Further study on the textures of neutrino mass matrix for maximal atmospherical mixing angle and Dirac *CP* phase

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In this paper, we derive in a novel approach the possible textures of neutrino mass matrix that can lead to maximal atmospherical mixing angle ($\theta_{23} = \pi/4$) and Dirac *CP* phase ($\delta = -\pi/2$) in two phenomenologically appealing scenarios: (1) one neutrino mass matrix element vanishing (2) one neutrino mass vanishing. For the obtained textures, some neutrino mass sum rules that relate the neutrino masses and mixing parameters emerge. With the help of these sum rules, the unknown absolute neutrino mass scale and Majorana *CP* phases can be determined. Some discussions about the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and maximal Majorana *CP* phases ($\rho, \sigma = \pi/4$ or $3\pi/4$) as well as the model realization and breakings of the obtained textures are also given.

I. INTRODUCTION

Thanks to the various neutrino oscillation experiments, it has been established that neutrinos have small but non-vanishing masses and mix among different flavors [1]. On the one hand, the smallness of neutrino masses m_i (for $i = 1, 2, 3$) can be naturally explained by the seesaw mechanism [2]. And the neutrino masses generated via this mechanism are of Majorana nature in most cases[1] for which the mass matrix M_ν is a complex symmetric one. On the other hand, the neutrino mixing matrix is given by $U = U_l^\dagger U_\nu$ where U_l and U_ν respectively result from diagonalization of the charged lepton mass matrix M_l and M_ν [4]. In the commonly used basis of M_l being diagonal that is also adopted here, U can be identified with the unitary matrix (i.e., U_ν) for diagonalizing M_ν,

$$U^\dagger M_\nu U^* = \text{Diag}(m_1, m_2, m_3). \tag{1}$$

In the standard way, it is parametrized as

$$U = P_l O_{23} U_{13} O_{12} P_\nu, \tag{2}$$

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[1]It is noteworthy that the neutrino masses resulting from the seesaw mechanism can be of Dirac nature in some special cases [3].

where $P_l = \text{Diag}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau})$ and $P_\nu = \text{Diag}(e^{i\rho}, e^{i\sigma}, 1)$ are two diagonal phase matrices, and

$$O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix},$$

$$O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ for the mixing angles θ_{ij} (for $ij = 12, 13, 23$). As for the phases, δ is (ρ and σ are) the Dirac (Majorana) *CP* phase(s), while $\phi_{e,\mu,\tau}$ are unphysical phases that can be removed by the redefinitions of charged lepton fields. In addition, neutrino oscillations are also governed by two independent neutrino mass squared differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$ (for $ij = 21, 31$).

So far, the neutrino oscillation experiments give us the following results for the neutrino mass squared differences [5]:

$$\Delta m_{21}^2 = (7.50_{-0.17}^{+0.19}) \times 10^{-5} \text{ eV}^2,$$
$$|\Delta m_{31}^2| = (2.524_{-0.040}^{+0.039}) \times 10^{-3} \text{ eV}^2. \tag{4}$$

Note that the sign of Δm_{31}^2 has not yet been determined, thereby allowing for two possible neutrino mass orderings: $m_1 < m_2 < m_3$ (the normal neutrino mass ordering, NO for short) and $m_3 < m_1 < m_2$ (the inverted neutrino mass ordering, IO for short). On the other side, the mixing parameters θ_{13}, θ_{23}, and δ take the values of

$$\sin^2\theta_{13} = 0.02166 \pm 0.00075, \quad \sin^2\theta_{23} = 0.441 \pm 0.024,$$
$$\delta = 261° \pm 55°, \tag{5}$$

in the NO case, or

$$\sin^2\theta_{13} = 0.02179 \pm 0.00076, \quad \sin^2\theta_{23} = 0.587 \pm 0.022,$$
$$\delta = 277° \pm 43°, \tag{6}$$

in the IO case, while θ_{12} takes the value of $\sin^2\theta_{12} = 0.306 \pm 0.012$ in either mass ordering case [5].

However, the neutrino oscillation experiments are insensitive to the absolute neutrino mass scale and Majorana CP phases. Information about them can only be inferred from nonoscillatory experiments: (1) The beta decay experiments can probe the effective electron neutrino mass $m_\beta = \sqrt{\sum m_i |U_{ei}|^2}$ (with U_{ei} being the ith element in the first row of U) by measuring the end point of the spectrum of electrons in beta decays. The current upper limit for it is around 2 eV [6], while the future KATRIN experiment is expected to achieve a sensitivity of 0.2 eV at 90% C.L. [7]. (2) The cosmological measurements can probe the sum of neutrino masses $\Sigma = \sum m_i$ by virtue of its effects on cosmic structure formation [8]. And the current upper limit for it is 0.12 eV [9]. But it should be noted that the neutrino mass limit obtained from the cosmological measurements is strongly dependent on the cosmological model and observation data used. (3) The lepton number violating (LNV) processes can even directly probe the magnitudes of neutrino mass matrix elements, as an LNV process with the charged leptons α and β (for $\alpha, \beta = e, \mu, \tau$) in the final state is governed by $|M_{\alpha\beta}|$ (with $M_{\alpha\beta}$ being the $\alpha\beta$ element of M_ν). At present, the neutrinoless double beta decay, which is governed by $|M_{ee}|$ [10], is the only feasible process to probe LNV. The current upper limit for $|M_{ee}|$ is 0.2–0.4 eV, where the large uncertainty is due to the inconclusive nuclear physics calculations [11].

On the theoretical side, one of the most important goals of neutrino physics research is to identify the flavor structure of M_ν and its origin [12]. Because of the particular observed neutrino mixing pattern (some parameters of which, as explained soon, are close to certain special values), it is widely expected that M_ν probably has a special texture that may originate from some underlying flavor theory (especially flavor symmetry [13]). The first step to a convincing flavor theory is to reconstruct M_ν with the help of existing experimental results [14]. However, since the absolute neutrino mass scale and Majorana CP phases remain unknown, the existing experimental results are not sufficient for reconstructing M_ν completely, leaving us with a large variety of possible forms for it. Guided by the principle of using as few parameters as possible to explain the observed neutrino physics [15], a number of approaches to restricting the form of M_ν and reducing the number of free parameters have been adopted in the

literature, among which the impositions of some vanishing neutrino mass matrix element(s) [16] or one vanishing neutrino mass [17] are two very popular ones. These two scenarios are both well motivated from the theoretical point of view and highly predictive from the phenomenological point of view: (1) The vanishing of neutrino mass matrix element(s) can naturally find an origin from the Abelian flavor symmetries [18], while one neutrino mass necessarily vanishes if only two right-handed neutrinos take effect in the popular type-I seesaw mechanism (the so-called minimal seesaw [17]). (2) The vanishing of neutrino mass matrix element(s) would result in some testable relations between the neutrino masses and mixing parameters [16], while the vanishing of one neutrino mass would additionally lead one Majorana CP phase to be ineffective [17]. Furthermore, the vanishing of a neutrino mass matrix element (e.g., M_{ee}) would render the associated LNV process (e.g., the neutrinoless double beta decay) impotent. If any of these interesting consequences turns out to be favored by the future measurements, the corresponding specific neutrino mass matrix texture will stand out and shed some light on the underlying flavor theory. Therefore, it makes a lot of sense to phenomenologically study the possible neutrino mass matrix textures featuring vanishing elements or mass eigenvalue.

It is interesting to note that the current neutrino oscillation data are consistent with maximal atmospherical mixing angle ($\theta_{23} = \pi/4$) and Dirac CP phase ($\delta = -\pi/2^2$). As mentioned above, these remarkable values may point towards some special texture of M_ν. In this connection, the specific texture given by the μ-τ reflection symmetry [20] serves as a unique example. This symmetry is defined as follows: M_ν keeps invariant under a combination of the μ-τ interchange and CP conjugate operations

$$\nu_e \leftrightarrow \nu_e^c, \qquad \nu_\mu \leftrightarrow \nu_\tau^c, \qquad \nu_\tau \leftrightarrow \nu_\mu^c, \tag{7}$$

and is characterized by

$$M_{e\mu} = M_{e\tau}^*, \quad M_{\mu\mu} = M_{\tau\tau}^*, \quad M_{ee} \text{ and } M_{\mu\tau} \text{ being real.} \tag{8}$$

Such a texture leads to the following predictions for the neutrino mixing parameters [21]

$$\phi_e = \frac{\pi}{2}, \qquad \phi_\mu = -\phi_\tau, \qquad \theta_{23} = \frac{\pi}{4},$$
$$\delta = \pm\frac{\pi}{2}, \qquad \rho, \sigma = 0 \quad \text{or} \quad \frac{\pi}{2}. \tag{9}$$

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[2]It should be noted that the current measurement error for δ is quite large. So we still need a precise measurement for it to see if it is really close to $-\pi/2$. It is expected that this problem will be cleared up over the next few years by the long baseline neutrino oscillation experiments [19].

However, the μ-τ reflection symmetry is over-restrictive in the sense that its predictions for trivial Majorana CP phases (i.e., ρ, $\sigma = 0$ or $\pi/2$) are not promised by the experimental results. Although this symmetry deserves particular attention due to its interesting properties, it is reasonable to have an open mind for other possible values of ρ and σ. For this consideration, in a previous work [22] we did an exercise to derive the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ [23] where ρ and σ are allowed to take values other than 0 and $\pi/2$.

In this work we attempt to derive the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ in two phenomenologically appealing scenarios: (1) one neutrino mass matrix element vanishing and (2) one neutrino mass vanishing. The rest part of this paper is organized as follows. In Sec. II, we briefly recapitulate our approach and some useful results developed in Sec. II and subsections 3.1–3.4 of Ref. [22] as a basis for the study performed here. The studies on the scenarios of one neutrino mass matrix element and one neutrino mass vanishing are carried out in Secs. III and IV, respectively. Some discussions about the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and maximal Majorana CP phases (i.e., ρ, $\sigma = \pi/4$ or $3\pi/4$) as well as the model realization and breakings of the obtained textures are given in Sec. V. Finally, in Sec. VI we summarize our main results.

II. THE APPROACH AND BASIS RESULTS

In order to avoid the uncertainties due to the unphysical phases, we choose to work on $\bar{M}_\nu = P_l^\dagger M_\nu P_l^*$ instead of M_ν. One can recover the results for M_ν from those for $\bar{M}_\nu$ by simply making the replacements $\bar{M}_{\alpha\beta} = M_{\alpha\beta} e^{-i(\phi_\alpha + \phi_\beta)}$ (with $\bar{M}_{\alpha\beta}$ being the $\alpha\beta$ element of $\bar{M}_\nu$). By definition, $\bar{M}_\nu$ can be diagonalized as

$$O_{12}^T U_{13}^\dagger O_{23}^T \bar{M}_\nu O_{23} U_{13}^* O_{12} = \text{Diag}(m_1 e^{2i\rho}, m_2 e^{2i\sigma}, m_3).$$

$$(10)$$

In light of the purpose of this study, we take $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ in O_{23} and U_{13}. To simplify the expressions in the following discussions, we define the following three matrices in order,

$$M_X = O_{23}^T \bar{M}_\nu O_{23}, \qquad M_Y = U_{13}^\dagger M_X U_{13}^*,$$
$$M_Z = O_{12}^T M_Y O_{12}, \qquad (11)$$

whose elements appear as

$$M_{X11} = \bar{M}_{ee}, \quad M_{X12} = \frac{\bar{M}_{e\mu} - \bar{M}_{e\tau}}{\sqrt{2}},$$

$$M_{X13} = \frac{\bar{M}_{e\mu} + \bar{M}_{e\tau}}{\sqrt{2}}, \quad M_{X22} = \frac{\bar{M}_{\mu\mu} + \bar{M}_{\tau\tau}}{2} - \bar{M}_{\mu\tau},$$

$$M_{X23} = \frac{\bar{M}_{\mu\mu} - \bar{M}_{\tau\tau}}{2}, \quad M_{X33} = \frac{\bar{M}_{\mu\mu} + \bar{M}_{\tau\tau}}{2} + \bar{M}_{\mu\tau},$$

$$M_{Y11} = c_{13}^2 M_{X11} - i \sin 2\theta_{13} M_{X13} - s_{13}^2 M_{X33},$$

$$M_{Y12} = c_{13} M_{X12} - i s_{13} M_{X23},$$

$$M_{Y13} = \cos 2\theta_{13} M_{X13} - \frac{i}{2} \sin 2\theta_{13} (M_{X11} + M_{X33}),$$

$$M_{Y22} = M_{X22}, \quad M_{Y23} = c_{13} M_{X23} - i s_{13} M_{X12},$$

$$M_{Y33} = c_{13}^2 M_{X33} - i \sin 2\theta_{13} M_{X13} - s_{13}^2 M_{X11},$$

$$M_{Z11} = c_{12}^2 M_{Y11} - \sin 2\theta_{12} M_{Y12} + s_{12}^2 M_{Y22}, \quad M_{Z33} = M_{Y33},$$

$$M_{Z12} = \cos 2\theta_{12} M_{Y12} + \frac{1}{2} \sin 2\theta_{12} (M_{Y11} - M_{Y22}),$$

$$M_{Z13} = c_{12} M_{Y13} - s_{12} M_{Y23},$$

$$M_{Z22} = s_{12}^2 M_{Y11} + \sin 2\theta_{12} M_{Y12} + c_{12}^2 M_{Y22},$$

$$M_{Z23} = s_{12} M_{Y13} + c_{12} M_{Y23}, \qquad (12)$$

with M_{X11} being the 11 element of M_X and so on. By comparing the two sides of Eq. (10), one gets the following seven diagonalization conditions (which are referred to as A-G in order),

A: $\text{Re}(M_{Y13}) = 0$
  $\Rightarrow 2 \cos 2\theta_{13} R_{X13} = -\sin 2\theta_{13}(I_{X11} + I_{X33})$,

B: $\text{Im}(M_{Y13}) = 0$
  $\Rightarrow 2 \cos 2\theta_{13} I_{X13} = \sin 2\theta_{13}(R_{X11} + R_{X33})$,

C: $\text{Re}(M_{Y23}) = 0 \Rightarrow c_{13} R_{X23} = -s_{13} I_{X12}$,

D: $\text{Im}(M_{Y23}) = 0 \Rightarrow c_{13} I_{X23} = s_{13} R_{X12}$,

E: $\text{Re}(M_{Z12}) = 0$
  $\Rightarrow 2 \cos 2\theta_{12} R_{Y12} = -\sin 2\theta_{12}(R_{Y11} - R_{Y22})$,

F: $\text{Im}(M_{Z12}) = 0$
  $\Rightarrow 2 \cos 2\theta_{12} I_{Y12} = -\sin 2\theta_{12}(I_{Y11} - I_{Y22})$,

G: $\text{Im}(M_{Z33}) = 0$
  $\Rightarrow \sin 2\theta_{13} R_{X13} = c_{13}^2 I_{X33} - s_{13}^2 I_{X11}$, (13)

with $R_{X13} = \text{Re}(M_{X13})$, $I_{X11} = \text{Im}(M_{X11})$, and so on, and

$$M_{Z11} = m_1 e^{2i\rho}, \quad M_{Z22} = m_2 e^{2i\sigma}, \quad \text{Re}(M_{Z33}) = m_3. \quad (14)$$

Because we only have two parameters (i.e., θ_{12} and θ_{13}) at hand to diagonalize $\bar{M}_\nu$ by making the seven diagonalization conditions hold, there exist five constraint equations for $\bar{R}_{\alpha\beta} = \text{Re}(\bar{M}_{\alpha\beta})$ and $\bar{I}_{\alpha\beta} = \text{Im}(\bar{M}_{\alpha\beta})$: By relating the

expressions for θ_{13} derived from diagonalization conditions A–D, one obtains the following constraint equations,

$$AB: (\bar{R}_{e\mu} + \bar{R}_{e\tau})\left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)$$
$$= -(\bar{I}_{e\mu} + \bar{I}_{e\tau})\left(\bar{I}_{ee} + \bar{I}_{\mu\tau} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{2}\right),$$

$$AC: (\bar{I}_{e\mu} - \bar{I}_{e\tau})(\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})\left(\bar{I}_{ee} + \bar{I}_{\mu\tau} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{2}\right)$$
$$= (\bar{R}_{e\mu} + \bar{R}_{e\tau})\left[(\bar{I}_{e\mu} - \bar{I}_{e\tau})^2 - \frac{1}{2}(\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})^2\right],$$

$$AD: (\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})(\bar{R}_{e\mu} - \bar{R}_{e\tau})\left(\bar{I}_{ee} + \bar{I}_{\mu\tau} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{2}\right)$$
$$= -(\bar{R}_{e\mu} + \bar{R}_{e\tau})\left[(\bar{R}_{e\mu} - \bar{R}_{e\tau})^2 - \frac{1}{2}(\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})^2\right],$$

$$BC: (\bar{I}_{e\mu} - \bar{I}_{e\tau})(\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})\left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)$$
$$= -(\bar{I}_{e\mu} + \bar{I}_{e\tau})\left[(\bar{I}_{e\mu} - \bar{I}_{e\tau})^2 - \frac{1}{2}(\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})^2\right],$$

$$BD: (\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})(\bar{R}_{e\mu} - \bar{R}_{e\tau})\left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)$$
$$= (\bar{I}_{e\mu} + \bar{I}_{e\tau})\left[(\bar{R}_{e\mu} - \bar{R}_{e\tau})^2 - \frac{1}{2}(\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})^2\right],$$

$$CD: (\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})(\bar{R}_{e\mu} - \bar{R}_{e\tau}) = -(\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})(\bar{I}_{e\mu} - \bar{I}_{e\tau}), \quad (15)$$

where the symbol AB (and so on) is used to indicate that the referred constraint equation results from a combination of diagonalization conditions A and B (and so on). It is easy to see that only three of these six constraint equations are independent ones. By relating the expressions for θ_{12} derived from diagonalization conditions E and F, one arrives at the constraint equation

$$EF: R_{Y12}(I_{Y11} - I_{Y22}) = I_{Y12}(R_{Y11} - R_{Y22}), \quad (16)$$

with

$$R_{Y12} = \text{sgn}(\bar{R}_{e\mu} - \bar{R}_{e\tau})\sqrt{\frac{1}{2}(\bar{R}_{e\mu} - \bar{R}_{e\tau})^2 + \frac{1}{4}(\bar{I}_{\mu\mu} - \bar{I}_{\tau\tau})^2},$$

$$I_{Y12} = \text{sgn}(\bar{I}_{e\mu} - \bar{I}_{e\tau})\sqrt{\frac{1}{2}(\bar{I}_{e\mu} - \bar{I}_{e\tau})^2 + \frac{1}{4}(\bar{R}_{\mu\mu} - \bar{R}_{\tau\tau})^2},$$

$$I_{Y11} - I_{Y22} = \bar{I}_{ee} - \bar{I}_{\mu\mu} - \bar{I}_{\tau\tau},$$

$$R_{Y11} - R_{Y22} = \frac{\bar{R}_{ee} + \bar{R}_{\mu\tau}}{2} - \frac{3}{4}(\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}) + \text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})$$
$$\times \sqrt{\frac{1}{2}(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \frac{1}{4}\left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2}.$$
$$(17)$$

Finally, a combination of diagonalization conditions A and G yields

$$AG: \bar{I}_{\mu\tau} - \bar{I}_{ee} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{2}$$
$$= \text{sgn}(\bar{R}_{e\mu} + \bar{R}_{e\tau})\sqrt{2(\bar{R}_{e\mu} + \bar{R}_{e\tau})^2 + \left(\bar{I}_{ee} + \bar{I}_{\mu\tau} + \frac{\bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}}{2}\right)^2}.$$
$$(18)$$

It should be noted that $\bar{M}_\nu$ might have such a special texture that some diagonalization condition(s) can hold automatically independent of the value of θ_{12} or θ_{13}. (For example, when one has $I_{X13} = R_{X11} + R_{X33} = 0$, diagonalization condition B holds automatically, in which case we do not need a particular θ_{13} to make such a diagonalization condition hold). In total, there are four basic cases of diagonalization condition(s) holding automatically [22]: (1) Diagonalization conditions A and G simultaneously hold automatically under the conditions of

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \qquad \bar{I}_{ee} = 0, \qquad -2\bar{I}_{\mu\tau} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}, \quad (19)$$

which combined with the constraint equations in Eqs. (15)–(18) result in a relation between the neutrino masses and mixing parameters (the so-called neutrino mass sum rule [24])

$$m_1 c_{12}^2 \sin 2\rho + m_2 s_{12}^2 \sin 2\sigma = 0. \quad (20)$$

(2) Diagonalization condition B holds automatically under the conditions of

$$\bar{I}_{e\mu} = -\bar{I}_{e\tau}, \qquad -2(\bar{R}_{ee} + \bar{R}_{\mu\tau}) = \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}, \quad (21)$$

which combined with the constraint equations in Eqs. (15)–(18) result in the neutrino mass sum rule

$$m_1 c_{12}^2 \cos 2\rho + m_2 s_{12}^2 \cos 2\sigma + m_3 = 0. \quad (22)$$

(3) Diagonalization conditions C and F simultaneously hold automatically under the conditions of

$$\bar{I}_{ee} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}, \qquad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau}, \qquad \bar{I}_{e\mu} = \bar{I}_{e\tau}, \quad (23)$$

which combined with the constraint equations in Eqs. (15)–(18) result in the neutrino mass sum rule

$$m_1 \sin 2\rho - m_2 \sin 2\sigma = 0. \quad (24)$$

(4) Diagonalization conditions D and E simultaneously hold automatically under the conditions of

$$\bar{R}_{ee} + \bar{R}_{\mu\tau} - \frac{3}{2}(\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau})$$

$$= -\text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})$$

$$\times \sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},$$

$$\bar{I}_{\mu\mu} = \bar{I}_{\tau\tau}, \qquad \bar{R}_{e\mu} = \bar{R}_{e\tau}, \tag{25}$$

which combined with the constraint equations in Eqs. (15)–(18) result in the neutrino mass sum rule

$$m_1 \cos 2\rho - m_2 \cos 2\sigma = 0. \tag{26}$$

As pointed out in Ref. [22], these cases can find a motivation from the partial μ-τ symmetry [25,26]. In the previous work [22], we have studied the various combinations of these cases themselves. In this work, we study the various combinations of these cases with two phenomenologically appealing scenarios: (1) one neutrino mass matrix element vanishing and (2) one neutrino mass vanishing.

III. ONE NEUTRINO MASS MATRIX ELEMENT VANISHING

In this section, we perform a study on the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ in the scenario of one neutrino mass matrix element vanishing. As we see shortly, the vanishing of a neutrino mass matrix element gives two neutrino mass sum rules. Therefore, we just need one more neutrino mass sum rule arising from the requirement of some diagonalization condition(s) holding automatically to completely determine the three unknown physical parameters (i.e., the absolute neutrino mass scale and two Majorana CP phases).

For later use, we give the expressions for $\bar{M}_{\alpha\beta}$ in terms of the neutrino masses and mixing parameters,

$$\bar{M}_{ee} = m_1 e^{2i\rho} c_{12}^2 c_{13}^2 + m_2 e^{2i\sigma} s_{12}^2 c_{13}^2 - m_3 s_{13}^2,$$

$$\bar{M}_{e\mu} = \frac{1}{\sqrt{2}}[m_1 e^{2i\rho} c_{12}(-s_{12} + ic_{12}s_{13})$$
$$+ m_2 e^{2i\sigma} s_{12}(c_{12} + is_{12}s_{13}) + im_3 s_{13}]c_{13},$$

$$\bar{M}_{e\tau} = \frac{1}{\sqrt{2}}[m_1 e^{2i\rho} c_{12}(s_{12} + ic_{12}s_{13})$$
$$+ m_2 e^{2i\sigma} s_{12}(-c_{12} + is_{12}s_{13}) + im_3 s_{13}]c_{13},$$

$$\bar{M}_{\mu\mu} = \frac{1}{2}[m_1 e^{2i\rho}(s_{12} - ic_{12}s_{13})^2$$
$$+ m_2 e^{2i\sigma}(c_{12} + is_{12}s_{13})^2 + m_3 c_{13}^2],$$

$$\bar{M}_{\mu\tau} = \frac{1}{2}[-m_1 e^{2i\rho}(s_{12}^2 + c_{12}^2 s_{13}^2)$$
$$- m_2 e^{2i\sigma}(c_{12}^2 + s_{12}^2 s_{13}^2) + m_3 c_{13}^2],$$

$$\bar{M}_{\tau\tau} = \frac{1}{2}[m_1 e^{2i\rho}(s_{12} + ic_{12}s_{13})^2$$
$$+ m_2 e^{2i\sigma}(c_{12} - is_{12}s_{13})^2 + m_3 c_{13}^2], \tag{27}$$

which are reconstructed as

$$\bar{M}_\nu = O_{23} U_{13} O_{12} \text{Diag}(m_1 e^{2i\rho}, m_2 e^{2i\sigma}, m_3) O_{12}^T U_{13}^T O_{23}^T. \tag{28}$$

In the calculations, $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ have been input.

A. $\bar{M}_{ee} = 0$

In the case of $\bar{M}_{ee} = 0$, one has $I_{X11} = \bar{I}_{ee} = 0$, which leads diagonalization conditions A(G) to hold automatically. [Here and in the following we use the phrases A(G), C(F), and D(E) to make it evident that diagonalization conditions A and G, C and F, and D and E always simultaneously hold automatically, respectively]. Hence there are four new constraint equations [i.e., those in Eq. (19) and $\bar{R}_{ee} = 0$] in addition to those in Eqs. (15)–(18). As a result, only three (i.e., EF and two of BC, BD, and CD) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are seven independent constraint equations in total, more than those in Eqs. (15)–(18) by two. It is thus natural to expect that there are two more predictions for the neutrino masses and mixing parameters in addition to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$, which are directly obtained as

$$m_1 c_{12}^2 \sin 2\rho + m_2 s_{12}^2 \sin 2\sigma = 0,$$
$$(m_1 c_{12}^2 \cos 2\rho + m_2 s_{12}^2 \cos 2\sigma)c_{13}^2 - m_3 s_{13}^2 = 0, \tag{29}$$

from Eq. (27) by taking $\bar{M}_{ee} = 0$. Not surprisingly, one of them is the same as that in Eq. (20). In the following, we study the various cases in which one more neutrino mass sum rule arises from the requirement of some diagonalization condition(s) in addition to A(G) also holding automatically so that the three unknown physical parameters can be completely determined.

In the case of diagonalization condition B holding automatically in combination with $\bar{M}_{ee} = 0$, there are the following six new constraint equations [i.e., those in Eqs. (19) and (21) and $\bar{R}_{ee} = 0$] in addition to those in Eqs. (15)–(18),

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \qquad \bar{I}_{ee} = 0, \qquad -2\bar{I}_{\mu\tau} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau},$$
$$\bar{I}_{e\mu} = -\bar{I}_{e\tau}, \qquad \bar{R}_{ee} = 0, \qquad -2\bar{R}_{\mu\tau} = \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}, \tag{30}$$

which can be recombined into

$$\bar{M}_{e\mu} = -\bar{M}_{e\tau}, \qquad -2\bar{M}_{\mu\tau} = \bar{M}_{\mu\mu} + \bar{M}_{\tau\tau}, \qquad \bar{M}_{ee} = 0. \tag{31}$$

As a result, only two (i.e., CD and EF) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (22) and (29). It is found that these sum rules

have no chance of being in agreement with the realistic results. [Note that the sum rule in Eq. (22) can only be fulfilled in the IO case while the second one in Eq. (29) can only be fulfilled in the NO case].

In the case of diagonalization conditions C(F) holding automatically in combination with $\bar{M}_{ee} = 0$, there are the following seven new constraint equations [i.e., those in Eqs. (19) and (23) and $\bar{R}_{ee} = 0$] in addition to those in Eqs. (15)–(18),

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \quad \bar{I}_{e\mu} = \bar{I}_{e\tau}, \quad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau}, \quad \bar{I}_{\mu\mu} = -\bar{I}_{\tau\tau},$$
$$\bar{R}_{ee} = \bar{I}_{ee} = \bar{I}_{\mu\tau} = 0, \tag{32}$$

which can be recombined into

$$\bar{M}_{e\mu} = -\bar{M}_{e\tau}^*, \quad \bar{M}_{\mu\mu} = \bar{M}_{\tau\tau}^*,$$
$$\bar{M}_{ee} = 0, \quad \bar{M}_{\mu\tau} \text{ being real.} \tag{33}$$

As a result, only one (i.e., BD) of the constraint equations in Eqs. (15)–(18) is still an independent one. So there are three neutrino mass sum rules in total, which are given by Eqs. (24) and (29). By solving these equations, one obtains $m_1 = 0.006$ eV with $[\rho, \sigma] = [0, \pi/2]$ or 0.002 eV with $[\rho, \sigma] = [\pi/2, 0]$ in the NO case. For these two possible results, the effective electron neutrino mass m_β takes a value of 0.011 or 0.010 eV while the neutrino mass sum Σ takes a value of 0.068 or 0.062 eV. On the other hand, $\bar{M}_\nu$ and the magnitudes of its elements (e.g., $\bar{M}_{ee}$) that govern the associated LNV processes (e.g., the neutrinoless double beta decay) are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0 & -0.006 - 0.005i & 0.006 - 0.005i \\ \times & 0.022 + 0.001i & 0.028 \\ \times & \times & 0.022 - 0.001i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0 & 0.008 & 0.008 \\ \times & 0.022 & 0.028 \\ \times & \times & 0.022 \end{pmatrix}, \tag{34}$$

or

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0 & 0.004 - 0.005i & -0.004 - 0.005i \\ \times & 0.027 - 0.001i & 0.022 \\ \times & \times & 0.027 + 0.001i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0 & 0.006 & 0.006 \\ \times & 0.027 & 0.022 \\ \times & \times & 0.027 \end{pmatrix}. \tag{35}$$

From these results, one can in a more intuitive way appreciate the texture of $\bar{M}_\nu$.

In the case of diagonalization conditions D(E) holding automatically in combination with $\bar{M}_{ee} = 0$, there are the

following seven new constraint equations [i.e., those in Eqs. (19) and (25) and $\bar{R}_{ee} = 0$],

$$\bar{R}_{\mu\tau} - \frac{3}{2}(\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau})$$
$$= -\text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})\sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \left(\bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},$$
$$\bar{R}_{ee} = \bar{R}_{e\mu} = \bar{R}_{e\tau} = \bar{I}_{ee} = 0, \quad \bar{I}_{\mu\mu} = \bar{I}_{\tau\tau} = -\bar{I}_{\mu\tau}, \tag{36}$$

in addition to those in Eqs. (15)–(18). As a result, only one (i.e., BC) of the constraint equations in Eqs. (15)–(18) is still an independent one. So there are three neutrino mass sum rules in total, which are given by Eqs. (26) and (29). By solving these equations, one obtains $m_1 = 0.004$ eV with $[\rho, \sigma] = [0.79\pi, 0.23\pi]$ in the NO case. For such a result, m_β and Σ respectively take a value of 0.010 and 0.065 eV, while $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0 & -0.001i & -0.010i \\ \times & 0.026 + 0.003i & 0.024 - 0.003i \\ \times & \times & 0.024 + 0.003i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0 & 0.001 & 0.010 \\ \times & 0.026 & 0.024 \\ \times & \times & 0.024 \end{pmatrix}. \tag{37}$$

B. $\bar{M}_{e\mu} = 0$ or $\bar{M}_{e\tau} = 0$

In the case of $\bar{M}_{e\mu} = 0$ ($\bar{M}_{e\tau} = 0$), we simply get two new constraint equations $\bar{R}_{e\mu} = \bar{I}_{e\mu} = 0$ ($\bar{R}_{e\tau} = \bar{I}_{e\tau} = 0$) in addition to those in Eqs. (15)–(18). The resulting two neutrino mass sum rules are directly obtained as

$$m_1 c_{12}^2 s_{13} \cos 2\rho - (+)m_1 c_{12} s_{12} \sin 2\rho + m_2 s_{12}^2 s_{13} \cos 2\sigma$$
$$+ (-)m_2 c_{12} s_{12} \sin 2\sigma + m_3 s_{13} = 0,$$
$$m_1 c_{12} s_{12} \cos 2\rho + (-)m_1 c_{12}^2 s_{13} \sin 2\rho - m_2 c_{12} s_{12} \cos 2\sigma$$
$$+ (-)m_2 s_{12}^2 s_{13} \sin 2\sigma = 0, \tag{38}$$

from Eq. (27) by taking $\bar{M}_{e\mu} = 0$ ($\bar{M}_{e\tau} = 0$). In the following, we study the various cases where one more neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the three unknown physical parameters can be completely determined. As a result of $\bar{M}_{e\mu} = 0$ ($\bar{M}_{e\tau} = 0$), one has $R_{X12} = -(+)R_{X13}$ and $I_{X12} = -(+)I_{X13}$, which leads diagonalization conditions A(G) and D(E) and B and C(F) to simultaneously hold automatically, respectively.

In the case of diagonalization conditions A(G) and D(E) holding automatically in combination with $\bar{M}_{e\mu} = 0$, there

are the following seven new constraint equations [i.e., those in Eqs. (19) and (25) and $\bar{I}_{e\mu} = 0$],

$$\bar{R}_{ee} + \bar{R}_{\mu\tau} - \frac{3}{2}(\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau})$$

$$= -\text{sgn}(\bar{I}_{e\tau})\sqrt{2\bar{I}_{e\tau}^2 + \left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},$$

$$\bar{R}_{e\mu} = \bar{R}_{e\tau} = \bar{I}_{ee} = \bar{I}_{e\mu} = 0, \qquad \bar{I}_{\mu\mu} = \bar{I}_{\tau\tau} = -\bar{I}_{\mu\tau}, \quad (39)$$

in addition to those in Eqs. (15)–(18). As a result, only one (i.e., BC) of the constraint equations in Eqs. (15)–(18) is still an independent one. So there are three neutrino mass sum rules in total, which are given by three independent ones of Eqs. (20), (26), and (38). By solving these equations, one obtains $m_1 = 0.009$ eV with $[\rho, \sigma] = [0.41\pi, 0.65\pi]$ in the NO case or $m_3 = 0.006$ eV with $[\rho, \sigma] = [0.51\pi, 0.47\pi]$ in the IO case. For these two possible results, m_β takes a value of 0.012 or 0.050 eV while Σ takes a value of 0.072 or 0.11 eV. On the other hand, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.008 & 0 & 0.009i \\ \times & 0.022 - 0.003i & 0.029 + 0.003i \\ \times & \times & 0.020 - 0.003i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.008 & 0 & 0.009 \\ \times & 0.022 & 0.029 \\ \times & \times & 0.020 \end{pmatrix}, \quad (40)$$

or

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.049 & 0 & -0.009i \\ \times & -0.022 + 0.003i & 0.028 - 0.003i \\ \times & \times & -0.020 + 0.003i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.049 & 0 & 0.009 \\ \times & 0.022 & 0.028 \\ \times & \times & 0.020 \end{pmatrix}. \quad (41)$$

In the case of diagonalization conditions B and C(F) holding automatically in combination with $\bar{M}_{e\mu} = 0$, there are the following six new constraint equations [i.e., those in Eqs. (21) and (23) and $\bar{R}_{e\mu} = 0$],

$$\bar{I}_{ee} = \bar{I}_{\mu\mu} + \bar{I}_{\tau\tau}, \qquad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau} = -(\bar{R}_{ee} + \bar{R}_{\mu\tau}),$$
$$\bar{R}_{e\mu} = \bar{I}_{e\mu} = \bar{I}_{e\tau} = 0, \quad (42)$$

in addition to those in Eqs. (15)–(18). As a result, only two (i.e., AD and AG) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by three independent ones of Eqs. (22), (24), and (38).

By solving these equations, one obtains $m_3 = 0.0007$ eV with $[\rho, \sigma] = [0.27\pi, 0.22\pi]$ in the IO case. For such a result, m_β and Σ respectively take a value of 0.049 and 0.10 eV, while $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.001 + 0.048i & 0 & -0.010 \\ \times & 0.003 + 0.025i & -0.002 - 0.025i \\ \times & \times & 0.003 + 0.023i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.048 & 0 & 0.010 \\ \times & 0.025 & 0.025 \\ \times & \times & 0.023 \end{pmatrix}. \quad (43)$$

In the case of some diagonalization condition(s) holding automatically in combination with $\bar{M}_{e\tau} = 0$, from the above results in the case of the same diagonalization condition(s) holding automatically in combination with $\bar{M}_{e\mu} = 0$, the constraint equations and the resulting $\bar{M}_\nu$ can be obtained by making the interchanges $\bar{R}_{e\mu} \leftrightarrow -\bar{R}_{e\tau}$, $\bar{I}_{e\mu} \leftrightarrow \bar{I}_{e\tau}$, $\bar{R}_{\mu\mu} \leftrightarrow \bar{R}_{\tau\tau}$, and $\bar{I}_{\mu\mu} \leftrightarrow -\bar{I}_{\tau\tau}$ and a sign change for $\bar{I}_{ee}$ and $\bar{I}_{\mu\tau}$, while the predictions for the three unknown physical parameters can be obtained by making the replacements $\rho \to \pi - \rho$ and $\sigma \to \pi - \sigma$. This reflects a symmetry between the μ and τ flavors.

C. $\bar{M}_{\mu\mu} = 0$ or $\bar{M}_{\tau\tau} = 0$

In the case of $\bar{M}_{\mu\mu} = 0$ ($\bar{M}_{\tau\tau} = 0$), we simply get two new constraint equations $\bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0$ ($\bar{R}_{\tau\tau} = \bar{I}_{\tau\tau} = 0$) in addition to those in Eqs. (15)–(18). The resulting two neutrino mass sum rules are directly obtained as

$$m_1(s_{12}^2 - c_{12}^2 s_{13}^2)\cos 2\rho + (-)2m_1 c_{12} s_{12} s_{13}\sin 2\rho$$
$$+ m_2(c_{12}^2 - s_{12}^2 s_{13}^2)\cos 2\sigma - (+)2m_2 c_{12} s_{12} s_{13}\sin 2\sigma$$
$$+ m_3 c_{13}^2 = 0,$$

$$2m_1 c_{12} s_{12} s_{13}\cos 2\rho - (+)m_1(s_{12}^2 - c_{12}^2 s_{13}^2)\sin 2\rho$$
$$- 2m_2 c_{12} s_{12} s_{13}\cos 2\sigma - (+)m_2(c_{12}^2 - s_{12}^2 s_{13}^2)\sin 2\sigma = 0,$$
$$(44)$$

from Eq. (27) by taking $\bar{M}_{\mu\mu} = 0$ ($\bar{M}_{\tau\tau} = 0$). In the following, we study the various cases where one more neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the three unknown physical parameters can be completely determined.

In the case of diagonalization conditions A(G) holding automatically in combination with $\bar{M}_{\mu\mu} = 0$, there are the following five new constraint equations [i.e., those in Eq. (19) and $\bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0$],

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \quad \bar{I}_{ee} = \bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0, \quad -2\bar{I}_{\mu\tau} = \bar{I}_{\tau\tau}, \quad (45)$$

in addition to those in Eqs. (15)–(18). As a result, only three (i.e., EF and two of BC, BD, and CD) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are totally three neutrino mass sum rules, which are given by Eqs. (20) and (44). By solving these equations, one obtains $m_3 = 0.024$ eV with $[\rho, \sigma] = [0.97\pi, 0.43\pi]$ in the IO case. For such a result, m_β and Σ respectively take a value of 0.055 and 0.13 eV, while $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.021 & -0.033 + 0.016i & 0.033 - 0.007i \\ \times & 0 & 0.020 - 0.007i \\ \times & \times & 0.005 + 0.014i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.021 & 0.037 & 0.034 \\ \times & 0 & 0.021 \\ \times & \times & 0.015 \end{pmatrix}. \quad (46)$$

In the case of diagonalization condition B holding automatically in combination with $\bar{M}_{\mu\mu} = 0$, there are the following four new constraint equations [i.e., those in Eq. (21) and $\bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0$],

$$\bar{I}_{e\mu} = -\bar{I}_{e\tau}, \quad -2(\bar{R}_{ee} + \bar{R}_{\mu\tau}) = \bar{R}_{\tau\tau}, \quad \bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0, \quad (47)$$

in addition to those in Eqs. (15)–(18). As a result, only four (i.e., AG, EF, and two of AC, AD, and CD) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (22) and (44). By solving these equations, one obtains $m_3 = 0.050$ eV with $[\rho, \sigma] = [0.35\pi, 0.54\pi]$ in the IO case. For such a result, m_β and Σ respectively take a value of 0.070 and 0.19 eV, while $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.050 + 0.033i & -0.012 - 0.024i & 0.005 + 0.024i \\ \times & 0 & 0.055 - 0.003i \\ \times & \times & -0.010 + 0.004i \end{pmatrix}, \quad \frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.060 & 0.027 & 0.025 \\ \times & 0 & 0.055 \\ \times & \times & 0.011 \end{pmatrix}. \quad (48)$$

In the case of diagonalization conditions C(F) holding automatically in combination with $\bar{M}_{\mu\mu} = 0$, there are the following five new constraint equations [i.e., those in Eq. (23) and $\bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0$],

$$\bar{I}_{ee} = \bar{I}_{\tau\tau}, \qquad \bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = \bar{R}_{\tau\tau} = 0,$$
$$\bar{I}_{e\mu} = \bar{I}_{e\tau}, \qquad (49)$$

in addition to those in Eqs. (15)–(18). As a result, only three (i.e., AG and two of AB, AD, and BD) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (24) and (44). By solving these equations, one obtains $m_3 = 0.022$ eV with $[\rho, \sigma] = [0.04\pi, 0.46\pi]$ in the IO case. For such a result, m_β and Σ respectively take a value of 0.055 and 0.13 eV, while $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.019 + 0.014i & -0.035 + 0.004i & 0.032 + 0.004i \\ \times & 0 & 0.021 - 0.007i \\ \times & \times & 0.014i \end{pmatrix}, \quad \frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.024 & 0.035 & 0.032 \\ \times & 0 & 0.022 \\ \times & \times & 0.014 \end{pmatrix}. \quad (50)$$

In the case of diagonalization conditions D(E) holding automatically in combination with $\bar{M}_{\mu\mu} = 0$, there are five new constraint equations [i.e., those in Eq. (25) and $\bar{R}_{\mu\mu} = \bar{I}_{\mu\mu} = 0$] in addition to those in Eqs. (15)–(18). As a result, only three (i.e., AG and two of AB, AC, and BC) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (26) and (44). It is found that these sum rules have no chance to be in agreement with the realistic results.

In the case of some diagonalization condition(s) holding automatically in combination with $\bar{M}_{\tau\tau} = 0$, from the above results in the case of the same diagonalization condition(s)

holding automatically in combination with $\bar{M}_{\mu\mu} = 0$, the constraint equations and the resulting $\bar{M}_\nu$ can be obtained by making the interchanges $\bar{R}_{e\mu} \leftrightarrow -\bar{R}_{e\tau}$, $\bar{I}_{e\mu} \leftrightarrow \bar{I}_{e\tau}$, $\bar{R}_{\mu\mu} \leftrightarrow \bar{R}_{\tau\tau}$, and $\bar{I}_{\mu\mu} \leftrightarrow -\bar{I}_{\tau\tau}$ and a sign change for $\bar{I}_{ee}$ and $\bar{I}_{\mu\tau}$, while the predictions for the three unknown physical parameters can be obtained by making the replacements $\rho \to \pi - \rho$ and $\sigma \to \pi - \sigma$. This reflects a symmetry between the μ and τ flavors.

D. $\bar{M}_{\mu\tau} = 0$

In the case of $\bar{M}_{\mu\tau} = 0$, we simply get two new constraint equations $\bar{R}_{\mu\tau} = \bar{I}_{\mu\tau} = 0$ in addition to those in

Eqs. (15)–(18). The resulting two neutrino mass sum rules are directly obtained as

$$m_1 \cos 2\rho(s_{12}^2 + c_{12}^2 s_{13}^2) + m_2 \cos 2\sigma(c_{12}^2 + s_{12}^2 s_{13}^2)$$
$$- m_3 c_{13}^2 = 0,$$
$$m_1 \sin 2\rho(s_{12}^2 + c_{12}^2 s_{13}^2) + m_2 \sin 2\sigma(c_{12}^2 + s_{12}^2 s_{13}^2) = 0, \quad (51)$$

from Eq. (27) by taking $\bar{M}_{\mu\tau} = 0$. In the following, we study the various cases where one more neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the three unknown physical parameters can be completely determined. As a result of $\bar{M}_{\mu\tau} = 0$, one has $I_{Y11} - I_{Y22} = I_{X11} - 2I_{X33}$, which leads diagonalization conditions A(G) and C(F) to always simultaneously hold automatically.

In the case of diagonalization conditions A(G) and C(F) holding automatically in combination with $\bar{M}_{\mu\tau} = 0$, there are the following seven new constraint equations [i.e., those in Eqs. (19) and (23) and $\bar{R}_{\mu\tau} = 0$] in addition to those in Eqs. (15)–(18),

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \qquad \bar{I}_{e\mu} = \bar{I}_{e\tau}, \qquad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau}, \qquad \bar{I}_{\mu\mu} = -\bar{I}_{\tau\tau},$$
$$\bar{I}_{ee} = \bar{R}_{\mu\tau} = \bar{I}_{\mu\tau} = 0, \qquad (52)$$

which can be recombined into

$$\bar{M}_{e\mu} = -\bar{M}_{e\tau}^*, \qquad \bar{M}_{\mu\mu} = \bar{M}_{\tau\tau}^*,$$
$$\bar{M}_{\mu\tau} = 0, \qquad \bar{M}_{ee} \text{ being real.} \qquad (53)$$

As a result, only one (i.e., BD) of the constraint equations in Eqs. (15)–(18) is still an independent one. So there are three neutrino mass sum rules in total, which are given by three independent ones of Eqs. (20), (24), and (51). By solving these equations, one obtains $m_1 = 0.17$ eV with $[\rho, \sigma] = [0, 0]$ in the NO case or $m_3 = 0.021$ eV with $[\rho, \sigma] = [\pi/2, 0]$ in the IO case. For these two possible results, m_β takes a value of 0.17 or 0.054 eV while Σ takes a value of 0.51 or 0.13 eV. Note that the result of $m_1 = 0.17$ eV is strongly disfavored by the cosmological measurements for the neutrino mass sum. On the other hand, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.158 & 0.035\text{i} & 0.035\text{i} \\ \times & 0.165 & 0 \\ \times & \times & 0.165 \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.158 & 0.035 & 0.035 \\ \times & 0.165 & 0 \\ \times & \times & 0.165 \end{pmatrix}, \qquad (54)$$

or

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.021 & 0.035 & -0.035 \\ \times & 0.021 + 0.007\text{i} & 0 \\ \times & \times & 0.021 - 0.007\text{i} \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.021 & 0.035 & 0.035 \\ \times & 0.022 & 0 \\ \times & \times & 0.022 \end{pmatrix}. \qquad (55)$$

In the case of diagonalization condition B holding automatically in combination with $\bar{M}_{\mu\tau} = 0$, there are four new constraint equations [i.e., those in Eq. (21) and $\bar{R}_{\mu\tau} = \bar{I}_{\mu\tau} = 0$] in addition to those in Eqs. (15)–(18). As a result, only four (i.e., AG, EF, and two of AC, AD, and CD) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (22) and (51). It is found that these sum rules have no chance to be in agreement with the realistic results.

In the case of diagonalization conditions D(E) holding automatically in combination with $\bar{M}_{\mu\tau} = 0$, there are five new constraint equations [i.e., those in Eq. (25) and $\bar{R}_{\mu\tau} = \bar{I}_{\mu\tau} = 0$] in addition to those in Eqs. (15)–(18). As a result, only three (i.e., AG and two of AB, AC, and BC) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are three neutrino mass sum rules in total, which are given by Eqs. (26) and (51). It is found that these sum rules have no chance to be in agreement with the realistic results.

IV. ONE NEUTRINO MASS VANISHING

In this section, we perform a study on the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ in the scenario of one neutrino mass vanishing. Given a vanishing neutrino mass, the neutrino mass spectrum can be fixed with the help of measured neutrino mass squared differences. On the other hand, there is only one effective Majorana CP phase that is specified to be σ. So we just need one neutrino mass sum rule arising from the requirement of some diagonalization condition(s) holding automatically to completely determine the neutrino physical parameters.

A. $m_1 = 0$

In the case of $m_1 = 0$, the vanishing of $\text{Re}(m_1 e^{2i\rho})$ and $\text{Im}(m_1 e^{2i\rho})$ gives the following two new conditions,

$$\text{H}: \ \sin 2\theta_{12} R_{Y12} = c_{12}^2 R_{Y11} + s_{12}^2 R_{Y22},$$
$$\text{I}: \ \sin 2\theta_{12} I_{Y12} = c_{12}^2 I_{Y11} + s_{12}^2 I_{Y22}, \qquad (56)$$

in addition to those in Eq. (13). By relating the expressions for θ_{12} derived from diagonalization conditions E and H and F and I, we obtain the following two new constraint equations in addition to those in Eqs. (15)–(18),

EH: $R_{Y11} + R_{Y22} = \text{sgn}(R_{Y12})\sqrt{4(R_{Y12})^2 + (R_{Y11} - R_{Y22})^2}$,

FI: $I_{Y11} + I_{Y22} = \text{sgn}(I_{Y12})\sqrt{4(I_{Y12})^2 + (I_{Y11} - I_{Y22})^2}$,

$$(57)$$

where the expressions for R_{Y12}, I_{Y12}, $R_{Y11} - R_{Y22}$, and $I_{Y11} - I_{Y22}$ have been given in Eq. (17), while $R_{Y11} + R_{Y22}$ and $I_{Y11} + I_{Y22}$ are given by

$$R_{Y11} + R_{Y22} = \frac{\bar{R}_{ee} - 3\bar{R}_{\mu\tau}}{2} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{4} + \text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})$$

$$\times \sqrt{\frac{1}{2}(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \frac{1}{4}\left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},$$

$$I_{Y11} + I_{Y22} = \bar{I}_{ee} - 2\bar{I}_{\mu\tau}. \tag{58}$$

So there are seven independent constraint equations in total. In the following, we study the various cases where one neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the only unknown physical parameter σ can be determined. Before proceeding, we make two observations: (1) It is easy to see that diagonalization conditions E and H and F and I always simultaneously hold automatically, respectively. (2) When diagonalization conditions A(G) hold automatically, one has $I_{Y11} = 0$, which leads diagonalization conditions F and I to hold automatically too. So diagonalization conditions A(G), C(F), and I always simultaneously hold automatically.

In the case of diagonalization conditions A(G), C(F), and I holding automatically, there are the following six new constraint equations [i.e., those in Eqs. (19) and (23)] in addition to those in Eqs. (15)–(18) and (57),

$$\bar{R}_{e\mu} = -\bar{R}_{e\tau}, \qquad \bar{I}_{e\mu} = \bar{I}_{e\tau}, \qquad \bar{R}_{\mu\mu} = \bar{R}_{\tau\tau},$$

$$\bar{I}_{\mu\mu} = -\bar{I}_{\tau\tau}, \qquad \bar{I}_{ee} = \bar{I}_{\mu\tau} = 0, \tag{59}$$

which can be recombined into

$$\bar{M}_{e\mu} = -\bar{M}_{e\tau}^*, \qquad \bar{M}_{\mu\mu} = \bar{M}_{\tau\tau}^*,$$

$$\bar{M}_{ee} \quad \text{and} \quad \bar{M}_{\mu\tau} \text{ being real.} \tag{60}$$

As a result, only two (i.e., BD and EH) of the constraint equations in Eqs. (15)–(18) and (57) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$\sigma = 0 \quad \text{or} \quad \frac{\pi}{2}, \tag{61}$$

from Eqs. (20) and (24) by taking $m_1 = 0$. For such a result, m_β and Σ respectively take a value of 0.009 and 0.059 eV.

On the other hand, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.002 & 0.003 + 0.005i & -0.003 + 0.005i \\ \times & 0.028 + 0.001i & 0.022 \\ \times & \times & 0.028 - 0.001i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.002 & 0.006 & 0.006 \\ \times & 0.028 & 0.022 \\ \times & \times & 0.028 \end{pmatrix}, \tag{62}$$

for $\sigma = 0$, or

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.004 & -0.003 + 0.005i & 0.003 + 0.005i \\ \times & 0.022 - 0.001i & 0.028 \\ \times & \times & 0.022 + 0.001i \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.004 & 0.006 & 0.006 \\ \times & 0.022 & 0.028 \\ \times & \times & 0.022 \end{pmatrix}, \tag{63}$$

for $\sigma = \pi/2$.

In the case of diagonalization condition B holding automatically, there are two new constraint equations given by Eq. (21) in addition to those in Eqs. (15)–(18) and (57). As a result, only six (i.e., AG, EF, EH, FI, and two of AC, AD, and CD) of the constraint equations in Eqs. (15)–(18) and (57) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$m_2 s_{12}^2 \cos 2\sigma + m_3 = 0, \tag{64}$$

from Eq. (22) by taking $m_1 = 0$. Apparently, this sum rule can never be fulfilled in the NO case.

In the case of diagonalization conditions D(E) and H holding automatically, there are the following four new constraint equations [i.e., those in Eq. (25) and $R_{Y11} = R_{Y22} = 0$],

$$\bar{R}_{ee} - 2\bar{R}_{\mu\tau} = -\text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})\sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + (\bar{R}_{ee} + 2\bar{R}_{\mu\tau})^2},$$

$$\bar{I}_{\mu\mu} = \bar{I}_{\tau\tau}, \quad \bar{R}_{e\mu} = \bar{R}_{e\tau}, \quad \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} = 2\bar{R}_{\mu\tau}, \tag{65}$$

in addition to those in Eqs. (15)–(18) and (57). As a result, only four (i.e., equations AG, FI, and two of AB, AC, and BC) of the constraint equations in Eqs. (15)–(18) and (57) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$\sigma = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}, \tag{66}$$

from Eq. (26) by taking $m_1 = 0$. For such a result, m_β and Σ respectively take a value of 0.009 and 0.059 eV. In the case

of $\sigma = \pi/4$, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$
\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.001+0.003i & 0.008i & 0.002i \\ \times & 0.024+0.003i & 0.025-0.003i \\ \times & \times & 0.025+0.003i \end{pmatrix},
$$

$$
\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.003 & 0.008 & 0.002 \\ \times & 0.024 & 0.025 \\ \times & \times & 0.025 \end{pmatrix}. \tag{67}
$$

In the case of $\sigma = 3\pi/4$, $\bar{M}_\nu$ can be obtained by making the interchanges $\bar{I}_{e\mu} \leftrightarrow \bar{I}_{e\tau}$, $\bar{R}_{\mu\mu} \leftrightarrow \bar{R}_{\tau\tau}$, and $\bar{I}_{\mu\mu} \leftrightarrow -\bar{I}_{\tau\tau}$ and a sign change for $\bar{I}_{ee}$ and $\bar{I}_{\mu\tau}$ in the $\bar{M}_\nu$ given by Eq. (67).

B. $m_3 = 0$

In the case of $m_3 = 0$, the vanishing of $\text{Im}(m_1 e^{2i\rho})$ and $\text{Re}(m_3)$ gives the following two new conditions,

$$
\text{I: } \sin 2\theta_{12} I_{Y12} = c_{12}^2 I_{Y11} + s_{12}^2 I_{Y22},
$$
$$
\text{J: } \sin 2\theta_{13} I_{X13} = s_{13}^2 R_{X11} - c_{13}^2 R_{X33}, \tag{68}
$$

in addition to those in Eq. (13). By relating the expressions for θ_{13} and θ_{12} derived from diagonalization conditions B and J and F and I, we obtain the following two new constraint equations,

$$
\text{BJ: } \bar{R}_{ee} - \bar{R}_{\mu\tau} - \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}
$$
$$
= \text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau}) \sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + \left(\bar{R}_{ee} + \bar{R}_{\mu\tau} + \frac{\bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}}{2}\right)^2},
$$

$$
\text{FI: } I_{Y11} + I_{Y22}
$$
$$
= \text{sgn}(I_{Y12}) \sqrt{4(I_{Y12})^2 + (I_{Y11} - I_{Y22})^2}, \tag{69}
$$

in addition to those in Eqs. (15)–(18). So there are seven independent constraint equations in total. In the following, we study the various cases where one neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the only unknown physical parameter σ can be determined. Before proceeding, we make two observations: (1) It is easy to see that diagonalization conditions B and J and F and I always simultaneously hold automatically, respectively. (2) When diagonalization conditions A(G) hold automatically, one has $I_{Y11} = 0$, which leads diagonalization conditions F and I to hold automatically too. So diagonalization conditions A(G), C(F), and I always simultaneously hold automatically.

In the case of diagonalization conditions A(G), C(F), and I holding automatically, there are six new constraint equations given by Eq. (59) in addition to those in

Eqs. (15)–(18) and (69). As a result, only two (i.e., BD and BJ) of the constraint equations in Eqs. (15)–(18) and (69) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained from Eqs. (20) and (24) by taking $\rho = 0$ and is the same as that in Eq. (61). For such a result, m_β and Σ respectively take a value of 0.049 and 0.10 eV. On the other hand, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$
\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.049 & 0.005i & 0.005i \\ \times & 0.024 & -0.025 \\ \times & \times & 0.024 \end{pmatrix},
$$

$$
\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.049 & 0.005 & 0.005 \\ \times & 0.024 & 0.025 \\ \times & \times & 0.024 \end{pmatrix}, \tag{70}
$$

for $\sigma = 0$, or

$$
\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.019 & -0.032+0.002i & 0.032+0.002i \\ \times & -0.010-0.007i & 0.010 \\ \times & \times & -0.010+0.007i \end{pmatrix},
$$

$$
\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.019 & 0.032 & 0.032 \\ \times & 0.012 & 0.010 \\ \times & \times & 0.012 \end{pmatrix}, \tag{71}
$$

for $\sigma = \pi/2$.

In the case of diagonalization conditions B and J holding automatically, there are the following three new constraint equations [i.e., those in Eq. (21) and $R_{X11} = R_{X33} = 0$],

$$
\bar{I}_{e\mu} = -\bar{I}_{e\tau}, \quad -2\bar{R}_{\mu\tau} = \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau}, \quad \bar{R}_{ee} = 0, \tag{72}
$$

in addition to those in Eqs. (15)–(18) and (69). As a result, only five (i.e., AG, EF, FI, and two of AC, AD, and CD) of the constraint equations in Eqs. (15)–(18) and (69) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$
m_1 c_{12}^2 + m_2 s_{12}^2 \cos 2\sigma = 0, \tag{73}
$$

from Eq. (22) by taking $\rho = m_3 = 0$. Apparently, this sum rule can never be fulfilled in the IO case.

In the case of diagonalization conditions D(E) holding automatically, there are three new constraint equations given by Eq. (25) in addition to those in Eqs. (15)–(18) and (69). As a result, only five (i.e., AG, BJ, FI, and two of AB, AC, and BC) of the constraint equations in Eqs. (15)–(18) and (69) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$
m_1 - m_2 \cos 2\sigma = 0, \tag{74}
$$

from Eq. (26) by taking $\rho = m_3 = 0$. By solving this equation, one obtains $\sigma = 0.03\pi$ or 0.97π. For such a result, m_β and Σ respectively take a value of 0.049 and 0.10 eV. In the case of $\sigma \simeq 0.03\pi$, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} 0.048+0.003\text{i} & 0.008\text{i} & 0.002\text{i} \\ \times & 0.024+0.003\text{i} & -0.025-0.003\text{i} \\ \times & \times & 0.025+0.003\text{i} \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.048 & 0.008 & 0.002 \\ \times & 0.024 & 0.025 \\ \times & \times & 0.025 \end{pmatrix}. \qquad (75)$$

In the case of $\sigma \simeq 0.97\pi$, $\bar{M}_\nu$ can be obtained by making the interchanges $\bar{I}_{e\mu} \leftrightarrow \bar{I}_{e\tau}$, $\bar{R}_{\mu\mu} \leftrightarrow \bar{R}_{\tau\tau}$, and $\bar{I}_{\mu\mu} \leftrightarrow -\bar{I}_{\tau\tau}$ and a sign change for $\bar{I}_{ee}$ and $\bar{I}_{\mu\tau}$ in the $\bar{M}_\nu$ given by Eq. (75).

V. DISCUSSIONS

In this section, we give some discussions about the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and maximal Majorana CP phases as well as the model realization and breakings of the obtained textures.

A. ρ, $\sigma = \pi/4$ or $3\pi/4$

Motivated by the μ-τ reflection symmetry that predicts $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and trivial Majorana CP phases, we make an attempt to derive the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and maximal Majorana CP phases. In this scenario, the vanishing of $\text{Re}(m_1 e^{2i\rho})$ and $\text{Re}(m_2 e^{2i\sigma})$ gives the following two new conditions,

$$\text{H}: \quad \sin 2\theta_{12} R_{Y12} = c_{12}^2 R_{Y11} + s_{12}^2 R_{Y22},$$
$$\text{K}: \quad \sin 2\theta_{12} R_{Y12} = -s_{12}^2 R_{Y11} - c_{12}^2 R_{Y22}, \qquad (76)$$

in addition to those in Eq. (13). A combination of diagonalization conditions E, H, and K results in $R_{Y11} = R_{Y22} = R_{Y12} = 0$, which lead diagonalization conditions D(E), H, and K to hold automatically. Hence there are four new constraint equations given by Eq. (65) in addition to those in Eqs. (15)–(18). As a result, only three (i.e., AG and two of AB, AC, and BC) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there are seven independent constraint equations in total. In the following, we study the various cases where one neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that the only unknown physical parameter (the absolute neutrino mass scale) can be determined.

In the case of diagonalization conditions A(G) [in addition to D(E), H, and K] holding automatically, there

are the following seven new constraint equations [i.e., those in Eqs. (19) and (65)],

$$\bar{R}_{ee} - 2\bar{R}_{\mu\tau} = -\text{sgn}(\bar{I}_{e\mu} + \bar{I}_{e\tau})\sqrt{2(\bar{I}_{e\mu} + \bar{I}_{e\tau})^2 + (\bar{R}_{ee} + 2\bar{R}_{\mu\tau})^2},$$
$$\bar{I}_{\mu\mu} = \bar{I}_{\tau\tau} = -\bar{I}_{\mu\tau}, \quad \bar{R}_{e\mu} = \bar{R}_{e\tau} = \bar{I}_{ee} = 0, \quad \bar{R}_{\mu\mu} + \bar{R}_{\tau\tau} = 2\bar{R}_{\mu\tau},$$
$$\qquad (77)$$

in addition to those in Eqs. (15)–(18). As a result, only one (i.e., BC) of the constraint equations in Eqs. (15)–(18) is still an independent one. So there is one neutrino mass sum rule, which is directly obtained as

$$m_1 c_{12}^2 \pm m_2 s_{12}^2 = 0, \qquad (78)$$

from Eq. (20) by taking $\rho, \sigma = \pi/4$ or $3\pi/4$. By solving this equation, one obtains $m_1 = 0.004$ eV with $[\rho, \sigma] = [\pi/4, 3\pi/4]$ or $[3\pi/4, \pi/4]$. For such a result, m_β and Σ respectively take a value of 0.010 and 0.064 eV. In the case of $[\rho, \sigma] = [\pi/4, 3\pi/4]$, $\bar{M}_\nu$ and the magnitudes of its elements are given by

$$\frac{\bar{M}_\nu}{\text{eV}} \simeq \begin{pmatrix} -0.001 & 0.001\text{i} & 0.010\text{i} \\ \times & 0.026-0.003\text{i} & 0.025+0.003\text{i} \\ \times & \times & 0.024-0.003\text{i} \end{pmatrix},$$

$$\frac{|\bar{M}_\nu|}{\text{eV}} \simeq \begin{pmatrix} 0.001 & 0.001 & 0.010 \\ \times & 0.026 & 0.025 \\ \times & \times & 0.024 \end{pmatrix}. \qquad (79)$$

In the case of $[\rho, \sigma] = [3\pi/4, \pi/4]$, $\bar{M}_\nu$ can be obtained by making the interchanges $\bar{I}_{e\mu} \leftrightarrow \bar{I}_{e\tau}$, $\bar{R}_{\mu\mu} \leftrightarrow \bar{R}_{\tau\tau}$, and $\bar{I}_{\mu\mu} \leftrightarrow -\bar{I}_{\tau\tau}$ and a sign change for $\bar{I}_{\mu\tau}$ in the $\bar{M}_\nu$ given by Eq. (79).

In the case of diagonalization condition B [in addition to D(E), H, and K] holding automatically, there are six new constraint equations given by Eqs. (21) and (65) in addition to those in Eqs. (15)–(18). As a result, only two (i.e., AC and AG) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$m_3 = 0, \qquad (80)$$

from Eq. (22) by taking $\rho, \sigma = \pi/4$ or $3\pi/4$. However, as discussed in Sec. IV, ρ would be fixed to 0 in the case of $m_3 = 0$. So this sum rule has no chance to be in agreement with the realistic results.

In the case of diagonalization conditions C(F) [in addition to D(E), H, and K] holding automatically, there are six new constraint equations given by Eqs. (23) and (65) in addition to those in Eqs. (15)–(18). As a result, only two (i.e., AB and AG) of the constraint equations in Eqs. (15)–(18) are still independent ones. So there is one neutrino mass sum rule, which is directly obtained as

$$m_1 \pm m_2 = 0, \qquad (81)$$

from Eq. (24) by taking ρ, $\sigma = \pi/4$ or $3\pi/4$. Apparently, this sum rule has no chance of being in agreement with the realistic results.

B. Model realization

The approach adopted by us is a bottom-up one in the sense that we start from the experimental hints for $\theta_{23} \simeq \pi/4$ and $\delta \simeq -\pi/2$ and then derive the possible textures of neutrino mass matrix that can lead to maximal θ_{23} and δ. (In a top-down approach one starts from a specific flavor theory and then derives its phenomenological consequences.) Although all the obtained textures are on an equal footing in our approach, a particular texture gains a more solid foundation if it can find an origin from some flavor symmetry. From the model realization point of view, the obtained textures can be classified into the following three categories.

First of all, we note that some of the obtained textures can find a connection with the μ-τ reflection symmetry. One can see that the resulting neutrino mass sum rules in the case of diagonalization conditions A(G) and C(F) holding automatically in combination with $\bar{M}_{ee} = 0$ ($\bar{M}_{\mu\tau} = 0$) are the same as those in the case of μ-τ reflection symmetry with $M_{ee} = 0$ ($M_{\mu\tau} = 0$) [27]. In fact, if one restores the unphysical phases with the help of $\bar{M}_{\alpha\beta} = M_{\alpha\beta}e^{-i(\phi_\alpha + \phi_\beta)}$ and takes $\phi_e = \pi/2$ and $\phi_\mu = -\phi_\tau$, the texture obtained in the former case reproduces that in the latter case. For example, under such a specification for the unphysical phases, the texture of $\bar{M}_\nu$ in Eq. (33) gives the following texture of M_ν:

$$M_{e\mu} = M_{e\tau}^*, \qquad M_{\mu\mu} = M_{\tau\tau}^*,$$
$$M_{ee} = 0, \qquad M_{\mu\tau} \text{ being real.} \qquad (82)$$

(BD just gives an expression for the unphysical phase ϕ_μ in terms of the neutrino mass matrix elements.) As a comparison, if we take $\phi_e = 0$ and $\phi_\mu = -\phi_\tau$, Eq. (33) gives

$$M_{e\mu} = -M_{e\tau}^*, \qquad M_{\mu\mu} = M_{\tau\tau}^*,$$
$$M_{ee} = 0, \qquad M_{\mu\tau} \text{ being real.} \qquad (83)$$

These two textures give the same results for the physical parameters. But the former one has the advantage of having a connection with the μ-τ reflection symmetry. Of course, one can choose any other specification for the unphysical phases, which does not alter the results for the physical parameters but may lead to a different texture. Similarly, the texture obtained in the case of diagonalization conditions A(G) and C(F) holding automatically in combination with m_1 or $m_3 = 0$ can reproduce that from the μ-τ reflection symmetry embedded in the minimal seesaw [28].

A texture belonging to the second category is one that cannot be connected to a known flavor symmetry like the μ-τ reflection symmetry but only possesses linear relations among the neutrino mass matrix elements. It may be realized with the help of some flavor symmetry. An example is the texture obtained in the case of diagonalization conditions A(G) and B holding automatically in combination with $\bar{M}_{ee} = 0$. If one restores the unphysical phases and takes $\phi_\mu = \phi_\tau + \pi$, Eq. (31) gives

$$M_{e\mu} = M_{e\tau}, \qquad 2M_{\mu\tau} = M_{\mu\mu} + M_{\tau\tau}, \qquad M_{ee} = 0. \qquad (84)$$

(CD and EF just give the expressions for the unphysical phases ϕ_e and ϕ_μ in terms of the neutrino mass matrix elements.) The texture zero $M_{ee} = 0$ can be attributed to an Abelian flavor symmetry [18] while the linear relations $M_{e\mu} = M_{e\tau}$ and $2M_{\mu\tau} = M_{\mu\mu} + M_{\tau\tau}$ may find an origin from some non-Abelian flavor symmetry [13].

A texture belonging to the third category is one that possesses not only linear but also nonlinear relations among the neutrino mass matrix elements. It can only be partially realized with the help of some flavor symmetry. Let us take the texture obtained in the case of diagonalization conditions A(G) and D(E) holding automatically in combination with $\bar{M}_{ee} = 0$ as an example. If one restores the unphysical phases and takes $\phi_e = 0$ and $\phi_\mu = -\phi_\tau = \pi/2$, Eq. (36) gives

$$R_{\mu\tau} + \frac{3}{2}(R_{\mu\mu} + R_{\tau\tau})$$
$$= \text{sgn}(R_{e\mu} - R_{e\tau})\sqrt{2(R_{e\mu} - R_{e\tau})^2 + \left(R_{\mu\tau} - \frac{R_{\mu\mu} + R_{\tau\tau}}{2}\right)^2},$$
$$R_{ee} = I_{e\mu} = I_{e\tau} = I_{ee} = 0, \qquad I_{\mu\mu} = I_{\tau\tau} = I_{\mu\tau}, \qquad (85)$$

with $R_{\alpha\beta} = \text{Re}(M_{\alpha\beta})$ and $I_{\alpha\beta} = \text{Im}(M_{\alpha\beta})$, while BC gives

$$(R_{e\mu} + R_{e\tau})(R_{\mu\mu} - R_{\tau\tau})\left(R_{\mu\tau} - \frac{R_{\mu\mu} + R_{\tau\tau}}{2}\right)$$
$$= (R_{e\mu} - R_{e\tau})\left[(R_{e\mu} + R_{e\tau})^2 - \frac{1}{2}(R_{\mu\mu} - R_{\tau\tau})^2\right]. \qquad (86)$$

In this case, the imaginary part of M_ν has a very simple structure, which may be easily realized with the help of some flavor symmetry. However, to our knowledge, flavor symmetries are unable to give nonlinear relations like those two for the real part of M_ν [i.e., the first one in Eq. (85) and that in Eq. (86)].

C. Breakings

In order to accommodate the deviations of θ_{23} and δ from $\pi/4$ and $-\pi/2$, one has to consider the breakings of the obtained textures. In the literature, at least four scenarios for the breakings of flavor symmetries (neutrino mass

matrix textures) have been considered: (1) The flavor symmetries are usually introduced at an extremely high energy scale (e.g., the seesaw scale). So the renormalization group (RG) running effect should be taken into account when one confronts the flavor symmetry models with the low energy data [29], which may induce the breakings of flavor symmetries. For example, in the RG running process the significant hierarchy between the mass of muon and that of tau can give rise to the breaking of μ-τ symmetry [30]. (2) As in this work, most of the studies on the textures of neutrino mass matrix have been performed in the basis of charged lepton mass matrix M_l being diagonal. But in a realistic model this may not be the case. (For example, in a grand unified theory inspired model, M_l is associated with the down-type quark mass matrix, which is generally treated as a nondiagonal one [31].) In this situation, even if the special texture of M_ν is realized exactly, the neutrino mixing receives corrections from the charged lepton sector [32]. (3) The LSND experiment [33] and reactor antineutrino anomaly [34] indicate the existence of eV scale sterile neutrinos mixing with the three active neutrinos. If this turns out to be true, the sterile neutrino sector may provide a source for the breakings of flavor symmetries in the active neutrino sector [35]. (4) In concrete flavor symmetry models, it is common that the special texture of M_ν only holds at the leading order but breaks to some extent at the higher orders [13].

A comprehensive study about the breakings of all the obtained textures in the above scenarios is model dependent and beyond the scope of this paper. As an example, we give a model-independent phenomenological study of the breakings of the texture given by the μ-τ reflection symmetry with $M_{ee} = 0$ ($M_{\mu\tau} = 0$). (A discussion about the breakings of the texture given by the μ-τ reflection symmetry embedded in the minimal seesaw can be found in Ref. [28].) Corresponding to the four symmetry conditions in Eq. (8) one by one, the following four dimensionless parameters can be introduced to measure the breaking strengths of μ-τ reflection symmetry [36],

$$\epsilon_1 = \frac{M_{e\mu} - M_{e\tau}^*}{M_{e\mu} + M_{e\tau}^*}, \qquad \epsilon_2 = \frac{M_{\mu\mu} - M_{\tau\tau}^*}{M_{\mu\mu} + M_{\tau\tau}^*},$$

$$\epsilon_3 = \frac{I_{ee}}{R_{ee}}, \qquad \epsilon_4 = \frac{I_{\mu\tau}}{R_{\mu\tau}}, \qquad (87)$$

whose magnitudes should be small enough (e.g., ≤ 0.1) in order to keep the symmetry as an approximate one. Two immediate comments are given as follows: (1) A model-specific breaking is characterized by a given pattern of these four symmetry-breaking parameters. For example, a RG induced breaking of μ-τ reflection symmetry is characterized by $I_{1,2} = \epsilon_{3,4} = 0$ [for $I_{1,2} = \text{Im}(\epsilon_{1,2})$] and $R_2 = 2R_1$ [for $R_{1,2} = \text{Re}(\epsilon_{1,2})$] [36]. Correspondingly, the implications of a model-specific breaking can be inferred from the general model-independent results. (2) It has been noted that ϵ_3 and ϵ_4 are respectively equivalent to an $I_1 \simeq -\epsilon_3/2$ and an $I_1 \simeq -\epsilon_4/2$ plus an $I_2 \simeq -\epsilon_4$ to a good approximation, thereby allowing one to pay attention to $\epsilon_{1,2}$ [36]. But for the condition of $R_{ee} = 0$ ($R_{\mu\tau} = 0$), there is not a well-defined dimensionless parameter that can be used to measure its breaking strength, so we preserve it. In the following, given a small value of $R_{1,2}$ or $I_{1,2}$, we study the deviations of physical parameters

$$\Delta m_{1,3} = m_{1,3} - m_{1,3}^{(0)}, \qquad \Delta\theta_{23} = \theta_{23} - \theta_{23}^{(0)}, \qquad \Delta\delta = \delta - \delta^{(0)},$$
$$\Delta\rho = \rho - \rho^{(0)}, \qquad \Delta\sigma = \sigma - \sigma^{(0)}, \qquad (88)$$

from their values in the symmetry limit [which are labeled by a superscript (0)].

In the case of the μ-τ reflection symmetry with $M_{ee} = 0$, the deviations of physical parameters induced by a $R_{1,2}$ or $I_{1,2}$ of the benchmark value 0.1 are given in Table I. The results outside (inside) the brackets are obtained in the case of $m_1^{(0)} = 0.006$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [0, \pi/2]$ ($m_1^{(0)} = 0.002$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [\pi/2, 0]$). It is found that $\Delta m_1/m_1^{(0)}$ is quite small, which can be understood in a way as follows: The μ-τ reflection symmetry itself is unable to give a prediction for the neutrino masses. It is $M_{ee} = 0$ that helps us fix the neutrino mass spectrum. So the preservation of $M_{ee} = 0$ ensures a small $\Delta m_1/m_1^{(0)}$. On the other hand, a considerable $\Delta\theta_{23}$ (significant $\Delta\delta$) may arise from R_2 (R_1 and I_2). Because of having distinct main origins, a considerable $\Delta\theta_{23}$ does not necessarily signify a significant $\Delta\delta$, and vice versa. In magnitude, given small values of $R_{1,2}$ and $I_{1,2}$, $\Delta\theta_{23}$ is not more than a few degrees while $\Delta\delta$ can reach dozens of degrees. Furthermore, all the deviations of CP phases induced by I_2 can be significant,

TABLE I. In the case of the μ-τ reflection symmetry with $M_{ee} = 0$, the deviations of physical parameters induced by a $R_{1,2}$ or $I_{1,2}$ of the benchmark value 0.1.

	$\Delta m_1/m_1^{(0)}$	$\Delta\theta_{23}$	$\Delta\delta$	$\Delta\rho$	$\Delta\sigma$
$R_1 = 0.1$	−0.04(0.07)	0.00 (0.00)	−0.29(0.14)	0.09 (0.11)	0.02 (0.01)
$I_1 = 0.1$	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)
$R_2 = 0.1$	0.00 (0.00)	0.04 (0.06)	−0.01(−0.03)	0.01(−0.03)	0.01(−0.01)
$I_2 = 0.1$	−0.05(0.08)	0.00 (0.00)	0.57(−0.38)	−0.33(0.11)	−0.26(0.21)

TABLE II. In the case of the μ-τ reflection symmetry with $M_{\mu\tau} = 0$, the deviations of physical parameters induced by a $R_{1,2}$ or $I_{1,2}$ of some benchmark values.

	$\Delta m_{1,3}/m_{1,3}^{(0)}$	$\Delta\theta_{23}$	$\Delta\delta$	$\Delta\rho$	$\Delta\sigma$
$R_1 = 0.1(0.01)$	0.00 (0.02)	0.10(−0.01)	0.02(−0.21)	0.00 (0.15)	0.00 (0.07)
$I_1 = 0.005$	0.02 (0.05)	0.00 (0.00)	−0.22(0.31)	0.00(−0.22)	0.00(−0.09)
$R_2 = 0.01$	−0.06 (0.01)	0.20 (0.00)	−0.01(−0.18)	0.05 (0.10)	−0.02(0.05)
$I_2 = 0.01$	0.02 (0.05)	0.00 (0.00)	0.22(−0.31)	0.00 (0.22)	0.00 (0.09)

while all the deviations of physical parameters induced by I_1 are negligibly small.

In the case of the μ-τ reflection symmetry with $M_{\mu\tau} = 0$, the deviations of physical parameters induced by a $R_{1,2}$ or $I_{1,2}$ of some benchmark values are given in Table II. The results outside (inside) the brackets are obtained in the case of $m_1^{(0)} = 0.165$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [0, 0]$ ($m_3^{(0)} = 0.021$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [\pi/2, 0]$). In most cases, even a tiny $R_{1,2}$ or $I_{1,2}$ (for which we specify 0.01 or 0.005 as the benchmark value) can induce rather large deviations of physical parameters. For a similar reason (i.e., the preservation of $M_{\mu\tau} = 0$), $\Delta m_{1,3}/m_{1,3}^{(0)}$ are quite small. In the case of $m_1^{(0)} = 0.165$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [0, 0]$, $R_{1,2}$ can induce significant $\Delta\theta_{23}$ but much smaller $\Delta\delta$, $\Delta\rho$, and $\Delta\sigma$, while $I_{1,2}$ can induce significant $\Delta\delta$ but negligibly small deviations of other physical parameters. In the case of $m_3^{(0)} = 0.021$ eV with $[\rho^{(0)}, \sigma^{(0)}] = [\pi/2, 0]$, all of $R_{1,2}$ and $I_{1,2}$ can induce significant deviations of CP phases. Unfortunately, such a case cannot accommodate a considerable $\Delta\theta_{23}$: One could obtain a sizable $\Delta\theta_{23}$ by increasing the value of R_1 from the benchmark value 0.01. But at the same time δ, ρ, and σ would go far away from their values in the symmetry limit.

Finally, we point out that the above results can be understood in an analytical approximation way by expanding the parameters around their values in the symmetry limit [36]. At the first order of parameter deviations, $\Delta m_{1,3}$ are vanishing while the deviations of other parameters are linear functions of $R_{1,2}$ and $I_{1,2}$. Only up to the second order of parameter deviations can $\Delta m_{1,3}$ receive some nonvanishing contributions, making its dependence on $R_{1,2}$ and $I_{1,2}$ be of a quadratic form.[3] Provided that the parameter deviations remain small enough, the results for another value of $R_{1,2}$ or $I_{1,2}$ can be approximately obtained from those in Tables I and II by invoking the quadratic (linear) dependence of $\Delta m_{1,3}$ (other parameters) on $R_{1,2}$ and $I_{1,2}$ [36]. When a given $R_{1,2}$ or $I_{1,2}$ changes its sign, $\Delta m_{1,3}$ remain invariant while the deviations of other parameters also undergo a sign change.

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[3]This explains why $m_{1,3}$ are stable against the symmetry breakings.

VI. SUMMARY

To summarize, the purpose of this work is to derive the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ in two phenomenologically appealing scenarios: (1) one neutrino mass matrix element vanishing (2) one neutrino mass vanishing. In the former scenario, there are two neutrino mass sum rules arising, which can be directly read from Eq. (27). In the latter scenario, one has $m_1 = \rho = 0$ in the NO case or $m_3 = \rho = 0$ in the IO case, leaving us with only one unknown physical parameter σ to determine. Furthermore, we also make an attempt to derive the possible textures of neutrino mass matrix that can lead to $\theta_{23} = \pi/4$, $\delta = -\pi/2$ and maximal Majorana CP phases, in which scenario there is

TABLE III. A summary of the results for the unknown physical parameters in the various cases. In the second column, the phrase A(G), B (and so on) is used to stand for the cases of diagonalization conditions A(G) and B (and so on) holding automatically.

$\bar{M}_{ee} = 0$	A(G), B	✗
	A(G), C(F)	$m_1 = 0.006$ eV, $\rho = 0$, $\sigma = \pi/2$
		$m_1 = 0.002$ eV, $\rho = \pi/2$, $\sigma = 0$
	A(G), D(E)	$m_1=0.004$ eV, $\rho=0.79\pi$, $\sigma=0.23\pi$
$\bar{M}_{e\mu} = 0$	A(G), D(E)	$m_1=0.009$ eV, $\rho=0.41\pi$, $\sigma=0.65\pi$
		$m_3=0.006$ eV, $\rho=0.51\pi$, $\sigma=0.47\pi$
	B, C(F)	$m_3=0.0007$ eV, $\rho=0.27\pi$, $\sigma=0.22\pi$
$\bar{M}_{\mu\mu} = 0$	A(G)	$m_3=0.024$ eV, $\rho=0.97\pi$, $\sigma=0.43\pi$
	B	$m_3=0.050$ eV, $\rho=0.35\pi$, $\sigma=0.54\pi$
	C(F)	$m_3=0.022$ eV, $\rho=0.04\pi$, $\sigma=0.46\pi$
	D(E)	✗
$\bar{M}_{\mu\tau} = 0$	A(G), C(F)	$m_1 = 0.165$ eV, $\rho = 0$, $\sigma = 0$
		$m_3 = 0.021$ eV, $\rho = \pi/2$, $\sigma = 0$
	B	✗
	D(E)	✗
$m_1 = 0$	A(G), C(F)	$\sigma = 0$ or $\pi/2$
	B	✗
	D(E)	$\sigma = \pi/4$ or $3\pi/4$
$m_3 = 0$	A(G), C(F)	$\sigma = 0$ or $\pi/2$
	B	✗
	D(E)	$\sigma = 0.03\pi$ or 0.97π
Maximal ρ/σ	A(G), D(E)	$m_1 = 0.004$ eV
	B, D(E)	✗
	C(F), D(E)	✗

only the absolute neutrino mass scale to be determined. In these three scenarios, we study the various cases where one more neutrino mass sum rule arises from the requirement of some diagonalization condition(s) holding automatically so that all the three unknown physical parameters (i.e., the absolute neutrino mass scale and two Majorana CP phases) can be determined. A summary of the results for the various cases is given in Table III. [The results in the case of $\bar{M}_{e\tau} = 0$ ($\bar{M}_{\tau\tau} = 0$) can be obtained from those in the case of $\bar{M}_{e\mu} = 0$ ($\bar{M}_{\mu\mu} = 0$) by making the replacements $\rho \to \pi - \rho$ and $\sigma \to \pi - \sigma$.] These results are useful for investigating which, if any, specific texture of neutrino mass matrix is realized by the nature.

We also give some discussions about the model realization of the obtained textures. It is found that the textures obtained in the case of diagonalization conditions A(G) and C(F) holding automatically in combination with $\bar{M}_{ee}$, $\bar{M}_{\mu\tau}$, m_1, or m_3 vanishing can reproduce those in the case of μ-τ reflection symmetry with M_{ee}, $M_{\mu\tau}$, m_1, or m_3 being vanishing, if one restores the unphysical phases with the help of $\bar{M}_{\alpha\beta} = M_{\alpha\beta}e^{-i(\phi_\alpha+\phi_\beta)}$ and takes $\phi_e = \pi/2$ and $\phi_\mu = -\phi_\tau$. This in some sense indicates that the approach adopted by us is reasonable. Although the textures obtained in other cases cannot be connected to a similar flavor symmetry, they are on an equal footing in our approach and deserve some attention from the phenomenological point of view. Finally, we give some discussions about the breakings of the obtained textures so as to accommodate the deviations of θ_{23} and δ from $\pi/4$ and $-\pi/2$. From the example cases we have studied, it is found that small deviations of the physical parameters from their values in the symmetry limit can be accommodated by small breakings of the given textures. And the results obtained in the symmetry limit would still be viable to some extent provided that the given textures hold to a good approximation. Furthermore, because of having distinct main origins, a considerable $\Delta\theta_{23}$ does not necessarily signify a considerable $\Delta\delta$, $\Delta\rho$, or $\Delta\sigma$, and vice versa.

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