Dihedral symmetries of gauge theories from dual Calabi-Yau threefolds

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Recent studies [J. High Energy Phys. 07 (2017) 112; Phys. Rev. D **97**, 046004 (2018); J. High Energy Phys. 11 (2018) 016] of six-dimensional supersymmetric gauge theories that are engineered by a class of toric Calabi-Yau threefolds $X_{N,M}$ have uncovered a vast web of dualities. In this paper we analyze the consequences of these dualities from the perspective of the partition functions $\mathcal{Z}_{N,M}$ (or the free energy $\mathcal{F}_{N,M}$) of these theories. Focusing on the case $M = 1$, we find that the latter is invariant under the group $\mathbb{G}(N) \times S_N$, where S_N corresponds to the Weyl group of the largest gauge group that can be engineered from $X_{N,1}$ and $\mathbb{G}(N)$ is a dihedral group, which acts in an intrinsically nonperturbative fashion and is of infinite order for $N \geq 4$. We give an explicit representation of $\mathbb{G}(N)$ as a matrix group that is freely generated by two elements which act naturally on a specific basis of the Kähler moduli space of $X_{N,1}$. While we show the invariance of $\mathcal{Z}_{N,1}$ under $\mathbb{G}(N) \times S_N$ in full generality, we provide explicit checks by series expansions of $\mathcal{F}_{N,1}$ for a large number of examples. We also comment on the relation of $\mathbb{G}(N)$ to the modular group that arises due to the geometry of $X_{N,1}$ as a double elliptic fibration, as well as the T duality of little string theories that are constructed from $X_{N,1}$.

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I. INTRODUCTION

The engineering of supersymmetric gauge theories [1,2] in dimensions ≤ 6 through string- and M-theory constructions has been an active and fruitful field of study throughout the years. Indeed, the numerous dual approaches and formulations that are available on the string theory side provide us with a large range of tools (both computationally as well as conceptually) to explore hidden symmetries, dualities, and even more sophisticated structures on the gauge theory side that would be very difficult to study otherwise. An important feature of this approach is that in many cases string theory methods give us access to nonperturbative aspects of the gauge theories and allow us to study them in an efficient manner [3–5]. One very rich subclass of theories which has attracted a lot of attention recently [6–9] are supersymmetric, $U(M)$ circular quiver gauge theories on $\mathbb{R}^5 \times S^1$, which can (among other methods) be approached through F-theory compactifications on a class of toric Calabi-Yau threefolds $X_{N,M}$.[1] The latter give rise to a quiver theory comprised of N nodes of type $U(M)$ (which we shall denote as $[U(M)]^N$ in the following). A particularity of these theories is the fact that their UV completion in general contains not only pointlike particles, but also stringy degrees of freedom, although gravity remains decoupled. Such theories are called little string theories (LSTs), which were originally introduced over a decade ago [11–19] and have recently received a lot of renewed interest [8,20–27]. The fully refined, nonperturbative partition function $\mathcal{Z}_{N,M}$ of this theory is captured by the (refined) topological string partition function on $X_{N,M}$ and can be computed very efficiently [3–6,8,25] with the help of the (refined) topological vertex [28–31] (see Refs. [32,33] for a general discussion of the topological string partition function on elliptic Calabi-Yau threefolds). Since the latter (for technical reasons) requires a choice of preferred direction in the web diagram of $X_{N,M}$, this method provides different, but completely equivalent expansions of $\mathcal{Z}_{N,M}$, which can be interpreted as instanton expansions of different but dual gauge theories. While it is straightforward to see [4,5,8] that in this fashion the theory $[U(M)]^N$ is dual to $[U(N)]^M$, it was argued in Ref. [24] that it is also dual to $[U(\frac{NM}{k})]^k$, where $k = \gcd(N, M)$, thus

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[1]The numbers N, $M \in \mathbb{N}$ refer to the fact that $X_{N,M}$ has the structure of a double elliptic fibration, where the two fibrations have Kodaira singularities of types I_{N-1} and I_{M-1}, respectively [10].

leading to a *triality* of gauge theories that are engineered by $X_{N,M}$.

The Calabi-Yau manifolds $X_{N,M}$ depend on $NM+2$ independent Kähler parameters and the corresponding moduli space takes the form of a cone. The faces of the latter (which we shall call walls in the following) are (among others) comprised of singular loci where the area of one or more of the curves in the web diagram of $X_{N,M}$ vanish. From the perspective of the geometry of $X_{N,M}$, crossing such a wall (i.e., continuing to negative area) gives rise to a new Calabi-Yau manifold, which corresponds to a different (but dual) resolution of the singularity. With the help of such *flop transitions* [34,35], the Kähler moduli space of $X_{N,M}$ can be extended to include further regions that allow the engineering of yet new gauge theories. Indeed, it was argued in Ref. [22] that the Calabi-Yau manifolds $X_{N,M}$ and $X_{N',M'}$ can be related through a series of flop transformations if $NM = N'M'$ and $\gcd(N,M) = \gcd(N',M')$. Furthermore, nontrivial checks were presented in Ref. [22] that the topological string partition functions associated with $X_{N,M}$ and $X_{N',M'}$ are the same upon taking into account the nontrivial duality map. This was shown explicitly in Ref. [25] for the cases $\gcd(N,M) = 1$ and a suitable basis of independent Kähler parameters was presented which is adapted to the invariance under a series of flop transformations that is instrumental in the duality $X_{N,M} \sim X_{N',M'}$.[2] Combining this invariance of $\mathcal{Z}_{N,M}$ with the triality of gauge theories proposed in Ref. [24], it was argued in Ref. [26] that the theory $[U(M)]^N$ is in fact dual to all theories of the form $[U(M')]^{N'}$ for any N', M' with $NM = N'M'$ and $\gcd(N,M) = \gcd(N',M')$. It was furthermore argued in Ref. [27] that the extended moduli space [36–40] of $X_{N,M}$ contains different decompactification regions, which engineer different five-dimensional gauge theories with various gauge structures and matter content.

While previous works have focused on interpreting different expansions of $\mathcal{Z}_{N,M}$ as instanton partition functions of different gauge theories, thereby establishing a large network of dual theories, in this paper we discuss the consequences of these dualities from the perspective of symmetries of $\mathcal{Z}_{N,M}$. Focusing on the cases $M=1$, rather than switching between different expansions of the partition function $\mathcal{Z}_{N,1}$ (or more concretely the free energy $\mathcal{F}_{N,1}$), we shall focus on one particular expansion (as a power series in a suitable basis of Kähler parameters of $X_{N,1}$) and recast the results of Refs. [22,24,26] in the form of highly nontrivial identities among the expansion coefficients of $\mathcal{F}_{N,1}$. From the perspective of any of the gauge theories of the type $[U(M')]^{N'}$, where (N',M') are relative primes and $N'M' = N$, these correspond to generically nonperturbative symmetries that act in a highly nontrivial fashion on the spectrum of Bogomol'nyi-Prasad-Sommerfield (BPS) states of the theory. Furthermore, since the combination of any two of these symmetries itself has to be another symmetry, they have the structure of a group $\hat{\mathbb{G}}(N)$ which acts naturally on the vector space spanned by the independent Kähler parameters of $X_{N,1}$.

We shall analyze $\hat{\mathbb{G}}(N)$ first with the help of the explicit examples $N = 1, 2, 3, 4$, where we can study it (or its subgroups) explicitly as a matrix group. Based on these examples, we find a pattern, which allows us to prove for generic N that $\hat{\mathbb{G}}(N)$ has a subgroup of the form

$$\tilde{\mathbb{G}}(N) \cong \mathbb{G}(N) \times S_N \quad \text{with} \quad \tilde{\mathbb{G}}(N) \subset \hat{\mathbb{G}}(N), \tag{1.1}$$

where S_N is the Weyl group of the largest simple gauge group that can be engineered from $X_{N,1}$ [i.e., $U(N)$] and $\mathbb{G}(N)$ is isomorphic to a dihedral group,[3] namely,

$$\mathbb{G}(N) \cong \begin{cases} \mathrm{Dih}_3 & \text{if } N=1, \\ \mathrm{Dih}_2 & \text{if } N=2, \\ \mathrm{Dih}_3 & \text{if } N=3, \\ \mathrm{Dih}_\infty & \text{if } N\geq 4. \end{cases} \tag{1.2}$$

Here Dih_∞ is a finitely generated group of infinite order [while $\mathrm{ord}(\mathrm{Dih}_n) = 2n$ for finite $2 \leq n \in \mathbb{N}$].

In particular the group $\mathbb{G}(N)$ in Eq. (1.1) combines nontrivially with other known symmetries and dualities of $X_{N,1}$.

(1) Modularity: Owing to the fact that $X_{N,1}$ has the structure of a double elliptic fibration, the partition function transforms as a Jacobi form under two copies of the modular group $SL(2,\mathbb{Z})_\tau$ and $SL(2,\mathbb{Z})_\rho$.[4] Since $\tilde{\mathbb{G}}(N)$ acts nontrivially on the modular parameters (τ,ρ) the combined symmetry group is in general larger than simply $\tilde{\mathbb{G}}(N) \times SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$. In the simplest case $N=1$, which we shall discuss in Sec. III, we are in fact able to explicitly analyze the resulting group and we can show that it is isomorphic to $Sp(4,\mathbb{Z})$, which is the automorphism group of the genus-2 curve that is the geometric mirror of the Calabi-Yau manifold $X_{1,1}$ (see Refs. [10,29]). For $N>1$, the symmetry is more difficult to analyze, and we are only be able to make statements about a specific region in the moduli space.

[2]This transformation is explained in detail in Appendix A and the basis is reviewed in the following section.

[3]For $n \in \mathbb{N}$ the dihedral group Dih_n is freely generated by two elements a, b of order 2 that satisfy a certain braid relation: $\mathrm{Dih}_n = \langle \{a,b | a^2 = b^2 = \mathbb{1} \text{ and } (ab)^n = \mathbb{1}\} \rangle$. The group Dih_∞ corresponds to the limit $n \to \infty$ and is of infinite order.

[4]Our notation follows the naming convention of the modular parameters as in, e.g., Ref. [8].

(2) T duality: As mentioned above, the UV completion of the gauge theory $[U(1)]^N$ is an LST with eight supercharges, which was called type IIb little string theory in Ref. [8]. The latter is T dual to type IIa little string theory, whose low-energy behaviour is described by the dual gauge theory $[U(N)]^1$ (see Refs. [8,20,21,41] for the discussion of T duality of LSTs engineered from double elliptic Calabi-Yau threefolds). Denoting the partition functions of these little string theories by Z_{IIb} and Z_{IIa}, respectively, it was proposed in Ref. [8] that the partition functions of these two little string theories are captured by $\mathcal{Z}_{N,1}$,

$$Z_{\mathrm{IIa}}(\tau,\rho,\mathbf{K}) = \mathcal{Z}_{N,1}(\tau,\rho,\mathbf{K}), \quad \text{and} \\ Z_{\mathrm{IIb}}(\tau,\rho,\mathbf{K}') = \mathcal{Z}_{N,1}(\rho,\tau,\mathbf{K}'), \tag{1.3}$$

where for simplicity we have only explicitly displayed the dependence on the modular parameters (τ,ρ) and only schematically indicated the dependence on the remaining Kähler parameters through $\mathbf{K}$ and $\mathbf{K}'$, respectively. Furthermore, in Ref. [8] it was proposed that the T duality of the IIa and IIb LSTs simply amounts to

$$Z_{\mathrm{IIa}}(\tau,\rho,\mathbf{K}) = Z_{\mathrm{IIb}}(\rho,\tau,\mathbf{K}'), \tag{1.4}$$

which, from the perspective of the Calabi-Yau manifold $X_{N,1}$, corresponds to an exchange of the two elliptic curves: one in the fiber and one in the base (with a duality map relating $\mathbf{K}$ and $\mathbf{K}'$). Since the group $\tilde{\mathbb{G}}(N)$ in Eq. (1.1) acts nontrivially on the modular parameters (τ,ρ) (and in general mixes them in a nontrivial fashion), it extends the incarnation (1.4) of T duality to a nontrivial group acting on the full spectrum of the LSTs.

This paper is organized as follows. In Sec. II we first review the important aspects of the computation of the partition function $\mathcal{Z}_{N,1}$, in particular the choice of basis of the independent Kähler parameters. Furthermore, we discuss in more detail our strategy for finding the group $\tilde{\mathbb{G}}(N)$ in Eq. (1.1). Finally, for the sake of readability, we also give a summary of the results obtained in the subsequent sections. In Secs. III–VI we discuss in detail the examples $N = 1, 2, 3, 4$, respectively. For each of these cases we construct $\tilde{\mathbb{G}}(N)$ and provide nontrivial evidence that it is a symmetry of the $\mathcal{F}_{N,1}$ by computing the leading orders in the expansion of the former as a power series of the Kähler parameters. In Sec. VII we generalize a pattern that emerges from the previous examples and which allows us to prove Eq. (1.2) for generic $N \in \mathbb{N}$. Finally, Sec. VIII contains our conclusions and directions for future research. Furthermore, this paper is accompanied by two Appendixes, which review a particular duality transformation for the web diagrams of $X_{N,1}$ and a finite representation of the group $Sp(4,\mathbb{Z})$, respectively. These technical details are relevant for the computations performed in the main body of this work.

II. REVIEW, GENERAL STRATEGY, AND SUMMARY OF RESULTS

A. Review: Partition function and free energy

The web diagram for a general $X_{N,1}$ is shown in Fig. 1. Each line is labeled by the area of the curve that they represent: horizontal lines are labeled by $h_{1,\dots,N}$, vertical lines by $v_{1,\dots,N}$, and diagonal lines by $m_{1,\dots,N}$. Not all of these areas are independent of one another, but they are subject to $2N$ consistency conditions (for $i = 1,\dots,N$), related to the N hexagons S_i of the web diagram

$$S_i\colon\ h_i + m_i = h_i + m_{i+1}, \\ v_i + m_i = v_{i+\delta} + m_{i+1}, \tag{2.1}$$

where $m_{i+N} = m_i$ and $v_{i+N} = v_i$. A general solution of these conditions is given by $v_i = v_{i+1}$ and $m_i = m_{i+1}$ for $i = 1,\dots,N-1$. Another solution, which is more adapted to the computations in the remainder of this work, is provided by the blue parameters in Fig. 1, which equally represent an independent set of Kähler parameters of the Calabi-Yau manifold $X_{N,1}$. Physically, from the perspective of (one particular) gauge theory engineered by $X_{N,1}$, the parameters $\hat{a}_{1,\dots,N}$ correspond to the (affine) roots of the gauge group $U(N)$ (i.e., the vacuum expectation values of the vector multiplet scalars), while the parameter R is related to the coupling constant and S to the mass parameter

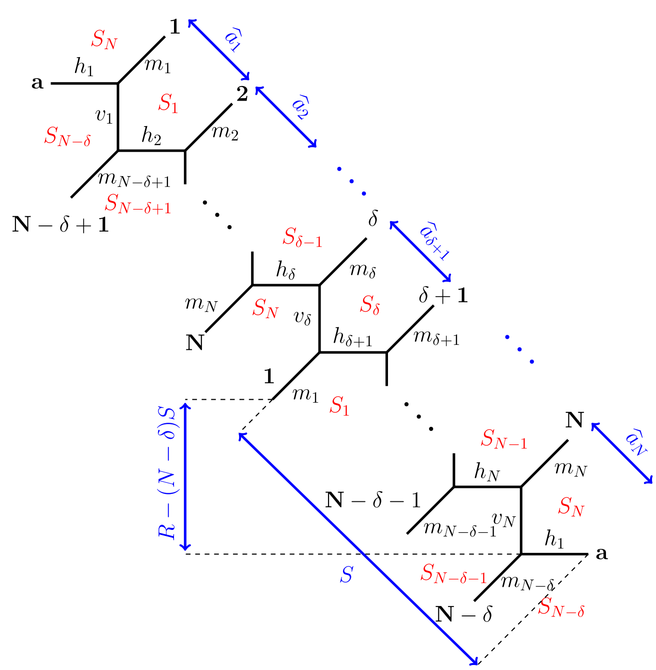


FIG. 1. Web diagram of $X^{(\delta)}_{N,1}$.

of the matter sector. As shown in Ref. [24], however, this assignment is not unique and the Calabi-Yau manifold $X_{N,1}$ in fact engineers several different gauge theories with different gauge groups[5] and possibly different matter content. In the following we will therefore not be too concerned with the physical interpretation of the parameters $(\hat{a}_{1,\ldots,N}, S)$. Instead, we shall treat the dependences of the partition function $\mathcal{Z}_{N,1}(\hat{a}_{1,\ldots,N}, S, R; \epsilon_{1,2})$ (associated with $X_{N,1}$) on all of these parameters on equal footing. The former can be computed from the web diagram in Fig. 1 with the help of the refined topological string. Here the constants $\epsilon_{1,2} \in \mathbb{R}$ represent the refinement and can be thought of as a means of regularizing the partition function, which would otherwise be ill defined.

An efficient method of computing $\mathcal{Z}_{N,1}$ from Fig. 1 (for arbitrary δ) was given in Ref. [25] (see also Refs. [3,5]) by computing a general building block $W^{\alpha_1,\ldots,\alpha_N}_{\beta_1,\ldots,\beta_N}$ that depends on the Kähler parameters $(\hat{a}_{1,\ldots,N}, S)$ and is labeled by $2N$ integer partitions $\alpha_{1,\ldots,N}$ and $\beta_{1,\ldots,N}$, which encode how the legs of the (various) building block(s) are glued together (for details, see Ref. [25]). While the formalism developed in Ref. [25] is more general and allows the computation of a much larger class of partition functions, in the present case we have

$$\mathcal{Z}_{N,1}(\hat{a}_{1,\ldots,N}, S, R; \epsilon_{1,2}) = \sum_{\{\alpha\}} \left(\prod_{i=1}^{N} Q_{m_i}^{|\alpha_i|} \right) W^{\alpha_1,\ldots,\alpha_N}_{\alpha_{N-\delta+1},\ldots,\alpha_{N-\delta}}(\hat{a}_{1,\ldots,N}, S; \epsilon_{1,2}), \quad (2.2)$$

with (our conventions for the normalization of $W^{\alpha_1,\ldots,\alpha_N}_{\beta_1,\ldots,\beta_N}$ are adapted to Fig. 1)

$$W^{\alpha_1,\ldots,\alpha_N}_{\alpha_{N-\delta+1},\ldots,\alpha_{N-\delta}}(\hat{a}_{1,\ldots,N}, S; \epsilon_{1,2}) = W^N_{\varnothing}(\hat{a}_{1,\ldots,N}) \left[\frac{(t/q)^{\frac{N-1}{2}}}{Q_\rho^{N-\delta-1}} \right]^{|\alpha_1|+\cdots+|\alpha_N|} \times \prod_{i,j=1}^{N} \frac{\vartheta_{\alpha_i\alpha_j}(\hat{Q}_{i,j};\rho)}{\vartheta_{\alpha_i,\alpha_j}(\bar{Q}_{i,j}\sqrt{q/t};\rho)}.$$

Here we have used the following notation:

$$Q_{m_i} = e^{-m_i}, \qquad \rho = \frac{i}{2\pi} \sum_{k=1}^{N} \hat{a}_k, \qquad Q_\rho = e^{-\sum_{k=1}^{N} \hat{a}_k},$$
$$q = e^{2\pi i \epsilon_1}, \qquad t = e^{-2\pi i \epsilon_2},$$

where $m_{i=1,\ldots,N}$ refer to the areas of the diagonal lines in Fig. 1 expressed as functions of $(\hat{\alpha}_{1,\ldots,N}, S, R)$ with the help of the consistency conditions (2.1). Furthermore, $W^N_{\varnothing}$ is a normalization factor (which from a physical perspective in particular encodes the perturbative contribution to the partition function) and $\vartheta_{\mu\nu}$ is a class of theta functions that is labeled by two integer partitions μ and ν

$$\vartheta_{\mu\nu}(x;\rho) = \prod_{(i,j)\in\mu} \vartheta(x^{-1} q^{-\nu_j^t+i-\frac{1}{2}} t^{-\mu_i+j-\frac{1}{2}};\rho) \times \prod_{(i,j)\in\nu} \vartheta(x^{-1} q^{\mu_j^t-i+\frac{1}{2}} t^{\nu_i-j+\frac{1}{2}};\rho),$$

with the further definition

$$\vartheta(x;\rho) = (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \prod_{k=1}^{\infty} (1 - x Q_\rho^k)(1 - x^{-1} Q_\rho^k). \quad (2.3)$$

Finally, the arguments of the ϑ functions can be defined as $\hat{Q}_{i,j} = e^{-z_{ij}}$ and $\bar{Q}_{i,j} = e^{-w_{ij}}$, where z_{ij} and w_{ij} are implicitly defined in Fig. 2 with respect to (part of) the web diagram [the labels on the diagonal and horizontal lines in Fig. 2 (and Fig. 1) indicate how they are glued together].

With the partition function $\mathcal{Z}_{N,1}$, we can define the free energy as the plethystic logarithm

$$\mathcal{F}_{N,1}(\hat{a}_{1,\ldots,N}, S, R; \epsilon_{1,2}) = \mathrm{PLog}\mathcal{Z}_{N,1} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \mathcal{Z}_{N,1}(k\hat{a}_{1,\ldots,N}, kS, kR; k\epsilon_{1,2}), \quad (2.4)$$

where $\mu(k)$ is the Möbius function. We can expand the free energy in the following fashion:

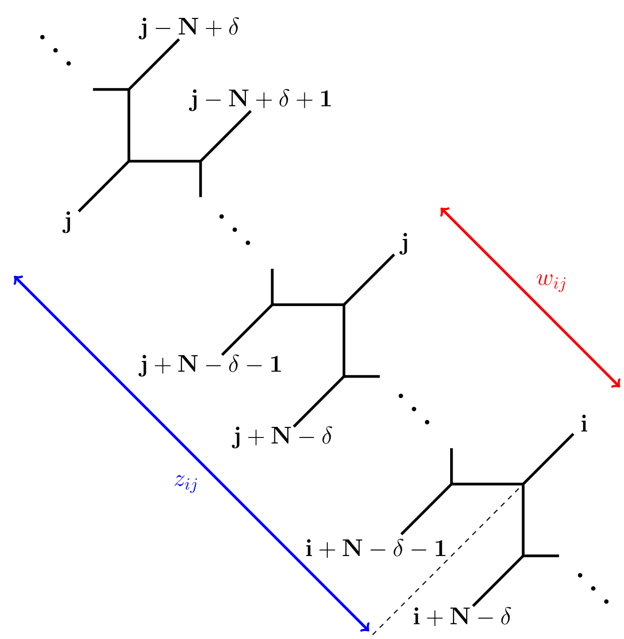


FIG. 2. Definition of the arguments of the ϑ functions appearing in $\mathcal{Z}_{N,1}$.

[5]The nonaffine part of the gauge groups, however, is in general a subgroup of $U(N)$.

$$\mathcal{F}_{N,1}(\hat{a}_{1,\ldots,N},S,R;\epsilon_1,\epsilon_2) = \sum_{n=0}^{\infty}\sum_{i_1,\ldots,i_N=0}^{\infty}\sum_{k\in\mathbb{Z}} f_{i_1,\ldots,i_N,k,n}(\epsilon_1,\epsilon_2)\hat{Q}_1^{i_1}\ldots\hat{Q}_N^{i_N}Q_S^kQ_R^n, \tag{2.5}$$

with $\hat{Q}_i = e^{-\hat{a}_i}$ (for $i=1,\ldots,N$), $Q_S = e^{-S}$, and $Q_R = e^{-R}$. Apart from a first-order pole, $\mathcal{F}_{N,1}$ has a power-series expansion in $\epsilon_{1,2}$, which allows to compute the Nekrasov-Shatashvili limit [42,43] and the unrefined limit in a straightforward fashion. For later convenience we therefore also introduce the expansion of the leading term in both parameters (which we simply denote as NS),

$$\lim_{\epsilon_{1,2}\to 0}\epsilon_1\epsilon_2\mathcal{F}_{N,1}(\hat{a}_{1,\ldots,N},S,R;\epsilon_1,\epsilon_2) = \sum_{n=0}^{\infty}\sum_{i_1,\ldots,i_N=0}^{\infty}\sum_{k\in\mathbb{Z}} f^{\rm NS}_{i_1,\ldots,i_N,k,n}\hat{Q}_1^{i_1}\ldots\hat{Q}_N^{i_N}Q_S^kQ_R^n, \tag{2.6}$$

where $f^{\rm NS}_{i_1,\ldots,i_N,k,n}\in\mathbb{Z}$.

B. Symmetry transformations: Strategy and summary of results

In Refs. [22,24,25] different duality transformations were discussed that involve flop transformations [34,35] of various curves of $X_{N,1}$, $SL(2,\mathbb{Z})$ transformations as well as cutting and regluing of the web diagram. While these duality transformations were shown in Ref. [25] to leave $\mathcal{Z}_{N,1}$ (and thus also $\mathcal{F}_{N,1}$) invariant, they generically act in a rather nontrivial fashion on the web diagram in Fig. 1. Indeed, a particular example of such a transformation is reviewed in Appendix A, which shifts $\delta\to\delta+1$ and transforms the areas of all curves $\{h_{1,\ldots,N},v_{1,\ldots,N},m_{1,\ldots,N}\}$ in a nontrivial fashion. In general, the web diagram in Fig. 1 is transformed to a similar "staircase" diagram as shown in Fig. 3 (possibly with $\delta'\neq\delta$), where the areas of the new curves can be rewritten as functions of the old areas,

$$\begin{aligned}\{h'_{1,\ldots,N},v'_{1,\ldots,N},m'_{1,\ldots,N}\} &= \{h'_{1,\ldots,N}(h_{1,\ldots,N},v_{1,\ldots,N},m_{1,\ldots,N}),\\ &\quad v'_{1,\ldots,N}(h_{1,\ldots,N},v_{1,\ldots,N},m_{1,\ldots,N}),\\ &\quad m'_{1,\ldots,N}(h_{1,\ldots,N},v_{1,\ldots,N},m_{1,\ldots,N})\}.\end{aligned} \tag{2.7}$$

Furthermore, since both $(\hat{a}_{1,\ldots,N},S,R)$ (as defined in Fig. 1) and $(\hat{a}'_{1,\ldots,N},S',R')$ (as defined in Fig. 3) are a maximal set of independent Kähler parameters, the areas $\{h_{1,\ldots,N},v_{1,\ldots,N},m_{1,\ldots,N}\}$ can be expressed as linear combinations of both of these bases. Therefore, Eq. (2.7) gives a set of linear equations which have a unique solution of the form

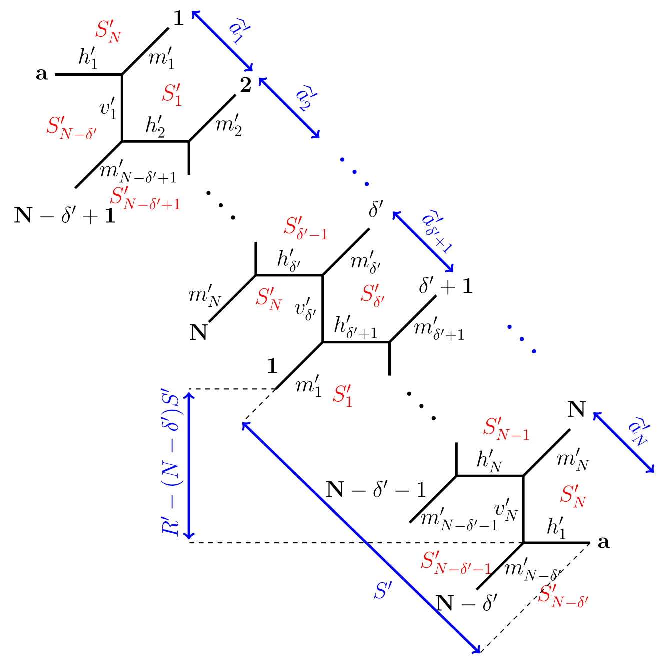


FIG. 3. Web diagram of $X_{N,1}^{(\delta')}$ after a duality transformation of Fig. 1.

$$(\hat{a}_1,\ldots,\hat{a}_N,S,R)^T = G\cdot(\hat{a}'_1,\ldots,\hat{a}'_N,S',R')^T, \tag{2.8}$$

where G is an invertible $(N+2)\times(N+2)$ matrix with integer entries. Finally, using the result [25] that the partition function $\mathcal{Z}_{N,1}$ is invariant under the duality transformation, i.e., $\mathcal{Z}_{N,1}(\hat{a}_{1,\ldots,N},S,R)=\mathcal{Z}_{N,1}(\hat{a}'_{1,\ldots,N},S',R')$, the matrix G in Eq. (2.8) is a symmetry of the partition function. More concretely, at the level of the free energy, we have the following relations for the expansion coefficients appearing in Eq. (2.5):

$$f_{i_1,\ldots,i_N,k,n}(\epsilon_1,\epsilon_2) = f_{i'_1,\ldots,i'_N,k',n'}(\epsilon_1,\epsilon_2)\quad \text{for } (i'_1,\ldots,i'_N,k',n')^T = G^T\cdot(i_1,\ldots,i_N,k,n)^T. \tag{2.9}$$

The transposition of G in this relation is due to the fact that the transformation (2.8) is a passive one from the perspective of the coefficients $f_{i_1,\ldots,i_N,k,n}$.

For given $X_{N,1}$ there are in general numerous different transformations G of the type described above. Since the concatenation of two such transformations defines a new transformation, the latter form a group. In the following sections we shall determine at least a subgroup of this group for the simplest examples $N=1$, 2, 3, 4, which in Sec. VII can be generalized to generic $N\in\mathbb{N}$. However, before doing so and for ease of readability, we summarize our results. For generic $N\in\mathbb{N}$, we identify a finitely generated group of symmetry transformations of the type (2.8), which can be written as

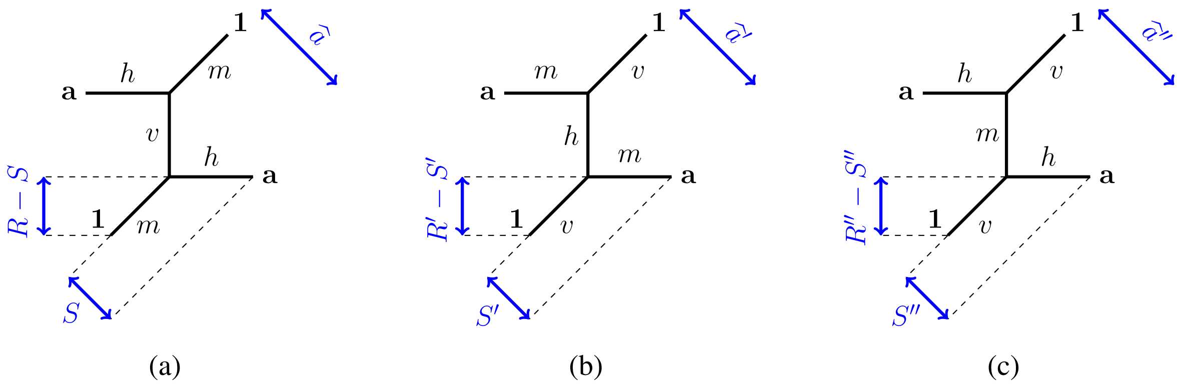


FIG. 4. Three different representations of the web diagram of $X_{1,1}$ with a parametrization of the areas of all curves. The parameters (h, v, m) are independent of each other and the blue parameters represent an alternative parametrization in line with Fig. 1.

$$\tilde{\mathbb{G}}(N) \cong \mathbb{G}(N) \times S_N \quad \text{with}$$
$$\mathbb{G}(N) \cong \begin{cases} \text{Dih}_3 & \text{if } N = 1, \\ \text{Dih}_2 & \text{if } N = 2, \\ \text{Dih}_3 & \text{if } N = 3, \\ \text{Dih}_\infty & \text{if } N \geq 4. \end{cases} \tag{2.10}$$

The group S_N is generated by simple relabelings of the web diagram of $X_{N,1}$ and physically corresponds to the Weyl group of $U(N)$, which is the largest gauge group that can be engineered by $X_{N,1}$. For generic N, the group $\mathbb{G}(N)$ is freely generated by two $(N+2) \times (N+2)$ matrices of order 2, which satisfy a specific braid relation,[6]

$$\mathbb{G}(N) \cong \langle \{\mathcal{G}_2(N), \mathcal{G}_2'(N) | (\mathcal{G}_2(N))^2 = (\mathcal{G}_2'(N))^2 = (\mathcal{G}_2(N) \cdot \mathcal{G}_2'(N))^n = \mathbb{1}\} \rangle, \tag{2.11}$$

where $n = 3$ for $N = 1$, 3 and $n = 2$ for $N = 2$, but for $N \geq 4$ we find $n \to \infty$, which means that there is no braid relation in these cases. Explicitly, the generators are given by the following lower and upper triangular matrices:

$$\mathcal{G}_2(N) = \begin{pmatrix} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{pmatrix}, \quad \text{and}$$
$$\mathcal{G}_2'(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

These matrices are symmetry transformations of the partition function $\mathcal{Z}_{N,1}$ and the free energy $\mathcal{F}_{N,1}$ in the sense of Eq. (2.9), which can be checked in explicit examples. In the case $N = 1$, combining the group $\tilde{\mathbb{G}}(N)$ with the modular group $SL(2,\mathbb{Z})$ acting on one of the modular parameters of $X_{1,1}$ generates the group $Sp(4,\mathbb{Z})$. For the cases with $N > 1$, the combination with the modular group is more difficult to analyze at a general point in the moduli space of $X_{N,1}$. However, in the region in moduli space where $\hat{a}_{1,\ldots,N} = \hat{a}$ in Fig. 1, this analysis is simpler and we can prove that the combination of $\mathbb{G}(N)$ with the modular group is a subgroup of $Sp(4,\mathbb{Z})$. This is in line with the checks performed in Ref. [22] to provide evidence for the duality $X_{N,M} \sim X_{N',M'}$ [for $NM = N'M'$ and $\gcd(N,M) = \gcd(N',M')$] of Calabi-Yau threefolds.

III. EXAMPLE: $(N,M) = (1,1)$

A. Dualities and Dih_3 group action

The simplest (albeit somewhat trivial) example to illustrate the idea explained in Sec. II B is the configuration $(N,M) = (1,1)$. The corresponding web diagram is shown in Fig. 4(a). Through simple $SL(2,\mathbb{Z})$ transformations (as well as cutting and regluing) the former can also be presented (among other ways) in the form of Figs. 4(b) and 4(c).

Each diagram can be parametrized in terms of the parameters (h, v, m) or, respectively, $(\hat{a}, S, R)$, $(\hat{a}', S', R')$, or $(\hat{a}'', S'', R'')$. The latter can be expressed in terms of (h, v, m) as

$$\begin{aligned} \hat{a} &= h + v, & S &= h, & R - S &= m, \\ \hat{a}' &= h + m, & S' &= m, & R' - S' &= v, \\ \hat{a}'' &= h + m, & S'' &= h, & R'' - S'' &= v. \end{aligned} \tag{3.1}$$

Inverting these relations, (h, v, m) can be expressed as linear combinations of $(\hat{a}, S, R)$, $(\hat{a}', S', R')$, or $(\hat{a}'', S'', R'')$, respectively,

[6]In the following, $\langle \mathcal{E} \rangle$ denotes the group freely generated by the ensemble $\mathcal{E}$.

$$h = S = \hat{a}' - S' = S'', \qquad v = \hat{a} - S = R' - S' = R'' - S'',$$
$$m = R - S = S' = \hat{a}'' - S''. \tag{3.2}$$

These equations also furnish linear transformations between $(\hat{a}, S, R)$, $(\hat{a}', S', R')$, or $(\hat{a}'', S'', R'')$,

$$\begin{pmatrix} \hat{a} \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \hat{a}' \\ S' \\ R' \end{pmatrix} = G_2 \cdot \begin{pmatrix} \hat{a}'' \\ S'' \\ R'' \end{pmatrix}, \quad \text{with}$$
$$G_1 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

The matrix G_1 is of order 3 (i.e., $G_1 \cdot G_1 \cdot G_1 = \mathbb{1}_{3\times 3}$), while G_2 is of order 2 (i.e., $G_2 \cdot G_2 = \mathbb{1}_{3\times 3}$). Thus, by also introducing the matrices[7]

$$E = \mathbb{1}_{3\times 3}, \quad G_3 = G_1 \cdot G_1, \quad G_4 = G_1 \cdot G_2, \quad G_5 = G_2 \cdot G_1, \tag{3.4}$$

the ensemble $\mathbb{G}(1) = \{E, G_1, G_2, G_3, G_4, G_5\}$ forms a finite group, whose multiplication table is

	E	G_1	G_2	G_3	G_4	G_5
E	E	G_1	G_2	G_3	G_4	G_5
G_1	G_1	G_3	G_4	E	G_5	G_2
G_2	G_2	G_5	E	G_4	G_3	G_1
G_3	G_3	E	G_5	G_1	G_2	G_4
G_4	G_4	G_2	G_1	G_5	E	G_3
G_5	G_5	G_4	G_3	G_2	G_1	E

(3.5)

from which we can read off $\mathbb{G}(1) = \{E, G_1, G_2, G_3, G_4, G_5\} \cong \mathrm{Dih}_3 \cong S_3$. The latter can be formulated more elegantly as the free group generated by the elements

$$a = G_4 = G_1 \cdot G_2 = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$
$$b = G_5 = G_2 \cdot G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \tag{3.6}$$

[7]In the same manner as G_1 and G_2, these matrices can also be read off from web diagrams as in Fig. 4 with a suitable exchange of (h, v, m), which, however, we do not show explicitly.

furnishing the following representation:

$$\mathbb{G}(1) \cong \mathrm{Dih}_3$$
$$\cong \langle \{a, b | a^2 = b^2 = \mathbb{1}_{3\times 3}, (ab)^3 = \mathbb{1}_{3\times 3}\} \rangle. \tag{3.7}$$

B. Invariance of the nonperturbative free energy

As a check of the fact that $G_{1,2}$ defined in Eq. (3.3) are indeed symmetry transformations of $\mathcal{Z}_{1,1}$, we can consider the coefficients in the expansion of the associated free energy $\mathcal{F}_{1,1}$. Indeed, for $N = 1$, the expansion (2.5) can be written as

$$\mathcal{F}_{1,1}(\hat{a}, S, R; \epsilon_1, \epsilon_2) = \sum_{n,i=0}^{\infty} \sum_{k\in\mathbb{Z}} f_{i,k,n}(\epsilon_1, \epsilon_2) \hat{Q}^i Q_S^k Q_R^n, \tag{3.8}$$

with $\hat{Q} = e^{-\hat{a}}$. As explained in Sec. II B, in order to be a symmetry, the coefficients $f_{i,k,n}(\epsilon_1, \epsilon_2)$ (which are functions of $\epsilon_{1,2}$ with a first-order pole) need to satisfy

$$f_{i,k,n}(\epsilon_1, \epsilon_2) = f_{i',k',n'}(\epsilon_1, \epsilon_2)$$
$$\text{for } (i', k', n')^T = G_\ell^T \cdot (i, k, n)^T, \quad \forall\ \ell = 1, 2. \tag{3.9}$$

Below we tabulate examples of the coefficients $f_{i,k,n}$ with $i \leq 8$ for $n = 1$, $i \leq 4$ for $n = 2$, and $i \leq 2$ for $n = 3$ that are related by $G_{1,2}$: Tables I and II show the relations for G_1 and G_2, respectively.

C. Modularity and $Sp(4,\mathbb{Z})$ symmetry

The action of $\mathbb{G}(1)$ as presented in Eq. (3.7) combines with $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ to become $Sp(4,\mathbb{Z})$, which is (a subgroup of) the automorphism group of $X_{1,1}$. To see this, instead of considering the action of $\mathbb{G}(1)$ on the vector space spanned by $(\hat{a}, S, R)$, we consider the vector space spanned by ($\tau = h + v$, $\rho = m + v$, v). Arranging the latter in the period matrix

$$\Omega = \begin{pmatrix} \tau & v \\ v & \rho \end{pmatrix}, \tag{3.10}$$

there is a natural action of $Sp(4,\mathbb{Z})$, as reviewed in Appendix B. The action of $G_{1,2}$ on Ω is

$$G_1\colon\ \Omega \to \begin{pmatrix} -2v + \rho + \tau & \tau - v \\ \tau - v & \tau \end{pmatrix},$$
$$G_2\colon\ \Omega \to \begin{pmatrix} \tau & \tau - v \\ \tau - v & -2v + \rho + \tau \end{pmatrix}. \tag{3.11}$$

Based on this action, we can equivalently represent the action of $\mathbb{G}(1)$ by $G'_{1,2} \in Sp(4,\mathbb{Z})$,

TABLE I. Action of G_1: The indices are related by $(i_1', i_2', k', n')^T = G_1^T \cdot (i_1, i_2, k, n)^T$.

(i,k,n)	(i',k',n')	$f_{i,k,n}(\epsilon_{1,2}) = f_{i',k',n'}(\epsilon_{1,2})$
(1,0,1)	$(2,-2,1)$	$\frac{(qt+1)(q^2t+q(t+1)^2+t)}{(q-1)q(t-1)t}$
(1,1,1)	$(3,-3,1)$	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
(1,2,1)	$(4,-4,1)$	$\frac{qt+1}{(q-1)(t-1)}$
$(2,-2,1)$	$(1,-2,2)$	$\frac{q^3t^2+q^2t(t^2+2t+2)+q(2t^2+2t+1)+t}{(q-1)q(t-1)t}$
(2,1,1)	$(4,-5,2)$	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
$(3,-3,1)$	$(1,-3,3)$	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(1,-1,2)$	$(2,-1,1)$	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
(1,1,2)	$(4,-3,1)$	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
(1,3,2)	$(6,-5,1)$	$-\frac{\sqrt{qt}}{(q-1)(t-1)}$
$(2,-3,2)$	$(1,-1,2)$	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
$(1,-2,3)$	(2,0,1)	$\frac{q^5t^3+q^4t^2(2t^2+3t+2)+q^3t(t^4+3t^3+8t^2+6t+2)+q^2(2t^4+6t^3+8t^2+3t+1)+qt(2t^2+3t+2)+t^2}{(q-1)q^2(t-1)t^2}$
(1,1,3)	$(5,-3,1)$	$-\frac{(q+1)(t+1)(q^5t^3+q^4t^2(t+1)^2+q^3t(t^4+2t^3+6t^2+4t+1)+q^2(t^4+4t^3+6t^2+2t+1)+qt(t+1)^2+t^2)}{(q-1)q^{5/2}(t-1)t^{5/2}}$
(1,2,3)	$(6,-4,1)$	$\frac{q^5t^3+q^4t^2(2t^2+3t+2)+q^3t(t^4+3t^3+8t^2+6t+2)+q^2(2t^4+6t^3+8t^2+3t+1)+qt(2t^2+3t+2)+t^2}{(q-1)q^2(t-1)t^2}$
(1,3,3)	$(7,-5,1)$	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$

TABLE II. Action of G_2: The indices are related by $(i_1', i_2', k', n')^T = G_2^T \cdot (i_1, i_2, k, n)^T$.

(i,k,n)	(i',k',n')	$f_{i,k,n}(\epsilon_{1,2}) = f_{i',k',n'}(\epsilon_{1,2})$
$(2,-3,1)$	$(1,-3,2)$	$-\frac{\sqrt{qt}}{(q-1)(t-1)}$
$(2,-2,1)$	$(1,-2,2)$	$\frac{q^3t^2+q^2t(t^2+2t+2)+q(2t^2+2t+1)+t}{(q-1)q(t-1)t}$
$(2,-1,1)$	$(1,-1,2)$	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
(2,0,1)	(1,0,2)	$\frac{q^5t^3+q^4t^2(2t^2+3t+2)+q^3t(t^4+3t^3+8t^2+6t+2)+q^2(2t^4+6t^3+8t^2+3t+1)+qt(2t^2+3t+2)+t^2}{(q-1)q^2(t-1)t^2}$
(2,1,1)	(1,1,2)	$\frac{q^4(-t^2)(t+1)-q^3t(t^3+3t^2+4t+1)-q^2(t^4+4t^3+7t^2+4t+1)-q(t^3+4t^2+3t+1)-t(t+1)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
(2,2,1)	(1,2,2)	$\frac{q^3t^2+q^2t(t^2+2t+2)+q(2t^2+2t+1)+t}{(q-1)q(t-1)t}$
(2,3,1)	(1,3,2)	$-\frac{\sqrt{qt}}{(q-1)(t-1)}$
$(3,-3,1)$	$(1,-3,3)$	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(3,-2,1)$	$(1,-2,3)$	$\frac{q^5t^3+q^4t^2(2t^2+3t+2)+q^3t(t^4+3t^3+8t^2+6t+2)+q^2(2t^4+6t^3+8t^2+3t+1)+qt(2t^2+3t+2)+t^2}{(q-1)q^2(t-1)t^2}$
$(3,-1,1)$	$(1,-1,3)$	$-\frac{(q+1)(t+1)(q^5t^3+q^4t^2(t+1)^2+q^3t(t^4+2t^3+6t^2+4t+1)+q^2(t^4+4t^3+6t^2+2t+1)+qt(t+1)^2+t^2)}{(q-1)q^{5/2}(t-1)t^{5/2}}$
(3,1,1)	(1,1,3)	$-\frac{(q+1)(t+1)(q^5t^3+q^4t^2(t+1)^2+q^3t(t^4+2t^3+6t^2+4t+1)+q^2(t^4+4t^3+6t^2+2t+1)+qt(t+1)^2+t^2)}{(q-1)q^{5/2}(t-1)t^{5/2}}$
(3,2,1)	(1,2,3)	$\frac{q^5t^3+q^4t^2(2t^2+3t+2)+q^3t(t^4+3t^3+8t^2+6t+2)+q^2(2t^4+6t^3+8t^2+3t+1)+qt(2t^2+3t+2)+t^2}{(q-1)q^2(t-1)t^2}$
(3,3,1)	(1,3,3)	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(1,-3,2)$	$(2,-3,1)$	$-\frac{\sqrt{qt}}{(q-1)(t-1)}$
$(1,-2,2)$	$(2,-2,1)$	$\frac{(qt+1)(q^2t+q(t+1)^2+t)}{(q-1)q(t-1)t}$
(1,2,2)	(2,2,1)	$\frac{(qt+1)(q^2t+q(t+1)^2+t)}{(q-1)q(t-1)t}$
(1,3,2)	(2,3,1)	$-\frac{\sqrt{qt}}{(q-1)(t-1)}$
$(1,-3,3)$	$(3,-3,1)$	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
(1,3,3)	(3,3,1)	$-\frac{(q+1)(t+1)(qt+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$

$$G'_1 = HK = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{and}$$

$$G'_2 = K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.12}$$

where K and H are defined as in Appendix B. This implies that $\mathbb{G}(1) \subset Sp(4,\mathbb{Z})$. Moreover, combining $\mathbb{G}(1)$ with the $SL(2,\mathbb{Z})_\rho$ symmetry[8] acting on the modular parameter[9] ρ as

$$S_\rho\colon\ (\tau,\rho,v) \mapsto \left(\tau - \frac{v^2}{\rho}, -\frac{1}{\rho}, \frac{v}{\rho}\right),$$
$$T_\rho\colon\ (\tau,\rho,v) \mapsto (\tau,\rho+1,v), \tag{3.13}$$

generates the complete action of $Sp(4,\mathbb{Z})$: the generators (S_ρ, T_ρ) can be expressed as $S_\rho = L^3$ and $T_\rho = L^9HL^{10}H = X_2$. Furthermore, we have $G'_2G'_1 = L^5KL^7$ such that we can write

$$X_1 = G'_2G'_1S_\rho^2, \qquad X_2 = T_\rho, \qquad X_3 = S_\rho G'_1G'_1S_\rho,$$
$$X_4 = G'_1G'_2T_\rho G'_1G'_2, \qquad X_5 = G'_1G'_2S_\rho^2,$$
$$X_6 = S_\rho^3G'_1G'_2S_\rho^2G'_1G'_2, \tag{3.14}$$

with $X_{1,2,3,4,5,6}$ defined in Eq. (B2). This indicates that

$$\langle G'_1, G'_2, S_\rho, T_\rho\rangle \supset \langle X_1, X_2, X_3, X_4, X_5, X_6\rangle \cong Sp(4,\mathbb{Z}), \tag{3.15}$$

where the last relation was shown in Ref. [44]. From Eq. (3.12), and using the representation of $Sp(4,\mathbb{Z})$ given in Ref. [45], it follows that

$$\langle G'_1, G'_2, S_\rho, T_\rho\rangle \subset \langle K, L\rangle \cong Sp(4,\mathbb{Z}), \tag{3.16}$$

which implies $\langle G'_1, G'_2, S_\rho, T_\rho\rangle \cong Sp(4,\mathbb{Z})$.

[8]Notice that the symmetry group is isomorphic to $SL(2,\mathbb{Z})$ rather than $PSL(2,\mathbb{Z})$, since $S_\rho^2 \neq \mathbb{1}$, as can be seen from the action of S_ρ^2 on the period matrix $\Omega \to (\tau - v - v\rho)$.

[9]We could also choose the modular group $SL(2,\mathbb{Z})_\tau$ which acts in a similar fashion on the modular parameter τ. More precisely, $SL(2,\mathbb{Z})_\tau$ is generated by $S_\tau = HS_\rho H$ and $T_\tau = HT_\rho H$.

IV. EXAMPLE: $(N,M) = (2,1)$

A. Dualities and Dih_2 group action

In this section we generalize the analysis of the previous section and, using the simplest nontrivial example [namely, $(N,M) = (2,1)$], explain how the duality transformations advocated in Refs. [24,26] lead to nontrivial symmetries at the level of the set of independent Kähler parameters of $X_{2,1}$. In the following subsection we give further evidence for this symmetry at the level of the partition function $\mathcal{Z}_{2,1}$. The starting point is the web diagram shown in Fig. 5 along with a parametrization of the areas of all curves involved. The latter are not all independent of one another, but for each of the two hexagons $S_{1,2}$, they have to satisfy the following consistency conditions:

$$S_1\colon\ h_2 + m_2 = m_1 + h_2, \quad v_1 + m_1 = m_2 + v_2,$$
$$S_2\colon\ h_1 + m_1 = m_2 + h_1, \quad m_1 + v_1 = m_2 + v_2. \tag{4.1}$$

A solution for these conditions was provided in Ref. [25] in the form of the parameters $(\hat{a}_{1,2}, S, R)$ as indicated in Fig. 5,

$$\hat{a}_1 = v_1 + h_2, \qquad \hat{a}_2 = v_2 + h_1,$$
$$S = h_2 + v_2 + h_1, \qquad R - 2S = m_1 - v_2. \tag{4.2}$$

Indeed, all of the areas $(h_{1,2}, v_{1,2}, m_{1,2})$ can be expressed as a linear combination of $(\hat{a}_1, \hat{a}_2, S, R)$:

$$h_1 = S - \hat{a}_1, \qquad h_2 = S - \hat{a}_2, \qquad v_1 = v_2 = \hat{a}_1 + \hat{a}_2,$$
$$m_1 = m_2 = \hat{a}_1 + \hat{a}_2 + R - 3S. \tag{4.3}$$

Mirroring the diagram and performing an $SL(2,\mathbb{Z})$ transformation, Fig. 5 can also be presented in the form of Fig. 6(a). Cutting the latter along the curve labeled $v_{1,2}$ and regluing along the curves labeled $m_{1,2}$ leads to the diagram in Fig. 6(b). The consistency conditions of this web are the

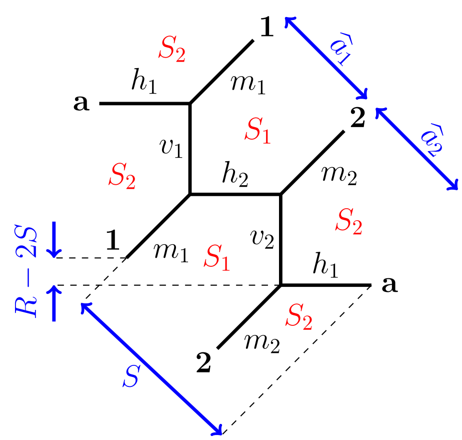


FIG. 5. Web diagram of $X_{2,1}$ with a parametrization of the areas of all curves. The blue parameters represent an independent set of Kähler parameters.

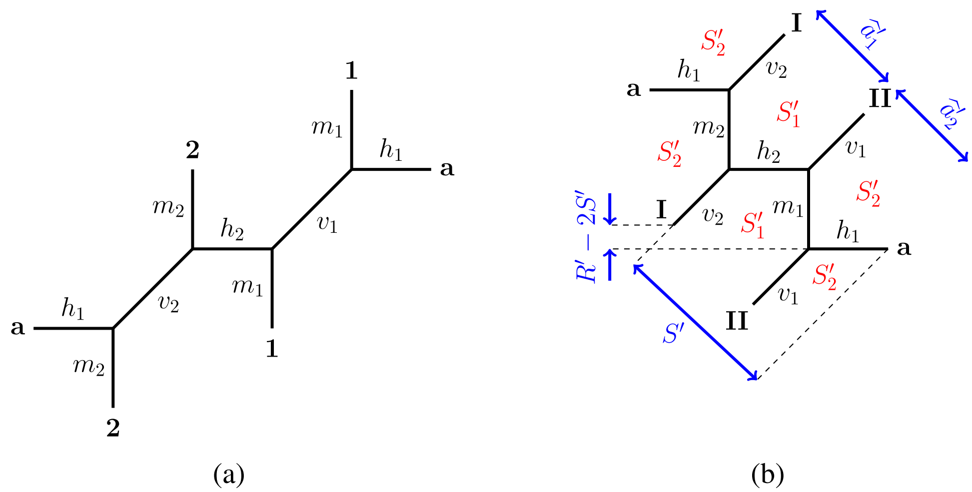


FIG. 6. (a) Web diagram of Fig. 5 after mirroring and an $SL(2,\mathbb{Z})$ transformation. (b) The same web diagram after cutting along the lines $v_{1,2}$ and regluing along the lines $m_{1,2}$.

same as Eq. (4.1). Furthermore, the web diagram in Fig. 6(b) is of the same form as Fig. 5 and thus allows for a solution of Eq. (4.1) in terms of the parameters $(\hat{a}'_1, \hat{a}'_2, S', R')$:

$$\hat{a}'_1 = m_1 + h_1, \qquad \hat{a}'_2 = m_2 + h_2,$$
$$S' = h_2 + m_1 + h_1, \qquad R' - 2S' = v_2 - m_1. \tag{4.4}$$

Indeed, we can express the areas $(h_{1,2}, v_{1,2}, m_{1,2})$ in terms of the latter,

$$\begin{aligned} &h_1 = S' - \hat{a}'_2, \qquad h_2 = S' - \hat{a}'_2, \\ &v_1 = v_2 = \hat{a}'_1 + \hat{a}'_2 - 3S + R', \\ &m_1 = m_2 = \hat{a}'_1 + \hat{a}'_2 - S'. \end{aligned} \tag{4.5}$$

Comparing Eq. (4.3) with Eq. (4.5) gives rise to a linear relation between $(\hat{a}_1, \hat{a}_2, S, R)$ and $(\hat{a}'_1, \hat{a}'_2, S', R')$:

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \hat{a}'_1 \\ \hat{a}'_2 \\ S' \\ R' \end{pmatrix}, \quad \text{where}$$
$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} \det G_1 = 1, \\ G_1 \cdot G_1 = \mathbb{1}_{4\times 4}. \end{matrix} \tag{4.6}$$

We can obtain another symmetry transformation by cutting the diagram in Fig. 5 along the line labeled v_2 and regluing it along the line h_1 to obtain Fig. 7(a). Mirroring the latter, it can also be presented in the form of Fig. 7(b) which takes the form of a web diagram with the shift $\delta = 1$. The latter can be parametrized by $(\hat{a}''_1, \hat{a}''_2, S'', R'')$,

$$\hat{a}''_1 = h_2 + v_2, \qquad \hat{a}''_2 = h_1 + v_1,$$
$$S'' = v_1, \qquad R'' - S'' = m_1, \tag{4.7}$$

which allows to uniquely express all areas $(h_{1,2}, v_{1,2}, m_{1,2})$

$$h_1 = \hat{a}''_2 - S'', \qquad h_2 = \hat{a}''_1 = S'', \qquad v_1 = v_2 = S'',$$
$$m_1 = m_2 = R'' - S''. \tag{4.8}$$

Comparing Eq. (4.8) with Eq. (4.5) gives rise to a transformation between $(\hat{a}_1, \hat{a}_2, S, R)$ and $(\hat{a}''_1, \hat{a}''_2, S'', R'')$,

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ S \\ R \end{pmatrix} = G_2 \cdot \begin{pmatrix} \hat{a}''_1 \\ \hat{a}''_2 \\ S'' \\ R'' \end{pmatrix}, \quad \text{where } G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 2 & -4 & 1 \end{pmatrix},$$
$$\text{with} \quad \begin{matrix} \det G_2 = -1, \\ G_2 \cdot G_2 = \mathbb{1}_{4\times 4}. \end{matrix} \tag{4.9}$$

Finally, cutting the diagram in Fig. 7(b) along the curve labeled v_1 and regluing it along the line m_2 yields the diagram in Fig. 8(a), which [after mirroring and performing an $SL(2,\mathbb{Z})$-transformation] can also be presented in the form of Fig. 8(b). This diagram is parametrized by $(\hat{a}'''_1, \hat{a}'''_2, S''', R''')$,

$$\hat{a}'''_1 = h_1 + m_1, \qquad \hat{a}'''_2 = h_2 + m_2,$$
$$S''' = m_2, \qquad R''' - S''' = v_2, \tag{4.10}$$

which provide a parametrization of all of the areas,

$$h_1 = \hat{a}'''_1 - S''', \qquad h_2 = \hat{a}'''_2 - S''',$$
$$v_1 = v_2 = R''' - S''', \qquad m_1 = m_2 = S'''. \tag{4.11}$$

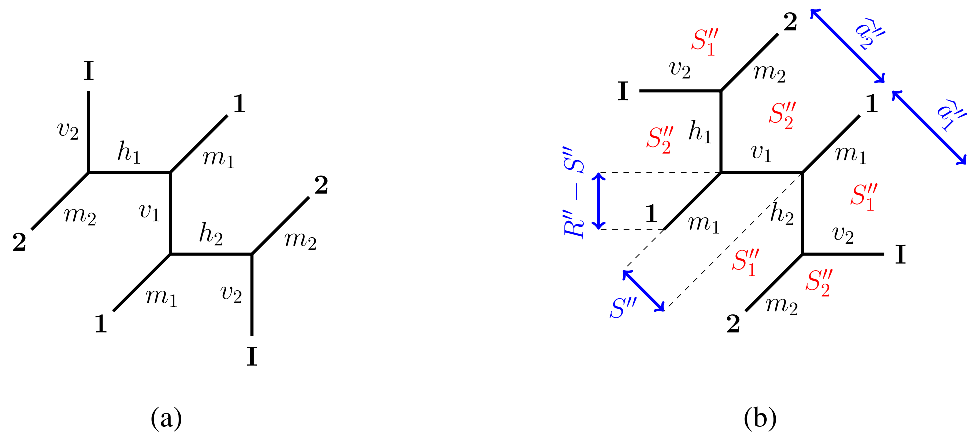


FIG. 7. (a) Web diagram obtained from Fig. 5 after cutting along the line labeled v_2 and regluing along h_1. (b) Alternative representation of the same diagram.

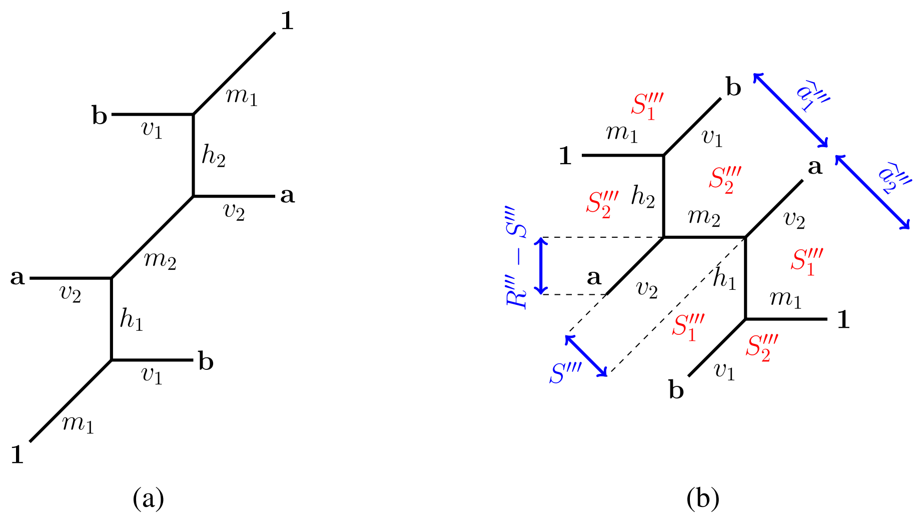


FIG. 8. (a) Web diagram obtained from Fig. 7(b) by cutting the curve labeled v_1 and regluing along m_2. (b) Alternative representation of the same diagram.

Comparing Eq. (4.11) with Eq. (4.3) provides a linear transformation between the parameters $(\hat{a}_1, \hat{a}_2, S, R)$ and $(\hat{a}'''_1, \hat{a}'''_2, S''', R''')$,

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ S \\ R \end{pmatrix} = G_3 \cdot \begin{pmatrix} \hat{a}'''_1 \\ \hat{a}'''_2 \\ S''' \\ R''' \end{pmatrix}, \quad \text{with} \quad G_3 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 \\ 2 & 2 & -4 & 1 \end{pmatrix},$$
$$\text{and} \quad \begin{array}{l} \det G_3 = -1, \\ G_3 \cdot G_3 = \mathbb{1}_{4\times 4}. \end{array} \tag{4.12}$$

The matrices $G_{1,2,3}$ together with the identity matrix $E = \mathbb{1}_{4\times 4}$ form a discrete group of order 4, whose multiplication table is given by

	E	G_1	G_2	G_3
E	E	G_1	G_2	G_3
G_1	G_1	E	G_3	G_2
G_2	G_2	G_3	E	G_1
G_3	G_3	G_2	G_1	E

(4.13)

The latter is identical to the multiplication table of Dih_2, i.e., the dihedral group of order 4 (which is isomorphic to the Klein four-group). We therefore have[10]

$$\mathbb{G}(2) \cong \{E, G_1, G_2, G_3\} \cong \mathrm{Dih}_2. \tag{4.14}$$

[10]For further reference, we remark that $\mathbb{G}(2)$ can also be presented as the group freely generated by $G_{1,2}$, i.e., $\mathbb{G}(2) \cong \langle\{G_1, G_2\}\rangle$, where $G_1^2 = \mathbb{1}_{4\times 4} = G_2^2$ and $(G_1 \cdot G_2)^2 = \mathbb{1}_{4\times 4}$.

An overview of $G_{1,2,3}$ and their relations to different representations of the web diagram in Fig. 5 is given in Fig. 9 (which corresponds to the cycle graph of Dih_2). We remark that all other representations of the web [including webs related by a transformation $\mathcal{F}$ (Appendix A)] only give rise to coordinate transformations that differ from $\{E, G_1, G_2, G_3\}$ by the action of

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.15}$$

which exchanges $\hat{a}_1 \longleftrightarrow \hat{a}_2$ and commutes with $G_{1,2,3}$. Since R generates the group S_2, we can define $\tilde{\mathbb{G}}(2) = \mathbb{G}(2) \times S_2$ as a nontrivial symmetry group of $\mathcal{F}_{2,1}$.

B. Invariance of the nonperturbative free energy

It was shown in Ref. [25] that the web diagrams in Figs. 5, 6(b), 7(b), and 8(b) give rise to the same partition function, and the linear transformations $G_{1,2,3}$ in Eqs. (4.6), (4.9), and (4.12) correspond to symmetries of the free energy $\mathcal{F}_{2,1}(\hat{a}_{1,2}, S, R; \epsilon_1, \epsilon_2)$, as defined in Eq. (2.4). In this section we provide evidence for this symmetry by considering the expansion

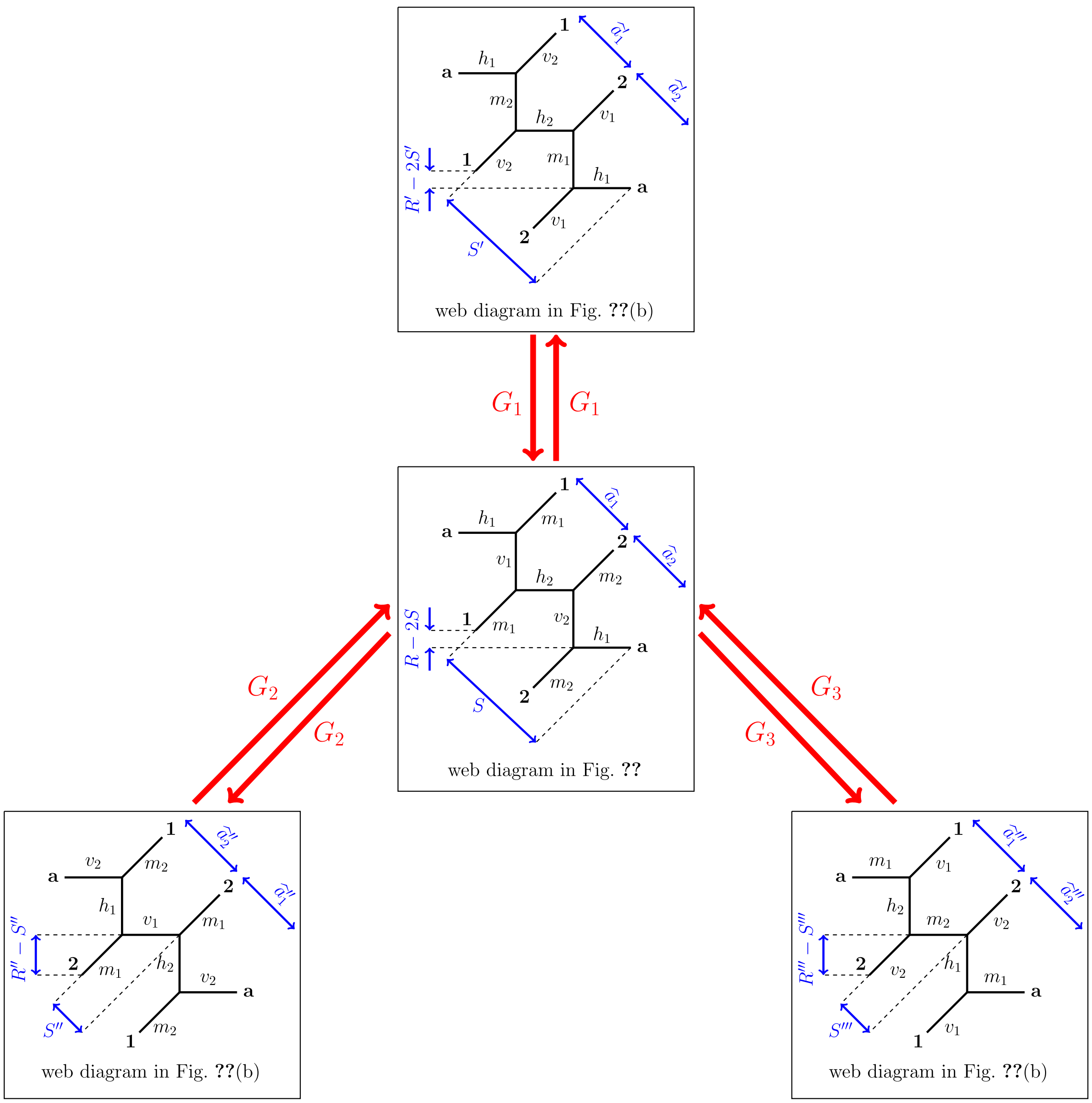


FIG. 9. Representations of web diagrams related to $X_{2,1}$. The transformations $G_{1,2,3}$ act on the basis of independent Kähler parameters $(\hat{a}_1, \hat{a}_2, S, R)$. The organization of web diagrams and transformations is reminiscent of the cycle graph of Dih_2.

$$\mathcal{F}_{2,1}(\hat{a}_1,\hat{a}_2,S,R;\epsilon_1,\epsilon_2) = \sum_{n=0}^{\infty}\sum_{i_1,i_2=0}^{\infty}\sum_{k\in\mathbb{Z}} f_{i_1,i_2,k,n}(\epsilon_1,\epsilon_2)\hat{Q}_1^{i_1}\hat{Q}_2^{i_2}Q_S^kQ_R^n, \qquad (4.16)$$

with $\hat{Q}_i = e^{-\hat{a}_i}$ (for $i = 1, 2$), $Q_S = e^{-S}$, and $Q_R = e^{-R}$. As explained in Sec. II B, we have

$$f_{i_1,i_2,k,n}(\epsilon_1,\epsilon_2) = f_{i'_1,i'_2,k',n'}(\epsilon_1,\epsilon_2)$$
$$\text{for } (i'_1,i'_2,k',n')^T = G_\ell^T\cdot(i_1,i_2,k,n)^T, \quad \forall\ \ell = 1,2,3. \qquad (4.17)$$

Below we tabulate the coefficients $f_{i_1,i_2,k,n}$ with $i_1 + i_2 \leq 3$ for $n = 1$ and $i_1 + i_2 \leq 2$ for $n = 2$ that are related by $G_{1,2,3}$: Tables III–V show the relations for G_1, G_2, and G_3, respectively.

C. Modularity at a particular point of the moduli space

For the case $N = 1$, we showed that the combination of $\mathbb{G}(1) \cong \mathrm{Dih}_3$ with the modular group acting as in Eq. (3.13) generates the group $Sp(4,\mathbb{Z})$. The case $N = 2$ is more complicated. However, in the following we shall show that in a particular region of the moduli space $\mathbb{G}(2) \cong \mathrm{Dih}_2$ in Eq. (4.14) can be understood as a subgroup of $Sp(4,\mathbb{Z})$. This region is characterized by imposing $\hat{a}_1^{(0)} = \hat{a}_2^{(0)} = \hat{a}$,[11] which implies $h_1 = h_2 = h$ [while the consistency conditions (4.1) already impose $v_1 = v_2 = v$ and $m_1 = m_2 = m$]. This region is also a fixed point of S_2 generated by R in Eq. (4.15). The remaining independent parameters can be organized in the period matrix

$$\Omega = \begin{pmatrix} \tau & v \\ v & \rho \end{pmatrix}, \quad \text{with} \quad \begin{matrix} \tau = m + v, \\ \rho = h + m. \end{matrix} \qquad (4.18)$$

Furthermore, the symmetry transformations G_1 in Eq. (4.6) and G_2 in Eq. (4.9) can be reduced to act on the subspace $(\hat{a}, S, R)$,

$$G_1^{(\mathrm{red})} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad G_2^{(\mathrm{red})} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{pmatrix}, \qquad (4.19)$$

or on the space (τ, ρ, v),

$$\tilde{G}_1^{(\mathrm{red})} = D_2^{-1}\cdot G_1^{(\mathrm{red})}\cdot D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix},$$
$$\text{with} \quad D_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & 4 & -4 \end{pmatrix},$$
$$\tilde{G}_2^{(\mathrm{red})} = D_2^{-1}\cdot G_2^{(\mathrm{red})}\cdot D_2 = \begin{pmatrix} 1 & 4 & -4 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

Rewriting the latter as elements of $Sp(4,\mathbb{Z})$ that act as in Eq. (B3) on the period matrix Ω in Eq. (4.18), they take the form

TABLE III. Action of G_1: The indices are related by $(i'_1,i'_2,k',n')^T = G_1^T\cdot(i_1,i_2,k,n)^T$.

(i_1,i_2,k,n)	(i'_1,i'_2,k',n')	$f_{i_1,i_2,k,n}(\epsilon_{1,2}) = f_{i'_1,i'_2,k',n'}(\epsilon_{1,2})$
(0,1,0,1)	$(0,1,-2,2)$	$\frac{(q+t)(q(1+t(q+t+2))+t)}{(q-1)q(t-1)t}$
$(0,2,-1,1)$	$(0,2,-3,2)$	$-\frac{(q+1)(t+1)(q+t)(q^2+t^2)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
(1,0,0,1)	$(1,0,-2,2)$	$\frac{(q+t)(q(t(q+t+2)+1)+t)}{(q-1)q(t-1)t}$
$(1,1,-1,1)$	$(1,1,-3,2)$	$-\frac{2(q^2(t(t+3)+1)+q(t(3t+7)+3)+t(t+3)+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(2,0,-1,1)$	$(2,0,-3,2)$	$-\frac{(q+1)(t+1)(q+t)(q^2+t^2)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
$(0,1,-2,2)$	(0,1,0,1)	$\frac{(q+t)(q(t(q+t+2)+1)+t)}{(q-1)q(t-1)t}$
$(0,2,-3,2)$	$(0,2,-1,1)$	$-\frac{(q+1)(t+1)(q+t)(q^2+t^2)}{(q-1)q^{3/2}(t-1)t^{3/2}}$
$(1,0,-2,2)$	(1,0,0,1)	$\frac{(q+t)(q(t(q+t+2)+1)+t)}{(q-1)q(t-1)t}$
$(1,1,-3,2)$	$(1,1,-1,1)$	$-\frac{2(q^2(t(t+3)+1)+q(t(3t+7)+3)+t(t+3)+1)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(2,0,-3,2)$	$(2,0,-1,1)$	$-\frac{(q+1)(t+1)(q+t)(q^2+t^2)}{(q-1)q^{3/2}(t-1)t^{3/2}}$

TABLE IV. Action of G_2: The indices are related by $(i'_1,i'_2,k',n')^T = G_2\cdot(i_1,i_2,k,n)^T$.

(i_1,i_2,k,n)	(i'_1,i'_2,k',n')	$f_{i_1,i_2,k,n}(\epsilon_{1,2}) = f_{i'_1,i'_2,k',n'}(\epsilon_{1,2})$
$(0,0,-1,1)$	$(1,1,-3,1)$	$-\frac{2\sqrt{q}\sqrt{t}}{(q-1)(t-1)}$
$(1,2,-3,1)$	$(0,1,-1,1)$	$-\frac{(q+1)(t+1)(q+t)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(2,1,-3,1)$	$(1,0,-1,1)$	$-\frac{(q+1)(t+1)(q+t)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$

TABLE V. Action of G_3: The indices are related by $(i'_1,i'_2,k',n')^T = G_3\cdot(i_1,i_2,k,n)^T$.

(i_1,i_2,k,n)	(i'_1,i'_2,k',n')	$f_{i_1,i_2,k,n}(\epsilon_{1,2}) = f_{i'_1,i'_2,k',n'}(\epsilon_{1,2})$
$(0,2,-2,1)$	$(0,2,-2,1)$	$\frac{(q+t)(q^2+t^2)}{(q-1)q(t-1)t}$
$(1,1,-2,1)$	$(1,1,-2,1)$	$\frac{4(q+1)(t+1)}{(q-1)(t-1)}$
$(1,2,-3,1)$	$(0,1,-1,1)$	$-\frac{(q+1)(t+1)(q+t)}{(q-1)\sqrt{q}(t-1)\sqrt{t}}$
$(2,0,-2,1)$	$(2,0,-2,1)$	$\frac{(q+t)(q^2+t^2)}{(q-1)q(t-1)t}$

[11]This is the same region in the moduli space that was used in Ref. [22] for a nontrivial check that $\mathcal{Z}_{N,M} = \mathcal{Z}_{N',M'}$ for $NM = N'M'$ and $\gcd(N,M) = \gcd(N',M')$.

$$\tilde{G}_1^{(\text{red,Sp})} = K \quad \text{and} \quad \tilde{G}_2^{(\text{red,Sp})} = HKL^6KH, \qquad (4.20)$$

where K, L, and H are defined in Appendix B. This implies that the restriction of $\mathbb{G}(2)$ to the particular region of the Kähler moduli space explained above is a subgroup of $Sp(4,\mathbb{Z})$. However, unlike the case $N=1$, we cannot conclude that the group freely generated as $\langle \tilde{G}_1^{(\text{red,Sp})}, \tilde{G}_2^{(\text{red,Sp})}, S_\rho, T_\rho, S_\tau, T_\tau \rangle$ is isomorphic to $Sp(4,\mathbb{Z})$.

V. EXAMPLE: $(N,M)=(3,1)$

A. Dualities and Dih_3 group action

Following the previous example of $X_{2,1}^{(\delta=0)}$, we can also analyze $X_{3,1}^{(\delta=0)}$ in a similar fashion. The starting point is the web diagram shown in Fig. 10, which includes labels for the areas of all of the curves. The consistency conditions associated with the three hexagons $S_{1,2,3}^{(0)}$ take the forms

$$\begin{aligned}
&S_1^{(0)}:\ h_2+m_2=m_1+h_2, \quad v_1+m_1=m_2+v_2,\\
&S_2^{(0)}:\ h_3+m_3=m_2+h_3, \quad v_2+m_2=m_3+v_3,\\
&S_3^{(0)}:\ h_1+m_1=m_3+h_1, \quad m_1+v_1=v_3+m_3. \qquad (5.1)
\end{aligned}$$

A solution of these conditions is provided by the parameters $(\hat{a}_{1,2,3}^{(0)}, S^{(0)}, R^{(0)})$,

$$\begin{aligned}
&\hat{a}_1^{(0)}=v_1+h_2, \quad \hat{a}_2^{(0)}=v_2+h_3, \quad \hat{a}_3^{(0)}=v_3+h_1,\\
&S^{(0)}=h_2+v_2+h_3+v_3+h_1,\\
&R^{(0)}-3S^{(0)}=m_1-v_2-v_3, \qquad (5.2)
\end{aligned}$$

such that the areas $(h_{1,2,3}, v_{1,2,3}, m_{1,2,3})$ can be expressed as the linear combinations

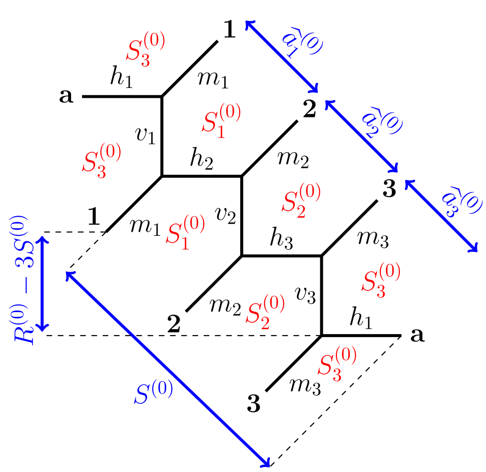


FIG. 10. Web diagram of $X_{3,1}$ with a parametrization of the areas of all curves. The blue parameters represent an independent set of Kähler parameters, as explained in Eq. (5.2).

$$\begin{aligned}
&h_1=S^{(0)}-\hat{a}_1^{(0)}-\hat{a}_2^{(0)}, \qquad h_2=S^{(0)}-\hat{a}_2^{(0)}-\hat{a}_3^{(0)},\\
&h_3=S^{(0)}-\hat{a}_1^{(0)}-\hat{a}_3^{(0)},\\
&m_1=m_2=m_3=2(\hat{a}_1^{(0)}+\hat{a}_2^{(0)}+\hat{a}_3^{(0)})+R^{(0)}-5S^{(0)},\\
&v_1=v_2=v_3=\hat{a}_1^{(0)}+\hat{a}_2^{(0)}+\hat{a}_3^{(0)}-S^{(0)}. \qquad (5.3)
\end{aligned}$$

The web diagram of $X_{3,1}^{(\delta=0)}$ allows various other representations: by mirroring the diagram and performing an $SL(2,\mathbb{Z})$ transformation, the web can be drawn in the form of Fig. 11(a). Furthermore, cutting the diagram along the lines labeled $v_{1,2,3}$ and regluing them along the lines labeled $m_{1,2,3}$ gives Fig. 11(b). The latter is again a web diagram with $\delta=0$, which can thus be parametrized by $(\hat{a}_{1,2,3}^{(1)}, S^{(1)}, R^{(1)})$, as indicated in Fig. 11(b),

$$\begin{aligned}
&\hat{a}_1^{(1)}=m_3+h_3, \qquad \hat{a}_2^{(1)}=m_2+h_2,\\
&\hat{a}_3^{(1)}=m_1+h_1, \qquad S^{(1)}=h_3+m_2+h_2+m_1+h_1,\\
&R^{(1)}=v_3-m_2-m_1, \qquad (5.4)
\end{aligned}$$

such that the areas can be expressed in the following manner:

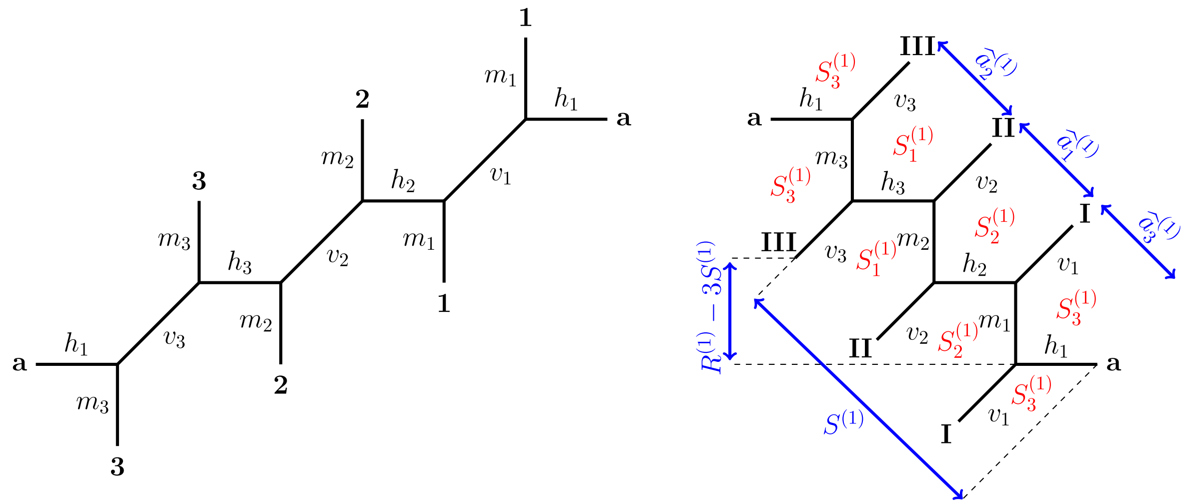
$$\begin{aligned}
&h_1=S^{(1)}-\hat{a}_1^{(1)}-\hat{a}_2^{(1)}, \qquad h_2=S^{(1)}-\hat{a}_1^{(1)}-\hat{a}_3^{(1)},\\
&h_1=S^{(1)}-\hat{a}_2^{(1)}-\hat{a}_3^{(1)},\\
&v_1=v_2=v_3=R^{(1)}+2(\hat{a}_1^{(1)}+\hat{a}_2^{(1)}+\hat{a}_3^{(1)})-5S^{(1)},\\
&m_1=m_2=m_3=\hat{a}_1^{(1)}+\hat{a}_2^{(1)}+\hat{a}_3^{(1)}-S^{(1)}. \qquad (5.5)
\end{aligned}$$

Moreover, as explained in Sec. II B, comparing Eq. (5.5) with Eq. (5.3) gives rise to a symmetry of the partition function as a linear transformation relating $(\hat{a}_{1,2,3}^{(1)}, S^{(1)}, R^{(1)})$ to $(\hat{a}_{1,2,3}^{(0)}, S^{(0)}, R^{(0)})$,

$$\begin{pmatrix}\hat{a}_1^{(0)}\\ \hat{a}_2^{(0)}\\ \hat{a}_3^{(0)}\\ S^{(0)}\\ R^{(0)}\end{pmatrix} = G_1\cdot\begin{pmatrix}\hat{a}_1^{(1)}\\ \hat{a}_2^{(1)}\\ \hat{a}_3^{(1)}\\ S^{(1)}\\ R^{(1)}\end{pmatrix}, \quad \text{where } G_1=\begin{pmatrix}2&1&1&-4&1\\1&2&1&-4&1\\1&1&2&-4&1\\2&2&2&-7&2\\3&3&3&-12&4\end{pmatrix}$$

$$\text{with} \quad \begin{aligned}&\det G_1=1,\\ &G_1\cdot G_1=\mathbb{1}_{5\times5}.\end{aligned} \qquad (5.6)$$

In order to obtain another symmetry generator we first perform a transformation $\mathcal{F}$ as explained in Appendix A. The corresponding geometry is of the type $X_{3,1}^{(\delta=1)}$ and a parametrization of the various curves through an independent set of Kähler parameters is shown in Fig. 12. The duality map of $\mathcal{F}$ is explicitly given by



(a) (b)

FIG. 11. (a) Alternative representation of the web diagram in Fig. 10. (b) The web diagram obtained by cutting along the lines labeled $v_{1,2,3}$ and regluing them along the curves labeled $m_{1,2,3}$.

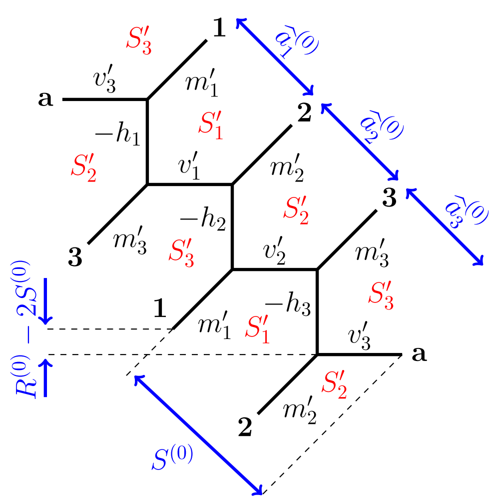


FIG. 12. Web diagram after a transformation $\mathcal{F}$ of Fig. 10. The blue parameters are the same as defined in Eq. (5.2).

$$
\begin{aligned}
v_1' &= v_1 + h_1 + h_2, & v_2' &= v_2 + h_2 + h_3, \\
v_3' &= v_3 + h_1 + h_3, & m_1' &= m_1 + h_1 + h_2, \\
m_2' &= m_2 + h_2 + h_3, & m_3' &= m_3 + h_1 + h_3. \qquad (5.7)
\end{aligned}
$$

As was shown in Ref. [25] for generic $X_{N,1}^{(\delta)}$, the independent parameters $(\hat{a}_{1,2,3}^{(0)}, S^{(0)}, R^{(0)})$ are invariants of $\mathcal{F}$ in the sense that the parameters appearing in Fig. 12 are the same as the ones defined in Eq. (5.2).[12]

[12]The only δ dependence (and thus dependence on $\mathcal{F}$) appears in the coefficient of $S^{(0)}$ in the defining equation of $R^{(0)}$ (see the generic parametrization of $X_{N,1}^{(\delta)}$ in Fig. 1).

While the transformation $\mathcal{F}$ itself therefore does not generate a new nontrivial symmetry transformation, one can consider different representations of Fig. 12. Indeed, by mirroring the latter and performing an $SL(2,\mathbb{Z})$ transformation, one obtains Fig. 13(a). Cutting the latter along the lines labeled $-h_{1,2,3}$ and regluing them along the lines labeled $m_{1,2,3}$ yields the representation in Fig. 13(b). The set of independent parameters $(\hat{a}_{1,2,3}^{(2)}, S^{(2)}, R^{(2)})$,

$$
\begin{aligned}
&\hat{a}_1^{(2)} = v_2' + m_3', \qquad \hat{a}_2^{(2)} = v_1' + m_2', \qquad \hat{a}_3^{(2)} = v_3' + m_1', \\
&S^{(2)} = v_2' + m_1' + v_3', \qquad R^{(2)} - 2S^{(2)} = -h_3 - m_1', \qquad (5.8)
\end{aligned}
$$

gives rise to a new parametrization of all of the curves of the original diagram in Fig. 13,

$$
\begin{aligned}
h_1 &= -\hat{a}_1^{(2)} - \hat{a}_2^{(2)} - R^{(2)} + 3S^{(2)}, \\
h_2 &= -\hat{a}_1^{(2)} - \hat{a}_3^{(2)} - R^{(2)} + 3S^{(2)}, \\
h_3 &= -\hat{a}_2^{(2)} - \hat{a}_3^{(2)} - R^{(2)} + 3S^{(2)}, \\
v_1 &= v_2 = v_3 = \hat{a}_1^{(2)} + \hat{a}_2^{(2)} + \hat{a}_3^{(2)} + 2R^{(2)} - 5S^{(2)}, \\
m_1 &= m_2 = m_3 = 2(\hat{a}_1^{(2)} + \hat{a}_2^{(2)} + \hat{a}_3^{(2)}) + 2R^{(2)} - 7S^{(2)}. \qquad (5.9)
\end{aligned}
$$

Comparing Eq. (5.12) with Eq. (5.3) gives rise to a symmetry of the partition function as a linear transformation relating $(\hat{a}_{1,2,3}^{(2)}, S^{(2)}, R^{(2)})$ to $(\hat{a}_{1,2,3}^{(0)}, S^{(0)}, R^{(0)})$,

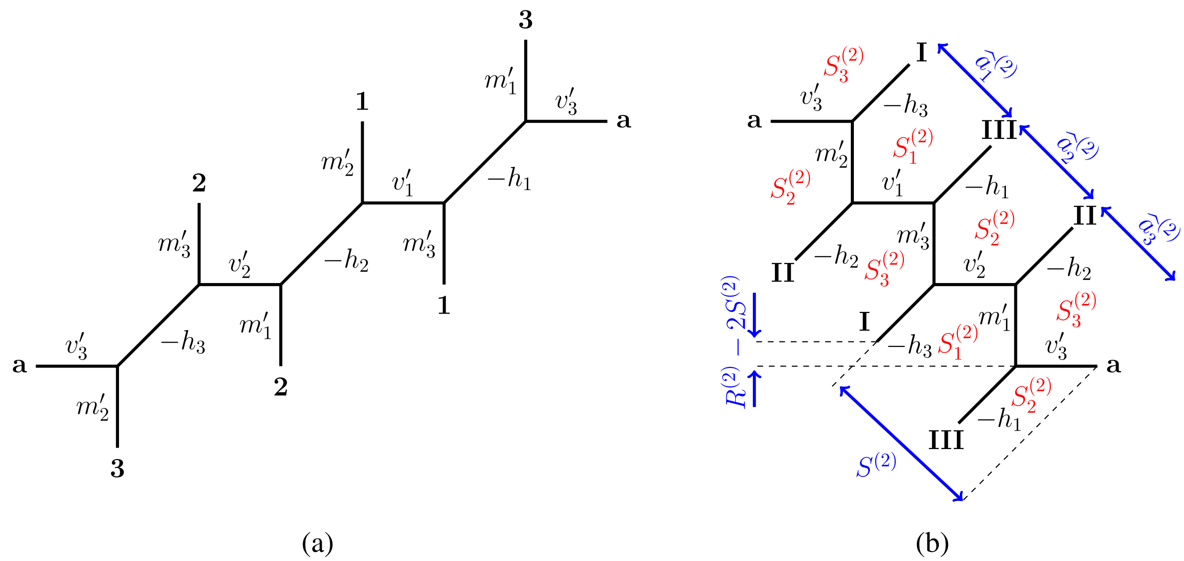


FIG. 13. (a) Alternative representation of the web diagram in Fig. 12. (b) The web diagram obtained by cutting the lines labeled $-h_{1,2,3}$ and regluing along the lines $m_{1,2,3}$.

$$\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_2 \cdot \begin{pmatrix} \hat{a}_1^{(2)} \\ \hat{a}_2^{(2)} \\ \hat{a}_3^{(2)} \\ S^{(2)} \\ R^{(2)} \end{pmatrix}, \quad \text{where } G_2 = \begin{pmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{with} \quad \begin{aligned} &\det G_2 = 1, \\ &G_2 \cdot G_2 = \mathbb{1}_{5\times 5}. \end{aligned} \tag{5.10}$$

One can find another symmetry transformation by cutting the diagram in Fig. 12 along the line labeled $-h_1$ and regluing it along the line labeled v'_3. After mirroring the diagram, it can also be presented in the form of Fig. 14, which corresponds to a web diagram of the form $X_{3,1}^{(\delta=1)}$. The latter can thus be parametrized by $(\hat{a}_{1,2,3}^{(3)}, S^{(3)}, R^{(3)})$, as shown in Fig. 8:

$$\begin{aligned} \hat{a}_1^{(3)} &= v'_3 - h_1, \qquad & \hat{a}_2^{(3)} &= v'_2 - h_3, \\ \hat{a}_3^{(3)} &= v'_1 - h_2, \qquad & S^{(2)} &= v'_2 - h_2 - h_3, \end{aligned}$$
$$R^{(2)} - 2S^{(2)} = m'_2 - v'_2. \tag{5.11}$$

Indeed, the areas $(h_{1,2,3}, v_{1,2,3}, m_{1,2,3})$ can be expressed in terms of $(\hat{a}_{1,2,3}^{(3)}, S^{(3)}, R^{(3)})$,

$$\begin{aligned} h_1 &= \hat{a}_3^{(3)} - S^{(3)}, \qquad & h_2 &= \hat{a}_2^{(3)} - S^{(3)}, \\ h_3 &= \hat{a}_1^{(3)} - S^{(3)}, \qquad & v_1 &= v_2 = v_3 = S^{(3)}, \end{aligned}$$
$$m_1 = m_2 = m_3 = R^{(3)} - S^{(3)}. \tag{5.12}$$

Since the partition functions computed from Figs. 14 and 10 are the same [25], comparing Eq. (5.12) to Eq. (5.3) gives rise to a linear transformation that is a symmetry of $\mathcal{Z}_{3,1}$.

Explicitly, one finds

$$\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_3 \cdot \begin{pmatrix} \hat{a}_1^{(3)} \\ \hat{a}_2^{(3)} \\ \hat{a}_3^{(3)} \\ S^{(3)} \\ R^{(3)} \end{pmatrix}, \tag{5.13}$$

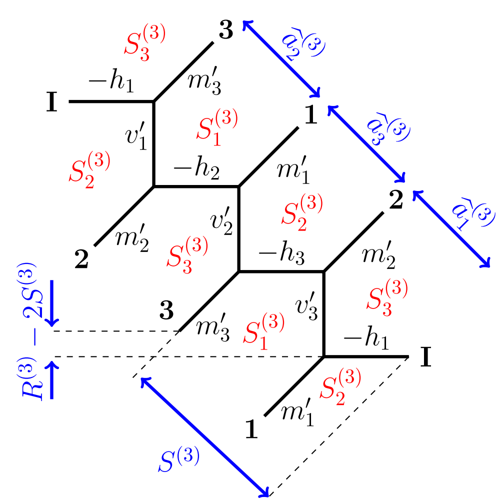


FIG. 14. Representation of the web diagram obtained by cutting Fig. 12 along the line $-h_1$ and gluing along the line v'_3.

where the 5×5 matrix G_3 is given by

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 3 & 3 & 3 & -6 & 1 \end{pmatrix} \quad \text{with} \quad \begin{array}{l} \det G_3 = 1, \\ G_3 \cdot G_3 = \mathbb{1}_{5\times 5}. \end{array} \tag{5.14}$$

From Fig. 12 one can extract yet another symmetry generator. Indeed, by cutting the diagram along the curves $v'_{1,2,3}$ and regluing it along the lines $m'_{1,2,3}$ one obtains Fig. 15(a). Furthermore, by cutting along the line labeled $-h_1$ and regluing along the line m'_3 one obtains Fig. 15(b) after performing an $SL(2,\mathbb{Z})$ transformation.

An independent set of parameters is given by

$$\begin{aligned} \hat{a}_1^{(4)} &= m'_2 - h_3, \qquad \hat{a}_2^{(4)} = m'_3 - h_1, \\ \hat{a}_3^{(4)} &= m'_1 - h_2, \qquad S^{(4)} = m'_2 - h_2 - h_3, \\ R^{(4)} - 2S^{(2)} &= v'_2 - m'_2, \end{aligned} \tag{5.15}$$

which allows to express $(h_{1,2,3}, v_{1,2,3}, m_{1,2,3})$ in the following fashion:

$$\begin{aligned} h_1 &= \hat{a}_3^{(4)} - S^{(4)}, \qquad h_2 = \hat{a}_1^{(4)} - S^{(4)}, \\ h_3 &= \hat{a}_2^{(4)} - S^{(4)}, \qquad v_1 = v_2 = v_3 = R^{(4)} - S^{(4)}, \\ m_1 &= m_2 = m_3 = S^{(4)}. \end{aligned} \tag{5.16}$$

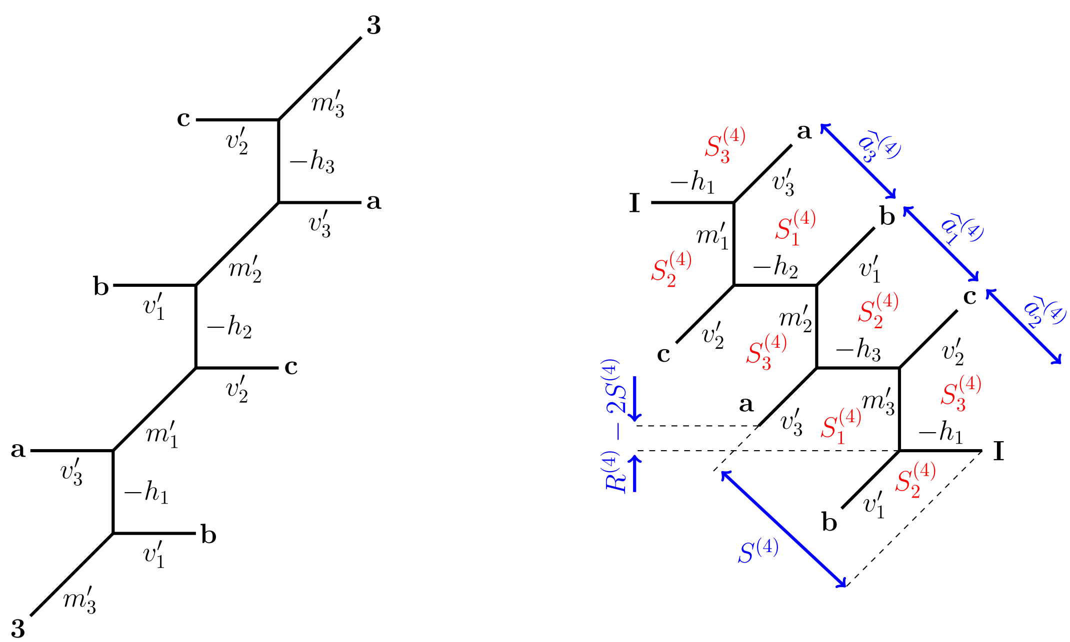
Comparing Eq. (5.16) to Eq. (5.3) gives rise to the following linear transformation:

$$\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_4 \cdot \begin{pmatrix} \hat{a}_1^{(4)} \\ \hat{a}_2^{(4)} \\ \hat{a}_3^{(4)} \\ S^{(4)} \\ R^{(4)} \end{pmatrix}, \quad \text{where } G_4 = \begin{pmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & -5 & 2 \\ 3 & 3 & 3 & -12 & 4 \end{pmatrix}$$

$$\text{with} \quad \begin{array}{l} \det G_4 = 1, \\ G_4 \cdot G_4 \cdot G_4 = \mathbb{1}_{5\times 5}. \end{array} \tag{5.17}$$

The matrix G_4 is of order 3, which means that $G_5 = G_4 \cdot G_4$ is a new symmetry element. It can also be associated to a particular representation of the web diagram of $X_{3,1}$. To see this, we first perform a transformation $\mathcal{F}$ on the web diagram in Fig. 12 to obtain Fig. 16.

Since $\mathcal{F}$ leaves the partition function invariant, the parameters $(\hat{a}_{1,2,3}^{(0)}, S^{(0)}, R^{(0)})$ are the same as those introduced in Eq. (5.2). Furthermore, we have introduced the areas



(a) (b)

FIG. 15. (a) Web diagram obtained by cutting the lines labeled $v'_{1,2,3}$ in Fig. 12 and regluing along the lines $m'_{1,2,3}$. (b) Alternative representation of the same web diagram after cutting along the line $-h_1$ and regluing along the line m'_3.

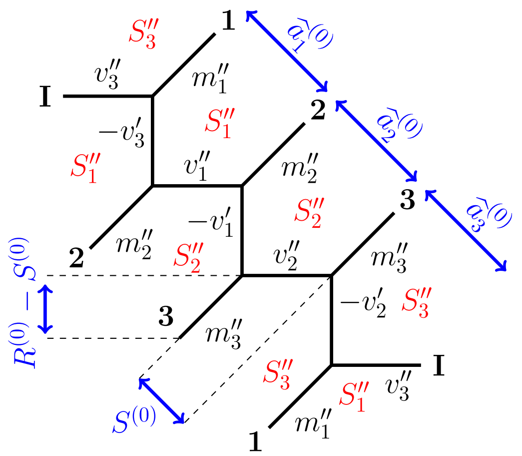


FIG. 16. Web diagram after a transformation $\mathcal{F}$ of Fig. 12. The blue parameters are the same as defined in Eq. (5.2).

$$v_1'' = -h_1 + v_1' + v_3', \qquad v_2'' = -h_2 + v_1' + v_2',$$
$$v_3'' = -h_3 + v_2' + {}'_3, \qquad m_1'' = m_1' + v_2' + v_3',$$
$$m_2'' = m_2' + v_1' + v_3', \qquad m_3'' = m_3' + v_1' + v_2', \qquad (5.18)$$

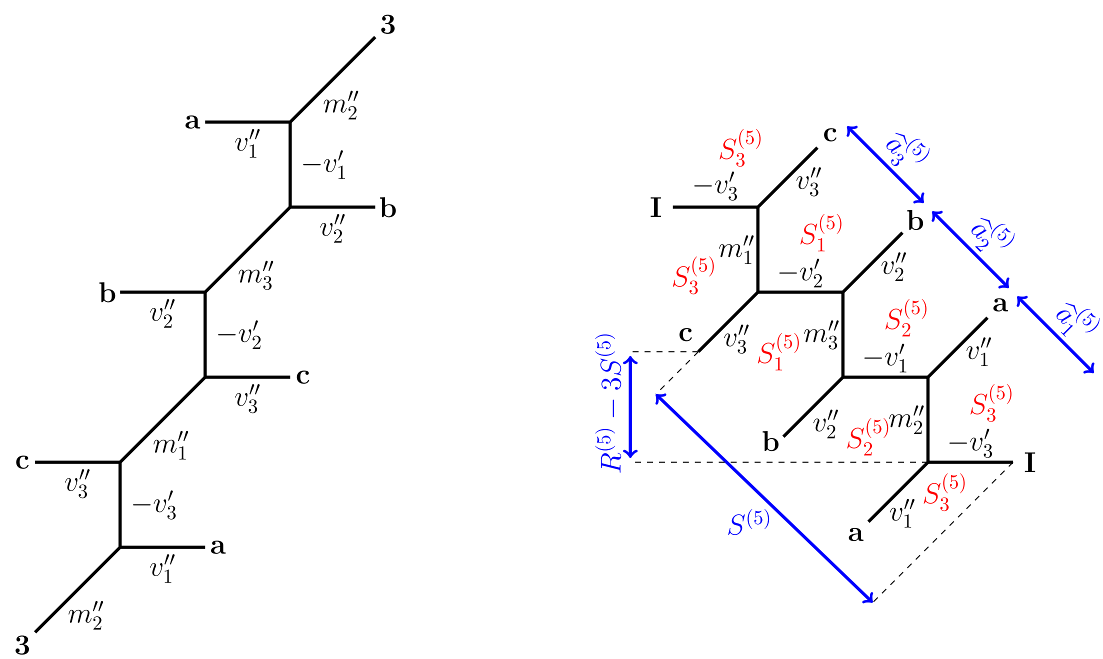
where we have used the definitions (5.7). Next, we cut the diagram in Fig. 16 along the lines labeled $v''_{1,2,3}$ and reglue it along the lines labeled $m''_{1,2,3}$ to obtain Fig. 17(a). Cutting the diagram again along the line $-v_3'$, it can also be presented in the form of Fig. 17(b), which is a diagram with shift $\delta = 0$. It can be parametrized by $(\hat{a}^{(5)}_{1,2,3}, S^{(5)}, R^{(5)})$,

$$
\begin{aligned}
h_1 &= 3S^{(5)} - \hat{a}_1^{(5)} - \hat{a}_2^{(5)} - R^{(5)},\\
h_2 &= 3S^{(5)} - \hat{a}_2^{(5)} - \hat{a}_3^{(5)} - R^{(5)},\\
h_3 &= 3S^{(5)} - \hat{a}_1^{(5)} - \hat{a}_3^{(5)} - R^{(5)},\\
v_1 &= v_2 = v_3 = 2(\hat{a}_1^{(5)} + \hat{a}_2^{(5)} + \hat{a}_3^{(5)}) - 7S^{(5)} + 2R^{(5)},\\
m_1 &= m_2 = m_3 = \hat{a}_1^{(5)} + \hat{a}_2^{(5)} + \hat{a}_3^{(5)} - 5S^{(5)} + 2R^{(5)}. \qquad (5.19)
\end{aligned}
$$

Comparing Eq. (5.19) to Eq. (5.3) indeed gives rise to the following symmetry transformation:

$$
\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_5 \cdot \begin{pmatrix} \hat{a}_1^{(5)} \\ \hat{a}_2^{(5)} \\ \hat{a}_3^{(5)} \\ S^{(5)} \\ R^{(5)} \end{pmatrix}, \quad \text{where } G_5 = \begin{pmatrix} 2 & 1 & 1 & -4 & 1 \\ 1 & 2 & 1 & -4 & 1 \\ 1 & 1 & 2 & -4 & 1 \\ 2 & 2 & 2 & -5 & 1 \\ 3 & 3 & 3 & -6 & 1 \end{pmatrix}
$$
$$
\text{with} \quad \begin{aligned} &\det G_5 = 1, \\ &G_5 \cdot G_5 \cdot G_5 = \mathbb{1}_{5\times 5}. \end{aligned} \qquad (5.20)
$$

Other representations of the web diagram of $X_{3,1}$ do not give rise to symmetries other than $G_{1,2,3,4,5}$, apart from a permutation of the parameters $\hat{a}_{1,2,3}$. These latter



(a) (b)

FIG. 17. (a) The web diagram in Fig. 16 after cutting the lines $v''_{1,2,3}$ and regluing along $m''_{1,2,3}$. (b) The web diagram after cutting along the line $-v_3'$ and gluing along m_2''.

symmetries form the group S_3, which, from the point of view of the gauge theory engineered by $X_{3,1}$, corresponds to the Weyl group of the gauge group $U(3)$. Factoring out this S_3 (the 5×5 identity matrix $E = \mathbb{1}_{5\times 5}$), the linear transformations $G_{1,2,3,4,5}$ form a finite group of order 6, which commute with S_3 and whose multiplication table is given by

	E	G_1	G_2	G_3	G_4	G_5
E	E	G_1	G_2	G_3	G_4	G_5
G_1	G_1	E	G_5	G_4	G_3	G_2
G_2	G_2	G_4	E	G_5	G_1	G_3
G_3	G_3	G_5	G_4	E	G_2	G_1
G_4	G_4	G_2	G_3	G_1	G_5	E
G_5	G_5	G_3	G_1	G_2	E	G_4

(5.21)

This table is the same as that of the dihedral group Dih_3, such that we have

$$\mathbb{G}(3) \cong \{E, G_1, G_2, G_3, G_4, G_5\} \cong \mathrm{Dih}_3. \tag{5.22}$$

An overview of the group elements $G_{1,2,3,4,5}$ and their relations to different representations of the web diagram in Fig. 10 are shown in Fig. 18.

For later use, we remark that the dihedral group (5.22) can also be represented as the group that is freely generated by G_2 and G_3,

$$\mathbb{G}(3) \cong \langle\{G_2, G_3\}\rangle, \ \text{with} \ \begin{aligned} &G_2\cdot G_2 = \mathbb{1}_{5\times5} = G_3\cdot G_3, \\ &(G_2\cdot G_3)^3 = \mathbb{1}_{5\times 5}. \end{aligned} \tag{5.23}$$

B. Invariance of the nonperturbative free energy

As in the previous example, following the result of Ref. [25], the linear transformations $G_{1,2,3,4,5}$ in Eqs. (5.6), (5.10), (5.14), (5.17), and (5.20) correspond to symmetries of the free energy $\mathcal{F}_{3,1}(\hat{a}_{1,2,3}, S, R; \epsilon_1, \epsilon_2)$, as defined in Eq. (2.4). In this section we provide evidence for this symmetry; however, for simplicity we limit ourselves to checking the leading limit in $\epsilon_{1,2}$ of the free energy. To this end, we introduce the following expansion:

$$\lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \mathcal{F}_{3,1}(\hat{a}_{1,2,3}, S, R; \epsilon_1, \epsilon_2) = \sum_{n=0}^{\infty} \sum_{i_1,i_2,i_3=0}^{\infty} \sum_{k\in\mathbb{Z}} f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n} \hat{Q}_1^{i_1}\hat{Q}_2^{i_2}\hat{Q}_3^{i_3} Q_S^k Q_R^n, \tag{5.24}$$

where $f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n} \in \mathbb{Z}$ and $\hat{Q}_i = e^{-\hat{a}_i}$ (for $i = 1, 2, 3$), $Q_S = e^{-S}$, and $Q_R = e^{-R}$. As explained in Sec. II B, the fact that the (shifted) web diagrams in Fig. 9 all give rise to the same partition functions implies

$$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n} = f^{\mathrm{NS}}_{i'_1,i'_2,i'_3,k',n'}$$

$$\text{for } (i'_1, i'_2, i'_3, k', n')^T = G_\ell^T\cdot(i_1, i_2, i_3, k, n)^T \quad \forall\, \ell = 1,2,3,4,5. \tag{5.25}$$

In Tables VI–VIII we tabulate the coefficients $f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$ with $i_1 + i_2 + i_3 \leq 7$ for $n = 1$ and $n = 2$ that are related by $G_{1,2,3,4,5}$.[13]

C. Modularity at a particular point of the moduli space

Similarly to the case $N = 2$ above, we can analyze how the group $\mathbb{G}(3)$ is related to $Sp(4,\mathbb{Z})$ in the particular region of the moduli space that is characterized by $\hat{a}_1^{(0)} = \hat{a}_2^{(0)} = \hat{a}_3^{(0)} = \hat{a}$, which implies $h_1 = h_2 = h_3 = h$ [while the consistency conditions (4.1) already impose $v_1 = v_2 = v_3 = v$ and $m_1 = m_2 = m_3 = m$]. As in the previous section, we can introduce the period matrix

$$\Omega = \begin{pmatrix} \tau & v \\ v & \rho \end{pmatrix}, \quad \text{with} \quad \begin{aligned} \tau &= m + v, \\ \rho &= h + m. \end{aligned} \tag{5.26}$$

Using the parametrization (5.23) of $\mathbb{G}(3)$, it is sufficient to analyze the relation of the generators G_2 and G_3 to $Sp(4,\mathbb{Z})$. The restriction of these generators to the subspace $(\hat{a}, S, R)$ can be written in the form

$$G_2^{(\mathrm{red})} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad G_3^{(\mathrm{red})} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 9 & -6 & 1 \end{pmatrix}. \tag{5.27}$$

Furthermore, by rewriting them to act as elements of $Sp(4,\mathbb{Z})$ in the form of Eq. (B3) on the period matrix Ω in Eq. (5.26), they take the form

$$\begin{aligned} \tilde{G}_2^{(\mathrm{red,Sp})} &= HKL^6HKHL^6KH, \quad \text{and} \\ \tilde{G}_3^{(\mathrm{red,Sp})} &= HKL^6KL^6KH, \end{aligned} \tag{5.28}$$

where K, L, and H are defined in Appendix B. As in the case of $N = 2$, this implies that the restriction of $\mathbb{G}(3)$ to the particular region of the Kähler moduli space $(\hat{a}, S, R)$ is a subgroup of $Sp(4,\mathbb{Z})$. However, unlike the case $N = 1$, we cannot conclude that the freely generated group $\langle \tilde{G}_2^{(\mathrm{red,Sp})}, \tilde{G}_3^{(\mathrm{red,Sp})}, S_\rho, T_\rho, S_\tau, T_\tau\rangle$ is isomorphic to $Sp(4,\mathbb{Z})$.

VI. EXAMPLE: $(N,M) = (4,1)$

A. Dualities and Dih_∞ group action

Using the previous examples, we next consider $X_{4,1}^{(\delta=0)}$, whose web diagram is shown in Fig. 19. While the method we employ to study it is the same as in the previous cases,

[13]We do not display symmetries between coefficients that also involve purely S_3 transformations.

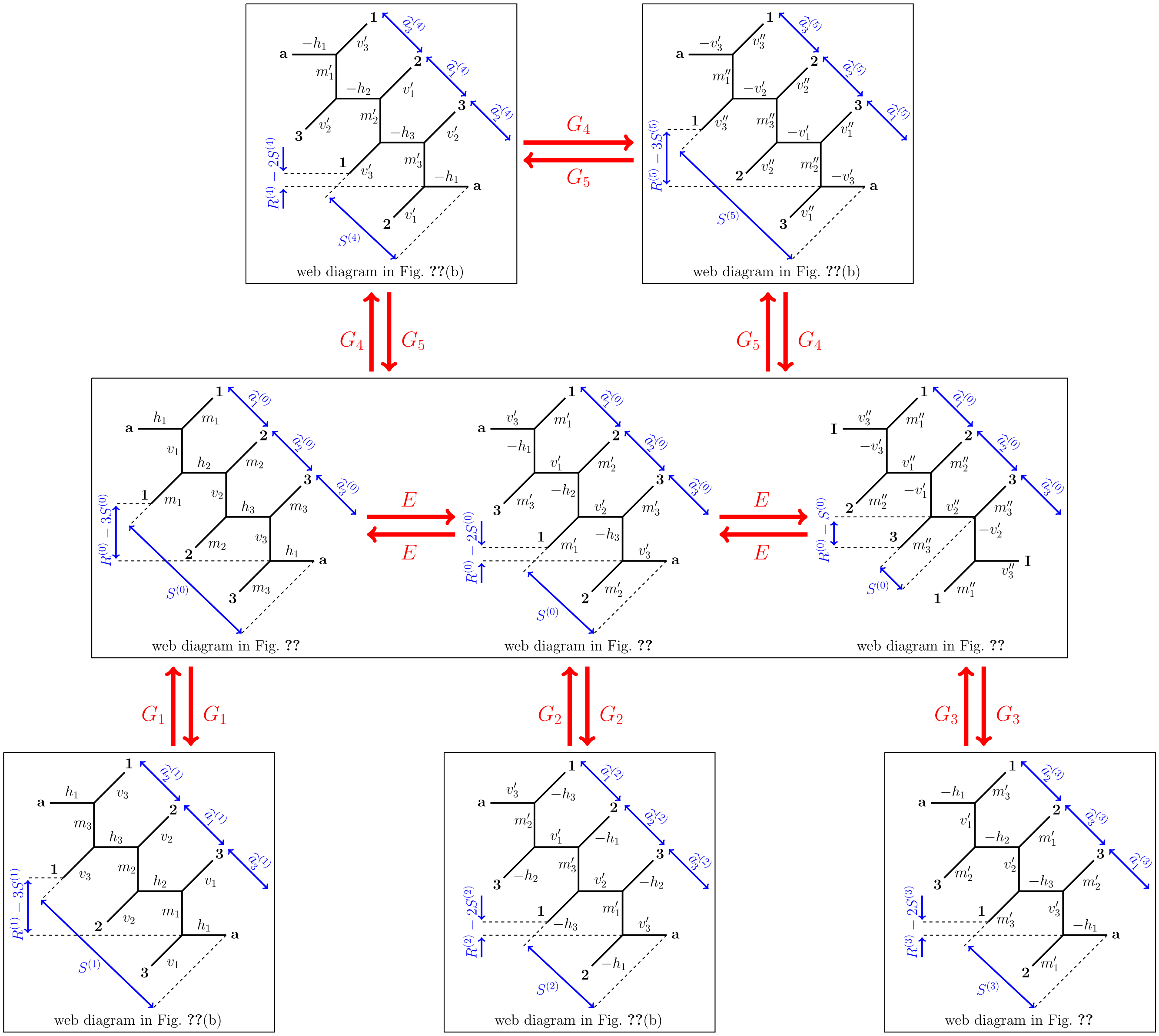


FIG. 18. Representations of web diagrams related to $X_{3,1}$. The transformations $G_{1,2,3,4,5}$ act on the basis of independent Kähler parameters $(\hat{a}_1^{(0)}, \hat{a}_2^{(0)}, \hat{a}_3^{(0)}, S^{(0)}, R^{(0)})$. The organization of web diagrams and transformations is reminiscent of the cycle graph of Dih_3.

we shall encounter a novel twist. The consistency conditions stemming from the web diagram are

$$
\begin{aligned}
S_1^{(0)}&: \; h_2 + m_2 = m_1 + h_2, v_1 + m_1 = m_2 + v_2,\\
S_2^{(0)}&: \; h_3 + m_3 = m_2 + h_3, v_2 + m_2 = m_3 + v_3,\\
S_3^{(0)}&: \; h_4 + m_4 = m_3 + h_4, v_3 + m_3 = m_4 + v_4,\\
S_4^{(0)}&: \; h_1 + m_1 = m_4 + h_1, m_1 + v_1 = m_4 + v_4, \qquad (6.1)
\end{aligned}
$$

while a solution is provided by the parameters $(\hat{a}_{1,2,3,4}^{(0)}, S^{(0)}, R^{(0)})$,

$$
\begin{aligned}
\hat{a}_1^{(0)} &= v_1 + h_2, \qquad \hat{a}_2^{(0)} = v_2 + h_3,\\
\hat{a}_3^{(0)} &= v_3 + h_4, \qquad \hat{a}_4^{(0)} = v_4 + h_1,\\
S^{(0)} &= h_2 + v_2 + h_3 + v_3 + h_4 + v_4 + h_1,\\
R^{(0)} - 4S^{(0)} &= m_1 - v_2 - v_3 - v_4. \qquad (6.2)
\end{aligned}
$$

The dihedral groups found in the previous examples were generated by two transformations. The latter can in fact be obtained in a simple fashion by considering two diagrams that are obtained from Fig. 19 through a rearrangement and a flop transformation, respectively.

TABLE VI. Action of G_1 (left) and G_2 (right): The indices are related by $(i_1', i_2', i_3', k', n')^T = G_1^T \cdot (i_1, i_2, i_3, k, n)^T$ and $(i_1'', i_2'', i_3'', k'', n')^T = G_2^T \cdot (i_1, i_2, i_3, k, n)^T$.

(i_1, i_2, i_3, k, n)	$(i_1', i_2', i_3', k', n')$	$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$
(0, 0, 0, −1, 1)	(1, 1, 1, −5, 2)	−3
(0, 0, 2, −2, 1)	(1, 1, 3, −6, 2)	4
(0, 1, 1, −2, 1)	(1, 2, 2, −6, 2)	8
(0, 1, 3, −3, 1)	(1, 2, 4, −7, 2)	−5
(0, 2, 2, −3, 1)	(1, 3, 3, −7, 2)	−4
(1, 1, 2, −3, 1)	(2, 2, 3, −7, 2)	−25

(i_1, i_2, i_3, k, n)	$(i_1'', i_2'', i_3'', k'', n'')$	$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$
(0,0,1,0,1)	(0, 0, 1, −2, 2)	12
(0, 0, 2, −1, 1)	(0, 0, 2, −3, 2)	−16
(0, 0, 3, −2, 1)	(0, 0, 3, −4, 2)	6
(0, 1, 1, −1, 1)	(0, 1, 1, −3, 2)	−23
(0, 1, 2, −2, 1)	(0, 1, 2, −4, 2)	18
(0, 1, 3, −3, 1)	(0, 1, 3, −5, 2)	−5
(0, 2, 2, −3, 1)	(0, 2, 2, −5, 2)	−4
(1, 1, 1, −2, 1)	(1, 1, 1, −4, 2)	42
(1, 1, 2, −3, 1)	(1, 1, 2, −5, 2)	−25
(1, 1, 3, −4, 1)	(1, 1, 3, −6, 2)	4
(2, 2, 2, −5, 1)	(2, 2, 2, −7, 2)	−3

1. Rearrangement

A simple rearrangement of Fig. 19 is shown in Fig. 20(a). The parametrization in terms of $(\hat{a}^{(1)}_{1,2,3,4}, S^{(1)}, R^{(1)})$ as indicated in Fig. 20(b) is distinct to the one in Fig. 19 by $(\hat{a}^{(0)}_{1,2,3,4}, S^{(0)}, R^{(0)})$. Indeed, the two bases are related through a linear transformation given by

$$\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ \hat{a}_4^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_1 \cdot \begin{pmatrix} \hat{a}_1^{(1)} \\ \hat{a}_2^{(1)} \\ \hat{a}_3^{(1)} \\ \hat{a}_4^{(1)} \\ S^{(1)} \\ R^{(1)} \end{pmatrix},$$

$$\text{where } G_1 = \begin{pmatrix} 3 & 2 & 2 & 2 & -6 & 1 \\ 2 & 3 & 2 & 2 & -6 & 1 \\ 2 & 2 & 3 & 2 & -6 & 1 \\ 2 & 2 & 2 & 3 & -6 & 1 \\ 6 & 6 & 6 & 6 & -17 & 3 \\ 16 & 16 & 16 & 16 & -48 & 9 \end{pmatrix}. \tag{6.3}$$

The matrix G_1 satisfies $\det G_1 = -1$ and $G_1^2 = \mathbb{1}_{6\times 6}$.

TABLE VII. Action of G_3: The indices are related by $(i_1', i_2', i_3', k', n')^T = G_3^T \cdot (i_1, i_2, i_3, k, n)^T$.

(i_1, i_2, i_3, k, n)	$(i_1', i_2', i_3', k', n')$	$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$
(0, 0, 0, −1, 1)	(2, 2, 2, −5, 1)	−3
(0, 0, 1, −2, 1)	(1, 1, 2, −4, 1)	2
(0, 0, 1, −1, 1)	(2, 2, 3, −5, 1)	−8
(0, 0, 2, −2, 1)	(1, 1, 3, −4, 1)	4
(0, 0, 3, −2, 1)	(1, 1, 4, −4, 1)	6
(0, 0, 4, −2, 1)	(1, 1, 5, −4, 1)	8
(0, 1, 2, −2, 1)	(1, 2, 3, −4, 1)	18
(0, 1, 3, −2, 1)	(1, 2, 4, −4, 1)	30
(0, 2, 2, −2, 1)	(1, 3, 3, −4, 1)	28
(1, 1, 1, −2, 1)	(2, 2, 2, −4, 1)	42
(1, 1, 2, −2, 1)	(2, 2, 3, −4, 1)	112
(1, 1, 6, −4, 1)	(0, 0, 5, −2, 1)	10
(1, 1, 7, −4, 1)	(0, 0, 6, −2, 1)	12
(1, 2, 6, −4, 1)	(0, 1, 5, −2, 1)	54
(1, 3, 4, −4, 1)	(0, 2, 3, −2, 1)	48

(i_1, i_2, i_3, k, n)	$(i_1', i_2', i_3', k', n')$	$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$
(1, 3, 5, −4, 1)	(0, 2, 4, −2, 1)	72
(1, 4, 4, −4, 1)	(0, 3, 3, −2, 1)	60
(2, 2, 4, −5, 1)	(0, 0, 2, −1, 1)	−16
(2, 2, 4, −4, 1)	(1, 1, 3, −2, 1)	208
(2, 2, 5, −5, 1)	(0, 0, 3, −1, 1)	−24
(2, 2, 5, −4, 1)	(1, 1, 4, −2, 1)	312
(2, 3, 3, −5, 1)	(0, 1, 1, −1, 1)	−23
(2, 3, 3, −4, 1)	(1, 2, 2, −2, 1)	286
(2, 3, 4, −5, 1)	(0, 1, 2, −1, 1)	−45
(2, 3, 4, −4, 1)	(1, 2, 3, −2, 1)	540
(3, 3, 3, −4, 1)	(2, 2, 2, −2, 1)	948
(0, 1, 3, −5, 2)	(1, 2, 4, −7, 2)	−5
(0, 2, 2, −5, 2)	(1, 3, 3, −7, 2)	−4
(1, 1, 1, −5, 2)	(2, 2, 2, −7, 2)	−3
(1, 1, 2, −5, 2)	(2, 2, 3, −7, 2)	−25

2. Transformation $\mathcal{F}$

Another symmetry transformation can be obtained after performing a transformation $\mathcal{F}$ on Fig. 19, as shown in Fig. 21.

TABLE VIII. Action of G_4 and G_5: The indices are related by $(i_1', i_2', i_3', k', n')^T = G_4^T \cdot (i_1, i_2, i_3, k, n)^T$ and $(i_1'', i_2'', i_3'', k'', n'')^T = G_4^T \cdot G_4^T \cdot (i_1, i_2, i_3, k, n)^T$, as well as $(i_1, i_2, i_3, k, n)^T = G_5^T \cdot (i_1', i_2', i_3', k', n')^T$ and $(i_1, i_2, i_3, k, n)^T = G_5^T \cdot G_5^T \cdot (i_1'', i_2'', i_3'', k'', n'')^T$.

(i_1, i_2, i_3, k, n)	$(i_1', i_2', i_3', k', n')$	$(i_1'', i_2'', i_3'', k'', n'')$	$f^{\mathrm{NS}}_{i_1,i_2,i_3,k,n}$
(0, 0, 2, −2, 1)	(1, 1, 3, −6, 2)	(1, 1, 3, −4, 1)	4
(0, 1, 1, −2, 1)	(1, 2, 2, −6, 2)	(1, 2, 2, −4, 1)	8
(0, 1, 3, −3, 1)	(0, 1, 3, −5, 2)	(1, 2, 4, −7, 2)	−5
(0, 2, 2, −3, 1)	(0, 2, 2, −5, 2)	(1, 3, 3, −7, 2)	−4
(1, 1, 2, −3, 1)	(1, 1, 2, −5, 2)	(2, 2, 3, −7, 2)	−25

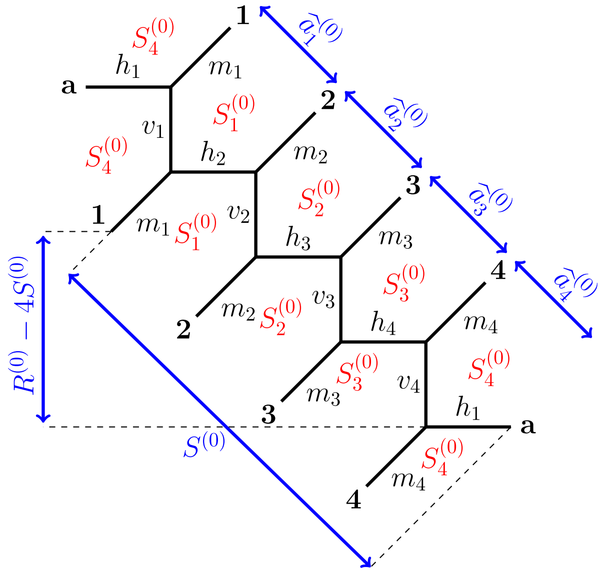


FIG. 19. Web diagram of $X_{4,1}$. An independent set of Kähler parameters is shown in blue.

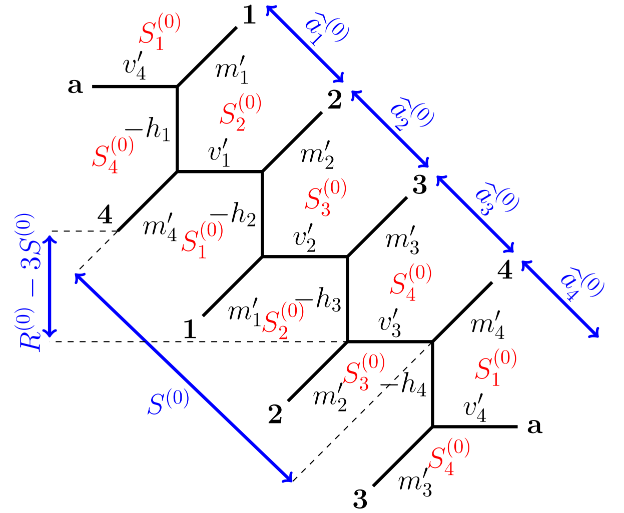


FIG. 21. Web diagram after a transformation $\mathcal{F}$ of Fig. 19. The blue parameters are the same as defined in Eq. (6.2).

Here we have introduced the variables

$$\begin{aligned} v'_1 &= v_1 + h_1 + h_2, & m'_1 &= m_1 + h_1 + h_2, \\ v'_2 &= v_2 + h_2 + h_3, & m'_2 &= m_2 + h_2 + h_3, \\ v'_3 &= v_3 + h_3 + h_4, & m'_3 &= m_3 + h_3 + h_4, \\ v'_4 &= v_4 + h_4 + h_1, & m'_4 &= m_4 + h_4 + h_1. \end{aligned} \tag{6.4}$$

The parameters $(\hat{a}_{1,2,3,4}^{(0)}, S^{(0)}, R^{(0)})$, shown in blue in Fig. 21, are the same as those appearing in Fig. 19, such that the flop transformation alone does not lead to a nontrivial symmetry transformation. However, starting from the web diagram in Fig. 21, we can present it in the form of Fig. 22. The parametrization in terms of the variables $(\hat{a}_{1,2,3,4}^{(2)}, S^{(2)}, R^{(2)})$ used in Fig. 22(b) can be related to $(\hat{a}_{1,2,3,4}^{(0)}, S^{(0)}, R^{(0)})$ in Fig. 19 through the transformation

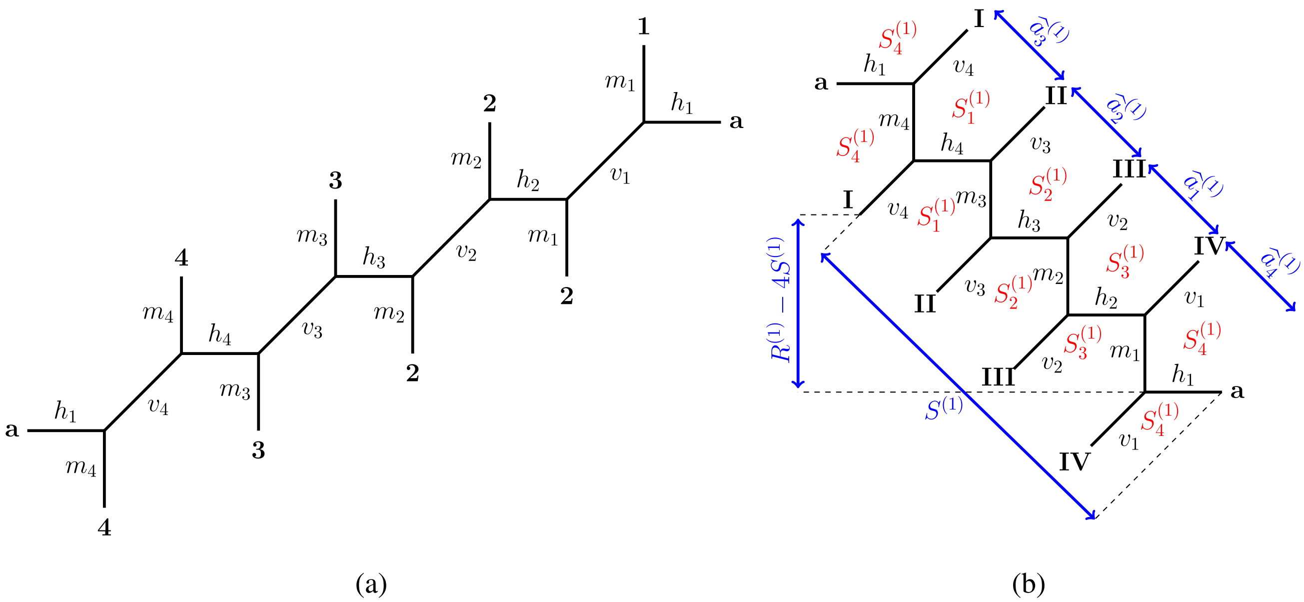


FIG. 20. (a) The mirrored web diagram in Fig. 19 after an $SL(2,\mathbb{Z})$ transformation. (b) The same diagram after cutting the lines $v_{1,2,3,4}$ and regluing the lines $m_{1,2,3,4}$ [and performing an $SL(2,\mathbb{Z})$ transformation].

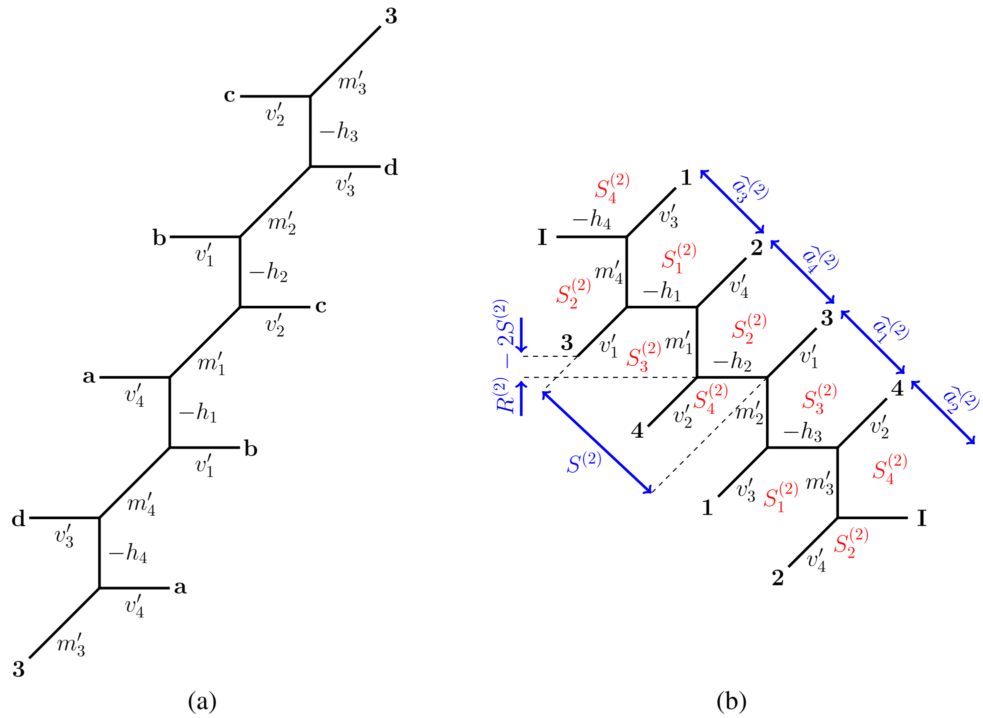


FIG. 22. (a) The web diagram in Fig. 21 after cutting the lines $m'_{1,2,3,4}$ and regluing along the lines $v'_{1,2,3,4}$. (b) Representation of the web diagram after cutting along the line $-h_4$ and gluing along the line m'_3.

$$\begin{pmatrix} \hat{a}_1^{(0)} \\ \hat{a}_2^{(0)} \\ \hat{a}_3^{(0)} \\ \hat{a}_4^{(0)} \\ S^{(0)} \\ R^{(0)} \end{pmatrix} = G_2 \cdot \begin{pmatrix} \hat{a}_1^{(2)} \\ \hat{a}_2^{(2)} \\ \hat{a}_3^{(2)} \\ \hat{a}_4^{(2)} \\ S^{(2)} \\ R^{(2)} \end{pmatrix}, \quad \text{where } G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 & -7 & 3 \\ 4 & 4 & 4 & 4 & -24 & 9 \end{pmatrix}. \tag{6.5}$$

The matrix G_2 has $\det G_2 = 1$ but does not have finite order.[14] This implies that the matrices G_1 and G_2 freely generate the group Dih_∞,

$$\mathbb{G}(4) = \langle \{G_1, G_2 \cdot G_1\} \rangle \cong \mathrm{Dih}_\infty. \tag{6.7}$$

[14]Indeed, by complete induction one can show that

$$G_2^n = \mathbb{1}_{6\times 6} + n \begin{pmatrix} n-1 & n-1 & n-1 & n-1 & 2-4n & n \\ n-1 & n-1 & n-1 & n-1 & 2-4n & n \\ n-1 & n-1 & n-1 & n-1 & 2-4n & n \\ n-1 & n-1 & n-1 & n-1 & 2-4n & n \\ 2n-1 & 2n-1 & 2n-1 & 2n-1 & -8n & 2n+1 \\ 4n & 4n & 4n & 4n & -8(2n+1) & 4(n+1) \end{pmatrix}, \quad \text{for } n \in \mathbb{N}. \tag{6.6}$$

which only resembles the identity matrix for $n = 0$.

B. A remark on infinite order

We have seen in the previous section that the symmetry transformation G_2 is of infinite order, which is markedly different than what we have seen in the previous examples. While we will present explicit checks that G_2 is indeed a symmetry of the free energy in the next subsection, we first want to provide an intuitive explanation of what makes the case $(N, 1) = (4, 1)$ different than all of the preceding ones. Indeed, we will provide some indication that the extended moduli space of $X_{4,1}$ contains many more regions that are represented by (*a priori*) very different looking web diagrams. While this will not prove that G_2 is of infinite order (as we have already done in the previous section by purely algebraic means), it will indicate the novel aspect of $X_{4,1}$ (in comparison to the previous examples).

Returning to Fig. 22(b), the latter is a web diagram of the form $X_{4,1}^{(\delta=2)}$. Another way of obtaining such a diagram is to perform two transformations of the form $\mathcal{F}$ on Fig. 19, as is shown in Fig. 23, with the new parameters

$$
\begin{aligned}
h'_1 &= -h_1 + v'_1 + v'_4 = h_1 + h_2 + h_4 + v_1 + v_4,\\
h'_2 &= -h_2 + v'_1 + v'_2 = h_1 + h_2 + h_3 + v_1 + v_2,\\
h'_3 &= -h_3 + v'_2 + v'_3 = h_2 + h_3 + h_4 + v_2 + v_3,\\
h'_4 &= -h_4 + v'_3 + v'_4 = h_1 + h_3 + h_4 + v_3 + v_4,
\end{aligned}
$$

as well as

$$
\begin{aligned}
m''_1 &= m'_1 + v'_4 + v'_2 = 2h_1 + 2h_2 + h_3 + h_4 + m_1 + v_2 + v_4,\\
m''_2 &= m'_2 + v'_1 + v'_3 = h_1 + 2h_2 + 2h_3 + h_4 + m_2 + v_1 + v_3,\\
m''_3 &= m'_3 + v'_2 + v'_4 = h_1 + h_2 + 2h_3 + 2h_4 + m_3 + v_2 + v_4,\\
m''_4 &= m'_4 + v'_4 + v'_1 = 2h_1 + h_2 + h_3 + 2h_4 + m_4 + v_1 + v_3.
\end{aligned}
$$

Notice that even upon imposing the consistency conditions (6.1), the parametrization of the web diagram in Fig. 23 is different than that of the web diagram in Fig. 22(b).[15] Thus, there is a duality transformation that transforms the web $X_{4,1}^{(2)} \longmapsto X_{4,1}^{(2)}$, however, with a nontrivial duality map $\mathcal{D}$ acting on the areas of all curves involved. The duality $\mathcal{D}$ can be repeatedly applied to $X_{4,1}^{(2)}$ in Fig. 22(b), thus producing an infinite number of diagrams of the type $X_{4,1}^{(2)}$, each one with an *a priori* different parametrization of individual curves.

Moreover, since the blue parameters $(\hat{a}^{(0)}_{1,2,3,4}, S^{(0)}, R^{(0)})$ in Fig. 23 are the same as in Fig. 19, the duality map $\mathcal{D}$ from the perspective of the independent Kähler parameters precisely corresponds to the symmetry transformation G_2. Therefore, the transition from Fig. 23 to Fig. 22(b) gives a (new) geometric representation of G_2 at the level of web diagrams, which readily allows to also compute arbitrary powers of G_2.

Finally, notice that the above discussion does not generalize to the cases $N = 2, 3$ (but can be extended to $N > 4$). Indeed, web diagrams with shifts $\delta \geq 2$ for $N = 2$, 3 can readily be related (possibly through simple cutting and regluing operations) to web diagrams with $\delta \in \{0, 1\}$, which only gave rise to symmetry transformations of finite order.[16] In other words, in the cases $N = 2$, 3, the equivalents of Figs. 22 and 23 are of the type $\delta \leq 1$, which we have seen provide only transformations of finite order.

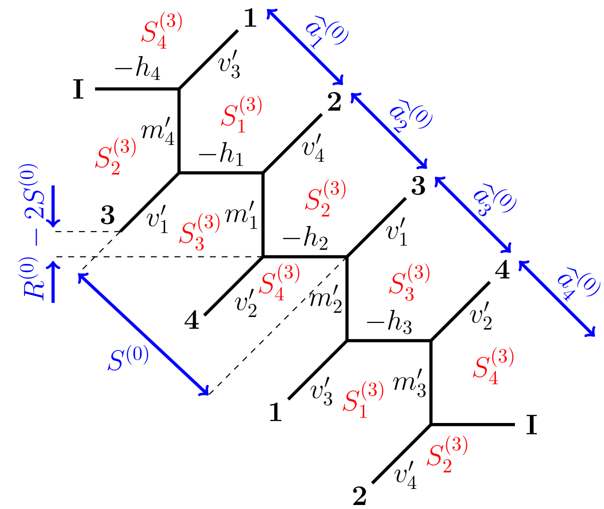


FIG. 23. Web diagram after two transformations $\mathcal{F}$ of Fig. 19. The blue parameters are the same as defined in Eq. (6.2).

C. Invariance of the nonperturbative free energy

As a nontrivial check of the fact that G_1 and G_2 are indeed symmetries of $\mathcal{Z}_{4,1}$, we consider the nonperturbative free energy associated with the latter. For simplicity, we restrict ourselves to the leading term in $\epsilon_{1,2}$. To this end, we define

$$
\lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \mathcal{F}_{4,1}(\hat{a}_{1,2,3,4}, S, R; \epsilon_1, \epsilon_2) = \sum_{n,i_a=0}^{\infty} \sum_{k\in\mathbb{Z}} f^{\mathrm{NS}}_{i_1,i_2,i_3,i_4,k,n} \hat{Q}_1^{i_1} \hat{Q}_2^{i_2} \hat{Q}_3^{i_3} \hat{Q}_3^{i_4} Q_S^k Q_R^n, \qquad (6.8)
$$

where $f^{\mathrm{NS}}_{i_1,i_2,i_3,i_4,k,n} \in \mathbb{Z}$ and $\hat{Q}_i = e^{-\hat{a}_i}$ (for $i = 1, 2, 3, 4$), $Q_S = e^{-S}$, and $Q_R = e^{-R}$. In the same manner as explained in Sec. II B, the symmetry transformations G_1 and G_2 act in the following manner on the coefficients $f^{\mathrm{NS}}_{i_1,i_2,i_3,i_4,k,n}$:

[15] This can be seen by choosing the solution $v_1 = v_2 = v_3 = v_4 = v$ and $m_1 = m_2 = m_3 = m_4 = m$.

[16] Notice, for example, that the only web diagrams in Figs. 9 and 18 that give rise to nontrivial symmetry transformations have either $\delta = 0$ or $\delta = 1$. Thus, in these cases, there is in fact no nontrivial equivalent of Fig. 22(b).

$$f^{\rm NS}_{i_1,i_2,i_3,i_4,k,n} = f^{\rm NS}_{i'_1,i'_2,i'_3,i'_4,k',n'}$$
$$\text{for } (i'_1,i'_2,i'_3,i'_4,k',n')^T = G_\ell^T\cdot(i_1,i_2,i_3,i_4,k,n)^T \quad \forall\ \ell = 1,2. \tag{6.9}$$

We can explicitly check the relations (6.9) by computing the relevant expansions of the free energies. However, since the matrix G_1 in Eq. (6.3) contains very large numbers, the relations are easier to check for the matrices $G_1\cdot G_2$ and G_2, with

$$G_1\cdot G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \\ 4 & 4 & 4 & 4 & -8 & 1 \end{pmatrix}. \tag{6.10}$$

In Tables IX and X we tabulate examples of the coefficients $f^{\rm NS}_{i_1,i_2,i_3,i_4,k,n}$ with $i_1+i_2+i_3+i_4\le 6$ for $n=1$ and $n=2$ that are related by $G_1\cdot G_2$ and G_2, respectively.

D. Modularity at a particular point of the moduli space

Similarly to the cases $N=2, 3$, we can analyze how the group $\mathbb{G}(4)$ is related to $Sp(4,\mathbb{Z})$ in the particular region in the moduli space characterized by $\hat{a}_1^{(0)} = \hat{a}_2^{(0)} = \hat{a}_3^{(0)} = \hat{a}_4^{(0)} = \hat{a}$, which implies $h_1=h_2=h_3=h_4=h$ [while the consistency conditions (4.1) impose $v_1=v_2=v_3=v_4=v$ and $m_1=m_2=m_3=m_4=m$]. We can introduce the period matrix

$$\Omega = \begin{pmatrix} \tau & v \\ v & \rho \end{pmatrix}, \quad \text{with} \quad \begin{matrix} \tau = m+v, \\ \rho = h+m. \end{matrix} \tag{6.11}$$

TABLE IX. Action of $G_1\cdot G_2$: $(i'_1,i'_2,i'_3,i'_4,k',n')^T = (G_1\cdot G_2)^T\cdot(i_1,i_2,i_3,i_4,k,n)^T$.

(i_1,i_2,i_3,i_4,k,n)	$(i'_1,i'_2,i'_3,i'_4,k',n')$	$f^{\rm NS}_{i_1,i_2,i_3,i_4,k,n}$
(0, 0, 1, 0, −2, 1)	(2, 2, 2, 3, −6, 1)	2
(0, 0, 1, 0, −1, 1)	(3, 3, 4, 3, −7, 1)	−8
(0, 0, 1, 1, −3, 1)	(1, 1, 2, 2, −5, 1)	−1
(0, 0, 1, 2, −2, 1)	(2, 2, 3, 4, −6, 1)	18
(0, 0, 1, 2, −1, 1)	(3, 3, 4, 5, −7, 1)	−45
(0, 0, 1, 3, −3, 1)	(1, 1, 2, 4, −5, 1)	−5
(0, 0, 1, 3, −2, 1)	(2, 2, 3, 5, −6, 1)	30
(0, 0, 1, 4, −3, 1)	(1, 1, 2, 5, −5, 1)	−7
(0, 0, 1, 4, −2, 1)	(2, 2, 3, 6, −6, 1)	42
(0, 0, 1, 5, −3, 1)	(1, 1, 2, 6, −5, 1)	−9
(0, 0, 1, 5, −2, 1)	(2, 2, 3, 7, −6, 1)	54
(0, 0, 0, 6, −2, 1)	(2, 2, 2, 8, −6, 1)	12

TABLE X. Action of G_2: $(i''_1,i''_2,i''_3,i''_4,k'',n')^T = G_2^T\cdot(i_1,i_2,i_3,i_4,k,n)^T$.

(i_1,i_2,i_3,i_4,k,n)	$(i'_1,i'_2,i'_3,i'_4,k',n')$	$f^{\rm NS}_{i_1,i_2,i_3,i_4,k,n}$
(0, 0, 1, 1, −3, 1)	(1, 1, 2, 2, −7, 2)	−1
(0, 1, 2, 2, −4, 1)	(0, 1, 1, 2, −6, 2)	2
(1, 1, 1, 2, −4, 1)	(1, 1, 1, 2, −6, 2)	4
(1, 1, 2, 3, −5, 1)	(0, 0, 1, 2, −3, 1)	−3
(1, 1, 2, 4, −5, 1)	(0, 0, 1, 3, −5, 2)	−5
(1, 1, 3, 3, −5, 1)	(0, 0, 2, 2, −5, 2)	−4

Using the parametrization (6.7) of $\mathbb{G}(4)$, it is sufficient to analyze the relation of the generators G_1 and $G'_2 = G_2\cdot G_1$ to $Sp(4,\mathbb{Z})$. The restriction of these generators to the subspace $(\hat{a}, S, R)$ can be written in the form

$$G_1^{(\rm red)} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad G_2^{\prime,(\rm red)} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 16 & -8 & 1 \end{pmatrix}. \tag{6.12}$$

Furthermore, by rewriting them to act as elements of $Sp(4,\mathbb{Z})$ in the form of Eq. (B3) on the period matrix Ω in Eq. (6.11), they take the form

$$\tilde{G}_1^{(\rm red,Sp)} = HKL^6KL^6HKHL^6KL^6KH, \quad \text{and}$$
$$\tilde{G}_2^{\prime,(\rm red,Sp)} = HKL^6KL^6KL^6KH, \tag{6.13}$$

where K, L, and H are defined in Appendix B. As in the cases of $N=2,3$, this implies that the restriction of $\mathbb{G}(3)$ to the particular region of the Kähler moduli space $(\hat{a}, S, R)$ is a subgroup of $Sp(4,\mathbb{Z})$. However, unlike the case $N=1$, we cannot conclude that the freely generated group $\langle \tilde{G}_1^{(\rm red,Sp)}, \tilde{G}_2^{\prime,(\rm red,Sp)}, S_\rho, T_\rho, S_\tau, T_\tau\rangle$ is isomorphic to $Sp(4,\mathbb{Z})$.

VII. GENERAL CASE $(N,1)$

A. Symmetry transformations of generic webs

We can summarize all previous examples by introducing the following matrices:

$$\mathcal{G}_2(N) = \begin{pmatrix} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{pmatrix}, \tag{7.1}$$

as well as

$$\mathcal{G}_\infty(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 1 & \cdots & 1 & -2N+1 & N-1 \\ N & \cdots & N & -2N(N-1) & (N-1)^2 \end{pmatrix}. \tag{7.2}$$

The matrices $\mathcal{G}_2(N)$ and $\mathcal{G}_\infty(N)$ for the examples previously studied are given explicitly as

N	$\mathcal{G}_2(N)$	$\mathcal{G}_\infty(N)$	Defined in
1	b	G_1	Eqs. (3.6) and (3.3)
2	G_2	G_3	Eqs. (4.6) and (4.12)
3	G_3	$G_3 \cdot G_2$	Eqs. (5.10) and (5.14)
4	$G_1 \cdot G_2$	G_2	Eqs. (6.3) and (6.5)

where the equation numbers refer to the definitions of the matrices in the individual cases. The matrices $\mathcal{G}_2$ and $\mathcal{G}_\infty(N)$ furnish two symmetry relations for a web diagram of the type $(N, 1)$. To see this, in the following we shall check explicitly the combinations of $\mathcal{G}_\infty(N) \cdot \mathcal{G}_2(N)$ and $\mathcal{G}_\infty(N)$, which at the level of the web diagrams are generated by the same transformations we already discussed in the example of $(N, 1) = (4, 1)$ and which can be generalized for generic N.

1. *Rearrangement:*

We first verify that $\mathcal{G}_\infty(N) \cdot \mathcal{G}_2(N)$ is a symmetry. To this end, we start from the configuration shown in Fig. 1 for $\delta = 0$, which [after mirroring and performing an $SL(2,\mathbb{Z})$ transformation] can be presented as in Fig. 24(a). The latter in turn can alternatively be presented in the form of Fig. 24(b). The matrix $\mathcal{G}_\infty(N) \cdot \mathcal{G}_2(N)$ [defined in Eqs. (7.1) and (7.2), respectively] relates the parameters in the web diagram in Fig. 1 to those in Fig. 24(b) in the following way:

$$(\hat{a}_1, \ldots, \hat{a}_N, S, R)^T = \mathcal{G}_2(N) \cdot \mathcal{G}_\infty(N) \cdot (\hat{a}'_1, \ldots, \hat{a}'_N, S', R')^T, \tag{7.3}$$

where

$$\mathcal{G}_\infty(N) \cdot \mathcal{G}_2(N) = \begin{pmatrix} & & & -2N+2 & 1 \\ & \mathbb{A}_{N\times N} & & \vdots & \vdots \\ & & & -2N+2 & 1 \\ N^2-3N+2 & \cdots & N^2-3N+2 & -2N^2+4N-1 & N-1 \\ N(N-2)^2 & \cdots & N(N-2)^2 & -2N(2-3N+N^2) & (N-1)^2 \end{pmatrix}, \tag{7.4}$$

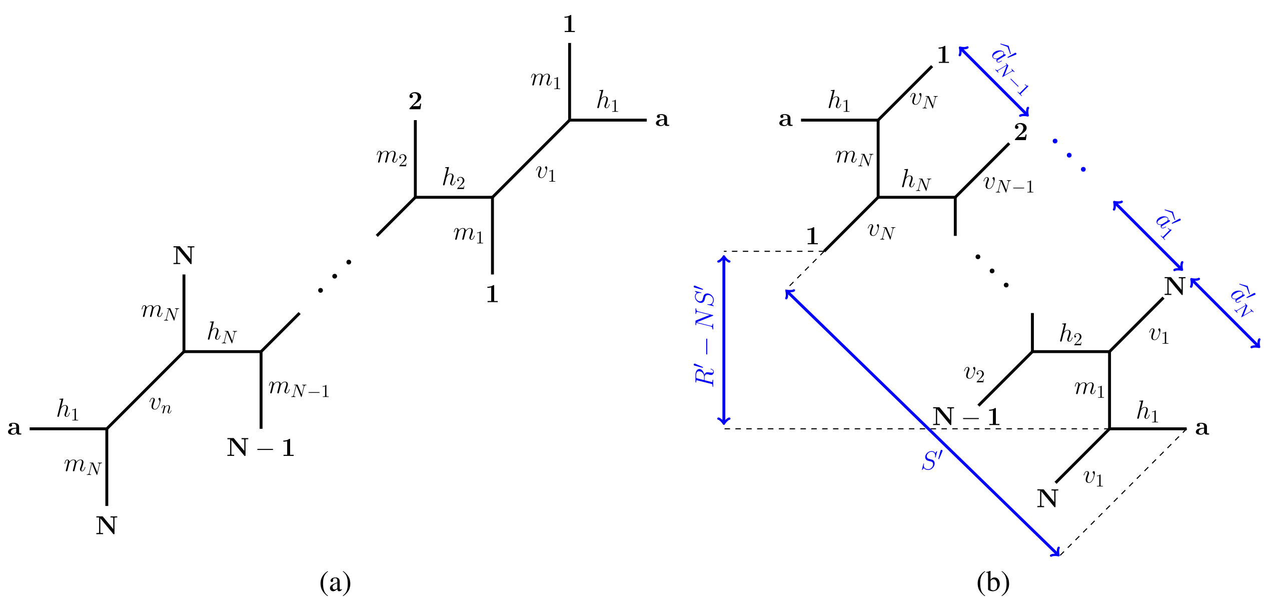


FIG. 24. Alternative representations of the web diagram of $X_{N,1}^{(\delta=0)}$ from Fig. 1 for $\delta = 0$.

with

$$\mathbb{A}_{N\times N} = (N-2)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \mathbb{1}_{N\times N}.$$

Upon using the solution of the consistency conditions in Eq. (2.1)

$$v_1 = v_2 = \ldots = v_N = v \quad \text{and} \quad m_1 = m_2 = \ldots = m_N = m, \tag{7.5}$$

which from Figs. 1 and 24(b) implies that (for $i = 1, \ldots, N$)

$$\begin{aligned} \hat{a}_i &= h_{i+1} + v, \qquad \hat{a}'_i = h_{i+1} + m, \\ S &= \sum_{k=1}^{N} h_k + (N-1)v, \\ S' &= \rho' - m = \sum_{k=1}^{N} \hat{a}'_k = \sum_{k=1}^{N} h_k + (N-1)m, \\ R - NS &= m - (N-1)v, \\ R' - NS' &= v - (N-1)m, \end{aligned} \tag{7.6}$$

we have indeed (with $\rho' = \sum_{k=1}^{N} \hat{a}'_k = \sum_{k=1}^{N} h_k + Nm$)

$$\begin{aligned} \hat{a}_i &= \hat{a}'_i + (N-2)\rho' - (2N-2)S' + R' = h_{i+1} + v, \\ S &= (N^2 - 3N + 2)\rho' - (2N^2 - 4N + 1)S' + (N-1)R' \\ &= \sum_{k=1}^{N} h_k + (N-1)v, \\ R &= N(N-2)^2\rho' - 2N(2 - 3N + N^2)S' + (N-1)^2 R' \\ &= N\sum_{k=1}^{N} h_k + m + (N-1)^2 v, \end{aligned}$$

which proves Eq. (7.3).

2. *Transformation $\mathcal{F}$:*

In a similar fashion we can show that $\mathcal{G}_\infty(N)$ is a symmetry transformation. To this end, we first consider a transformation of the type $\mathcal{F}$ acting on the web diagram in Fig. 1 for $\delta = 0$ which results in the web diagram shown in Fig. 25, representing $X_{N,1}^{(\delta=1)}$. The blue parameters in Fig. 25 are the same as in Fig. 1, and we have also introduced

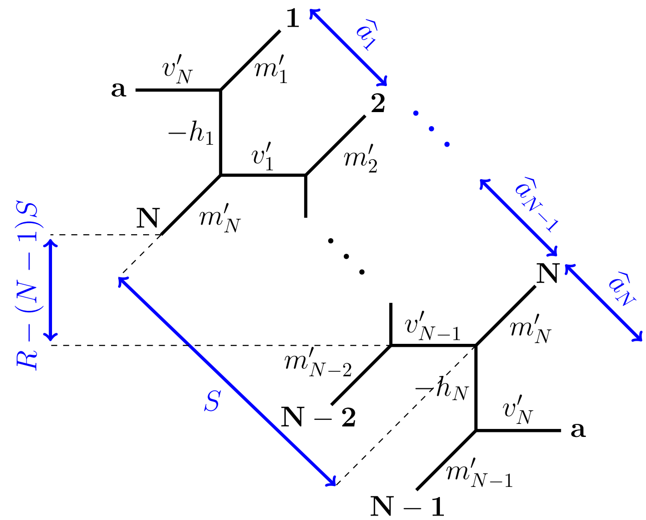


FIG. 25. Web diagram of $X_{N,1}^{(1)}$.

$$\begin{aligned} v'_1 &= v_1 + h_1 + h_2, \qquad & m'_1 &= m_1 + h_1 + h_2, \\ v'_2 &= v_2 + h_2 + h_3, \qquad & m'_2 &= m_2 + h_2 + h_3, \\ &\vdots & &\vdots \\ v'_N &= v_N + h_N + h_1, \qquad & m'_N &= m_N + h_N + h_1. \end{aligned}$$

Cutting the diagram in Fig. 25 along the lines $v'_{1,\ldots,N-1}$ and regluing it along the lines $m'_{1,\ldots,N}$, we obtain the web diagram shown in Fig. 26(a). Furthermore, by cutting the latter diagram along the line $-h_N$ it can also be represented in the form Fig. 26(b), which corresponds to a staircase diagram with shift $\delta = N-2$. The set of independent Kähler parameters $(\hat{a}''_{1,\ldots,N}, S'', R'')$ can be related to $(\hat{a}_{1,\ldots,N}, S, R)$ in the following manner:

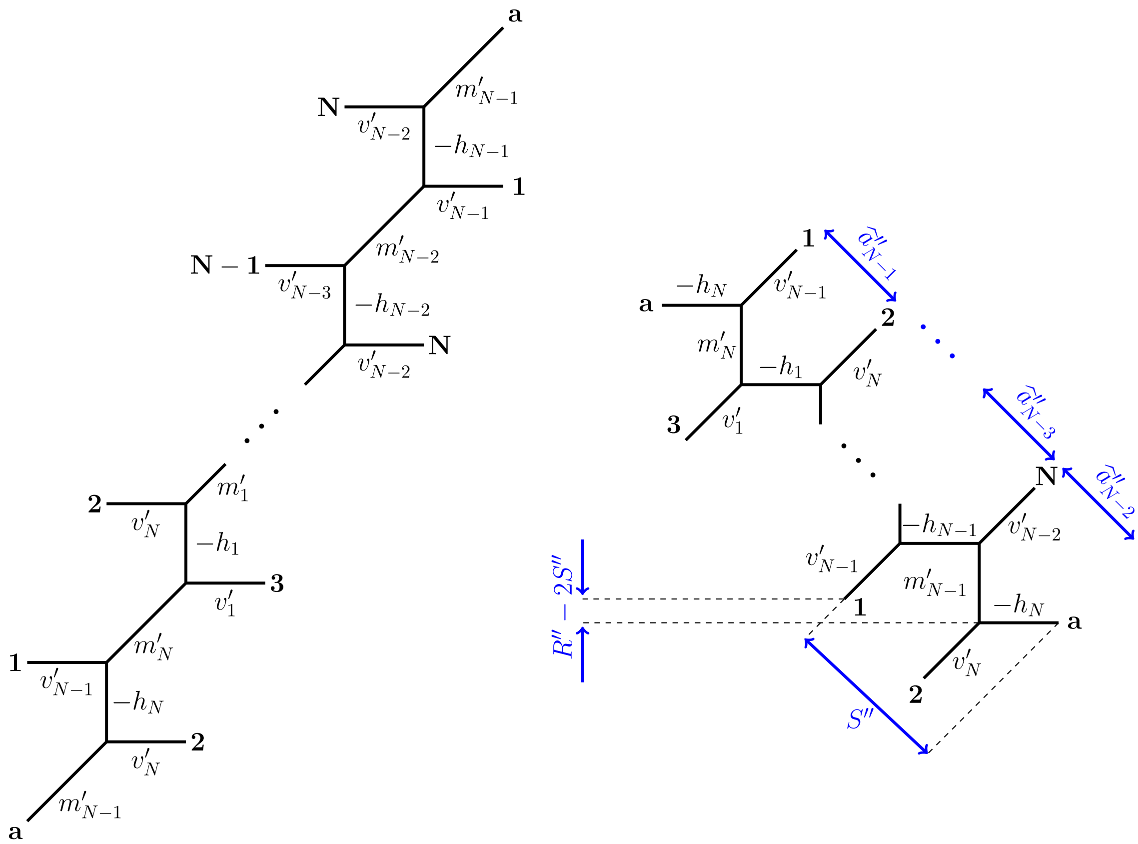
$$(\hat{a}_1, \ldots, \hat{a}_N, S, R)^T = \mathcal{G}_\infty(N) \cdot (\hat{a}''_1, \ldots, \hat{a}''_N, S'', R'')^T. \tag{7.7}$$

To show this, we use Eqs. (7.2) and (7.5) along with

$$\begin{aligned} \hat{a}''_i &= m'_{i+1} - h_{i+2} = m + h_{i+1}, \qquad S'' = m, \\ R'' - 2S'' &= v - m \end{aligned} \tag{7.8}$$

to compute [with $\rho'' = \sum_{k=1}^{N}(m'_k - h_k) = Nm + \sum_{k=1}^{N} h_k$]

$$\begin{aligned} \hat{a}_i &= \hat{a}''_i - 2S'' + R'' = m + h_{i+1} + v - m = h_{i+1} + v, \\ S &= \rho'' - (2N-1)S'' + (N-1)R'' = \sum_{k=1}^{N} h_k + (N-1)v, \\ R &= N\rho'' - 2N(N-1)S'' + (N-1)^2 R'' \\ &= N\sum_{k=1}^{N} h_k + m + (N-1)^2 v, \end{aligned} \tag{7.9}$$



(a) (b)

FIG. 26. (a) Alternative representations of the web diagram in Fig. 25. (b) Another representation in the form of a shifted web diagram with $\delta = N - 2$.

which matches Eq. (7.6) and therefore shows that $\mathcal{G}_\infty(N)$ is a symmetry transformation.

B. Generators of the dihedral group

Having shown that the transformations $\mathcal{G}_\infty(N) \cdot \mathcal{G}_2(N)$ and $\mathcal{G}_\infty(N)$ [and thus also $\mathcal{G}_2(N)$] are symmetry transformations of the partition function $\mathcal{Z}_{N,1}$, we shall now discuss the group structure that they generate. The matrix $\mathcal{G}_2(N)$ has order 2 [i.e., $\mathcal{G}_2(N).\mathcal{G}_2(N) = \mathbb{1}_{(N+2)\times(N+2)}$], while $\mathcal{G}_\infty(N)$ has the following order:

$$\mathrm{ord}\,\mathcal{G}_\infty(N) = \begin{cases} 3 & \text{if } N = 1, \\ 2 & \text{if } N = 2, \\ 3 & \text{if } N = 3, \\ \infty & \text{if } N \geq 4. \end{cases} \tag{7.10}$$

Here, infinite order means $\nexists m \in \mathbb{N}$ such that $(\mathcal{G}_\infty(N))^m = \mathbb{1}_{(N+2)\times(N+2)}$. While we have shown all cases $N \leq 4$ explicitly in previous sections, for $N > 4$ it is sufficient to realize that

$$\vec{v}_N = \left(\underbrace{1, \ldots, 1}_{N\text{times}}, \frac{N + \sqrt{N(N-4)}}{2}, \frac{N}{2}\left(N - 2 + \sqrt{N(N-4)}\right)\right)^T \tag{7.11}$$

is an eigenvector of $\mathcal{G}_\infty(N)$ for the eigenvalue[17]

$$\lambda_N = \frac{1}{2}\left((N-2)^2 - 2 + \sqrt{N(N-4)}(N-2)\right) \in \mathbb{R}. \tag{7.12}$$

Since $\lambda_N > 1$ for $N \geq 5$ [and $\mathcal{G}_\infty(N)$ is diagonalizable for $N \geq 5$], it follows that $\mathcal{G}_\infty(N)$ is not of finite order in these cases. Thus, upon introducing the matrix

[17] The remaining eigenvalues are $+1$ (with degeneracy N) and λ_N^{-1}.

$$\mathcal{G}_2'(N) = \mathcal{G}_2(N)\cdot\mathcal{G}_\infty(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{7.13}$$

which is of order 2 [i.e., $\mathcal{G}_2'(N).\mathcal{G}_2'(N) = \mathbb{1}_{N+2\times N+2}$], we find that $\mathcal{G}_2(N)$ and $\mathcal{G}_2'(N)$ freely generate a dihedral group,

$$\mathbb{G}(N) = \langle\{\mathcal{G}_2(N), \mathcal{G}_2'(N)\}\rangle \cong \begin{cases} \mathrm{Dih}_3 & \text{if } N=1, \\ \mathrm{Dih}_2 & \text{if } N=2, \\ \mathrm{Dih}_3 & \text{if } N=3, \\ \mathrm{Dih}_\infty & \text{if } N\geq 4. \end{cases} \tag{7.14}$$

For $N\geq 4$, Eq. (7.10) shows that $\nexists n\in\mathbb{N}$ with $(\mathcal{G}_2(N)\cdot \mathcal{G}_2'(N))^n = \mathbb{1}_{(N+2)\times(N+2)}$ [which also implies $\nexists n\in\mathbb{N}$ with $(\mathcal{G}_2'(N)\cdot\mathcal{G}_2(N))^n = \mathbb{1}_{(N+2)\times(N+2)}$]. Furthermore, since $(\mathcal{G}_2(N))^2 = \mathbb{1}_{(N+2)\times(N+2)} = (\mathcal{G}_2'(N))^2$, this also implies $\nexists n\in\mathbb{N}$ with $\mathcal{G}_2'(N)\cdot(\mathcal{G}_2(N)\cdot\mathcal{G}_2'(N))^n = \mathbb{1}_{(N+2)\times(N+2)}$ or $(\mathcal{G}_2(N)\cdot\mathcal{G}_2'(N))^n\cdot\mathcal{G}_2(N) = \mathbb{1}_{(N+2)\times(N+2)}$.[18] This means that there are no nontrivial (braid) relations between $\mathcal{G}_2(N)$ and $\mathcal{G}_2'(N)$, which indeed shows that the group $\mathbb{G}(N)\cong\mathrm{Dih}_\infty$ for $N\geq 4$.

Notice that $\mathcal{G}_2(N)$ is a lower diagonal matrix, while $\mathcal{G}_2'(N)$ is an upper diagonal $(N+2)\times(N+2)$ matrix. Furthermore, the partition function is invariant under the action of the group S_N, which is generated by matrices of the form

$$R(M) = \begin{pmatrix} & & & 0 & 0 \\ & M & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.15}$$

where M is an $N\times N$ matrix that acts by permuting the $\hat{a}_{1,\dots,N}$. One can check that matrices of the form $R(M)$ commute with both $\mathcal{G}_2(N)$ and $\mathcal{G}_2'(N)$, such that we have the following symmetry group of the partition function: $\tilde{G}(N)\cong\mathbb{G}(N)\times S_N$.

C. Modularity at a particular point of the moduli space

Using the general parametrization of the group $\mathbb{G}(N)$ in Eq. (7.14), we once again ask the question of how the latter is related to $Sp(4,\mathbb{Z})$ in the particular region in the moduli space characterized by $\hat{a}^{(0)}_{1,\dots,N} = \hat{a}^{(0)}_4 = \hat{a}$, which implies $h_{1,\dots,N} = h$ (while the consistency conditions already impose $v_{1,\dots,N} = v$ and $m_{1,\dots,N} = m$). We can similarly introduce the period matrix

$$\Omega = \begin{pmatrix} \tau & v \\ v & \rho \end{pmatrix}, \quad \text{with} \quad \begin{array}{l} \tau = m+v, \\ \rho = h+m. \end{array} \tag{7.16}$$

Using the parametrization (7.14) of $\mathbb{G}(4)$, it is sufficient to analyze the relation of the generators $\mathcal{G}_2(N)$ and $\mathcal{G}_2'(N)$ to $Sp(4,\mathbb{Z})$. The restriction of these generators to the subspace $(\hat{a}, S, R)$ can be written in the form

$$\mathcal{G}_2^{(\mathrm{red})}(N) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$
$$\mathcal{G}_2^{\prime,(\mathrm{red})}(N) = \begin{pmatrix} 1 & 0 & 0 \\ N & -1 & 0 \\ N^2 & -2N & 1 \end{pmatrix}, \tag{7.17}$$

or in the space (τ, ρ, v)

$$\tilde{\mathcal{G}}_2^{(\mathrm{red})}(N) = D_N^{-1}\cdot\mathcal{G}_2^{(\mathrm{red})}(N)\cdot D_N = \begin{pmatrix} (N-1)^2 & (N-2)^2N^2 & -2N(N^2-3N+2) \\ 1 & (N-1)^2 & 2(1-N) \\ N-1 & N(N^2-3N+2) & -2N^2+4N-1 \end{pmatrix},$$
$$\tilde{\mathcal{G}}_2^{\prime,(\mathrm{red})}(N) = D_N^{-1}\cdot\mathcal{G}_2^{\prime,(\mathrm{red})}(N)\cdot D_N = \begin{pmatrix} 1 & 4 & -4 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}, \quad \text{with} \quad D_N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & N & -1 \\ 1 & N^2 & -2N \end{pmatrix}. \tag{7.18}$$

[18]For example, the former relation is equivalent to $(\mathcal{G}_2(N)\cdot\mathcal{G}_2'(N))^n = \mathcal{G}_2'(N)$. Squaring this relation [due to the fact that $\mathcal{G}_2'(N)$ is of order 2] would be equivalent to $(\mathcal{G}_2(N)\cdot\mathcal{G}_2'(N))^{2n} = \mathbb{1}_{(N+2)\times(N+2)}$, which does not agree with Eq. (7.10).

Furthermore, by rewriting these generators to act as elements of $Sp(4,\mathbb{Z})$ in the form of Eq. (B3) on the period matrix Ω in Eq. (7.16), they take the form

$$\tilde{\mathcal{G}}_2^{(\mathrm{red,Sp})}(N) = (HKL^6H)^{N-2}K(HL^6KH)^{N-2} = \begin{pmatrix} N-1 & 1-(N-1)^2 & 0 & 0 \\ 1 & 1-N & 0 & 0 \\ 0 & 0 & N-1 & 1 \\ 0 & 0 & 1-(N-1)^2 & 1-N \end{pmatrix},$$

$$\tilde{\mathcal{G}}_2^{\prime,(\mathrm{red,Sp})}(N) = HK(L^6K)^{N-1}H = \begin{pmatrix} -1 & N & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & N & 1 \end{pmatrix},$$

where K, L, and H are defined in Appendix B. For $N \in \mathbb{N}$, the restriction of $\mathbb{G}(N)$ to the particular region of the Kähler moduli space $(\hat{a}, S, R)$ is a subgroup of $Sp(4,\mathbb{Z})$. However, for $N > 1$, we cannot conclude that the freely generated group $\langle \tilde{\mathcal{G}}_2^{(\mathrm{red,Sp})}(N), \tilde{\mathcal{G}}_2^{\prime,(\mathrm{red})}(N), S_\rho, T_\rho, S_\tau, T_\tau \rangle$ is isomorphic to $Sp(4,\mathbb{Z})$.

VIII. CONCLUSIONS

In this paper, we studied the consequences of the web of dualities among certain supersymmetric quiver gauge theories on $\mathbb{R}^5 \times S^1$ which are engineered by a class of toric Calabi-Yau threefolds $X_{N,M}$. These dualities were established in Refs. [22,24–26]; however, rather than focusing on the different physical theories, here we have analyzed their consequences from the perspective of the partition function $\mathcal{Z}_{N,M}$. For the sake of simplicity, our analysis has been limited to the case $M = 1$. We found that the partition function $\mathcal{Z}_{N,1}$ associated to the geometries $X_{N,1}$ is invariant under the group $\tilde{\mathbb{G}}(N) \cong \mathbb{G}(N) \times S_N$ which acts on the vector space spanned by a maximal set of independent Kähler parameters. Here S_N has an intuitive interpretation as the largest gauge group that can be engineered by the given geometry, which is $U(N)$ in this case. The group $\mathbb{G}(N)$ was shown to depend on N as derived in Eq. (7.14) and was found by exploiting the fact that $X_{N,1}$ can be related to various other geometries (that are part of the same extended Kähler moduli space) through flop and symmetry transformations. These geometries are characterized by giving rise to the same topological string partition function (i.e., the same $\mathcal{Z}_{N,1}$), but they are described by web diagrams whose Kähler parameters are related through a nontrivial duality map to those of the initial geometry. By studying a collection of these "self-duality" maps we showed that they form the group $\tilde{\mathbb{G}}(N)$.

A notable feature is the appearance of the infinite dihedral group for $N \geq 4$. By using the matrix representations of the generating elements, we have explicitly shown in Sec. VII that for the cases $N \geq 4$, the group $\mathbb{G}(N)$ is generated by two matrices of order 2, which have no nontrivial braid relations (implying the existence of a group element of infinite order). An intuitive understanding of the appearance of the infinite-order generator can be gained by looking at the behavior under the series of flop transformations $\mathcal{F}$, reviewed in Appendix A. They can be used to relate web diagrams that look identical but have a nontrivial mapping between their Kähler parameters. By iterating this procedure, it is thus possible to generate an infinite series of inequivalent web diagrams, thus giving an intuitive argument for the appearance of an infinite-order group. For the cases with $N \leq 3$ there is no such iterative procedure for producing nontrivially related geometries, due to the simpler nature of the diagram.

Furthermore, we showed that $\mathbb{G}(N)$ combines nontrivially with other known symmetry groups of the partition function. For the case $N = 1$, we showed explicitly that $\mathbb{G}(1) \cong \mathrm{Dih}_3$ together with the modular group $SL(2,\mathbb{Z})$ freely generate $Sp(4,\mathbb{Z})$, which is known to be the automorphism group associated to the mirror curve of $X_{1,1}$ [10,29]. For $N > 1$, we showed that in a particular region of the Kähler moduli space, $\mathbb{G}(N)$ corresponds to a subgroup of $Sp(4,\mathbb{Z})$. Similarly, the group $\tilde{\mathbb{G}}(N)$ mixes nontrivially with the T duality (as specifically proposed in Ref. [8]) that relates the IIa and IIb little string theories that are engineered by $X_{N,1}$. In both cases, it would be interesting to extend this analysis and to characterize the full (nonperturbative) U-duality group of the LSTs. We leave this point for future work.

From the perspective of the various gauge theories engineered by $X_{N,1}$, the symmetry group $\tilde{\mathbb{G}}(N)$ also has important consequences: acting in the form of Eq. (2.9), it identifies the multiplicities of certain single-particle BPS states in the free energy. This symmetry acts *a priori* nonperturbatively, since in particular an element $G \in \mathbb{G}(N)$ mixes all Kähler parameters of $X_{N,1}$ (which from the perspective of the BPS states of the gauge theory correspond to various fugacities in the free energy) in an arbitrary fashion. It is also important to remember that, in general, there are several different gauge theories that are engineered by $X_{N,1}$: as argued in Ref. [26], the latter engineers circular quiver gauge theories with M' nodes of type $U(N')$ for any (N', M'), with $N'M' = N$ and $\gcd(N', M') = 1$. All of these theories are dual to one another, in the sense that they have the same partition function $\mathcal{Z}_{N,1}$ and thus also share the symmetry $\tilde{\mathbb{G}}(N)$. The main difference is that the latter acts very differently from the perspective of the BPS spectrum; indeed, these theories differ in how the physical parameters (like coupling constants or Coulomb branch parameters) are expressed in terms of the Kähler parameters of $X_{N,1}$. The action of $\tilde{\mathbb{G}}(N)$ on the latter thus leads to different (physical) symmetries from the perspective of the various gauge theories (in particular their BPS states).

Another important aspect concerns the relation of the symmetry group $\tilde{\mathbb{G}}(N)$ with other symmetries that have previously been observed in the literature:

(1) In Ref. [21] it was found that (in a particular region in the Kähler moduli space of $X_{N,1}$) the free energy $\mathcal{F}_{N,1}$ in the NS limit is fully captured by $\mathcal{F}_{1,1}$.

(2) In Ref. [23] it was argued that in the NS limit a particular part of $\mathcal{Z}_{N,M}$ (called the reduced partition function) can be written as the partition function of a symmetric orbifold conformal field theory, giving rise to numerous Hecke-like relations between various terms in the corresponding free energies.

(3) In Ref. [9] it was demonstrated through a large number of examples that (in the unrefined limit), for a particular choice of some of the Kähler parameters, the partition function $\mathcal{Z}_{N,M}$ can be written as the sum over the weights of a single integrable representation of the affine Lie algebra $\hat{\mathfrak{a}}_{N-1}$ associated with the gauge group $U(N)$.

It is important that in all of these cases it was necessary to choose particular values for (some of) the Kähler moduli and/or the regularization parameters $\epsilon_{1,2}$, in one way or another. The elements of the group $\tilde{\mathbb{G}}(N)$ we found in the current work are more general in the sense that they are symmetries of $\mathcal{Z}_{N,1}$ (or the corresponding free energy $\mathcal{F}_{N,1}$) at a generic point in the Kähler moduli space of $X_{N,1}$ and for generic values of $\epsilon_{1,2}$.[19] In the future, it will be interesting to analyze how $\tilde{\mathbb{G}}(N)$ combines with the additional symmetries mentioned above in the respective regions of the moduli space.

At a generic point in the moduli space, it would be interesting to analyze how $\tilde{\mathbb{G}}(N)$ combines with other symmetries of the partition function [such as the modular groups $SL(2,\mathbb{Z})_\tau$ and $SL(2,\mathbb{Z})_\rho$] to form an even larger symmetry group. As the symmetries discussed in this work impose severe constraints on the structure of $\mathcal{Z}_{N,1}$, it would be interesting to investigate how much perturbative information (from the perspective of either of the gauge theories engineered by $X_{N,1}$) on the spectrum is required to recover the whole nonperturbative partition function. Questions of this type were recently considered, e.g., in Ref. [46], where the authors showed that the partition function can be reconstructed by using information from the two-dimensional world-sheet theories of the little string in combination with T duality.

Another interesting implication of the symmetries discussed in this work concerns the consequences at the level of the gauge theories themselves. For example, in Ref. [47] the authors used the well-known fiber-base duality of (a limit of) $X_{N,1}$ in order to argue for an enhancement of the global symmetry group of a certain class of five-dimensional theories at their superconformal fixed point. They showed explicitly the appearance of characters of the enhanced global symmetry group when expanding the Nekrasov partition function in a specific set of Coulomb branch parameters that are invariant under fiber-base duality. While the theories we analyzed here are six-dimensional and also do not have a superconformal fixed point (rather, their UV completions are LSTs), one might hope to gain information about some enhanced global symmetry. We leave some of these points for future work.

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APPENDIX A: DUALITY TRANSFORMATION $\mathcal{F}$

Since it is frequently used in the main body of this paper, in this Appendix we review a particular duality transformation (called $\mathcal{F}$) that was first proposed in Ref. [22] (see also Ref. [25]) and which acts on a shifted web diagram as shown in Fig. 1 by changing $\delta \to \delta + 1$. We specifically recall the duality map.

Starting from the web diagram in Fig. 1 with shift $\delta \in \{0, \ldots, N-1\}$, the duality trans-formation $\mathcal{F}$ is comprised of flop transformations on the curves with areas $\{h_1, \ldots, h_N\}$, along with $SL(2,\mathbb{Z})$ transformations and cutting and regluing of the web diagram. The resulting web diagram can again be presented in the form of a shifted "staircase" diagram with shift $\delta + 1$, as shown in Fig. 27.

It is important to notice that the independent Kähler parameters $(\hat{a}_{1,\ldots,N}, S, R)$ (shown in blue in Fig. 27) are in fact the same parameters as in Fig. 1, which in Ref. [25] were indeed shown to be invariant under the duality transformation. Similarly, these parameters are a solution of the consistency conditions imposed by the hexagons $S'_{1,\ldots,N}$, the latter being equivalent to the conditions (2.1) stemming from the hexagons $S_{1,\ldots,N}$ in the web diagram in Fig. 1. While the basis of the Kähler parameters $(\hat{a}_{1,\ldots,N}, S, R)$ is invariant under $\mathcal{F}$, the individual curves $(h_{1,\ldots,N}, v_{1,\ldots,N}, m_{1,\ldots,N})$ are not invariant under the transformation $\mathcal{F}$. Indeed, with respect to Fig. 27 we have the following duality map:

[19] Indeed, the group $\tilde{\mathbb{G}}(N)$ is based on dualities among web diagrams, which themselves are blind to $\epsilon_{1,2}$. Furthermore, while we considered the NS limit (combined with the unrefined limit) in Secs. V B and VI C, the latter was only a convenience to minimize computational complexity when performing certain checks of the symmetry transformations. The latter, however, hold in full generality.

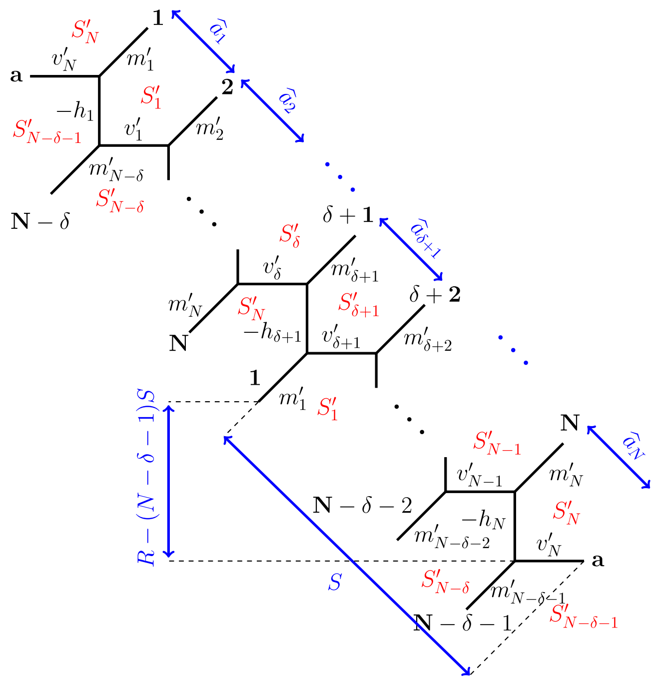

FIG. 27. Web diagram after a transformation $\mathcal{F}$ of $X_{N,1}^{(\delta)}$.

$$
\begin{aligned}
v'_1 &= v_1 + h_1 + h_2, \quad m'_1 = m_1 + h_1 + h_{\delta+2},\\
v'_2 &= v_2 + h_2 + h_3, \quad m'_2 = m_2 + h_2 + h_{\delta+3},\\
&\vdots \qquad\qquad\qquad \vdots\\
v'_{N-\delta} &= v_{N-\delta} + h_{N-\delta} + h_{N-\delta+1}, \quad m'_{N-\delta} = m_{N-\delta} + h_1 + h_{N-\delta}\\
&\vdots \qquad\qquad\qquad \vdots\\
v'_N &= v_N + h_1 + h_N, \quad m'_N = m_N + h_N + h_{\delta+1}, \qquad \text{(A1)}
\end{aligned}
$$

where $h_{i+N} = h_i$ for $i = 1, \ldots, N$.

APPENDIX B: REPRESENTATION OF $Sp(4,\mathbb{Z})$ AND MODULARITY

In Ref. [45] a representation of $Sp(4,\mathbb{Z})$ was given in terms of two generators (satisfying eight defining relations). The latter are of order 2 and 12, respectively,

$$
K = \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{and}
$$

$$
L = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \text{(B1)}
$$

which satisfy

$$
\begin{aligned}
&K^2 = L^{12} = \mathbb{1}_{4\times4}, \quad (KL^7KL^5K)L = L(KL^5KL^7K),\\
&(L^2KL^4)H = H(L^2KL^4),\\
&(L^3KL^3)H = H(L^3KL^3), \quad (L^2H)^2 = (HL^2)^2,\\
&L(L^6H)^2 = (L^6H)^2L, \quad (KL^5)^5 = (L^6H)^2,
\end{aligned}
$$

where $H = KL^5KL^7K$. We also mention that another representation [44] (in terms of six generators and 18 defining relations) is given by $X_{1,2,3,4,5,6}$, which can be expressed in terms of L and K as follows:

$$
\begin{aligned}
&X_1 = L^5KL, \quad X_2 = L^9HL^{10}H, \quad X_3 = L^8KL^{10},\\
&X_4 = HL^9HL^{10}, \quad X_5 = HL^6, \quad X_6 = L^9HL^6H. \qquad \text{(B2)}
\end{aligned}
$$

Furthermore, the group $Sp(4,\mathbb{Z})$ acts in a very natural form on the period matrix

$$
\Omega = \begin{pmatrix} \tau & v\\ v & \rho \end{pmatrix}
$$

of a genus-2 Riemann surface,

$$
\begin{pmatrix} A & B\\ C & D \end{pmatrix}\colon\ \Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}. \qquad \text{(B3)}
$$

Here A, B, C, D are 2×2 matrices that satisfy

$$
\begin{aligned}
A^TD - C^TB &= \mathbb{1}_{2\times2} = DA^T - CB^T,\\
A^TC &= C^TA, \quad B^TD = D^TB. \qquad \text{(B4)}
\end{aligned}
$$

For convenience, we provide the action of some of the generators on the period matrix Ω,

$$
\begin{aligned}
K&\colon\ \Omega \to \begin{pmatrix} \tau & \tau - v\\ \tau - v & -2v + \rho + \tau \end{pmatrix},\\
L^3&\colon\ \Omega \to \begin{pmatrix} \tau - \frac{v^2}{\rho} & \frac{v}{\rho}\\ \frac{v}{\rho} & -\frac{1}{\rho} \end{pmatrix},\\
L^6&\colon\ \Omega \to \begin{pmatrix} \tau & -v\\ -v & \rho \end{pmatrix},\\
L^9&\colon\ \Omega \to \begin{pmatrix} \tau - \frac{v^2}{\rho} & -\frac{v}{\rho}\\ -\frac{v}{\rho} & -\frac{1}{\rho} \end{pmatrix},\\
H&\colon\ \Omega \to \begin{pmatrix} \rho & v\\ v & \tau \end{pmatrix},\\
L^2KL^4&\colon\ \Omega \to \begin{pmatrix} \tau & v - 1\\ v - 1 & \rho \end{pmatrix},\\
L^9HL^{10}H&\colon\ \Omega \to \begin{pmatrix} \tau & v\\ v & \rho + 1 \end{pmatrix},\\
HL^9HL^{10}&\colon\ \Omega \to \begin{pmatrix} \tau + 1 & v\\ v & \rho \end{pmatrix}. \qquad \text{(B5)}
\end{aligned}
$$

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