. \tag{B12}$$

The lepton mixing angles and CP invariants can be read out as

$$\begin{aligned} \sin^2\theta_{13}&=\frac{(x^2-2x+4)\sin^2\theta}{3(x^2-2x+2)},\\ \sin^2\theta_{12}&=1-\frac{2(x-1)^2}{3(x^2-2x+2)-(x^2-2x+4)\sin^2\theta},\\ \sin^2\theta_{23}&=\frac{1}{2}-\frac{\sqrt{6}(x-1)\sqrt{x^2-2x+2}\sin 2\theta\sin\psi}{2(3(x^2-2x+2)-(x^2-2x+4)\sin^2\theta)},\\ J_{CP}&=\frac{(x^3-3x^2+6x-4)\sin 2\theta\cos\psi}{6\sqrt{6}(x^2-2x+2)^{3/2}},\\ I_1&=\frac{(x^2-2x+4)^2\sin^2 2\theta\sin(\rho-\sigma)}{36(x^2-2x+2)^2}. \end{aligned} \tag{B13}$$

As a consequence, the sum rules among the Dirac CP phase δ_{CP} and the mixing angles are as follows:

TABLE XIII. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$ and $X_{\rm sol} = \{1, SU\}$. Here we choose many benchmark values for the parameters x and η. The fixed column of the PMNS mixing matrix is $\left(3\sqrt{\frac{3}{41}}, \sqrt{\frac{7}{41}}, \sqrt{\frac{7}{41}}\right)^T$ for both $x = 10$ and $x = -8$. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 10, -8)^T$	10	0	8.682	0.0350	22.926	0.0207	0.328	0.5	0.5	1	8.618	50.223	3.806
		$\frac{\pi}{6}$	8.684	0.0350	23.417	0.0205	0.328	0.506	0.491	−0.833	8.621	50.220	3.713
		$-\frac{\pi}{6}$	8.680	0.0350	27.567	0.0205	0.328	0.494	0.509	0.833	8.618	50.224	3.711
$(1, -8, 10)^T$	−8	0	8.682	0.035	22.926	0.0207	0.328	0.5	−0.5	1	8.618	50.223	3.806
		$\frac{\pi}{6}$	8.680	0.0350	27.567	0.0205	0.328	0.494	−0.509	−0.833	8.618	50.224	3.711
		$-\frac{\pi}{6}$	8.684	0.0350	23.417	0.0205	0.328	0.506	−0.491	0.833	8.621	50.220	3.713

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{2(1-x)^2}{3(x^2-2x+2)},$$
$$\cos\delta_{CP} = \frac{\cot 2\theta_{23}(3x(x-2) - (5x^2-10x+8)\cos 2\theta_{13})}{4\sin\theta_{13}\sqrt{(1-x)^2(2+2x-x^2+3(x^2-2x+2)\cos 2\theta_{13})}}. \tag{B14}$$

For a given value of x, the possible range of $\sin^2\theta_{12}$ can be obtained from the above correlations by varying θ_{13} over its 3σ range and we also can obtain the prediction for $\cos\delta_{CP}$ from the 3σ ranges of the mixing angles θ_{13} and θ_{23}. For fixed x and η, all mixing parameters and neutrino masses depend on two input parameters m_a and r. The results of the χ^2 analysis for some benchmark values of x and η are reported in Table XIII. From Table XIII we find that the results for $\eta = 0$ are viable for both $x = -8$ and 10. Furthermore, a maximal atmospheric mixing angle, maximal Dirac CP phase and trivial Majorana CP phase are obtained for $\eta = 0$.

The admissible ranges of x, r and η can be obtained from the requirement that the three mixing angles and mass ratio m_2^2/m_3^2 are in the experimentally preferred 3σ ranges [1]. When all three mixing angles and the mass ratio m_2^2/m_3^2 lie in their 3σ ranges, we find that the allowed regions of the input parameters x, η and r are in the intervals $[-8.094, -6.351] \cup [8.351, 10.094]$, $[-\pi, \pi]$ and $[0.0324, 0.0550]$, respectively. Furthermore, the mixing angles $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ are predicted to be in the ranges $[0.326, 0.330]$ and $[0.486, 0.514]$, respectively. The Dirac CP phase δ_{CP} is found to lie in the rather narrow region around its maximal value, i.e., $|\delta_{CP}| \in [0.481\pi, 0.519\pi]$. While the Majorana CP phase β can take any value from $-\pi$ to π.

(ii) $X_{\rm sol} = \{U, S\}$

Analogous to previous cases, it is easy to obtain the light neutrino mass matrix m_ν from Eq. (2.8). Subsequently a unitary transformation $U_{\nu 1}$ is performed on the light neutrino fields, and then $m'_\nu = U_{\nu 1}^T m_\nu U_{\nu 1}$ becomes a block-diagonal matrix. The unitary transformation matrix $U_{\nu 1}$ can take the following form:

$$U_{\nu 1} = \begin{pmatrix} 0 & -\frac{i(x^2+1)}{\sqrt{(x^2+1)(x^2+3)}} & -\frac{i\sqrt{2}}{\sqrt{x^2+3}} \\ \frac{ix+1}{\sqrt{2(x^2+1)}} & \frac{-i+x}{\sqrt{(x^2+1)(x^2+3)}} & -\frac{-i+x}{\sqrt{2(x^2+3)}} \\ \frac{i(i+x)}{\sqrt{2(x^2+1)}} & -\frac{i+x}{\sqrt{(x^2+1)(x^2+3)}} & \frac{i+x}{\sqrt{2(x^2+3)}} \end{pmatrix}. \tag{B15}$$

The parameters y, z and w in the neutrino mass matrix m'_ν are

$$y = -\frac{(x^2+1)(m_a + 9m_s e^{i\eta})}{x^2+3}, \quad z = \frac{\sqrt{2(x^2+1)}(-m_a + 3x^2 m_s e^{i\eta})}{x^2+3}, \quad w = -\frac{2(m_a + x^4 m_s e^{i\eta})}{x^2+3}. \tag{B16}$$

Then the block-diagonal neutrino mass matrix m'_ν can be diagonalized by a unitary rotation matrix $U_{\nu 2}$ for the NO case given in Eq. (2.16). From the expressions of the matrices U_l in Eq. (3.3), $U_{\nu 1}$ and $U_{\nu 2}$, we find that the lepton mixing matrix is

$$U=\begin{pmatrix} \frac{\sqrt{2}x}{\sqrt{3(x^2+1)}} & -\sqrt{\frac{x^2+3}{3(x^2+1)}}\cos\theta & -\sqrt{\frac{x^2+3}{3(x^2+1)}}e^{i\psi}\sin\theta \\ \frac{x+\sqrt{3}}{\sqrt{6(x^2+1)}} & \frac{x(x+\sqrt{3})\cos\theta}{\sqrt{3(x^2+1)(x^2+3)}}+\frac{(x-\sqrt{3})e^{-i\psi}\sin\theta}{\sqrt{2(x^2+3)}} & \frac{(\sqrt{3}-x)\cos\theta}{\sqrt{2(x^2+3)}}+\frac{x(x+\sqrt{3})e^{i\psi}\sin\theta}{\sqrt{3(x^2+1)(x^2+3)}} \\ \frac{x-\sqrt{3}}{\sqrt{6(x^2+1)}} & \frac{x(x-\sqrt{3})\cos\theta}{\sqrt{3(x^2+1)(x^2+3)}}-\frac{(x+\sqrt{3})e^{-i\psi}\sin\theta}{\sqrt{2(x^2+3)}} & \frac{(x+\sqrt{3})\cos\theta}{\sqrt{2(x^2+3)}}+\frac{x(x-\sqrt{3})e^{i\psi}\sin\theta}{\sqrt{3(x^2+1)(x^2+3)}} \end{pmatrix}. \tag{B17}$$

The expressions for the lepton mixing angles are as follows:

$$\sin^2\theta_{13}=\frac{(x^2+3)\sin^2\theta}{3(x^2+1)},\qquad \sin^2\theta_{12}=1-\frac{2x^2}{2x^2+(x^2+3)\cos^2\theta},$$
$$\sin^2\theta_{23}=\frac{1}{2}-\frac{\sqrt{3}x}{x^2+3}+\frac{8\sqrt{3}x^3\sin^2\theta-\sqrt{6}x(x^2-3)\sqrt{x^2+1}\sin 2\theta\cos\psi}{2(x^2+3)(2x^2+(x^2+3)\cos^2\theta)}. \tag{B18}$$

These give a sum rule between the mixing angles θ_{12} and θ_{13} with

$$\cos^2\theta_{12}\cos^2\theta_{13}=\frac{2x^2}{3(1+x^2)}. \tag{B19}$$

On the one hand, for a fixed value of x, the possible values of θ_{12} will always be limited to a narrow range by varying the mixing angle θ_{13} over its 3σ range. On the other hand, from the 3σ ranges of the mixing angles θ_{13} and θ_{12}, we find that $x\leq -4.997$ or $x>5.392$ should be satisfied. From the PMNS matrix, we find that the two CP rephasing invariants J_{CP} and I_1 are predicted to be

$$J_{CP}=\frac{x(x^2-3)\sin 2\theta\sin\psi}{6\sqrt{6}(x^2+1)^{3/2}},\qquad I_1=\frac{(x^2+3)^2\sin^2 2\theta\sin(\rho-\sigma)}{36(x^2+1)^2}. \tag{B20}$$

We can derive the following sum rule among the Dirac CP phase δ_{CP} and mixing angles:

$$\cos\delta_{CP}=\frac{2\cos^2\theta_{13}(2\sqrt{3}x-3\cos 2\theta_{23})+x^2(3-5\cos 2\theta_{13})\cos 2\theta_{23}}{4\sin\theta_{13}\sin 2\theta_{23}\sqrt{x^2(3-x^2+3(1+x^2)\cos 2\theta_{13})}}. \tag{B21}$$

For a given value of x, the possible range of $\cos\delta_{CP}$ can be obtained from the above sum rule by varying θ_{13} and θ_{23} over their 3σ regions. Detailed numerical analyses show that all three mixing angles and the mass ratio m_2^2/m_3^2 can simultaneously lie in their respective 3σ ranges for input parameters $|x|$, η and r lying in the ranges [7.347, 9.104], $[-\pi,\pi]$ and [0.0324, 0.0549], respectively. Then θ_{12} is found to lie in the narrow interval $0.326\leq\sin^2\theta_{12}\leq 0.330$, the atmospheric mixing angle is constrained to lie in the interval $0.485\leq\sin^2\theta_{23}\leq 0.515$ and $|\delta_{CP}|$ is predicted to be in the range $[0, 0.0190\pi]\cup[0.981\pi,\pi]$. Any value between $-\pi$ and π is permitted for the Majorana CP phase β. Here we find that the Dirac CP phase is approximately trivial. Hence this breaking pattern would be ruled out if the signal of maximal δ_{CP} is confirmed by future neutrino facilities. In Table XIV we present the predictions for the mixing angles, CP-violating phases, light neutrino masses and the effective mass in neutrinoless double beta decay for some benchmark values of the parameters x and η. We find that the results of $\eta=0$ are viable. This is useful in model building. However, $\eta=0$ leads to a trivial Dirac CP phase and Majorana CP phase. The reason is that the parameters y, z and w in Eq. (B16) are all real. From the expressions of the parameters ψ, ρ and σ given in Eq. (2.19), we find that the three parameters can only take values of 0 or π. Then up to the diagonal phase matrix P_ν with entries ±1 or $\pm i$, it is easy to check that the PMNS matrix in Eq. (B17) is a real matrix. This mixing matrix gives a trivial Dirac CP phase and Majorana CP phase.

($\mathcal{N}_8$) $(G_l,G_{\rm atm},G_{\rm sol})=(K_4^{(S,TST^2)},Z_2^U,Z_2^{TU})$, $X_{\rm atm}=\{1,U\}$, $X_{\rm sol}=\{U,T\}$

For this breaking pattern, only the residual CP transformation $X_{\rm sol}=\{U,T\}$ is viable, and the VEV of the flavon $\phi_{\rm sol}$ is proportional to $(1,\omega x,\omega^2 x)^T$. It is easy to check that the neutrino mass matrix has an eigenvalue of 0 with the eigenvector $(x,1,1)^T$. The neutrino mass matrix m_ν can be block diagonalized by the unitary matrix $U_{\nu1}$, where $U_{\nu1}$ is

TABLE XIV. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$ and $X_{\rm sol} = \{U, S\}$. Here we choose many benchmark values for the parameters x and η. The first column of PMNS matrix are fixed to be $\left(\frac{16}{\sqrt{390}}, \frac{\sqrt{3}\pm 8}{\sqrt{390}}, \frac{8\mp\sqrt{3}}{\sqrt{390}}\right)^T$ and $\left(\frac{6\sqrt{3}}{2\sqrt{41}}, \frac{3\sqrt{3}\pm 1}{2\sqrt{41}}, \frac{3\sqrt{3}\mp 1}{2\sqrt{41}}\right)^T$ for $x = \pm 8$ and $x = \pm 9$ respectively. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 1+8i, 1-8i)^T$	8	π	8.639	0.0445	27.361	0.0208	0.330	0.489	0	1	8.582	50.268	1.727
		$\pm\frac{\pi}{2}$	8.662	0.0443	18.077	0.0234	0.328	0.502	$\mp$0.0178	$\pm$0.501	8.594	50.253	3.108
		$\pm\frac{2\pi}{3}$	8.658	0.0443	18.400	0.0221	0.329	0.496	$\mp$0.0158	$\pm$0.667	8.595	50.251	2.516
		$\pm\frac{3\pi}{4}$	8.652	0.0444	21.067	0.0215	0.329	0.493	$\mp$0.0131	$\pm$0.751	8.592	50.256	2.224
		$\pm\frac{2\pi}{5}$	8.657	0.0443	22.196	0.0243	0.327	0.506	$\mp$0.0166	$\pm$0.401	8.585	50.264	3.421
		$\pm\frac{3\pi}{5}$	8.661	0.0443	17.255	0.0226	0.328	0.498	$\mp$0.0172	$\pm$0.601	8.596	50.249	2.758
		$\pm\frac{4\pi}{5}$	8.648	0.0444	22.923	0.0213	0.329	0.492	$\mp$0.0109	$\pm$0.800	8.589	50.259	2.065
		$\pm\frac{5\pi}{6}$	8.645	0.0444	24.117	0.0211	0.329	0.491	$\mp$0.00932	$\pm$0.834	8.587	50.262	1.970
$(1, 1+9i, 1-9i)^T$	9	0	8.686	0.035	18.468	0.0207	0.328	0.513	0	0	8.623	50.218	3.807
		$\pm\frac{\pi}{6}$	8.686	0.035	21.429	0.0205	0.328	0.512	$\mp$0.00861	$\pm$0.167	8.623	50.217	3.713
$(1, 1-8i, 1+8i)^T$	−8	π	8.644	0.0444	19.680	0.0208	0.330	0.511	1	1	8.587	50.262	1.729
		$\pm\frac{\pi}{2}$	8.659	0.0443	19.667	0.0234	0.328	0.498	$\pm$0.982	$\pm$0.501	8.591	50.257	3.107
		$\pm\frac{2\pi}{3}$	8.659	0.0443	15.344	0.0221	0.329	0.504	$\pm$0.984	$\pm$0.667	8.597	50.249	2.517
		$\pm\frac{3\pi}{4}$	8.654	0.0444	16.091	0.0215	0.329	0.507	$\pm$0.987	$\pm$0.751	8.595	50.252	2.225
		$\pm\frac{2\pi}{5}$	8.652	0.0443	26.655	0.0243	0.327	0.494	$\pm$0.983	$\pm$0.401	8.580	50.271	3.419
		$\pm\frac{3\pi}{5}$	8.661	0.0443	15.973	0.0226	0.328	0.502	$\pm$0.983	$\pm$0.601	8.596	50.250	2.758
		$\pm\frac{4\pi}{5}$	8.651	0.0444	17.004	0.0213	0.329	0.508	$\pm$0.989	$\pm$0.800	8.593	50.254	2.066
		$\pm\frac{5\pi}{6}$	8.649	0.0444	17.671	0.0211	0.329	0.509	$\pm$0.991	$\pm$0.834	8.591	50.256	1.971
$(1, 1-9i, 1+9i)^T$	−9	0	8.676	0.035	28.194	0.0207	0.328	0.487	1	0	8.613	50.230	3.804
		$\pm\frac{\pi}{6}$	8.677	0.035	30.042	0.0205	0.328	0.488	$\pm$0.991	$\pm$0.167	8.614	50.228	3.710

$$U_{\nu 1} = \begin{pmatrix} -\frac{x}{\sqrt{2+x^2}} & 0 & -\frac{\sqrt{2}}{\sqrt{2+x^2}} \\ -\frac{1}{\sqrt{2+x^2}} & -\frac{1}{\sqrt{2}} & \frac{x}{\sqrt{2(2+x^2)}} \\ -\frac{1}{\sqrt{2+x^2}} & \frac{1}{\sqrt{2}} & \frac{x}{\sqrt{2(2+x^2)}} \end{pmatrix}. \tag{B22}$$

Then the three nonzero parameters y, z and w of m'_ν are

$$y = 2m_a - \frac{3}{2}x^2 m_s e^{i\eta}, \qquad z = -\frac{1}{2}ix\sqrt{3(2+x^2)}m_s e^{i\eta}, \qquad w = \frac{1}{2}(2+x^2)m_s e^{i\eta}. \tag{B23}$$

The neutrino mass matrix m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$ which is shown in Eq. (2.16). Then the lepton mixing matrix is determined to be

$$U = \begin{pmatrix} -\frac{2+x}{\sqrt{3(2+x^2)}} & -\frac{\sqrt{2}(x-1)e^{-i\psi}\sin\theta}{\sqrt{3(2+x^2)}} & \frac{\sqrt{2}(1-x)\cos\theta}{\sqrt{3(2+x^2)}} \\ \frac{1-x}{\sqrt{3(2+x^2)}} & \frac{i\cos\theta}{\sqrt{2}} + \frac{(2+x)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} & \frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} - \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} \\ \frac{1-x}{\sqrt{3(2+x^2)}} & -\frac{i\cos\theta}{\sqrt{2}} + \frac{(2+x)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} & \frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} + \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} \end{pmatrix}. \tag{B24}$$

TABLE XV. The predictions for the lepton mixing angles, CP violation phases, neutrino masses and the effective Majorana mass m_{ee} for the breaking pattern $\mathcal{N}_8$ with $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,TST^2)}, Z_2^U, Z_2^{TU})$ and $X_{\rm sol} = \{U, T\}$. Here we choose many benchmark values for the parameters x and η. The TM1 mixing matrix is reproduced in the case of $x = 4$, and the first column of the PMNS matrix are fixed to be $\left(\frac{11}{3\sqrt{19}}, \frac{5}{3\sqrt{19}}, \frac{5}{3\sqrt{19}}\right)^T$, $\left(\frac{23}{\sqrt{771}}, \frac{11}{\sqrt{771}}, \frac{11}{\sqrt{771}}\right)^T$ and $\left(\frac{29}{3\sqrt{137}}, \frac{14}{3\sqrt{137}}, \frac{14}{3\sqrt{137}}\right)^T$ for $x = 7/2$, $15/4$ and $19/5$ respectively. Notice that the lightest neutrino mass is vanishing $m_1 = 0$.

$\langle\phi_{\rm sol}\rangle/v_{\phi_s}$	x	η	m_a (meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	β/π	m_2 (meV)	m_3 (meV)	m_{ee} (meV)
$(1, 4\omega, 4\omega^2)^T$	4	$\pm\frac{\pi}{3}$	26.798	0.0335	10.716	0.0225	0.318	0.513	∓0.482	±0.401	8.628	50.212	2.694
$(1, \frac{7}{2}\omega, \frac{7}{2}\omega^2)^T$	$\frac{7}{2}$	$\pm\frac{\pi}{3}$	26.502	0.0444	30.264	0.0210	0.277	0.517	∓0.478	±0.397	8.922	49.846	2.453
		$\pm\frac{2\pi}{5}$	24.191	0.0503	19.851	0.0235	0.275	0.577	∓0.405	±0.313	8.278	50.656	2.534
$(1, \frac{15}{4}\omega, \frac{15}{4}\omega^2)^T$	$\frac{15}{4}$	$\pm\frac{\pi}{3}$	26.661	0.0384	13.801	0.0218	0.299	0.515	∓0.480	±0.399	8.766	50.044	2.582
$(1, \frac{19}{5}\omega, \frac{19}{5}\omega^2)^T$	$\frac{19}{5}$	$\pm\frac{\pi}{3}$	26.690	0.0374	12.276	0.0220	0.303	0.514	∓0.481	±0.399	8.737	50.079	2.606

One can straightforwardly extract the lepton mixing angles and the two CP rephasing invariants J_{CP} and I_1 as follows:

$$\sin^2\theta_{13} = \frac{2(1-x)^2\cos^2\theta}{3(2+x^2)}, \qquad \sin^2\theta_{12} = \frac{2(1-x)^2\sin^2\theta}{3(2+x^2) - 2(1-x)^2\cos^2\theta},$$
$$\sin^2\theta_{23} = \frac{1}{2} + \frac{\sqrt{3}(2+x)\sqrt{2+x^2}\sin 2\theta\sin\psi}{2(3(2+x^2) - 2(1-x)^2\cos^2\theta)},$$
$$J_{CP} = -\frac{(1-x)^2(2+x)\sin 2\theta\cos\psi}{6\sqrt{3}(2+x^2)^{3/2}}, \qquad I_1 = \frac{(1-x)^4\sin^2 2\theta\sin(\rho-\sigma)}{9(2+x^2)^2}. \tag{B25}$$

We see that the three mixing angles and Dirac CP phase only depend on two free parameters θ and ψ. Then the mixing parameters are strongly correlated such that the following sum rules among the mixing angles and Dirac CP phase are found to be satisfied:

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{(2+x)^2}{3(2+x^2)}, \qquad \cos\delta_{CP} = \frac{[3 + 6x - (5 + 2x + 2x^2)\cos 2\theta_{13}]\cot 2\theta_{23}}{2\sin\theta_{13}\sqrt{3(2+x)^2(2+x^2)\cos^2\theta_{13} - (2+x)^4}}. \tag{B26}$$

The former correlation implies that the solar mixing angle θ_{12} is restricted by the observed value of θ_{13} for a given x. From the sum rule among δ_{CP} and the mixing angles, we find that a maximal atmospheric mixing angle $\theta_{23} = 45°$ leads to a maximal Dirac CP phase, i.e., $\cos\delta_{CP} = 0$. For a fixed value of x, the possible values of δ_{CP} is determined by the 3σ ranges of the mixing angles θ_{13} and θ_{23}.

In order to see how well the lepton mixing angles can be described by this breaking pattern and its prediction for the CP phases, we perform a χ^2 analysis defined in Eq. (3.5) for some benchmark values of x and η. The results are listed in Table XV. Furthermore, we find that the mixing pattern with $x = 4$ is equivalent to the breaking pattern $\mathcal{N}_1$ with $X_{\rm sol} = \{1, SU\}$ and $x = -1$. In order to obtain all possible values of the mixing angles and CP phases, we consider the input parameters x, η and r as free parameters and require that all three mixing angles and the mass ratio m_2^2/m_3^2 lie in their 3σ ranges. Then we find that the allowed values of the input parameters x, $|\eta|$ and r are [3.472, 4.481], $[0.253\pi, 0.412\pi]$ and [0.0240, 0.0516], respectively. Moreover, any value of θ_{12} within its 3σ range can be achieved and θ_{23} is restricted to the range $0.450 \le \sin^2\theta_{23} \le 0.588$. The absolute values of the Dirac CP phase and Majorana CP phase are predicted to be in the ranges $[0.393\pi, 0.579\pi]$ and $[0.298\pi, 0.512\pi]$, respectively.

APPENDIX C: OTHER MIXING PATTERNS WITH IO

In this Appendix, we shall list the other possible choices for the residual symmetries G_l, $G_{\rm atm}$, $G_{\rm sol}$ and the resulting predictions for the lepton mixing parameters and neutrino masses.

($\mathcal{I}_6$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_3^T, Z_2^{TST^2}, Z_2^U)$, $X_{\rm atm} = \{SU, ST^2S, T^2, T^2STU\}$, $X_{\rm sol} = \{1, U\}$

Here the residual symmetries in the charged lepton sector, atmospheric neutrino sector and solar neutrino sector are the same as in the $\mathcal{N}_4$ case which is discussed in Sec. III. Then the light neutrino mass matrix m_ν takes the same form as in Ref. [61]. From the discussion below Eq. (2.21), the PMNS matrix of this case can be directly obtained from the PMNS matrix in $\mathcal{N}_4$:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 2i\sqrt{\frac{x^2+x+1}{5x^2+2x+2}}\cos\theta & 2i\sqrt{\frac{x^2+x+1}{5x^2+2x+2}}e^{i\psi}\sin\theta & \frac{\sqrt{6}x}{\sqrt{5x^2+2x+2}} \\ -e^{-i\psi}\sin\theta - \frac{i\sqrt{3}x\cos\theta}{\sqrt{5x^2+2x+2}} & \cos\theta - \frac{i\sqrt{3}xe^{i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} & \sqrt{\frac{2(x^2+x+1)}{5x^2+2x+2}} \\ e^{-i\psi}\sin\theta - \frac{i\sqrt{3}x\cos\theta}{\sqrt{5x^2+2x+2}} & -\cos\theta - \frac{i\sqrt{3}xe^{i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} & \sqrt{\frac{2(x^2+x+1)}{5x^2+2x+2}} \end{pmatrix}.$$

The lepton mixing angles and CP rephasing invariants can be read off as

$$\sin^2\theta_{13} = \frac{3x^2}{5x^2+2x+2}, \qquad \sin^2\theta_{12} = \sin^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = -\frac{\sqrt{3}x(x^2+x+1)\sin 2\theta\sin\psi}{2(5x^2+2x+2)^{3/2}}, \qquad I_1 = -\frac{(x^2+x+1)^2\sin^2 2\theta\sin(\rho-\sigma)}{(5x^2+2x+2)^2}. \tag{C1}$$

As in the case $\mathcal{I}_1$, θ_{23} is maximal and θ_{13} only depends on the real parameter x. For the 3σ interval $0.02068 \leq \sin^2\theta_{13} \leq 0.02463$, we have $x \in [-0.123, -0.113] \cup [0.127, 0.140]$. In order to know how well the predicted mixing patterns agree with the experimental data, we shall perform a χ^2 analysis for $x = \frac{1}{8}$. The numerical results are

$$\eta = 0.993\pi, \qquad m_a = 19.180\ \text{meV}, \qquad r = 2.890, \qquad \chi^2_{\text{min}} = 32.054, \qquad \sin^2\theta_{13} = 0.0201,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.876\pi, \qquad \beta = 0.510\pi,$$
$$m_1 = 49.377\ \text{meV}, \qquad m_2 = 50.120\ \text{meV}, \qquad m_3 = 0\ \text{meV}, \qquad m_{ee} = 36.250\ \text{meV}. \tag{C2}$$

For $x = \frac{1}{8}$, the absolute value of the third column of the PMNS matrix is fixed to be $\left(\sqrt{\frac{3}{149}}, \sqrt{\frac{73}{149}}, \sqrt{\frac{73}{149}}\right)^T$. Due to the requirement that the three mixing angles and mass ratio m_1^2/m_2^2 lie in their 3σ ranges [1], the input parameters $|\eta|$ and r are restricted to the ranges $[0.9913\pi, 0.9957\pi]$ and [2.862, 2.915], respectively. Then the values of δ_{CP} lie in the range $[-0.904\pi, -0.842\pi] \cup [-0.158\pi, -0.0962\pi] \cup [0.0666\pi, 0.106\pi] \cup [0.894\pi, 0.933\pi]$, and the allowed range of the absolute value of the Majorana phase is $[0.375\pi, 0.406\pi] \cup [0.492\pi, 0.546\pi]$. The other mixing angles can take any values in their 3σ ranges.

($\mathcal{I}_7$) $(G_l, G_{\text{atm}}, G_{\text{sol}}) = (Z_3^T, Z_2^U, Z_2^{TU})$, $X_{\text{atm}} = \{1, U\}$, $X_{\text{sol}} = \{U, T\}$

Here only the residual CP transformation $X_{\text{sol}} = \{U, T\}$ can give phenomenologically viable predictions. Then the VEV alignments of the flavons ϕ_{atm} and ϕ_{sol} are

$$\langle\phi_{\text{atm}}\rangle = v_{\phi_a}(0, 1, -1)^T, \qquad \langle\phi_{\text{sol}}\rangle = v_{\phi_s}(1, x\omega, x\omega^2)^T. \tag{C3}$$

The light neutrino mass matrix can be block diagonalized by the unitary matrix $U_{\nu 1}$, where the unitary matrix $U_{\nu 1}$ takes the following form:

$$U_{\nu 1} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{2+x^2}} & -\frac{x}{\sqrt{2+x^2}} \\ -\frac{1}{\sqrt{2}} & -\frac{x}{\sqrt{2(2+x^2)}} & -\frac{1}{\sqrt{2+x^2}} \\ \frac{1}{\sqrt{2}} & -\frac{x}{\sqrt{2(2+x^2)}} & -\frac{1}{\sqrt{2+x^2}} \end{pmatrix}. \tag{C4}$$

The three nonzero elements y, z and w are determined to be

$$y = 2m_a - \frac{3}{2}x^2 m_s e^{i\eta}, \qquad z = \frac{i}{2}x\sqrt{3(2+x^2)}m_s e^{i\eta}, \qquad w = \frac{1}{2}(2+x^2)m_s e^{i\eta}. \tag{C5}$$

Then the neutrino mass matrix m'_ν can be diagonalized by using the unitary transformation matrix $U_{\nu 2}$ which is given in Eq. (2.16). As a consequence, the PMNS matrix is

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{2e^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & \frac{2\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}x}{\sqrt{2+x^2}} \\ -\cos\theta-\frac{xe^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & e^{i\psi}\sin\theta-\frac{x\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}}{\sqrt{2+x^2}} \\ \cos\theta-\frac{xe^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & -e^{i\psi}\sin\theta-\frac{x\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}}{\sqrt{2+x^2}} \end{pmatrix}. \tag{C6}$$

Its predictions for the three mixing angles and the two CP rephasing invariants are

$$\sin^2\theta_{13}=\frac{x^2}{2+x^2},\qquad \sin^2\theta_{12}=\cos^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},$$
$$J_{CP}=-\frac{x\sin2\theta\sin\psi}{2(2+x^2)^{3/2}},\qquad I_1=-\frac{\sin^2 2\theta\sin(\rho-\sigma)}{(2+x^2)^2}. \tag{C7}$$

It predicts a maximal atmospheric mixing angle θ_{23}. The viable range of x can be obtained by varying θ_{13} over its 3σ range, i.e., $|x|\in[0.206,0.225]$. In order to see how well the lepton mixing angles can be described by this breaking pattern and its predictions for the CP phases, we shall perform a numerical analysis. When the experimentally allowed regions at the 3σ confidence level of the mixing parameters and mass ratio m_1^2/m_2^2 are considered, the viable ranges of the input parameters $|\eta|$ and r are $[0.0138\pi, 0.0201\pi]$ and [1.803, 1.831], respectively. Then the Dirac CP phase and the absolute value of the Majorana CP phase are limited to the narrow ranges $[0.0151\pi,0.0357\pi]\cup[0.964\pi,0.985\pi]$ and $[0.166\pi,0.189\pi]$, respectively.

Furthermore we perform a comprehensive numerical analysis for $x=\pm2/9$ which give relatively simple VEVs of ϕ_{sol}. From the PMNS matrix in Eq. (C6), we find that the fixed column of the PMNS matrix for $x=\pm2/9$ is $\left(\sqrt{\frac{2}{83}},\frac{9}{\sqrt{166}},\frac{9}{\sqrt{166}}\right)^T\simeq(0.155,0.699,0.699)^T$. It agrees with all measurements to date [1]. The usual χ^2 analysis results for $x=\pm2/9$ are

$$\eta=0.0162\pi,\qquad m_a=25.829\ \mathrm{meV},\qquad r=1.810,\qquad \chi^2_{\mathrm{min}}=22.509,\qquad \sin^2\theta_{13}=0.0241,$$
$$\sin^2\theta_{12}=0.310,\qquad \sin^2\theta_{23}=0.5,\qquad \delta_{CP}=0.0264\pi,\qquad \beta=0.181\pi,$$
$$m_1=49.377\ \mathrm{meV},\qquad m_2=50.120\ \mathrm{meV},\qquad m_3=0\ \mathrm{meV},\qquad m_{ee}=46.753\ \mathrm{meV}. \tag{C8}$$

($\mathcal{I}_8$) $(G_l,G_{\mathrm{atm}},G_{\mathrm{sol}})=(Z_3^T,Z_2^U,Z_2^{STSU})$, $X_{\mathrm{atm}}=\{1,U\}$, $X_{\mathrm{sol}}=\{U,STS\}$

For this combination of residual flavor symmetries, only the residual CP transformations $X_{\mathrm{sol}}=\{U,STS\}$ are viable. The vacuum alignments of the flavons ϕ_{atm} and ϕ_{sol} can be read from Table II,

$$\langle\phi_{\mathrm{atm}}\rangle=v_{\phi_a}(0,1,-1),\qquad \langle\phi_{\mathrm{sol}}\rangle=v_{\phi_s}\left(\frac{\sqrt{3}x-1}{2},1+ix,1-ix\right)^T. \tag{C9}$$

Before diagonalizing the neutrino mass matrix, we first perform a unitary transformation, where the unitary transformation matrix $U_{\nu1}$ takes the following form:

$$U_{\nu1}=\begin{pmatrix} 0 & \frac{1-\sqrt{3}x}{\sqrt{3x^2-2\sqrt{3}x+9}} & \frac{2\sqrt{2}}{\sqrt{3x^2-2\sqrt{3}x+9}} \\ -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3x^2-2\sqrt{3}x+9}} & \frac{1-\sqrt{3}x}{\sqrt{6x^2-4\sqrt{3}x+18}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3x^2-2\sqrt{3}x+9}} & \frac{1-\sqrt{3}x}{\sqrt{6x^2-4\sqrt{3}x+18}} \end{pmatrix}. \tag{C10}$$

The expressions of the parameters y, z and w are

$$y=2m_a-2x^2m_se^{i\eta},\qquad z=-ix\sqrt{\frac{9-2\sqrt{3}x+3x^2}{2}}m_se^{i\eta},\qquad w=\frac{1}{4}(9-2\sqrt{3}x+3x^2)m_se^{i\eta}. \tag{C11}$$

The unitary transformation $U_{\nu 2}$ diagonalizing the neutrino mass matrix m'_ν is of the form given in Eq. (2.16) for the IO case. Then the lepton mixing matrix is determined to be of the form

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{(\sqrt{3}x-1)e^{-i\psi}\sin\theta}{\sqrt{9-2\sqrt{3}x+3x^2}} & \frac{(\sqrt{3}x-1)\cos\theta}{\sqrt{9-2\sqrt{3}x+3x^2}} & \frac{2\sqrt{2}}{\sqrt{9-2\sqrt{3}x+3x^2}} \\ \frac{\cos\theta}{\sqrt{2}}+\frac{2e^{-i\psi}\sin\theta}{\sqrt{9-2\sqrt{3}x+3x^2}} & \frac{2\cos\theta}{\sqrt{9-2\sqrt{3}x+3x^2}}-\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1-\sqrt{3}x}{\sqrt{2(9-2\sqrt{3}x+3x^2)}} \\ -\frac{\cos\theta}{\sqrt{2}}+\frac{2e^{-i\psi}\sin\theta}{\sqrt{9-2\sqrt{3}x+3x^2}} & \frac{2\cos\theta}{\sqrt{9-2\sqrt{3}x+3x^2}}+\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1-\sqrt{3}x}{\sqrt{2(9-2\sqrt{3}x+3x^2)}} \end{pmatrix}. \tag{C12}$$

The mixing parameters extracted from the above PMNS matrix are

$$\sin^2\theta_{13}=\frac{8}{9-2\sqrt{3}x+3x^2},\qquad \sin^2\theta_{12}=\cos^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},$$
$$J_{CP}=-\frac{\sqrt{2}(1-\sqrt{3}x)^2\sin2\theta\sin\psi}{2(9-2\sqrt{3}x+3x^2)^{3/2}},\qquad I_1=-\frac{(1-\sqrt{3}x)^4\sin^22\theta\sin(\rho-\sigma)}{4(9-2\sqrt{3}x+3x^2)^2}. \tag{C13}$$

Inserting the 3σ ranges of the third column of the PMNS matrix, we find that the parameter x should vary in the interval $[-10.660,-9.699]\cup[10.854,11.815]$. As an example, we take $x=-6\sqrt{3}$. Then the fixed column of the PMNS matrix is $\frac{1}{3\sqrt{82}}(4,19,19)^T\simeq(0.147,0.699,0.699)^T$ which is not beyond the 3σ confidence level [1]. Furthermore, we perform a conventional χ^2 analysis and the numerical results are

$$\eta=0.00227\pi,\qquad m_a=45.595\ \text{meV},\qquad r=0.00645,\qquad \chi^2_{\min}=19.755,\qquad \sin^2\theta_{13}=0.0217,$$
$$\sin^2\theta_{12}=0.310,\qquad \sin^2\theta_{23}=0.5,\qquad \delta_{CP}=0.210\pi,\qquad \beta=0.716\pi,$$
$$m_1=49.377\ \text{meV},\qquad m_2=50.120\ \text{meV},\qquad m_3=0\ \text{meV},\qquad m_{ee}=26.550\ \text{meV}. \tag{C14}$$

In the case that all three input parameters x, η and r are free parameters, we find that the three mixing angles and mass ratio m_1^2/m_2^2 can lie in their 3σ ranges at the same time only when x, $|\eta|$ and r are restricted to the ranges $[-10.660,-9.699]\cup[10.854,11.815]$, $[0.00121\pi,0.00295\pi]$ and $[0.00530,\ 0.00738]$, respectively. We find that any values of θ_{12} and θ_{13} within their 3σ ranges can be achieved, and the two CP phases are predicted to be $\delta_{CP}\in[-0.827\pi,-0.681\pi]\cup[-0.319\pi,-0.173\pi]\cup[0.153\pi,0.274\pi]\cup[0.726\pi,0.847\pi]$ and $|\beta|\in[0.677\pi,0.825\pi]$.

$(\mathcal{I}_9)$ $(G_l,G_{\text{atm}},G_{\text{sol}})=(Z_3^T,Z_2^{SU},Z_2^{STSU})$, $X_{\text{atm}}=\{1,SU\}$, $X_{\text{sol}}=\{U,STS\}$

In this case, only the residual CP symmetry $X_{\text{sol}}=\{U,STS\}$ is viable. From Table II, we can read out the VEVs of the flavons ϕ_{atm} and ϕ_{sol} invariant under the residual symmetry. This neutrino mass matrix can become a block-diagonal matrix when we perform a unitary transformation $U_{\nu 1}$ with

$$U_{\nu 1}=\begin{pmatrix} 0 & -\frac{i(x(19x+6\sqrt{3})+9)}{\sqrt{3(9x^2+2\sqrt{3}x+3)(19x^2+6\sqrt{3}x+9)}} & \frac{2\sqrt{2}ix}{\sqrt{3(9x^2+2\sqrt{3}x+3)}} \\ \frac{x(4i+\sqrt{3})+3}{\sqrt{2(19x^2+6\sqrt{3}x+9)}} & \frac{2x((\sqrt{3}+4i)x+3)}{\sqrt{3(9x^2+2\sqrt{3}x+3)(19x^2+6\sqrt{3}x+9)}} & \frac{x(4i+\sqrt{3})+3}{\sqrt{6(9x^2+2\sqrt{3}x+3)}} \\ \frac{x(-4i+\sqrt{3})+3}{\sqrt{2(19x^2+6\sqrt{3}x+9)}} & \frac{2x((-\sqrt{3}+4i)x-3)}{\sqrt{3(9x^2+2\sqrt{3}x+3)(19x^2+6\sqrt{3}x+9)}} & \frac{x(4i-\sqrt{3})+3}{\sqrt{6(9x^2+2\sqrt{3}x+3)}} \end{pmatrix}. \tag{C15}$$

The parameters y, z and w take the following forms:

$$y=\frac{6(x^2+2\sqrt{3}x+3)m_a+2(16x^4+8\sqrt{3}x^3+27x^2+6\sqrt{3}x+9)m_s e^{i\eta}}{19x^2+6\sqrt{3}x+9},$$
$$z=\frac{i\sqrt{2(9x^2+2\sqrt{3}x+3)}(12(x+\sqrt{3})m_a+(-12x^3+\sqrt{3}x^2-6x+3\sqrt{3})m_s e^{i\eta})}{2(19x^2+6\sqrt{3}x+9)},$$
$$w=\frac{-48(9x^2+2\sqrt{3}x+3)m_a-3(27x^4-12\sqrt{3}x^3+6x^2-4\sqrt{3}x+3)m_s e^{i\eta}}{4(19x^2+6\sqrt{3}x+9)}. \quad (C16)$$

The neutrino mass matrix m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$. Then the PMNS matrix is

$$U=\begin{pmatrix} \sqrt{\frac{19x^2+6\sqrt{3}x+9}{3(9x^2+2\sqrt{3}x+3)}}e^{-i\psi}\sin\theta & -\sqrt{\frac{19x^2+6\sqrt{3}x+9}{3(9x^2+2\sqrt{3}x+3)}}\cos\theta & \frac{2\sqrt{2}x}{\sqrt{3(9x^2+2\sqrt{3}x+3)}} \\ \frac{\cos\theta}{\sqrt{2}}-\frac{2xe^{-i\psi}\sin\theta}{\sqrt{3(9x^2+2\sqrt{3}x+3)}} & \frac{2x\cos\theta}{\sqrt{3(9x^2+2\sqrt{3}x+3)}}+\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \sqrt{\frac{19x^2+6\sqrt{3}x+9}{6(9x^2+2\sqrt{3}x+3)}} \\ -\frac{\cos\theta}{\sqrt{2}}-\frac{2xe^{-i\psi}\sin\theta}{\sqrt{3(9x^2+2\sqrt{3}x+3)}} & \frac{2x\cos\theta}{\sqrt{3(9x^2+2\sqrt{3}x+3)}}-\frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \sqrt{\frac{19x^2+6\sqrt{3}x+9}{6(9x^2+2\sqrt{3}x+3)}} \end{pmatrix}. \quad (C17)$$

The three lepton mixing angles are predicted to be

$$\sin^2\theta_{13}=\frac{8x^2}{9+6\sqrt{3}x+27x^2},\qquad \sin^2\theta_{12}=\cos^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},$$
$$J_{CP}=\frac{x(19x^2+6\sqrt{3}x+9)\sin 2\theta\sin\psi}{3\sqrt{6}(9x^2+2\sqrt{3}x+3)^{3/2}},\qquad I_1=\frac{(19x^2+6\sqrt{3}x+9)^2\sin^2 2\theta\sin(\sigma-\rho)}{36(9x^2+2\sqrt{3}x+3)^2}. \quad (C18)$$

The atmospheric mixing angle is maximal and the reactor mixing angle only depends on the input parameter x. Inserting the 3σ ranges of the third column of the PMNS matrix, we find that the parameter x should vary in the interval $[-0.157,-0.144]\cup[0.173,0.192]$. As an example, we take $x=\sqrt{3}/10$. Then the third column of the PMNS matrix is $\frac{1}{3\sqrt{86}}(4,\sqrt{379},\sqrt{379})^T\simeq(0.144,0.700,0.700)^T$ which agrees with all measurements to date [1]. The χ^2 analysis results are

$$\eta=0.996\pi,\qquad m_a=12.428\ \text{meV},\qquad r=2.760,\qquad \chi^2_{\text{min}}=26.533,\qquad \sin^2\theta_{13}=0.0207,$$
$$\sin^2\theta_{12}=0.310,\qquad \sin^2\theta_{23}=0.5,\qquad \delta_{CP}=-0.954\pi,\qquad \beta=0.246\pi,$$
$$m_1=49.377\ \text{meV},\qquad m_2=50.120\ \text{meV},\qquad m_3=0\ \text{meV},\qquad m_{ee}=45.510\ \text{meV}. \quad (C19)$$

After calculation and analysis, we find that only when the input parameters $|\eta|$ and r lie in the ranges $[0.9954\pi,0.9976\pi]$ and $[2.450,2.472]\cup[2.755,2.763]$ respectively can the three mixing angles and mass ratio m_1^2/m_2^2 be in their 3σ ranges. Then the two CP phases δ_{CP} and β are predicted to be $\delta_{CP}\in[-0.966\pi,-0.942\pi]\cup[-0.0583\pi,-0.0343\pi]\cup[0.0150\pi,0.0260\pi]\cup[0.974\pi,0.985\pi]$ and $|\beta|\in[0.115\pi,0.127\pi]\cup[0.240\pi,0.260\pi]$.

($\mathcal{I}_{10}$) $(G_l,G_{\text{atm}},G_{\text{sol}})=(Z_4^{TSU},Z_2^S,Z_2^{TU})$, $X_{\text{atm}}=\{1,S,TST^2U,T^2STU\}$

(i) $X_{\text{sol}}=\{U,T\}$

The remnant symmetries determine that the alignments of ϕ_{atm} and ϕ_{sol} are along the directions $(1,1,1)^T$ and $(1,x\omega,x\omega^2)^T$, respectively. The general form of the neutrino mass matrix can be obtained from Eq. (2.8). This neutrino mass matrix can be diagonalized into a block-diagonal form by performing a unitary transformation $U_{\nu 1}$ on m_ν, where the unitary matrix $U_{\nu 1}$ takes the following form:

$$U_{\nu 1}=\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2+x}{\sqrt{3(2+2x+5x^2)}} & -\frac{i\sqrt{3}x}{\sqrt{2+2x+5x^2}} \\ \frac{1}{\sqrt{3}} & \frac{1-(\omega+i\sqrt{3})x}{\sqrt{3(2+2x+5x^2)}} & \frac{\omega x-1}{\sqrt{2+2x+5x^2}} \\ \frac{1}{\sqrt{3}} & \frac{1-(3\omega^2+1)x}{\sqrt{3(2+2x+5x^2)}} & \frac{1-\omega^2 x}{\sqrt{2+2x+5x^2}} \end{pmatrix}. \quad (C20)$$

Then the neutrino mass matrix m'_ν is a block-diagonal matrix with elements

$$y = 3m_a + \frac{(1-x)^2}{3} m_s e^{i\eta}, \qquad z = \frac{x-1}{3}\sqrt{2+2x+5x^2} m_s e^{i\eta}, \qquad w = \frac{2+2x+5x^2}{3} m_s e^{i\eta}. \tag{C21}$$

The diagonalization matrix of m'_ν can be written in the form of $U_{\nu 2}$ in Eq. (2.16) for the IO case. Then the PMNS matrix is determined to be

$$U = \begin{pmatrix} -\frac{i(1+2x)e^{-i\psi}\sin\theta}{\sqrt{2+2x+5x^2}} & -\frac{i(1+2x)\cos\theta}{\sqrt{2+2x+5x^2}} & \frac{1-x}{\sqrt{2+2x+5x^2}} \\ \frac{\cos\theta}{\sqrt{2}} - \frac{(1-x)e^{-i\psi} i \sin\theta}{\sqrt{2(2+2x+5x^2)}} & \frac{i(x-1)\cos\theta}{\sqrt{2(2+2x+5x^2)}} - \frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{-1-2x}{\sqrt{2(2+2x+5x^2)}} \\ \frac{\cos\theta}{\sqrt{2}} + \frac{i(1-x)e^{-i\psi}\sin\theta}{\sqrt{2(2+2x+5x^2)}} & \frac{(1-x) i \cos\theta}{\sqrt{2(2+2x+5x^2)}} - \frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1+2x}{\sqrt{2(2+2x+5x^2)}} \end{pmatrix}. \tag{C22}$$

Its predictions for the mixing angles and CP invariants are

$$\sin^2\theta_{13} = \frac{(1-x)^2}{2+2x+5x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = \frac{(x-1)(1+2x)^2 \sin 2\theta \cos\psi}{4(2+2x+5x^2)^{3/2}}, \qquad I_1 = -\frac{(1+2x)^4 \sin^2 2\theta \sin(\rho-\sigma)}{4(2+2x+5x^2)^2}. \tag{C23}$$

The atmospheric mixing angle θ_{23} is predicted to be maximal. We find that the correct value of θ_{13} can be obtained when x is restricted to the range $[0.638, 0.662] \cup [1.615, 1.699]$. If we require that all three mixing angles and the mass ratio m_1^2/m_2^2 lie in their 3σ ranges, the two CP phases δ_{CP} and $|\beta|$ are determined to take values in the intervals $[-\pi, -0.987\pi] \cup [-0.0128\pi, 0.00336\pi] \cup [0.997\pi, \pi]$ and $[0.0964\pi, 0.111\pi]$, respectively. Furthermore, we find that the viable ranges of $|\eta|$ and r are $[0.965\pi, 0.976\pi]$ and $[0.442, 0.478] \cup [1.595, 1.647]$.

For illustration, we shall give the χ^2 results for the typical value $x = 2/3$. The vacuum alignment of the flavon $\phi_{\rm sol}$ is proportional to the column vector $(1, \frac{2}{3}\omega, \frac{2}{3}\omega^2)^T$ and the third column of the PMNS matrix is $(\frac{1}{5\sqrt{2}}, \frac{7}{10}, \frac{7}{10})^T$. The χ^2 analysis results are

$$\eta = -0.969\pi, \qquad m_a = 16.794~\text{meV}, \qquad r = 1.579, \qquad \chi^2_{\rm min} = 33.640, \qquad \sin^2\theta_{13} = 0.02,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.997\pi, \qquad \beta = 0.0964\pi,$$
$$m_1 = 49.377~\text{meV}, \qquad m_2 = 50.120~\text{meV}, \qquad m_3 = 0~\text{meV}, \qquad m_{ee} = 48.137~\text{meV}. \tag{C24}$$

We see that θ_{13} is rather close to its 3σ lower limit 0.2068 [1]. Hence this example should be considered as a good leading-order approximation. The reason is that if subleading contributions are taken into account, accordance with experimental data is easily achieved.

(ii) $X_{\rm sol} = \{STS, T^2STU\}$

For the residual CP transformations $X_{\rm sol} = \{STS, T^2STU\}$ in the solar neutrino sector, the vacuum alignment of $\phi_{\rm sol}$ is fixed to be

$$\langle\phi_{\rm sol}\rangle = v_{\phi_s}(1+2ix, \omega(1-ix), \omega^2(1-ix))^T. \tag{C25}$$

After performing a $U_{\nu 1}$ transformation, m'_ν can be a block-diagonal matrix, where the unitary matrix $U_{\nu 1}$ takes the form

$$U_{\nu 1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{1-ix}{\sqrt{1+x^2}} & \frac{-x-i}{\sqrt{1+x^2}} \\ 1 & \frac{\omega - ix\omega^2}{\sqrt{1+x^2}} & \frac{-i\omega^2 - x\omega}{\sqrt{1+x^2}} \\ 1 & \frac{\omega^2 - ix\omega}{\sqrt{1+x^2}} & \frac{-x\omega^2 - i\omega}{\sqrt{1+x^2}} \end{pmatrix}. \tag{C26}$$

The nonzero elements y, z and w of the block-diagonal m'_ν are

$$y = 3m_a - 3x^2 m_s e^{i\eta}, \qquad z = -3ix\sqrt{1+x^2}m_s e^{i\eta}, \qquad w = 3(1+x^2)m_s e^{i\eta}. \tag{C27}$$

Then the neutrino mass matrix m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$ which is given in Eq. (2.16). As a consequence, the PMNS matrix can take the following form:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{\sqrt{2}e^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & -\frac{\sqrt{2}\cos\theta}{\sqrt{1+x^2}} & \frac{\sqrt{2}x}{\sqrt{1+x^2}} \\ \cos\theta - \frac{xe^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & -e^{i\psi}\sin\theta - \frac{x\cos\theta}{\sqrt{1+x^2}} & -\frac{1}{\sqrt{1+x^2}} \\ \cos\theta + \frac{xe^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & -e^{i\psi}\sin\theta + \frac{x\cos\theta}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}. \tag{C28}$$

The lepton mixing parameters are predicted to be

$$\sin^2\theta_{13} = \frac{x^2}{1+x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = \frac{x\sin 2\theta\sin\psi}{4(1+x^2)^{3/2}}, \qquad I_1 = -\frac{\sin^2 2\theta\sin(\rho-\sigma)}{4(1+x^2)^2}. \tag{C29}$$

This mixing pattern gives a maximal θ_{23}. The reactor mixing angle θ_{13} only depends on the parameter x which comes from the vacuum alignment of the flavon $\phi_{\rm sol}$. In order to obtain a value of θ_{13} allowed by experimental data, the input parameter $|x|$ must be restricted to the range [0.145, 0.159]. In order to accommodate the experimentally favored 3σ ranges [1] of the mixing angles and mass ratio m_1^2/m_2^2, we find that the allowed regions of the parameters $|\eta|$ and r are $[0.0241\pi, 0.0346\pi]$ and [0.945, 0.954], respectively. Any values of θ_{13} and θ_{23} in their 3σ ranges can be obtained. The two CP phases are predicted to be $\delta_{CP} \in [-\pi, -0.988\pi] \cup [-0.0120\pi, 0.00241\pi] \cup [0.997\pi, \pi]$ and $|\beta| \in [0.0973\pi, 0.111\pi]$. Detailed numerical analyses show that accordance with experimental data can be achieved for $x = \frac{1}{4\sqrt{3}}$ and the best-fit values of the mixing parameters and neutrino masses are

$$\eta = -0.0308\pi, \qquad m_a = 16.797\ \text{meV}, \qquad r = 0.955, \qquad \chi^2_{\rm min} = 29.075, \qquad \sin^2\theta_{13} = 0.0204,$$
$$\sin^2\theta_{12} = 0.307, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.00289\pi, \qquad \beta = -0.0973\pi,$$
$$m_1 = 49.377\ \text{meV}, \qquad m_2 = 50.120\ \text{meV}, \qquad m_3 = 0\ \text{meV}, \qquad m_{ee} = 48.108\ \text{meV}. \tag{C30}$$

Furthermore, the fixed column of the PMNS matrix is $\frac{1}{7}(1, 2\sqrt{6}, 2\sqrt{6})^T$ in the case of $x = \frac{1}{4\sqrt{3}}$.

($\mathcal{I}_{11}$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_4^{TSU}, Z_2^S, Z_2^{T^2U})$, $X_{\rm atm} = \{1, S, TST^2U, T^2STU\}$

(iii) $X_{\rm sol} = \{U, T^2\}$

In this case, both $X_{\rm sol} = \{U, T^2\}$ and $X_{\rm sol} = \{ST^2S, TST^2U\}$ are compatible with the residual flavor symmetry. For the former residual CP symmetry, the invariant vacuum of $\phi_{\rm sol}$ is

$$\langle\phi_{\rm sol}\rangle = v_{\phi_s}(1, x\omega^2, x\omega)^T. \tag{C31}$$

First, we perform a unitary transformation $U_{\nu 1}$ on the neutrino mass matrix m_ν, where $U_{\nu 1}$ takes the following form:

$$U_{\nu 1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2+x}{\sqrt{3(2+2x+5x^2)}} & \frac{i\sqrt{3}x}{\sqrt{2+2x+5x^2}} \\ \frac{1}{\sqrt{3}} & \frac{1-(3\omega^2+1)x}{\sqrt{3(2+2x+5x^2)}} & \frac{\omega^2 x-1}{\sqrt{2+2x+5x^2}} \\ \frac{1}{\sqrt{3}} & \frac{1-(\omega+i\sqrt{3})x}{\sqrt{3(2+2x+5x^2)}} & \frac{1-\omega x}{\sqrt{2+2x+5x^2}} \end{pmatrix}. \tag{C32}$$

Then the neutrino mass matrix m_ν is a block-diagonal matrix and the parameters y, z and w which are introduced in Eq. (2.13) are

$$y = 3m_a + \frac{(1-x)^2}{3} m_s e^{i\eta}, \qquad z = \frac{x-1}{3}\sqrt{2+2x+5x^2} m_s e^{i\eta}, \qquad w = \frac{2+2x+5x^2}{3} m_s e^{i\eta}. \tag{C33}$$

The neutrino mass matrix m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$ which is shown in Eq. (2.16). Then the lepton mixing matrix takes the following form:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i\sqrt{2}(x-1)e^{-i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} & \frac{i\sqrt{2}(x-1)\cos\theta}{\sqrt{5x^2+2x+2}} & \frac{\sqrt{2}(2x+1)}{\sqrt{5x^2+2x+2}} \\ \cos\theta - \frac{i(2x+1)e^{-i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} & -\frac{i(2x+1)\cos\theta}{\sqrt{5x^2+2x+2}} - e^{i\psi}\sin\theta & \frac{x-1}{\sqrt{5x^2+2x+2}} \\ \cos\theta + \frac{i(2x+1)e^{-i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} & \frac{i(2x+1)\cos\theta}{\sqrt{5x^2+2x+2}} - e^{i\psi}\sin\theta & \frac{1-x}{\sqrt{5x^2+2x+2}} \end{pmatrix}. \tag{C34}$$

The predictions for the lepton mixing angles and CP invariants are

$$\sin^2\theta_{13} = \frac{(1+2x)^2}{2+2x+5x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = -\frac{(1-x)^2(1+2x)\sin 2\theta\cos\psi}{4(2+2x+5x^2)^{3/2}}, \qquad I_1 = -\frac{(1-x)^4\sin^2 2\theta\sin(\rho-\sigma)}{4(2+2x+5x^2)^2}. \tag{C35}$$

The atmospheric mixing angle θ_{23} is maximal and θ_{13} only depends on x. Inserting the 3σ range of θ_{13}, we find that the parameter x should vary in the interval $[-0.629, -0.618] \cup [-0.398, -0.390]$. Let us give a relatively simple example which is easier to present in an explicit model, i.e., with $x = -2/5$. Then the vacuum alignment of the flavon field $\phi_{\rm sol}$ is proportional to the column vector $(1, -\frac{2}{5}\omega^2, -\frac{2}{5}\omega)^T$ and the third column of the PMNS matrix is $(\frac{1}{5\sqrt{2}}, \frac{7}{10}, \frac{7}{10})^T$. The χ^2 analysis results for this example are

$$\eta = -0.996\pi, \qquad m_a = 23.399\ \text{meV}, \qquad r = 2.260, \qquad \chi^2_{\rm min} = 33.640, \qquad \sin^2\theta_{13} = 0.02,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.872\pi, \qquad \beta = 0.548\pi,$$
$$m_1 = 49.377\ \text{meV}, \qquad m_2 = 50.120\ \text{meV}, \qquad m_3 = 0\ \text{meV}, \qquad m_{ee} = 34.550\ \text{meV}. \tag{C36}$$

Furthermore, we think it is necessary to give the predictions for the three mixing angles and two CP phases. We obtain the viable ranges of the mixing angles and CP phases by scanning the input parameters x, r and η in their ranges. Then we find that the mixing angles θ_{13} and θ_{12} can take any values in their 3σ ranges, while the absolute values of the two CP phases are restricted to $|\delta_{CP}| \in [0.0956\pi, 0.161\pi] \cup [0.839\pi, 0.904\pi]$ and $|\beta| \in [0.526\pi, 0.574\pi]$. Moreover, in order to obtain viable ranges of the mixing parameters and the mass ratio m_1^2/m_2^2, the input parameters $|\eta|$ and r should take values in the ranges $[0.9951\pi, 0.9966\pi]$ and $[1.661, 1.691] \cup [2.274, 2.282]$, respectively.

(iv) $X_{\rm sol} = \{ST^2S, TST^2U\}$

For this kind of residual symmetries, the vacuum alignment of the flavon field $\phi_{\rm sol}$ is

$$\langle\phi_{\rm sol}\rangle = v_{\phi_s}(1+2ix, \omega^2(1-ix), \omega(1-ix))^T. \tag{C37}$$

We first perform a unitary transformation $U_{\nu 1}$ with

$$U_{\nu 1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{ix-1}{\sqrt{x^2+1}} & \frac{i+x}{\sqrt{x^2+1}} \\ 1 & \frac{ix\omega-\omega^2}{\sqrt{x^2+1}} & \frac{x\omega^2+i\omega}{\sqrt{x^2+1}} \\ 1 & \frac{ix\omega^2-\omega}{\sqrt{x^2+1}} & \frac{i\omega^2+x\omega}{\sqrt{x^2+1}} \end{pmatrix}. \tag{C38}$$

Then the light neutrino mass matrix becomes block diagonal with the nonzero parameters y, z and w,

$$y = 3m_a - 3x^2 m_s e^{i\eta}, \qquad z = -3ix\sqrt{1+x^2}m_s e^{i\eta}, \qquad w = 3(1+x^2)m_s e^{i\eta}. \tag{C39}$$

Following the procedures presented in Sec. I, we know that the neutrino mass matrix m'_ν can be diagonalized by the unitary matrix $U_{\nu 2}$. Then the PMNS matrix is given by

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{\sqrt{2}xe^{-i\psi}\sin\theta}{\sqrt{x^2+1}} & \frac{\sqrt{2}x\cos\theta}{\sqrt{x^2+1}} & \frac{\sqrt{2}}{\sqrt{x^2+1}} \\ i\cos\theta + \frac{e^{-i\psi}\sin\theta}{\sqrt{x^2+1}} & ie^{i\psi}\sin\theta - \frac{\cos\theta}{\sqrt{x^2+1}} & \frac{x}{\sqrt{x^2+1}} \\ -i\cos\theta + \frac{e^{-i\psi}\sin\theta}{\sqrt{x^2+1}} & -ie^{i\psi}\sin\theta - \frac{\cos\theta}{\sqrt{x^2+1}} & \frac{x}{\sqrt{x^2+1}} \end{pmatrix}. \tag{C40}$$

One can straightforwardly extract the lepton mixing angles and CP phases as follows:

$$\sin^2\theta_{13} = \frac{1}{1+x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = -\frac{x^2\sin 2\theta\cos\psi}{4(1+x^2)^{3/2}}, \qquad I_1 = -\frac{x^4\sin^2 2\theta\sin(\rho-\sigma)}{4(1+x^2)^2}. \tag{C41}$$

θ_{23} is predicted to be maximal and θ_{13} only depends on the input parameter x which comes from the general VEV invariant under the action of the residual symmetry $Z_2^{T^2U} \times H_{CP}^{\mathrm{sol}}$ with a $\mathbf{3}'$ representation. When we take into account the current 3σ bounds of θ_{13}, we find that the parameter $|x|$ is constrained to be in the range [6.293, 6.882].

In order to give the predictions for the mixing parameters, we could focus on the admissible values of x, r and η in their ranges given $\mathcal{I}_1$. The admissible ranges of x, r and η can be obtained from the requirement that the three mixing angles and mass ratio m_1^2/m_2^2 lie in their experimentally preferred 3σ ranges, i.e., $|x| \in [6.293, 6.882]$, $|\eta| \in [0.0034\pi, 0.0049\pi]$ and $r \in [0.0105, 0.0120]$. Then the possible ranges of the absolute values of the two CP phases δ_{CP} and β are $[0.598\pi, 0.661\pi]$ and $[0.527\pi, 0.574\pi]$, respectively. This mixing pattern gives no predictions for the mixing angles θ_{12} and θ_{13}. Here we shall give an example ($x = 4\sqrt{3}$) to show how well the lepton mixing angles can be described by this mixing pattern and the predictions for the CP phases. In this example, the fixed column which only depends on the VEVs of ϕ_{atm} and ϕ_{sol} is determined to be $\frac{1}{7}(1, 2\sqrt{6}, 2\sqrt{6})^T$. Then the best-fit point and the predictions for various observable quantities obtained at the best-fit point are

$$\eta = 0.00408\pi, \qquad m_a = 23.396\ \mathrm{meV}, \qquad r = 0.0103, \qquad \chi^2_{\min} = 29.075, \qquad \sin^2\theta_{13} = 0.0204,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.372\pi, \qquad \beta = 0.548\pi,$$
$$m_1 = 49.377\ \mathrm{meV}, \qquad m_2 = 50.120\ \mathrm{meV}, \qquad m_3 = 0\ \mathrm{meV}, \qquad m_{ee} = 34.539\ \mathrm{meV}. \tag{C42}$$

($\mathcal{I}_{12}$) $(G_l, G_{\mathrm{atm}}, G_{\mathrm{sol}}) = (Z_4^{TSU}, Z_2^U, Z_2^{TU})$, $X_{\mathrm{atm}} = \{1, U\}$, $X_{\mathrm{sol}} = \{U, T\}$

Similar to the previous cases, we can obtain the neutrino mass matrix m_ν from the vacuum alignments of ϕ_{atm} and ϕ_{sol} invariant under the residual symmetry. Then we apply the unitary transformation $U_{\nu 1}$ to make $m'_\nu = U_{\nu 1}^T m_\nu U_{\nu 1}$ a block-diagonal matrix, where the unitary matrix $U_{\nu 1}$ is

$$U_{\nu 1} = \begin{pmatrix} -\frac{i\sqrt{3}}{\sqrt{2x^2+2x+5}} & \frac{x+2}{\sqrt{(x^2+2)(2x^2+2x+5)}} & -\frac{x}{\sqrt{x^2+2}} \\ \frac{\omega x-1}{\sqrt{2x^2+2x+5}} & \frac{i\sqrt{3}+x(\omega x-1)}{\sqrt{(x^2+2)(2x^2+2x+5)}} & -\frac{1}{\sqrt{x^2+2}} \\ \frac{1-\omega^2 x}{\sqrt{2x^2+2x+5}} & \frac{-i\sqrt{3}-x(1-\omega^2 x)}{\sqrt{(x^2+2)(2x^2+2x+5)}} & -\frac{1}{\sqrt{x^2+2}} \end{pmatrix}. \tag{C43}$$

The parameters y, z and w of m'_ν are given by

$$y = \frac{(x+2)^2 m_a - 3(x-1)^2 m_s e^{i\eta}}{2x^2+2x+5},$$
$$z = \frac{i\sqrt{3(x^2+2)}(-(x+2)m_a + (x-1)(2x+1)m_s e^{i\eta})}{2x^2+2x+5},$$
$$w = \frac{(x^2+2)(-3m_a + (2x+1)^2 m_s e^{i\eta})}{2x^2+2x+5}. \quad \text{(C44)}$$

The neutrino mass matrix m'_ν can be diagonalized by the unitary transformation matrix $U_{\nu 2}$. Then the following PMNS matrix can be obtained:

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{\frac{2(2x^2+2x+5)}{x^2+2}} e^{-i\psi} \sin\theta & \sqrt{\frac{2(2x^2+2x+5)}{x^2+2}} \cos\theta & -\frac{\sqrt{2}(x-1)}{\sqrt{x^2+2}} \\ \sqrt{3}\cos\theta - \frac{(x-1)e^{-i\psi}\sin\theta}{\sqrt{x^2+2}} & \frac{(x-1)\cos\theta}{\sqrt{x^2+2}} + \sqrt{3}e^{i\psi}\sin\theta & \sqrt{\frac{2x^2+2x+5}{x^2+2}} \\ -\sqrt{3}\cos\theta - \frac{(x-1)e^{-i\psi}\sin\theta}{\sqrt{x^2+2}} & \frac{(x-1)\cos\theta}{\sqrt{x^2+2}} - \sqrt{3}e^{i\psi}\sin\theta & \sqrt{\frac{2x^2+2x+5}{x^2+2}} \end{pmatrix}. \quad \text{(C45)}$$

The lepton mixing angles and CP phases can be read off as

$$\sin^2\theta_{13} = \frac{(1-x)^2}{3(2+x^2)}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = \frac{(2x^3+3x-5)\sin 2\theta \sin\psi}{12\sqrt{3}(x^2+2)^{3/2}}, \qquad I_1 = -\frac{(2x^2+2x+5)^2 \sin^2 2\theta \sin(\rho-\sigma)}{36(x^2+2)^2}. \quad \text{(C46)}$$

We see that this breaking pattern predicts a maximal θ_{23}. The reactor mixing angle θ_{13} only depends on the input parameter x, while the solar mixing angle θ_{12} and two CP phases depend on the input parameters x, η and r. Varying the mixing angle θ_{13} over its 3σ range [1], we obtain the allowed region of x, i.e., $x \in [0.584, 0.616] \cup [1.516, 1.575]$. As an example, we take $x = \frac{3}{5}$. Then the VEV alignment of the flavon $\phi_{\rm sol}$ is proportional to $(1, \frac{3}{5}\omega, \frac{3}{5}\omega^2)^T$ and the third column of the PMNS matrix is fixed to be $\left(\frac{2}{\sqrt{177}}, \sqrt{\frac{173}{354}}, \sqrt{\frac{173}{354}}\right)^T$. The χ^2 results are given by

$$\eta = 0.00408\pi, \qquad m_a = 30.164 \text{ meV}, \qquad r = 1.153, \qquad \chi^2_{\rm min} = 17.644, \qquad \sin^2\theta_{13} = 0.0226,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.228\pi, \qquad \beta = -0.729\pi,$$
$$m_1 = 49.377 \text{ meV}, \qquad m_2 = 50.120 \text{ meV}, \qquad m_3 = 0 \text{ meV}, \qquad m_{ee} = 25.931 \text{ meV}. \quad \text{(C47)}$$

Requiring that the three mixing angles lie in their experimentally preferred 3σ ranges, the allowed regions of $|\eta|$ and r are $[0.0027\pi, 0.0065\pi]$ and $[0.335, 0.354] \cup [1.127, 1.181]$, respectively. Any values of θ_{13} and θ_{12} in their allowed 3σ ranges can be taken in this mixing pattern. The two CP phases are determined to take values in the intervals

$$\delta_{CP} \in [-0.835, -0.695] \cup [-0.305, -0.165] \cup [0.0871, 0.140] \cup [0.860, 0.913],$$
$$|\beta| \in [0.465, 0.507] \cup [0.680, 0.800]. \quad \text{(C48)}$$

$(\mathcal{I}_{13})$ $(G_l, G_{\rm atm}, G_{\rm sol}) = (Z_4^{TSU}, Z_2^{TU}, Z_2^U)$, $X_{\rm atm} = \{U, T\}$, $X_{\rm sol} = \{1, U\}$

For this breaking pattern, we can read out the neutrino mass matrix m_ν from Eq. (2.8), and the unitary matrix $U_{\nu 1}$ is of the following form:

$$U_{\nu 1} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{2+x^2}} & \frac{x}{\sqrt{2+x^2}} \\ \frac{\omega}{\sqrt{2}} & \frac{\omega x}{\sqrt{2(2+x^2)}} & \frac{\omega}{\sqrt{2+x^2}} \\ -\frac{\omega^2}{\sqrt{2}} & \frac{\omega^2 x}{\sqrt{2(2+x^2)}} & \frac{\omega^2}{\sqrt{2+x^2}} \end{pmatrix}. \tag{C49}$$

The parameters y, z and w in m'_ν are

$$y = 2m_a - \frac{3}{2}x^2 m_s e^{i\eta}, \qquad z = -\frac{i}{2}x\sqrt{3(2+x^2)}m_s e^{i\eta}, \qquad w = \frac{1}{2}(2+x^2)m_s e^{i\eta}. \tag{C50}$$

Taking into account the diagonalization matrix of the charged lepton mass matrix in Eq. (3.3), we find that the lepton mixing matrix is fixed to be

$$U = \begin{pmatrix} \frac{\sqrt{\frac{2}{3}}(1-x)e^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & \frac{\sqrt{\frac{2}{3}}(1-x)\cos\theta}{\sqrt{2+x^2}} & \frac{2+x}{\sqrt{3(2+x^2)}} \\ \frac{(2+x)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} - \frac{\cos\theta}{\sqrt{2}} & \frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} + \frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{x-1}{\sqrt{3(2+x^2)}} \\ -\frac{(2+x)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} - \frac{\cos\theta}{\sqrt{2}} & -\frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} + \frac{e^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1-x}{\sqrt{3(2+x^2)}} \end{pmatrix}. \tag{C51}$$

The lepton mixing parameters are predicted to be

$$\sin^2\theta_{13} = \frac{(2+x)^2}{3(2+x^2)}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2},$$
$$J_{CP} = -\frac{(1-x)^2(2+x)\sin 2\theta\sin\psi}{6\sqrt{3}(2+x^2)^{3/2}}, \qquad I_1 = -\frac{(1-x)^4\sin^2 2\theta\sin(\rho-\sigma)}{9(2+x^2)^2}. \tag{C52}$$

These predict a maximal atmospheric mixing angle and the experimentally allowed 3σ range of θ_{13} requires that the input parameter x lies in the range $[-2.870, -2.776] \cup [-1.489, -1.450]$. For a certain value of $x = -\frac{14}{5}$, accordance with experimental data can be achieved, and the corresponding χ^2 results are

$$\eta = 0.00223\pi, \qquad m_a = 45.926\ \text{meV}, \qquad r = 0.119, \qquad \chi^2_{\min} = 19.755, \qquad \sin^2\theta_{13} = 0.0217,$$
$$\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.5, \qquad \delta_{CP} = -0.787\pi, \qquad \beta = 0.721\pi,$$
$$m_1 = 49.337\ \text{meV}, \qquad m_2 = 50.120\ \text{meV}, \qquad m_3 = 0\ \text{meV}, \qquad m_{ee} = 26.358\ \text{meV}. \tag{C53}$$

In the case $x = -\frac{14}{5}$, the third column of the PMNS matrix is predicted to be $\frac{1}{3\sqrt{82}}(4, 19, 19)^T$. If we require that the three mixing angles and mass ratio m_1^2/m_2^2 lie in their 3σ regions [1], we find that the other two input parameters $|\eta|$ and r lie in the rather narrow regions $[0.0017\pi, 0.0037\pi]$ and $[0.114, 0.121] \cup [0.367, 0.381]$, respectively. Then the two CP phases are predicted to be in the ranges $\delta_{CP} \in [-0.842\pi, -0.725\pi] \cup [-0.275\pi, -0.158\pi] \cup [0.124\pi, 0.211\pi] \cup [0.789\pi, 0.876\pi]$ and $|\beta| \in [0.608\pi, 0.674\pi] \cup [0.683\pi, 0.772\pi]$, respectively.

(v) $(\mathcal{I}_{14})$ $(G_l, G_{\text{atm}}, G_{\text{sol}}) = (K_4^{(S,TST^2)}, Z_3^T, Z_2^{SU})$, $X_{\text{atm}} = \{1, T, T^2\}$ $X_{\text{sol}} = \{1, SU\}$

This residual symmetry has been discussed in the case of $\mathcal{N}_7$ of NO. Consequently the predictions for the neutrino mass matrix and lepton flavor mixing can be read off from those of $\mathcal{N}_7$. For $X_{\text{sol}} = \{1, SU\}$, the lepton mixing matrix takes the following form:

$$U=\begin{pmatrix} -\sqrt{\frac{x^2-2x+4}{3(x^2-2x+2)}}\cos\theta & -\sqrt{\frac{x^2-2x+4}{3(x^2-2x+2)}}e^{i\psi}\sin\theta & \frac{\sqrt{2}(x-1)}{\sqrt{3(x^2-2x+2)}} \\ \frac{(x-1)\cos\theta}{\sqrt{3(x^2-2x+2)}}-\frac{e^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{(x-1)e^{i\psi}\sin\theta}{\sqrt{3(x^2-2x+2)}}+\frac{\cos\theta}{\sqrt{2}} & \sqrt{\frac{x^2-2x+4}{6(x^2-2x+2)}} \\ \frac{(x-1)\cos\theta}{\sqrt{3(x^2-2x+2)}}+\frac{e^{-i\psi}\sin\theta}{\sqrt{2}} & \frac{(x-1)e^{i\psi}\sin\theta}{\sqrt{3(x^2-2x+2)}}-\frac{\cos\theta}{\sqrt{2}} & \sqrt{\frac{x^2-2x+4}{6(x^2-2x+2)}} \end{pmatrix}. \tag{C54}$$

Moreover, the expressions of the neutrino masses m_1 and m_2 coincide with m_2 and m_3 of the $\mathcal{N}_7$ case, respectively. For the mixing matrix in Eq. (C54), we can extract the mixing angles and CP invariants as follows:

$$\sin^2\theta_{13}=\frac{2(x-1)^2}{3(x^2-2x+2)},\qquad \sin^2\theta_{12}=\sin^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2},$$
$$J_{CP}=\frac{(1-x)(x^2-2x+4)\sin 2\theta\sin\psi}{6\sqrt{6}(x^2-2x+2)^{3/2}},\qquad I_1=-\frac{(x^2-2x+4)^2\sin^2 2\theta\sin(\rho-\sigma)}{36(x^2-2x+2)^2}. \tag{C55}$$

We find that the atmospheric mixing angle θ_{23} is exactly 45°. The experimentally allowed region of x depends on the 3σ range of θ_{13}. We find that the viable range of x is $[0.804, 0.821]\cup[1.179, 1.196]$. Here we shall give the predictions for $x=\frac{4}{5}$. First, the third column of the PMNS matrix is $\frac{1}{\sqrt{39}}(1,\sqrt{19},\sqrt{19})^T$. Second, the results of the χ^2 analysis are

$$\eta=-0.994\pi,\qquad m_a=60.715\text{ meV},\qquad r=0.323,\qquad \chi^2_{\min}=38.315,\qquad \sin^2\theta_{13}=0.0256,$$
$$\sin^2\theta_{12}=0.310,\qquad \sin^2\theta_{23}=0.5,\qquad \delta_{CP}=0.895\pi,\qquad \beta=-0.451\pi,$$
$$m_1=49.377\text{ meV},\qquad m_2=50.120\text{ meV},\qquad m_3=0\text{ meV},\qquad m_{ee}=38.556\text{ meV}. \tag{C56}$$

We note that the best-fit value of $\sin^2\theta_{13}$ is rather close to its 3σ upper limit 0.02463. Hence we think that this breaking pattern with $x=\frac{4}{5}$ is a good leading-order approximation. Furthermore we perform a comprehensive numerical analysis. When the three mixing angles and mass ratio m_1^2/m_2^2 are restricted to their 3σ ranges, the input parameters $|\eta|$ and r have to lie in the ranges $[0.9929\pi, 0.9950\pi]$ and $[0.322, 0.325]$, respectively. Limiting the input parameters leads to $|\delta_{CP}|\in[0.0798\pi, 0.128\pi]\cup[0.872\pi, 0.920\pi]$ and $|\beta|\in[0.430\pi, 0.469\pi]$. The mixing angles θ_{12} and θ_{13} can take any values in their 3σ ranges.

(vi) $X_{\text{sol}}=\{S, U\}$

For this case, we can read out the lepton mixing matrix from Eq. (B17) as,

$$U=\begin{pmatrix} -\sqrt{\frac{x^2+3}{3(x^2+1)}}\cos\theta & -\sqrt{\frac{x^2+3}{3(x^2+1)}}e^{i\psi}\sin\theta & \frac{\sqrt{2}x}{\sqrt{3(x^2+1)}} \\ \frac{x(x+\sqrt{3})\cos\theta}{\sqrt{3(x^2+1)(x^2+3)}}+\frac{(x-\sqrt{3})e^{-i\psi}\sin\theta}{\sqrt{2(x^2+3)}} & \frac{(\sqrt{3}-x)\cos\theta}{\sqrt{2(x^2+3)}}+\frac{x(x+\sqrt{3})e^{i\psi}\sin\theta}{\sqrt{3(x^2+1)(x^2+3)}} & \frac{x+\sqrt{3}}{\sqrt{6(x^2+1)}} \\ \frac{x(x-\sqrt{3})\cos\theta}{\sqrt{3(x^2+1)(x^2+3)}}-\frac{(x+\sqrt{3})e^{-i\psi}\sin\theta}{\sqrt{2(x^2+3)}} & \frac{(x+\sqrt{3})\cos\theta}{\sqrt{2(x^2+3)}}+\frac{x(x-\sqrt{3})e^{i\psi}\sin\theta}{\sqrt{3(x^2+1)(x^2+3)}} & \frac{x-\sqrt{3}}{\sqrt{6(x^2+1)}} \end{pmatrix}. \tag{C57}$$

Then the predictions for the three mixing angles and two CP invariants are

$$\sin^2\theta_{13}=\frac{2x^2}{3(1+x^2)},\qquad \sin^2\theta_{12}=\sin^2\theta,\qquad \sin^2\theta_{23}=\frac{1}{2}+\frac{\sqrt{3}x}{3+x^2},$$
$$J_{CP}=\frac{x(x^2-3)\sin 2\theta\sin\psi}{6\sqrt{6}(x^2+1)^{3/2}},\qquad I_1=-\frac{(x^2+3)^2\sin^2 2\theta\sin(\rho-\sigma)}{36(x^2+1)^2}. \tag{C58}$$

We see that the third column of the PMNS matrix only depends on the parameter x which dictates the vacuum alignment of the flavon ϕ_{sol}. Since both mixing angles θ_{13} and θ_{23} depend on only one input parameter x, we can obtain the following sum rule:

$$\sin^2\theta_{23} = \frac{1}{2} \pm \frac{\tan\theta_{13}}{2}\sqrt{2-\tan^2\theta_{13}} \simeq \frac{1}{2} \pm \frac{\sqrt{2}\tan\theta_{13}}{2}, \tag{C59}$$

where the "+" sign in $\pm$ is satisfied for $x > 0$ and "−" is satisfied for $x < 0$. Given the experimental 3σ range of θ_{13}, we have $0.602 \le \sin^2\theta_{23} \le 0.612$ or $0.388 \le \sin^2\theta_{23} \le 0.398$. The latter range has been removed by the 3σ range of θ_{23}. The experimental data on the third column of the PMNS matrix at the 3σ level can be accommodated when the parameter $x \in [0.179, 0.196]$. The requirement that the three mixing angles and mass ratio m_1^2/m_2^2 lie in their 3σ ranges requires that the other two input parameters $|\eta|$ and r lie in the ranges $[0.9929\pi, 0.9950\pi]$ and $[0.3226, 0.3248]$, respectively. Then the mixing angle θ_{23} and two CP phases are predicted to be $0.602 \le \sin^2\theta_{23} \le 0.612$, $0.580\pi \le |\delta_{CP}| \le 0.628\pi$ and $0.431\pi \le |\beta| \le 0.469\pi$. The other two mixing angles can take any values in their 3σ ranges.

Now let us give the numerical results of a relatively simple example with $x = \frac{1}{3\sqrt{3}}$. In this example, the third column of the PMNS matrix is $\frac{1}{\sqrt{42}}(1,5,4)^T$ which agrees with all measurements to date [1]. When we perform a χ^2 analysis, the predictions for the various observable quantities are

$$\begin{gathered}\eta = -0.994\pi, \quad m_a = 60.743\ \text{meV}, \quad r = 0.323, \quad \chi^2_{\text{min}} = 5.731, \quad \sin^2\theta_{13} = 0.0238, \\ \sin^2\theta_{12} = 0.310, \quad \sin^2\theta_{23} = 0.610, \quad \delta_{CP} = -0.604\pi, \quad \beta = -0.449\pi, \\ m_1 = 49.377\ \text{meV}, \quad m_2 = 50.120\ \text{meV}, \quad m_3 = 0\ \text{meV}, \quad m_{ee} = 38.688\ \text{meV}. \end{gathered}\tag{C60}$$

$(\mathcal{I}_{15})$ $(G_l, G_{\text{atm}}, G_{\text{sol}}) = (K_4^{(S,TST^2)}, Z_2^U, Z_2^{TU})$, $X_{\text{atm}} = \{1, U\}$, $X_{\text{sol}} = \{U, T\}$

The vacuum alignment of the flavons ϕ_{atm} and ϕ_{sol} are given in Table II, and the neutrino mass matrix can be fixed. We perform the first unitary transformation $U_{\nu 1}$ on the light neutrino fields

$$U_{\nu 1} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{2+x^2}} & -\frac{x}{\sqrt{2+x^2}} \\ -\frac{1}{\sqrt{2}} & \frac{x}{\sqrt{2(2+x^2)}} & -\frac{1}{\sqrt{2+x^2}} \\ \frac{1}{\sqrt{2}} & \frac{x}{\sqrt{2(2+x^2)}} & -\frac{1}{\sqrt{2+x^2}} \end{pmatrix}. \tag{C61}$$

Then the neutrino mass matrix $m'_\nu = U_{\nu 1}^T m_\nu U_{\nu 1}$ is a block-diagonal matrix with nonzero elements

$$y = 2m_a - \frac{3}{2}x^2 m_s e^{i\eta}, \qquad z = -\frac{i}{2}x\sqrt{3(2+x^2)}m_s e^{i\eta}, \qquad w = \frac{1}{2}(2+x^2)m_s e^{i\eta}. \tag{C62}$$

Subsequently we find that the PMNS mixing matrix is of the following form:

$$U = \begin{pmatrix} -\frac{\sqrt{2}(1-x)e^{-i\psi}\sin\theta}{\sqrt{3(2+x^2)}} & -\frac{\sqrt{2}(1-x)\cos\theta}{\sqrt{3(2+x^2)}} & \frac{2+x}{\sqrt{3(2+x^2)}} \\ \frac{i\cos\theta}{\sqrt{2}} + \frac{(2a+b)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} & \frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} - \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1-x}{\sqrt{3(2+x^2)}} \\ -\frac{i\cos\theta}{\sqrt{2}} + \frac{(2+x)e^{-i\psi}\sin\theta}{\sqrt{6(2+x^2)}} & \frac{(2+x)\cos\theta}{\sqrt{6(2+x^2)}} + \frac{ie^{i\psi}\sin\theta}{\sqrt{2}} & \frac{1-x}{\sqrt{3(2+x^2)}} \end{pmatrix}. \tag{C63}$$

Then we can extract the expressions for the lepton mixing angles and CP invariants as follows:

$$\begin{gathered}\sin^2\theta_{13} = \frac{(2+x)^2}{3(2+x^2)}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2}, \\ J_{CP} = -\frac{(1-x)^2(2+x)\sin 2\theta\cos\psi}{6\sqrt{3}(2+x^2)^{3/2}}, \qquad I_1 = -\frac{(1-x)^4\sin^2 2\theta\sin(\rho-\sigma)}{9(2+x^2)^2}. \end{gathered}\tag{C64}$$

We see that the atmospheric mixing angle is maximal. Inserting the 3σ range of θ_{13}, we find that the parameter x should vary in the interval $[-2.870, -2.776] \cup [-1.489, -1.450]$. The mixing angle θ_{12} and the mass ratio m_1^2/m_2^2 depend on the three input parameters x, η and r. Hence if we require θ_{12} and m_1^2/m_2^2 to lie in their 3σ ranges, we can obtain that the restrictions on $|\eta|$ and r are $|\eta| \in [0.0017\pi, 0.0037\pi]$ and $r \in [0.114, 0.121] \cup [0.367, 0.381]$. Then the allowed values of the two CP phases would generically be constrained to the regions $|\delta_{CP}| \in [0.225\pi, 0.342\pi] \cup [0.624\pi, 0.711\pi]$ and $|\beta| \in [0.608\pi, 0.673\pi] \cup [0.683\pi, 0.772\pi]$. As an example easily achievable in a model, we consider the case of $x = -3/2$. Then the vacuum alignment of the flavon $\phi_{\rm sol}$ is proportional to the column vector $(1, -\frac{3}{2}\omega, -\frac{3}{2}\omega^2)^T$, and the fixed column of the PMNS matrix is $\frac{1}{\sqrt{51}}(1, 5, 5)^T$. The best-fit values of the mixing parameters read

$$\begin{aligned}
&\eta = -0.00301\pi, \quad m_a = 40.128\ \text{meV}, \quad r = 0.362, \quad \chi^2_{\rm min} = 38.746, \quad \sin^2\theta_{13} = 0.0196, \\
&\sin^2\theta_{12} = 0.310, \quad \sin^2\theta_{23} = 0.5, \quad \delta_{CP} = -0.668\pi, \quad \beta = -0.640\pi, \\
&m_1 = 49.377\ \text{meV}, \quad m_2 = 50.120\ \text{meV}, \quad m_3 = 0\ \text{meV}, \quad m_{ee} = 30.231\ \text{meV}. \qquad \text{(C65)}
\end{aligned}$$

We see that the best-fit value of θ_{13} is a bit smaller than its 3σ lower limit 0.02068 [1], and it should be a good leading-order approximation to the present data.

$(\mathcal{I}_{16})$ $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,U)}, Z_2^{TST^2}, Z_2^{TU})$, $X_{\rm atm} = \{SU, ST^2S, T^2, T^2STU\}$, $X_{\rm sol} = \{STS, T^2STU\}$

The neutrino mass matrix can be determined as in previous cases, and we choose the unitary matrix $U_{\nu 1}$ to be

$$U_{\nu 1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{ix-1}{\sqrt{3(1+x^2)}} & \frac{i+x}{\sqrt{3(1+x^2)}} \\ \frac{\omega^2}{\sqrt{3}} & \frac{ix-\omega}{\sqrt{3(1+x^2)}} & \frac{i+\omega x}{\sqrt{3(1+x^2)}} \\ \frac{\omega}{\sqrt{3}} & \frac{ix-\omega^2}{\sqrt{3(1+x^2)}} & \frac{i+\omega^2 x}{\sqrt{3(1+x^2)}} \end{pmatrix}. \qquad \text{(C66)}$$

The block-diagonal neutrino mass matrix m'_ν is parametrized by y, z and w with

$$y = 3m_a - 3x^2 m_s e^{i\eta}, \qquad z = -3ix\sqrt{1+x^2}\, m_s e^{i\eta}, \qquad w = 3(1+x^2) m_s e^{i\eta}. \qquad \text{(C67)}$$

Subsequently m'_ν is diagonalized by $U_{\nu 2}$ in Eq. (2.16). As a consequence, the PMNS matrix is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\sqrt{2}xe^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & -\frac{\sqrt{2}x\cos\theta}{\sqrt{1+x^2}} & \frac{\sqrt{2}}{\sqrt{1+x^2}} \\ \cos\theta + \frac{e^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & \frac{\cos\theta}{\sqrt{1+x^2}} - e^{i\psi}\sin\theta & \frac{x}{\sqrt{1+x^2}} \\ -\cos\theta + \frac{e^{-i\psi}\sin\theta}{\sqrt{1+x^2}} & \frac{\cos\theta}{\sqrt{1+x^2}} + e^{i\psi}\sin\theta & \frac{x}{\sqrt{1+x^2}} \end{pmatrix}. \qquad \text{(C68)}$$

We can extract the following results for the lepton mixing angles and CP invariants:

$$\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{1+x^2}, \qquad \sin^2\theta_{12} = \cos^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2}, \\
&J_{CP} = -\frac{x^2\sin 2\theta \sin\psi}{4(1+x^2)^{3/2}}, \qquad I_1 = -\frac{x^4 \sin^2 2\theta \sin(\rho-\sigma)}{4(1+x^2)^2}. \qquad \text{(C69)}
\end{aligned}$$

We see that θ_{23} is maximal. In order to obtain a viable θ_{13}, the absolute value of the input parameter x must lie in the range [6.293, 6.882]. Freely varying the three mixing angles and the mass ratio m_1^2/m_2^2 in their 3σ ranges, we find that the other input parameters $|\eta|$ and r are limited to the ranges $[0.0033\pi, 0.0049\pi]$ and [0.0104, 0.0124], respectively. Moreover, we find that the values of the CP phases $|\delta_{CP}|$ and $|\beta|$ are in the intervals $[0.839\pi, 0.905\pi] \cup [0.0955\pi, 0.162\pi]$ and $[0.526\pi, 0.574\pi]$, respectively. We consider the benchmark value of $x = -7$ for illustration. In this example, the third column of the PMNS matrix is determined to be $(\frac{1}{5\sqrt{2}}, \frac{7}{10}, \frac{7}{10})^T$. The best-fit values of the model parameters and mixing parameters are

$$\eta = 0.00408\pi, \quad m_a = 23.399 \text{ meV}, \quad r = 0.0100, \quad \chi^2_{\min} = 33.640, \quad \sin^2\theta_{13} = 0.02,$$
$$\sin^2\theta_{12} = 0.310, \quad \sin^2\theta_{23} = 0.5, \quad \delta_{CP} = -0.872\pi, \quad \beta = 0.548\pi,$$
$$m_1 = 49.377 \text{ meV}, \quad m_2 = 50.120 \text{ meV}, \quad m_3 = 0 \text{ meV}, \quad m_{ee} = 34.550 \text{ meV}. \tag{C70}$$

($\mathcal{I}_{17}$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,U)}, Z_2^{TU}, Z_2^U)$, $X_{\rm atm} = \{U, T\}$, $X_{\rm sol} = \{S, SU\}$

The light neutrino mass matrix can be straightforwardly obtained from Eq. (2.8) and the alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$. We choose the unitary transformation $U_{\nu 1}$ as

$$U_{\nu 1} = \begin{pmatrix} \frac{2ix^2+x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & -\frac{4x^2+1}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{1-ix}{\sqrt{3(3x^2+1)}} \\ \frac{i(1-2\omega^2)x^2+\omega x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & \frac{2x^2-\sqrt{3}x-\omega^2}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{\omega(2ix+1)}{\sqrt{3(3x^2+1)}} \\ \frac{i(1-2\omega)x^2+\omega^2 x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & \frac{2x^2+\sqrt{3}x-\omega}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{\omega^2(2ix+1)}{\sqrt{3(3x^2+1)}} \end{pmatrix}. \tag{C71}$$

The three parameters y, z and w in m'_ν are given by

$$y = \frac{(3x^2+1)(m_a - 3m_s e^{i\eta})}{2x^2+1},$$
$$z = \frac{(3x^2+1)((-2ix^2+x-i)m_a + 3x(4x^2+1)m_s e^{i\eta})}{(2x^2+1)\sqrt{(3x^2+1)(4x^2+1)}},$$
$$w = -\frac{((2x^2+ix+1)^2 m_a + 3x^2(4x^2+1)^2 m_s e^{i\eta})}{(2x^2+1)(4x^2+1)}. \tag{C72}$$

Following the procedures listed in Sec. II, we find that the lepton mixing matrix is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{\frac{2(2x^2+1)}{3x^2+1}}\cos\theta & -\sqrt{\frac{2(2x^2+1)}{3x^2+1}}e^{i\psi}\sin\theta & \frac{\sqrt{2}x}{\sqrt{3x^2+1}} \\ \frac{x\cos\theta}{\sqrt{(2x^2+1)(3x^2+1)}} + \sqrt{\frac{4x^2+1}{2x^2+1}}e^{-i\psi}\sin\theta & \frac{xe^{i\psi}\sin\theta}{\sqrt{(2x^2+1)(3x^2+1)}} - \sqrt{\frac{4x^2+1}{2x^2+1}}\cos\theta & \frac{1}{\sqrt{3x^2+1}} \\ x\sqrt{\frac{4x^2+1}{(2x^2+1)(3x^2+1)}}\cos\theta - \frac{e^{-i\psi}\sin\theta}{\sqrt{2x^2+1}} & \frac{\cos\theta}{\sqrt{2x^2+1}} + x\sqrt{\frac{4x^2+1}{(2x^2+1)(3x^2+1)}}e^{i\psi}\sin\theta & \sqrt{\frac{4x^2+1}{3x^2+1}} \end{pmatrix}. \tag{C73}$$

It is straightforward to extract the mixing angles and the two CP rephasing invariants as follows:

$$\sin^2\theta_{13} = \frac{x^2}{1+3x^2}, \qquad \sin^2\theta_{12} = \sin^2\theta, \qquad \sin^2\theta_{23} = \frac{1}{2+4x^2},$$
$$J_{CP} = \frac{x\sqrt{1+4x^2}\sin 2\theta \sin\psi}{4(1+3x^2)^{3/2}}, \qquad I_1 = -\frac{(2x^2+1)^2\sin^2 2\theta\sin(\rho-\sigma)}{4(3x^2+1)^2}. \tag{C74}$$

We see that both the atmospheric mixing angle and the reactor mixing angle only depend on one input parameter x which decides the vacuum alignment of the flavon $\phi_{\rm sol}$. Then a sum rule between the mixing angle θ_{13} and θ_{23} is obtained

$$\sin^2\theta_{23} = \frac{1}{2} - \tan^2\theta_{13}. \tag{C75}$$

This sum rule has also been obtained in Ref. [37]. It implies that the atmospheric mixing angle is in the first octant, i.e., $\theta_{23} < 45°$. For the fitted 3σ range of θ_{13}, the atmospheric mixing angle is constrained to be in the interval $0.475 \le \sin^2\theta_{23} \le 0.479$. This can be tested in future neutrino oscillation experiments. Inserting the 3σ range of θ_{13}, we find that the viable region of $|x|$ is [0.148, 0.163]. Detailed numerical analyses show that accordance with experimental data can be achieved for certain values of x, m_a, r and η. As an example, the fixed column of the PMNS matrix is $\left(\frac{1}{4\sqrt{3}}, \frac{3\sqrt{5}}{4\sqrt{6}}, \frac{7}{4\sqrt{6}}\right)^T$ for $x = \frac{\sqrt{5}}{15}$. The best-fit values of all parameters are

$$\begin{aligned} &\eta = 0.0969\pi, \qquad m_a = 34.963 \text{ meV}, \qquad r = 0.633, \qquad \chi^2_{\min} = 35.201, \qquad \sin^2\theta_{13} = 0.0208,\\ &\sin^2\theta_{12} = 0.310, \qquad \sin^2\theta_{23} = 0.479, \qquad \delta_{CP} = -0.184\pi, \qquad \beta = -0.546\pi,\\ &m_1 = 49.377 \text{ meV}, \qquad m_2 = 50.120 \text{ meV}, \qquad m_3 = 0 \text{ meV}, \qquad m_{ee} = 34.625 \text{ meV}. \end{aligned} \tag{C76}$$

Furthermore, it is necessary to give the allowed ranges of all mixing parameters. Freely varying the input parameters we find that $\sin^2\theta_{23}$ can take any value between 0.475 and 0.479. The other two mixing angles are restricted to their 3σ ranges. The two CP phases are predicted to be $|\delta_{CP}| \in [0.152\pi, 0.219\pi] \cup [0.881\pi, 0.949\pi]$ and $|\beta| \in [0.522\pi, 0.575\pi]$. The requirement that the mixing angles and mass ratio lie in their 3σ ranges also requires that the input parameters $|\eta|$ and r take values in the ranges $[0.0888\pi, 0.106\pi]$ and [0.626, 0.638], respectively.

($\mathcal{I}_{18}$) $(G_l, G_{\rm atm}, G_{\rm sol}) = (K_4^{(S,U)}, Z_2^{TU}, Z_2^{T^2U})$, $X_{\rm atm} = \{U, T\}$, $X_{\rm sol} = \{ST^2S, TST^2U\}$

In this case, we take the unitary matrix $U_{\nu 1}$ to be

$$U_{\nu 1} = \begin{pmatrix} \frac{2ix^2+x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & -\frac{4x^2+1}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{1-ix}{\sqrt{3(3x^2+1)}} \\ \frac{i(1-2\omega^2)x^2+\omega x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & \frac{2x^2-\sqrt{3}x-\omega^2}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{\omega(2ix+1)}{\sqrt{3(3x^2+1)}} \\ \frac{i(1-2\omega)x^2+\omega^2 x+i}{\sqrt{3(2x^2+1)(3x^2+1)}} & \frac{2x^2+\sqrt{3}x-\omega}{\sqrt{3(2x^2+1)(4x^2+1)}} & \frac{\omega^2(2ix+1)}{\sqrt{3(3x^2+1)}} \end{pmatrix}. \tag{C77}$$

The neutrino mass matrix m'_ν is block diagonal with nonzero elements

$$\begin{aligned} y &= \frac{(3x^2+1)(m_a + 3x^2 m_s e^{i\eta})}{2x^2+1},\\ z &= \frac{(2x-i)(3x^2+1)((1-ix)m_a + 3x(ix^2+x+i)m_s e^{i\eta})}{(2x^2+1)\sqrt{(3x^2+1)(4x^2+1)}},\\ w &= \frac{(i-2x)((x+i)^2 m_a + 3(x^2-ix+1)^2 m_s e^{i\eta})}{(2x+i)(2x^2+1)}. \end{aligned} \tag{C78}$$

We can diagonalize m'_ν with $U_{\nu 2}$ in Eq. (2.16), and consequently we find that the lepton mixing matrix is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{\frac{2(2x^2+1)}{3x^2+1}}\cos\theta & -\sqrt{\frac{2(2x^2+1)}{3x^2+1}}e^{i\psi}\sin\theta & \frac{\sqrt{2}x}{\sqrt{3x^2+1}} \\ x\sqrt{\frac{4x^2+1}{(2x^2+1)(3x^2+1)}}\cos\theta - \frac{e^{-i\psi}\sin\theta}{\sqrt{2x^2+1}} & \frac{\cos\theta}{\sqrt{2x^2+1}} + x\sqrt{\frac{4x^2+1}{(2x^2+1)(3x^2+1)}}e^{i\psi}\sin\theta & \sqrt{\frac{4x^2+1}{3x^2+1}} \\ \frac{x\cos\theta}{\sqrt{(2x^2+1)(3x^2+1)}} + \sqrt{\frac{4x^2+1}{2x^2+1}}e^{-i\psi}\sin\theta & \frac{xe^{i\psi}\sin\theta}{\sqrt{(2x^2+1)(3x^2+1)}} - \sqrt{\frac{4x^2+1}{2x^2+1}}\cos\theta & \frac{1}{\sqrt{3x^2+1}} \end{pmatrix}. \tag{C79}$$

Then the lepton mixing parameters read

$$\begin{aligned} &\sin^2\theta_{13} = \frac{x^2}{1+3x^2}, \qquad \sin^2\theta_{12} = \sin^2\theta, \qquad \sin^2\theta_{23} = 1 - \frac{1}{2(1+2x^2)},\\ &J_{CP} = -\frac{x\sqrt{1+4x^2}\sin 2\theta \sin\psi}{4(1+3x^2)^{3/2}}, \qquad I_1 = -\frac{(2x^2+1)^2 \sin^2 2\theta \sin(\rho-\sigma)}{4(3x^2+1)^2}. \end{aligned} \tag{C80}$$

Both the atmospheric mixing angle and the reactor mixing angle depend on only one input parameter x which comes from the vacuum alignment invariant under the action of the residual symmetry in the solar neutrino sector. As a consequence, the following sum rule between the reactor mixing angle and the atmospheric mixing angle is found to be satisfied:

$$\sin^2\theta_{23} = \frac{1}{2} + \tan^2\theta_{13}. \tag{C81}$$

We note that θ_{23} is constrained to lie in the second octant. Inserting the experimentally preferred 3σ range of θ_{13}, the atmospheric mixing angle is predicted to be $0.521 \le \sin^2\theta_{23} \le 0.525$. It is remarkable that a good fit to the experimental data can always be achieved for any $|x|$. When these two mixing angles are required to lie in their 3σ ranges, the input parameter x is restricted to the range [0.148, 0.163]. Similar to the example in $\mathcal{I}_{17}$, we also give the example with $x = \frac{\sqrt{5}}{15}$. The χ^2 analysis results are

$$\begin{aligned} &\eta = 0.0913\pi, \quad m_a = 34.659\ \text{meV}, \quad r = 0.644, \quad \chi^2_{\text{min}} = 17.329, \quad \sin^2\theta_{13} = 0.0208, \\ &\sin^2\theta_{12} = 0.310, \quad \sin^2\theta_{23} = 0.521, \quad \delta_{CP} = -0.748\pi, \quad \beta = 0.513\pi, \\ &m_1 = 49.377\ \text{meV}, \quad m_2 = 50.120\ \text{meV}, \quad m_3 = 0\ \text{meV}, \quad m_{ee} = 36.089\ \text{meV}. \end{aligned} \tag{C82}$$

The input parameters x, r and η are treated as random numbers, and the mixing angles and mass ratio are required to lie in their 3σ ranges. We find that the allowed regions of the parameters $|\eta|$ and r are $[0.0905\pi, 0.108\pi]$ and [0.6328, 0.6448], respectively. The allowed ranges of the two CP phases are determined to be $|\delta_{CP}| \in [0, 0.0619\pi] \cup [0.710\pi, 0.777\pi]$ and $|\beta| \in [0.485\pi, 0.536\pi]$.

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