Adjoint QCD_4, deconfined critical phenomena, symmetry-enriched topological quantum field theory, and higher symmetry extension

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Recent work explores the candidate phases of the 4D adjoint quantum chromodynamics (QCD_4) with an SU(2) gauge group and two massless adjoint Weyl fermions. Both Cordova-Dumitrescu and Bi-Senthil propose possible low energy 4D topological quantum field theories (TQFTs) to saturate the higher 't Hooft anomalies of adjoint QCD_4 under a renormalization-group flow from high energy. In this work, we generalize the symmetry-extension method of Wang-Wen-Witten [Phys. Rev. X **8**, 031048 (2018)] to higher symmetries, and formulate a higher group cohomology and cobordism theory approach to construct higher-symmetric TQFTs. We prove that the symmetry-extension method saturates certain anomalies, but also prove that *neither* $A\mathcal{P}_2(B_2)$ *nor* $\mathcal{P}_2(B_2)$ can be fully trivialized, with the background 1-form field A, Pontryagin square $\mathcal{P}_2$, and 2-form field B_2. Surprisingly, this indicates an obstruction to constructing a fully 1-form center and 0-form chiral symmetry (full discrete axial symmetry) preserving 4D TQFT with confinement, a *no-go* scenario via symmetry extension for specific higher anomalies. We comment on the implications and constraints on *deconfined quantum criticality* and ultraviolet-infrared duality in $3+1$ spacetime dimensions.

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I. INTRODUCTION AND SUMMARY OF MAIN RESULTS

Recent work explores the candidate phases of the adjoint quantum chromodynamics in four-dimensional (4D) spacetime (QCD_4) with an SU(2) gauge group and two massless adjoint Weyl fermions (equivalently, two massless adjoint Majorana fermions, or one massless adjoint Dirac fermion) [1–4].[1] This adjoint QCD_4 has a 1-form electric $\mathbb{Z}_2$ center global symmetry, which is a generalized global symmetry of a higher differential form [5]. This adjoint QCD_4 has the SU(2) gauge theory coupling to the matter fields in the adjoint representation; thus, it gains a 1-form electric $\mathbb{Z}_2$ center symmetry, while the usual fundamental QCD_4 has the gauge theory coupling to the matter fields in the fundamental representation, which lacks the 1-form symmetry. We will soon learn that this 1-form symmetry plays a crucial rule to constrain the higher 't Hooft anomaly matching [6] of the quantum phases of the adjoint QCD_4. (See Sec. II for more detailed information regarding the global symmetries and 't Hooft anomalies of this adjoint QCD_4.)

Given the adjoint QCD_4 at the high energy scale, it is known that this theory is weakly coupled and thus asymptotically free at ultraviolet (UV free) when the number of the Weyl fermion flavor $N_f \leq 5$. Viewing the adjoint QCD_4 as a UV completion of a quantum field theory (QFT), we should ask what this QFT flows to under a renormalization-group (RG) flow from UV to the low energy at infrared (IR). Both Cordova-Dumitrescu [2] and Bi-Senthil [3] propose its low energy candidate phases at IR, saturating the higher 't Hooft anomalies involving the 1-form symmetry.

In particular, Bi-Senthil [3] suggests a fully symmetric 4D TQFT to saturate higher 't Hooft anomalies without breaking any UV global symmetries of the adjoint QCD_4. Namely, an interesting RG flow from Bi-Senthil [3] speculates that

$$\text{Adjoint } QCD_4 \text{ at UV} \xrightarrow{\text{RG flow (?) to a long distance}} \text{Massless 1 Dirac (2 Weyl) fermion} + \text{4D TQFT at IR (?)}. \tag{1}$$

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[1]In this case, we denote the adjoint Weyl fermion flavor $N_f = 2$ and the gauge group $N_c = 2$ for $SU(N_c)$.

The IR theory only involves a massless free 1 Dirac (or 2 Weyl) fermion, and a decoupled 4D TQFT. Since the massless 1 Dirac fermion has only the ordinary 0-form symmetry but *no* 1-form symmetry, so the massless fermion sector alone *cannot* saturate the higher anomaly of the adjoint QCD_4. Thus, the crucial and nontrivial check on the Bi-Senthil [3] proposal of this UV-IR duality equation (1) relies on the explicit construction of the fully symmetric 4D TQFT to saturate all higher 't Hooft anomalies involving 1-form symmetry. One of the motivations of our present work is to rigorously verify the validity of this symmetric anomalous 4D TQFT.

In this work, we have two goals:

(1) We generalize the symmetry-extension method of Wang-Wen-Witten [7] to higher symmetries. We formulate a *higher group cohomology* or a *higher cobordism theory* approach of Ref. [7] to construct symmetric anomalous TQFTs that can live on the boundary of symmetry protected topological states (SPTs). The "symmetric anomalous TQFTs" is an abbreviation of "the TQFTs that saturate the (higher) 't Hooft anomalies of a given global symmetry by preserving the global symmetry." Previous works in condensed matter physics suggest that the *long-range entangled* anomalous topological order (whose effective low energy theory is a TQFT) can live on the boundary of a *short-range entangled* SPT state; see [8] and references therein on this exotic phenomenon. The boundary of SPTs protected by symmetry group G (called G-SPTs) has the 't Hooft anomaly of symmetry G. Reference [7] provides a systematic way to construct the symmetric anomalous TQFTs for a G-SPTs of a given symmetry G. In particular, among other results, Ref. [7] proves the following:

"For any bosonic G-SPTs protected by a finite group G (unitary or antiunitary time-reversal symmetry) in a two-dimensional spacetime (2D) or above ($\geq 2d$), there *always exists* a finite group K bosonic gauge theory which is a TQFT, saturating the G-'t Hooft anomaly, that can live on the boundary of G-SPTs, based on the symmetry-extension method via a short exact sequence $1 \to K \to H \to G \to 1$, where all G, K, and H are finite groups of 0-form symmetry."

In this article, we will explore the related phenomenon of Ref. [7] but we improve the formulation by replacing the 0-form G symmetry to include generalized higher symmetries of Ref. [5].

(2) We apply the above generalized higher symmetry-extension method from Ref. [7] either to construct the higher-symmetric anomalous TQFTs, for adjoint QCD_4, or to show the invalidity of the TQFTs via a symmetry-extension method.

Specifically, we find an *obstruction* to construct certain symmetric 4D TQFTs via symmetry extension, for the mixed anomaly mixing between the discrete axial symmetry (here the 0-form $\mathbb{Z}_{2N_cN_f} = \mathbb{Z}_8$ symmetry, with $N_c = N_f = 2$) and the 1-form electric center symmetry (denoted as $\mathbb{Z}^e_{2,[1]} = \mathbb{Z}_{2,[1]}$). This higher anomaly is abbreviated as the type I higher anomaly in Ref. [3]. The type I anomaly in 4D has a $\mathbb{Z}_4$ class (below $k \in \mathbb{Z}_4$ class) and one can explicitly write down the 5D topological (abbreviated as "topo.") invariant [2] which is a cobordism invariant (see mathematical details in [9] and Sec. A),

$$\text{Type I anomaly/topo. invariant:}\ \mathrm{e}^{\frac{\mathrm{i}k\pi}{2}\int A\cup\mathcal{P}_2(B_2)}. \quad (2)$$

Here A is the $\mathbb{Z}_4$-valued background 1-form gauge field coupling to the 0-form $\mathbb{Z}_8/\mathbb{Z}_2^F = \mathbb{Z}_4$ part of the axial global symmetry. The $\mathbb{Z}_2^F$ is the fermionic parity symmetry which is $(-1)^{N_F}$, assigning a minus to the state of the system when there is an odd number of the total number of fermions N_F. The B_2 is the $\mathbb{Z}_2$-valued background 2-form gauge field coupling to the 1-form $\mathbb{Z}^e_{2,[1]}$-symmetry. The $\cup$ is the cup product, and the $\mathcal{P}_2$ is the Pontryagin square; see more details in Sec. II. In Sec. A, we will prove the nonexistence of anomalous symmetric 4D TQFTs (of finite groups or higher groups) for this 4D higher anomaly (or equivalently, 5D higher SPTs) of Eq. (2), via the symmetry-extension method. However, we clarify that our proof does not necessarily imply a no-go theorem for the anomalous symmetric 4D TQFTs for Bi-Senthil [3]. In general, it could be due to the limitation of the symmetry extension [7] we used. Nevertheless, it is known that [7]'s method is general and systematic enough to construct symmetric TQFT for all bosonic anomalies of the ordinary 0-form finite group symmetries; thus, the obstruction from [7] is severe and interesting by itself to be presented here. This proof indicates a no-go scenario for anomalous symmetric 4D TQFTs if we *only* limit the construction under the symmetry-extension construction of TQFTs.

In contrast, we find that the generalized symmetry-extension method can indeed construct another symmetric 4D TQFT saturating a different higher mixed anomaly, mixing between the background gravity (or the curved spacetime geometry) and the 1-form center symmetry (denoted as $\mathbb{Z}_{2,[1]}$). This higher anomaly is abbreviated as the type II higher anomaly in Ref. [3]. We can explicitly write down the 5D topological invariant [2] as the following cobordism invariant (see mathematical details in [9] and Sec. A),

$$\text{Type II anomaly/topo. invariant:}$$
$$\mathrm{e}^{\mathrm{i}\pi\int w_2(TM)\mathrm{Sq}^1 B_2} = \mathrm{e}^{\mathrm{i}\pi\int w_3(TM) B_2}. \quad (3)$$

Here $w_j(TM)$ has the w_j as the jth Stiefel-Whitney (SW) class [10], as the probed background spacetime M connection over the spacetime tangent bundle TM. Sq^1 is the Steenrod operation. We demonstrate the explicit construction of the 4D symmetric anomalous TQFT for this 4D higher anomaly (or equivalently, 5D higher SPTs) of Eq. (3) in Sec. III.

Physically, the above description concerns the physics side of the story, relating to quantum field theory, QCD, or the strongly correlated systems in condensed matter physics.

Mathematically, we ask the following questions (corresponding to the physics story above) and find an obstruction to a positive answer for a Bi-Senthil's scenario [3] via the symmetry extension alone, generalizing the method of [7]:

Question 1. Can we trivialize the topological term $A \cup \mathcal{P}_2(B_2)$ via extending the global symmetry by the 0-form symmetry and 1-form symmetry? To answer this, we deal with the trivialization problem of the cobordism invariant $A \cup \mathcal{P}_2(B_2)$ of the bordism group $\Omega_5^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_8}(\mathrm{B}^2\mathbb{Z}_2)$.[2] We prove that the answer is negative.

Question 2. We also solve the trivialization problem of the cobordism invariant $\mathcal{P}_2(B_2)$ of the bordism group $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$: Can we trivialize the topological term $\mathcal{P}_2(B_2)$ via extending the global symmetry by the 0-form symmetry and 1-form symmetry? We prove that the answer is also negative.

The plan of the article goes as follows. In Sec. II, we detail the related global symmetries and higher anomalies relevant for our goal, following Ref. [2]. In Sec. III, we discuss the higher symmetry-extension generalization of [7], and successfully apply the method to construct a 4D symmetric anomalous TQFT for the type II anomaly equation (3). But this method shows an obstruction for the type I anomaly equation (2). We conclude in Sec. IV.

We leave the rigorous but more formal and mathematical details of the calculation to the Appendixes. In Appendix A, we find a potential obstruction: The type I anomaly equation (2) cannot be saturated by a symmetric anomalous finite group/higher group TQFT, at least by a symmetry-extension method. In Appendix B, we give a counter example as the proof for the failure of the symmetry-extension method applying to trivializing the 5D $A \cup \mathcal{P}_2(B_2)$. In Appendix C, we show a similar obstruction: The 4D $\mathcal{P}_2(B_2)$ cannot be saturated by a symmetric anomalous finite group/higher group TQFT, at least by a symmetry-extension method. We note that Appendixes A, B, and C are more technical and mathematically demanding. For readers who are not familiar with the mathematical background for these three sections, one can either consult [9,11] (e.g., the Appendix of [11]), or simply skip them and proceed to Sec. IV in which we summarize the physics interpretations of the above three sections.

The mathematical details of our cobordism calculations can be found in a companion paper [9].

[2] In this work, we will use the term dd "cobordism invariant" to describe the dd topological term or dd (higher) SPTs. On a manifold with a boundary, the boundary of such a cobordism invariant (or SPTs) has a 't Hooft anomaly. We denote the bordism group Ω_d^G, while we denote the cobordism group Ω_G^d.

II. THEORY OF ADJOINT QCD_4

We have an SU(2) gauge theory coupled to 2 **3** ($N_f = 2$ for the 2, and the **3** for the triplet) adjoint Weyl fermions in the adjoint representation of SU(2). The path integral (or partition function) of this adjoint QCD_4, in the Minkowski signature, viewed as a UV QFT theory can be written as

$$Z_{\mathrm{UV}} = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}][\mathcal{D}a] \exp(\mathrm{i}S_{\mathrm{UV}}), \tag{4}$$

$$S_{\mathrm{UV}} = \int d^4x \sum_{j=1,2} \frac{\mathrm{i}}{g^2} \bar{\psi}_j^{b'} \bar{\sigma}^\mu (\partial_\mu - \mathrm{i} g a_\mu^{a'} (T^{a'})_{b'b}) \psi^{jb} - \frac{1}{g^2} \int \mathrm{Tr}(F \wedge \star F) + \dots . \tag{5}$$

Equation (5) contains the first term as the Dirac Lagrangian, and the second term as the Yang-Mills Lagrangian. The $[\mathcal{D}\dots]$ is the path integral measure for the quantum fields. The $\bar{\sigma}^\mu \equiv (1, -\vec{\sigma})$ contains the standard Pauli sigma matrices $\vec{\sigma}$. Here the Weyl fermion ψ_α^{jb} has the following:

(i) the flavor index j [of the classical U(2) flavor symmetry, or more precisely the $\frac{\mathrm{SU}(2)\times\mathbb{Z}_8}{\mathbb{Z}_2^F}$ flavor symmetry in a quantum theory, see later],

(ii) the gauge index b of the gauge SU(2) of the adjoint triplet,

(iii) the Lorentz index α of the Lorentz group.

The Hermitian conjugation of the fermion field is $\bar{\psi}_j^{b'} = \psi^{jb'\dagger}$. With the Lorentz index, we have $\bar{\psi}_{\dot{\alpha}j}^{b'} = \psi_\alpha^{jb'\dagger}$, following the standard supersymmetry notation.

Here are some other comments:

(i) The g is the dimensionless Yang-Mills coupling, which is a running coupling in the quantum theory.

(ii) The F is the SU(2) gauge field a's 2-form field strength. The $\star F$ is the F's Hodge dual.

(iii) One can consider the deformation of the theory as extra terms in the …, such as the mass deformation [4], e.g., $(m_{ij}\delta_{b'b}\psi_\alpha^{ib'}\epsilon^{\alpha\beta}\psi_\beta^{jb} + \mathrm{c.c.})$. In the classical theory, we can add the θ-term,

$$\int \left(\frac{\theta}{8\pi^2} \mathrm{Tr} F \wedge F \right). \tag{6}$$

However, in the quantum theory, with the presence of the fermion fields ψ, we can rotate the θ away. If we have the mass term for the fermions, we can absorb the θ-term into the complex fermion mass matrix in the mass deformation.

A. Global symmetries

The global symmetries of the adjoint QCD_4 equation (4) has been analyzed systematically in [2]. Here we recap the results and will write the results suitable for the cobordism theory analysis later in Appendixes A to C.

(1) Flavor symmetry $\frac{\mathrm{SU}(2)\times\mathbb{Z}_8}{\mathbb{Z}_2^F}$: The classical flavor symmetry of 2 triplet Weyl fermions is the flavor $\mathrm{U}(2)=\frac{\mathrm{SU}(2)\times\mathrm{U}(1)_\mathrm{A}}{\mathbb{Z}_2^F}$. However, the axial symmetry $\mathrm{U}(1)_\mathrm{A}$ is broken down to a discrete axial symmetry $\mathbb{Z}_{2N_cN_f,\mathrm{A}}$, which is $\mathbb{Z}_{2N_cN_f,\mathrm{A}}=\mathbb{Z}_8$ here, due to the Adler-Bell-Jackiw anomaly.[3] It is a standard calculation of the $U(1)_\mathrm{A}$-axial symmetry that is explicitly broken by the dynamical $\mathrm{SU}(N_c)$-gauge instanton down to the $\mathbb{Z}_{2N_cN_f,\mathrm{A}}$-axial symmetry.

So the flavor symmetry is simply $\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}=\frac{\mathrm{SU}(2)\times\mathbb{Z}_8}{\mathbb{Z}_2^F}$ for the quantum theory. The SU(2) is also written as the $\mathrm{SU}(2)_R$ as the R symmetry thanks to the standard convention in $\mathcal{N}=2$ supersymmetric Yang-Mills theory (SYM) [13]. In the $\mathcal{N}=2$ SYM, the adjoint fermions are gauginos.

(2) The 1-form center symmetry $\mathbb{Z}_{2,[1]}^e\equiv\mathbb{Z}_{2,[1]}$: The adjoint QCD has the matter in adjoint representation, so the $\mathrm{SU}(N_c)$ [here SU(2)] fundamental Wilson line is charged under the 1-form electric center symmetry $\mathbb{Z}_{2,[1]}^e$ measured by a 2-surface "charge" operator. The "charged" fundamental Wilson line [spin-1/2 representation of SU(2)] has an odd $\mathbb{Z}_2$ charge. The odd half integer spin-$n/2$ representation of SU(2) has an odd $\mathbb{Z}_2$ charge of 1-form symmetry. Wilson lines of other integer spin-n representations (e.g., the adjoint) of SU(2) have a trivial (namely, even) $\mathbb{Z}_2$ charge of 1-form symmetry.

Importantly, that the 1-form center symmetry $\mathbb{Z}_{2,[1]}^e$ is preserved means that the electric Wilson loop (e-loop) is unbreakable, or called tension-ful [3]. Since the adjoint QCD has the 1-form center symmetry, we can use the 1-form center symmetry charged object to detect the following:

(i) Confinement: If 1-form symmetry is preserved, and all the Wilson loops (of all representations) obey the area law.

(ii) Deconfinement: If 1-form symmetry is spontaneously broken, then the Wilson loops of odd half integer spin-$n/2$ representation (e.g., fundamental representation) obey the perimeter law.

(3) Spacetime symmetry: In the Lorentz signature, we have the Poincaré group symmetry which contains the Lorentz group. We also have the discrete CPT symmetries. There is no charge conjugation C for SU(2) gauge theory due to the lack of SU(2) outer automorphism. So there is only T and P symmetry interchangeably thanks to the CPT theorem. If we focus on orientable spacetime for the adjoint QCD in dd, we can consider the $\mathrm{Spin}(d)$ spacetime symmetry, for the purpose of classifying the 't Hooft anomalies through the cobordism theory [9,14]. If we consider the nonorientable spacetime for the adjoint QCD in dd, we should consider the $\mathrm{Pin}^-(d)$ spacetime symmetry, for the purpose of classifying the 't Hooft anomalies through a cobordism theory; see Refs. [9,14]. This adjoint QCD is a fermionic theory, the spacetime symmetry $G_\text{spacetime}$ and the internal symmetry $\mathbb{G}_\text{internal}$ share the fermionic parity $\mathbb{Z}_2^F$, so the precise way to write the full global symmetry would be

$$\left(\frac{G_\text{spacetime}\times\mathbb{G}_\text{internal}}{\mathbb{Z}_2^F}\right)\equiv G_\text{spacetime}\times_{\mathbb{Z}_2^F}\mathbb{G}_\text{internal},\qquad(7)$$

where the common $\mathbb{Z}_2^F$ is mod out, while the "$\times_{\mathbb{Z}_2^F}$" notation follows [14].

By combining the internal global symmetry (flavor and 1-form center symmetries) and the spacetime global symmetry above, the overall global symmetry can be written as

$$\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\mathbb{Z}_{2,[1]}^e\qquad(8)$$

$$\mathrm{Pin}^-\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\mathbb{Z}_{2,[1]}^e.\qquad(9)$$

Below we follow Ref. [9], which generalizes a theorem in a remarkable work of Freed-Hopkins [14]. Freed-Hopkins [14] formulates a cobordism theory whose cobordism group, of the ordinary 0-form global symmetries, classifies a class of symmetric invertible TQFTs, which is relevant to the SPT classification. Reference [9] generalizes [14] to a cobordism theory of the higher global symmetries (e.g., including 0-form global symmetries and 1-form global symmetries) and computes some examples of such cobordism groups.

In terms of bordism group notation, which later will be helpful for identifying all the (higher) 't Hooft anomalies

[3]For a clarification of the different meanings of anomalies [such as the three different types of physics of anomalies: (1) classical global symmetry is violated at the quantum theory: the Adler-Bell-Jackiw anomaly; (2) quantum global symmetry is well defined and preserved but with the 't Hooft anomaly; (3) dynamical gauge anomaly] the readers can consult, e.g., the Sec. I Introduction of [12] and references therein.

and the SPT classes via the computations of [9], we write their corresponding bordism groups Ω_d as[4]

(i) Bordism group for Eq. (8):

$$\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times \mathrm{B}\mathbb{Z}^e_{2,[1]}}(pt) \equiv \Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)}(\mathrm{B}^2\mathbb{Z}^e_{2,[1]}). \quad (10)$$

(ii) Bordism group for Eq. (9):

$$\Omega_d^{\mathrm{Pin}^-\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times \mathrm{B}\mathbb{Z}^e_{2,[1]}}(pt) \equiv \Omega_d^{\mathrm{Pin}^-\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)}(\mathrm{B}^2\mathbb{Z}^e_{2,[1]}). \quad (11)$$

For adjoint QCD_4 in 4D, the higher 't Hooft anomalies are classified by the dimension $d = 5$ for the above bordism groups.[5] See more details in Ref. [12].

B. Anomalies

Now consider the $d = 5$ bordism groups above in Eqs. (10) and (11); we like to match their selective 5D cobordism invariants to the anomalies captured by the 4D adjoint QCD_4.

Cordova and Dumitrescu [2] have captured several anomalies, which we now overview:

(1) The SU(2) Witten anomaly [15] for the flavor $\mathrm{SU}(2)_R$ sector, because there is an odd number of the $\mathrm{SU}(2)_R$ flavor doublet. The appearance of the SU(2) Witten anomaly also indicates that the IR fate of this adjoint QCD_4 is gapless instead of fully gapped.

(2) The $(\mathbb{Z}_{8,\mathrm{A}})^3$ anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4D).

(3) The $(\mathbb{Z}_{8,\mathrm{A}})$-(gravity)^2 anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4D). The gravity part is due to the diffeomorphism of the background geometry.

(4) The $(\mathbb{Z}_{8,\mathrm{A}})$-$(\mathrm{SU}(2)_R)^2$ anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4D).

Reference [2] explains the two interesting mixed 't Hooft higher anomalies involving 1-form symmetry, the type I equation (2) and type II equation (3) anomalies earlier.

(5) Type I higher anomaly: mixing between the 1-form electric center symmetry ($\mathbb{Z}^e_{2,[1]}$) and the 0-form discrete axial symmetry ($\mathbb{Z}_{2N_cN_f} = \mathbb{Z}_8$). We can write Eq. (2) as

$$e^{i\frac{k\pi}{2}\int A\cup\mathcal{P}_2(B_2)} = e^{i\frac{k\pi}{2}\int A\cup(B_2\cup B_2+B_2\cup_1\delta B_2)} = e^{i\frac{k\pi}{2}\int A\cup(B_2\cup B_2+B_2(2\mathrm{Sq}^1 B_2))}, \quad (12)$$

see [9] for introducing the cup products, higher cup products, and the Steenrod square, Sq.

(6) Type II higher anomaly: mixing between the 1-form center symmetry (denoted as $\mathbb{Z}_{2,[1]}$) and the background gravity (or the curved spacetime geometry) in Eq. (3).

The UV theory as an adjoint QCD_4 has all of the above 't Hooft anomalies, captured also by a particular 5D cobordism invariant, in Eqs. (10) and (11).

Following our Introduction, in Sec. III, we formulate the higher symmetry extension generalizing [7], and successfully construct a 4D symmetric anomalous TQFT for the type II anomaly equation (3). But we will soon show an obstruction to construct symmetric TQFT for the type I anomaly equation (2).

III. HIGHER SYMMETRY EXTENSION

A. Summary of ordinary symmetry extension

Reference [7] sets up the symmetry-extension problem as follows. Consider the dd SPTs protected by an internal symmetry group G, whose boundary theory has the $(d-1)$d 't Hooft anomaly in G. There are three different ways to phrase the question asked by [7], but their underlying meanings are the same:

Q1. Condensed matter statement: Can we find a total group H such that G is its quotient group, and such that the G-SPTs becomes a trivial gapped vacua in H? More precisely, there is a local unitary transformation preserving the symmetry H (but breaking the symmetry G), such that when the G-SPTs is viewed as an H-SPTs, it can be deformed to a trivial gapped insulator in H via a local unitary transformation, without breaking H and without any phase transition.[6]

Q2. QFT or high energy particle physics statement: Given a $(d-1)$d 't Hooft anomaly in G, can we find an enlarged group H, with a total group H having G as its quotient group, such that the 't Hooft anomaly in G becomes *anomaly free* in H? (i.e., the G-anomaly becomes trivial in H.)

Q3. Mathematical and algebraic topology statement: Given a dd topological term of a group G, here the topological term can be the following:

[4]Here BG means the classifying space of G and pt means the point.

[5]On the other hand, if we aim to know the 4D SPTs compatible with the symmetry of adjoint QCD_4, then we need to consider the dimension $d = 4$ for the above bordism groups. This research direction is pursued by Ref. [11] for the related $\mathrm{SU}(N_c)$ Yang-Mills gauge theories.

[6]This procedure has been demonstrated explicitly in a many body quantum system recently in Ref. [16], which constructs an explicit path in the enlarged H-symmetric quantum Hilbert space.

(i) the dd cocycle for a dth cohomology group $\mathrm{H}^d(\mathrm{B}G, \mathrm{U}(1))$ in a group cohomology theory;
(ii) the dd co/bordism invariant for a dth cobordism group $\Omega^d_{-}(\mathrm{B}G, \mathrm{U}(1))$ or bordism group $\Omega_d^{-}(\mathrm{B}G)$ or the bordism group, in a cobordism theory[7];

can we find an extended group H with G its quotient group, via a short exact sequence

$$1 \to K \to H \to G \to 1, \tag{13}$$

such that the topological term of a group G can be pulled back to a trivial topological term of a group H?

Suppose the above answer is positive, and suppose that G, H, and K are finite groups, then Ref. [7] shows, valid for both the lattice Hamiltonian and the path integral construction, that the G-SPTs in dd can allow the following:

(i) H-symmetry-extended gapped boundary in any spacetime dimension $d \geq 2$,
(ii) G-symmetry-preserving and topological K-gauge theory gapped boundary: topological emergent K-gauge theory with preserving global symmetry G on a bulk $d \geq 3$.

Reference [7] addresses the above questions Q1, Q2, and Q3, by proving that, at least for a finite group G (with G a unitary symmetry group or anti-unitary symmetry group involving time-reversal symmetry), by the following positive answers, with the *always-existences* on the validity of the symmetric gapped boundary construction:

A1. For any bosonic SPT state with a finite on site symmetry group G, including both unitary and anti-unitary symmetry, there always exists an H-symmetry-extended (or G-symmetry-preserving) gapped boundary via a nontrivial group extension by a finite K, given the bulk spacetime dimension $d \geq 2$.

A2. For any G-anomaly in $(d-1)$d given by a cocycle $\nu_d^G \in \mathrm{H}^d(\mathrm{B}G, \mathrm{U}(1))$ of group cohomology of a finite group G, there always exists a pull back to a finite group H via a certain group extension $1 \to K \to H \overset{r}{\to} G \to 1$, extended by a finite K, such that the G-anomaly becomes H-anomaly free, given the dimension $d \geq 2$

A3. For any G-cocycle $\nu_d^G \in \mathrm{H}^d(\mathrm{B}G, \mathrm{U}(1))$ of a finite group G, there always exists a pull back to a finite group H via a certain short exact sequence of a group extension $1 \to K \to H \overset{r}{\to} G \to 1$ by a finite K, such that

$$r^* \nu_d^G = \nu_d^H = \delta \mu_{d-1}^H \in \mathrm{H}^d(H, \mathrm{U}(1)).$$

Here r is the pullback operation, and δ is the coboundary operation. Namely, a G-cocycle becomes an H-coboundary, which splits to one-lower dimensional H-cochains μ_{d-1}^H, given the dimension $d \geq 2$.

[7]Here the $-$ can be chosen as co/bordism with different structures such as special/orthogonal SO/O, spin/pin, or Spin/Pin$^\pm$ structures.

The proof of [7] has also been verified later by [17]. The related constructions similar to [7] are explored also in specific cases or from different perspectives in [18,19].

B. Higher symmetry generalization

Now we generalizes the approach in [7]. The short exact sequence of a group extension $1 \to K \to H \overset{r}{\to} G \to 1$ extended by a finite K given in [7] also implies an induced fiber sequence from the fibration

$$\mathrm{B}K \to \mathrm{B}H \to \mathrm{B}G, \tag{14}$$

where all G, K, and H are finite groups of 0-form symmetry such that the G-SPTs protected by a finite group G becomes trivial H-SPTs by pulling back G to H, under the above criteria A1, A2, and A3.

We consider the higher symmetry-extension problem. A simpler example is

$$\mathrm{B}K_{[0]} \times \mathrm{B}^2 K_{[1]} \to \mathrm{B}\mathbb{H} \to \mathrm{B}\mathbb{G},$$

where $K_{[0]}$ is an extension from a normal 0-form symmetry $K_{[0]}$, while $K_{[1]}$ is an extension from a less familiar and more exotic 1-form symmetry $K_{[1]}$. However, our goal is more ambitious to check a more general fibration

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2 K_{[1]} \to \mathrm{B}\mathbb{H} \to \mathrm{B}\mathbb{G} \tag{15}$$

where $\mathbb{G}$ and $\mathbb{H}$ are 2-groups, $K_{[0]}$ and $K_{[1]}$ are finite abelian groups of 0-form symmetry and 1-form symmetry respectively such that the higher $\mathbb{G}$-SPTs protected by a 2-group $\mathbb{G}$ becomes the trivial higher $\mathbb{H}$-SPTs by pulling back $\mathbb{G}$ to $\mathbb{H}$[8]. Here $\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2 K_{[1]}$ is the total space $\mathrm{B}\mathbb{K}$ of the fibration

$$\begin{array}{ccc} \mathrm{B}^2 K_{[1]} & \longrightarrow & \mathrm{B}\mathbb{K} \\ & & \downarrow \\ & & \mathrm{B}K_{[0]}. \end{array} \tag{16}$$

Similar to questions in Q1, Q2, and Q3 of Sec. III A, we ask a set of generalized questions:

Q4. Condensed matter statement: Can we find a total 2-group $\mathbb{H}$ as a total space such that $\mathrm{B}\mathbb{G}$ is $\mathrm{B}\mathbb{H}$'s orbit (or base space), and such that the $\mathbb{G}$-SPTs becomes a trivial gapped vacua in $\mathbb{H}$? More precisely, there is a local unitary transformation preserving the symmetry $\mathbb{H}$ (but breaking the symmetry $\mathbb{G}$), such that when the $\mathbb{G}$-SPTs are viewed as an $\mathbb{H}$-SPTs, they can be deformed to a trivial gapped insulator in $\mathbb{H}$ via a *local* unitary transformation (note that the locality also needs to be generalized to a higher dimensional

[8]For the related physics topics on higher group symmetries and higher SPTs, the readers can find from the recent developments in Refs. [20–25] and references therein.

extended object such as a line instead of just a point, due to the 2-group structure), without breaking $\mathbb{H}$ and without any phase transition in the enlarged $\mathbb{H}$-symmetric quantum Hilbert space.

Q5. QFT or high energy particle physics statement: Given a $(d-1)$d 't Hooft anomaly in a higher group $\mathbb{G}$, can we find an enlarged group $\mathbb{H}$, with a total group $\mathbb{H}$ obeying Eq. (15), such that the 't Hooft anomaly in $\mathbb{G}$ becomes anomaly free in $\mathbb{H}$ (i.e., the $\mathbb{G}$-anomaly becomes trivial in $\mathbb{H}$)?

Q6. Mathematical and algebraic topology statement: Given a dd topological term of a higher group $\mathbb{G}$, here the topological term can be the following:

(i) the dd cocycle for a dth cohomology group $\mathrm{H}^d(\mathrm{B}\mathbb{G}, \mathrm{U}(1))$ in a higher group cohomology theory,

(ii) the dd co/bordism invariant for a dth cobordism group $\Omega_-^d(\mathrm{B}\mathbb{G}, \mathrm{U}(1))$ or bordism group $\Omega^-_d(\mathrm{B}\mathbb{G})$ or bordism group, in a cobordism theory[9];

can we find an extended group $\mathbb{H}$ obeying Eq. (15) such that the topological term of a group $\mathbb{G}$ can be pulled back to a trivial topological term of a group $\mathbb{H}$?

In the next two subsections, we implement the strategy equation (15) by asking the questions in Q4, Q5, and Q6, for the two examples: the type I anomaly/topo. invariant in Eq. (2) and the type II anomaly/topo. invariant in Eq. (3).

We relegate more formal and mathematical details of the calculation of the above two subsections into Appendixes A, B, and C.

C. Saturate type II anomaly: Symmetric TQFTs

We first try to do a higher symmetry extension to trivialize the 4D type II higher anomaly (given by a 5D topological invariant) equation (3)

$$\mathrm{e}^{\mathrm{i}\pi\int w_2(TM)\mathrm{Sq}^1 B_2} = \mathrm{e}^{\mathrm{i}\pi\int w_3(TM) B_2}.$$

We have found that Eq. (3) is a topological invariant in $d=5$, for the following:

(i) $\mathrm{H}^d(\mathrm{B}^2\mathbb{Z}_2, \mathrm{U}(1))$ group cohomology of a higher classifying space finite group, as well as

(ii) $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ cobordism group of a higher classifying space finite group. Below we can either use the group cohomology or the cobordism group viewpoint to understand the trivialization of the 4D type II higher anomaly.

(1) The first way to trivialize this 4D type II higher anomaly is by extending the spacetime symmetry from the special orthogonal group $\mathrm{SO}(d) = \mathrm{Spin}(d)/\mathbb{Z}_2^F$ to $\mathrm{Spin}(d)$:

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{BSpin}(d) \times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{BSO}(d) \times \mathrm{B}^2\mathbb{Z}_2. \quad (17)$$

[9]Here the "−" follows the earlier footnote 7.

This extension works since $w_2(TM) = 0$ vanishes on the spin manifold. Thus, Eq. (3) is trivialized once we pull back Eq. (3) into $\mathrm{BSpin}(d) \times \mathrm{B}^2\mathbb{Z}_2$. According to the interpretation in Sec. III A and Ref. [7], the fibration $\mathrm{B}\mathbb{Z}_2$ contains an emergent 0-form global symmetry which is anomaly free and can be dynamically gauged. Indeed, the natural way to interpret Eq. (17) as the generalized construction of [7] is that there is an emergent 1-form $\mathbb{Z}_2$ gauge theory (dynamically gauged from emergent 0-form global symmetry $\mathrm{B}\mathbb{Z}_2$), such that the $\mathbb{Z}_2$ gauge theory has additional emergent *fermionic* particle excitations due to the emergent spin structure [the $\mathrm{Spin}(d)$ in the total space in Eq. (17)]. In terms of the full 4D symmetric TQFT saturating the higher 't Hooft anomaly (coupling to the 5D higher SPTs), we can write the involved QFT sectors into a partition function, which looks like the following *locally*:

$$\underbrace{\mathrm{e}^{\mathrm{i}\pi\int_{M^5} w_2(TM)\mathrm{Sq}^1 B_2}}_{\text{5D higher SPTs (4D higher anomaly)}} \cdot \underbrace{\sum_{\substack{a\in C^1((\partial M)^4,\mathbb{Z}_2),\\ b\in C^2((\partial M)^4,\mathbb{Z}_2)}} \exp\left(\mathrm{i}2\pi\int_{(\partial M)^4}\frac{1}{2}(b\delta a) + \ldots\right)}_{\text{locally a 4D}\mathbb{Z}_2\text{-TQFT with emergent fermions and spin-structure}}. \quad (18)$$

Here a is the $\mathbb{Z}_2$-valued 1-form gauge field (the standard notation as the 1-cochain in C^1), b is the $\mathbb{Z}_2$-valued 2-form gauge field (the standard notation as the 2-cochain in C^2), the δ is the coboundary operator here $\delta = 2\mathrm{Sq}^1$, and we use the cup product $\cup$. See also our previous explanations around Eq. (3) for notations. The … are additional coupling terms between dynamical gauge fields and background fields. The … also include additional sectors from the UV adjoint QCD_4 from Eq. (4), in order to saturate the other anomalies. Note that the similar emergent dynamical spin structure with the $\mathbb{Z}_2$ gauge field has been studied in Ref. [26]. The important thing is that the 1-form gauge field a can be regarded as the difference between two spin structures, while the gauge field a becomes dynamical.

Moreover, we can write the extension of Eq. (17) in terms of the full symmetry equation (8):

$$\begin{aligned}\mathrm{B}\mathbb{Z}_2 &\to \mathrm{B}\left(\mathrm{Spin} \times \left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right) \times \mathrm{B}^2\mathbb{Z}^e_{2,[1]} \\ &\to \mathrm{B}\left(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right) \times \mathrm{B}^2\mathbb{Z}^e_{2,[1]}, \end{aligned} \quad (19)$$

while the physical interpretation remains the same as Eqs. (17) and (18).

(2) The second way to trivialize this 4D type II higher anomaly is by extending the 1-form symmetry:

$$\mathrm{B}^2\mathbb{Z}_2 \to \mathrm{BSO}(d) \times \mathrm{B}^2\mathbb{Z}_4 \to \mathrm{BSO}(d) \times \mathrm{B}^2\mathbb{Z}_2. \tag{20}$$

This way works since B is pulled back to $\tilde{B} \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2)$ and $\mathrm{Sq}^1\tilde{B} = 0$ (see Appendix A 2 d).

According to the interpretation in Sec. III A and Ref. [7], the fibration $\mathrm{B}^2\mathbb{Z}_2$ is associated to an emergent 1-form global symmetry $\mathbb{Z}_{2,[1]}$ which is anomaly free and can be dynamically gauged. Indeed, the natural way to interpret Eq. (17) as the generalized construction of [7] is that there is an emergent 2-form $\mathbb{Z}_2$ gauge theory (dynamically gauged from the emergent 1-form global symmetry $\mathrm{B}\mathbb{Z}_2$) with a 2-form gauge field b'. The original 1-form $\mathbb{Z}^e_{2,[1]}$-symmetry acts projectively on the emergent 2-form $\mathbb{Z}_2$ gauge theory, but the extended 1-form $\mathbb{Z}^e_{4,[1]}$ symmetry acts on it faithfully.

We can write the involved QFT sectors into a following partition function, which looks like the following locally:

$$\underbrace{\mathrm{e}^{\mathrm{i}\pi\int_{M^5} w_2(TM)\mathrm{Sq}^1 B_2}}_{\text{5D higher SPTs (4D higher anomaly)}} \cdot \underbrace{\sum_{\substack{a'\in C^1((\partial M)^4,\mathbb{Z}_2),\\ b'\in C^2((\partial M)^4,\mathbb{Z}_2)}} \exp(\mathrm{i}2\pi\int_{(\partial M)^4}\frac{1}{2}(a'\delta b') + \ldots)}_{\text{locally a 4D}\mathbb{Z}_2\text{-TQFT, on which the 1-form } \mathbb{Z}^e_{2,[1]}\text{-symmetry acts projectively}}. \tag{21}$$

Here b' is the $\mathbb{Z}_2$-valued 2-form gauge field (the standard notation as the 2-cochain in C^2), a' is the $\mathbb{Z}_2$-valued 1-form gauge field (the standard notation as the 1-cochain in C^1), while other notations are explained around Eqs. (3) and (21). The … are additional coupling terms between dynamical gauge fields and background fields. The … also include additional sectors from the UV adjoint QCD_4 from Eq. (4), in order to saturate the other anomalies. We can also write the extension of Eq. (20) in terms of the full symmetry equation (8):

$$\begin{aligned}\mathrm{B}^2\mathbb{Z}_{2,[1]} &\to \mathrm{B}\left(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{4,[1]}\\ &\to \mathrm{B}\left(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{2,[1]},\end{aligned} \tag{22}$$

while the physical interpretation remains the same as Eq. (20).

D. Saturate type I anomaly: Obstruction

We now try to do higher symmetry extension to trivialize the 4D type I higher anomaly (given by a 5D topological invariant of higher SPTs) equation (2)

$$\mathrm{e}^{\mathrm{i}\frac{k\pi}{2}\int A\cup\mathcal{P}_2(B_2)} = \mathrm{e}^{\mathrm{i}\frac{k\pi}{2}\int A\cup(B_2\cup B_2+B_2\cup_1\delta B_2)}.$$

Below we show that

(1) When $k = 2 \in \mathbb{Z}_4$, the type I anomaly equation (2) can be trivialized, thanks to the fact that we can rewrite Eq. (2) as

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\pi\int A\cup\mathcal{P}_2(B_2)} &= \mathrm{e}^{\mathrm{i}\pi\int A\cup(B_2\cup B_2+B_2\cup_1\delta B_2)}\\ &= \mathrm{e}^{\mathrm{i}\pi\int A\cup(B_2\cup B_2+2B_2\cup_1\mathrm{Sq}^1 B_2)}\\ &= \mathrm{e}^{\mathrm{i}\pi\int A\cup(B_2\cup B_2)}\\ &= \mathrm{e}^{\mathrm{i}\pi\int \mathrm{Sq}^2(A\cup B_2)}\\ &= \mathrm{e}^{\mathrm{i}\pi\int (w_2(TM)+w_1(TM)^2)(A\cup B_2)},\end{aligned} \tag{23}$$

where we have used the fact that $\mathrm{Sq}^1\tilde{A} = 0$ where $\tilde{A} = A \bmod 2$ as well as the Wu formula. See also useful information in [9].

So when $k = 2 \in \mathbb{Z}_4$, if we extend the global symmetry by

$$\begin{aligned}\mathrm{B}\mathbb{Z}_2 &\to \mathrm{B}(\mathrm{Spin}\times(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\times\mathrm{B}^2\mathbb{Z}_2\\ &\to \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\times\mathrm{B}^2\mathbb{Z}_2,\end{aligned} \tag{24}$$

then the type I anomaly equation (2) vanishes. This extension works since $w_1(TM) = w_2(TM) = 0$ vanishes on spin manifolds. Thus, Eq. (2) is trivialized once we pull back Eq. (2) into $\mathrm{B}(\mathrm{Spin}\times(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\times\mathrm{B}^2\mathbb{Z}_2$.

(2) When $k = 1, 3 \in \mathbb{Z}_4$, or k odd, the type I anomaly equation (2) cannot be trivialized by extensions.

We have tried three approaches, which we relegate the details of in Appendix A 2 while we summarize the physics story and implication here.

(i) The first approach (Appendix A 2 b) is a breaking case since we set B to be zero. Physically this means that in order to saturate the 't Hooft anomaly, we can break 1-form $\mathbb{Z}_2$ symmetry to nothing. In comparison, this 1-form $\mathbb{Z}_2$-symmetry breaking is a different scenario from [2,3].

(ii) In the second approach (details and notations explained in Appendix A 2 c), we define $\mathbb{G}$ to be a group which sits in a homotopy pullback square

$$\begin{array}{ccccc} \mathrm{B}\mathbb{G} & \longrightarrow & & \longrightarrow & \mathrm{B}^2\mathbb{Z}_2 \\ \downarrow & & & & \downarrow x_2 \\ \mathrm{B}(\mathrm{Spin}\times \mathrm{SU}(2)\times \mathbb{Z}_8) & \xrightarrow{j_3} & \mathrm{B}\mathbb{Z}_8 & \xrightarrow{\tilde{b}} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{25}$$

Hence we have a fiber sequence

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{B}\mathbb{G} \to \mathrm{B}(\mathrm{Spin}\times \mathrm{SU}(2)\times \mathbb{Z}_8)\times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_2. \tag{26}$$

In this case, $B_2 = B$ is identified with $\beta_{(2,8)}A$ where $A \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$ and $\mathrm{Sq}^1 B = 0$, but $\tilde{A} \cup \frac{B^2}{2}$ is still not trivialized. This case is also a breaking case, since B is locked with A. In physics, the locking between two probed background fields means that the global symmetry between two sectors are locked together, which then results in global symmetry breaking.

Physically this means that in order to saturate the 't Hooft anomaly, we still need to break symmetry in some way.

(iii) In the third approach, we extend both the 0-form symmetry and the 1-form symmetry:

$$\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}(\mathrm{Spin}\times(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\times\mathrm{B}^2\mathbb{Z}_4 \to \mathrm{B}\left(\frac{\mathrm{Spin}\times(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}{\mathbb{Z}_2^F}\right) \times\mathrm{B}^2\mathbb{Z}_2. \tag{27}$$

But in this case, $\tilde{A} \cup \frac{B^2}{2}$ is still not, and cannot be, trivialized.

In summary, we finally conclude that when k is odd, $k = 1, 3 \in \mathbb{Z}_4$, the type I anomaly equation (2) cannot be trivialized by extensions and give a proof in Appendix B. In comparison, Ref. [3] proposes a full symmetry-preserving TQFT different from all of our scenarios above, which contradicts to our proof in Appendix B.

E. Saturate both type I (for even class in $\mathbb{Z}_4$) and II anomalies

When $k = 2$, such that the type I anomaly survives as only a $\mathbb{Z}_2$ subclass (even k) in the original $k \in \mathbb{Z}_4$ class (of $kA \cup \mathcal{P}_2(B_2)$), however, we can actually trivialize the $\mathbb{Z}_2$ subclass of the type I anomaly and the full type II anomaly together via the fibration:

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{B}\left(\mathrm{Spin}\times\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{2,[1]} \to \mathrm{B}\left(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{2,[1]}. \tag{28}$$

The above is achieved by combining both Eqs. (17) and (24) into Eq. (19). Since we only care about $k = 2$, this also means that the $\mathbb{Z}_{8,\mathrm{A}}$ symmetry only needs to be survived as a $\mathbb{Z}_{4,\mathrm{A}}$ symmetry. Physically this means that the $\mathbb{Z}_{8,\mathrm{A}}$ symmetry can be spontaneously broken down to a $\mathbb{Z}_{4,\mathrm{A}}$ symmetry. Thus, Eq. (28) really implies a fibration of a smaller symmetry (e.g., a smaller classifying space) as

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{B}\left(\mathrm{Spin}\times\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{4,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{2,[1]} \to \mathrm{B}\left(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{4,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\right)\times\mathrm{B}^2\mathbb{Z}^e_{2,[1]}. \tag{29}$$

For such a 4D TQFT preserving a $(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{4,\mathrm{A}}}{\mathbb{Z}_2^F})$-chiral symmetry and 1-form $\mathbb{Z}^e_{2,[1]}$ symmetry (from UV adjoint QCD_4), saturating the higher 't Hooft anomaly (coupling to the 5D higher SPTs), we can write the involved QFT sectors into a partition function, which looks like the following locally:

$$\underbrace{e^{i\pi\int A\cup(B_2\cup B_2)}\cdot e^{i\pi\int_{M^5} w_2(TM)\mathrm{Sq}^1 B_2}}_{\text{5D higher SPTs (4D higher anomaly)}} \cdot \underbrace{\sum_{\substack{a\in C^1((\partial M)^4,\mathbb{Z}_2),\\ b\in C^2((\partial M)^4,\mathbb{Z}_2)}} \exp\left(i2\pi\int_{(\partial M)^4}\frac{1}{2}(b\delta a)+\ldots\right)}_{\text{locally a 4D}\mathbb{Z}_2\text{-TQFT with emergent fermions and spin-structure}}. \tag{30}$$

Again the 1-form gauge field a can be regarded as the difference between two spin structures; the 1-form emergent dynamical $\mathbb{Z}_2$ gauge field a is associated to a dynamical spin structure (similar to a situation in Ref. [26]).

We note that the … terms can involve additional 't Hooft anomaly cancellation for the UV's adjoint QCD_4, such as the gapless sector proposed in [1–4]. Besides, the … terms also involve the coupling terms between dynamical gauge fields and background fields, so that the full partition function can be made gauge invariant. Although Eq. (29) already suggests a formation definition of TQFTs (based on the extension construction of bulk-boundary coupled TQFTs, see [7] and related constructions in [27,28]), it may be worthwhile to formulate a cochain or continuum TQFT description following [27,28]—which we leave for future work. It may also be worthwhile to give a continuum 4D TQFT formulation for the higher-form gauge theory analogous to the Dijkgraaf-Witten-like [29] gauge theory, which is similar to the continuum TQFT formulation given in [22,30].

F. Other examples

In our companion work [12], we consider a similar trivialization problem for 5D topological invariants of the 4D Yang-Mills SU(N) gauge theory (in particular at $\theta = \pi$) anomaly.

We find that many other examples of 5D topological invariants of the 4D Yang-Mills anomaly can be trivialized by extending the 0-form symmetry and 1-form symmetry. Hence, the higher symmetry-extension generalization, Ref. [7], is powerful enough to trivialize a lot of other higher bosonic types of anomalies, thus constructing exotic fully symmetric anomalous TQFTs, although it gives an obstruction to saturate the type I anomaly at an odd k while preserving the full symmetry.

IV. CONCLUSION

We conclude by summarizing the implications of the higher symmetry-extension construction of TQFTs on the low energy dynamics of QCD_4. Then we comment about the constraints on the deconfined quantum critical phenomena, or the so-called deconfined quantum critical point (dQCP) [31], in $3+1$ spacetime dimensions [3].

A. The fate of the dynamics of QCD_4

1. *Possible fates of the dynamics of fundamental QCD_4 with N_f Dirac fermions*

First, we recall the possible fates of the dynamics of QCD_4 with N_f Dirac fermions in fundamental representations of $SU(N_c)$. The conventional wisdom teaches us that the phase structure of the dynamics of QCD_4 via tuning N_f (with a fix N_c), shown in Fig. 1, is that

(i) At lower N_f, there should be a confinement (IR confinement) and IR ChSB.

(ii) At larger N_f, there is a range of N_f, such that at IR, the QFT flows to an interacting CFT; this is known as the range of *conformal window* phenomena studied by Bank-Zaks [32] and others.

(iii) Let $N_f^{\text{asym.free}} = \frac{11}{2} N_c$, when $N_f < N_f^{\text{asym.free}}$, the UV theory is weak coupling known as the asymptotic freedom (or UV free) [33,34]. When $N_f > N_f^{\text{asym.free}}$, the UV theory becomes strongly coupled while the coupling g flows weak at IR, at least perturbatively.

2. *Possible fates of the dynamics of adjoint QCD_4 with N_f Weyl fermions*

Now we organize the possible fates of the dynamics of QCD_4 with N_f Weyl fermions in adjoint representations of $SU(N_c)$. The possible phase structure of dynamics of QCD_4 via tuning N_f (with a fix N_c) is shown in Fig. 2. We remark that the candidate adjoint phases are summarized very elegantly in [2]. We recap concisely in Fig. 2, while we also list the related scenarios, 1, 2, 3, and 4, from Refs. [2,3], and from the list summarized in Sec. IVA 2.

The conventional wisdom teaches us that the phase structure of dynamics of adjoint QCD_4 via tuning N_f (with a fix N_c), shown in Fig. 2, is that

(i) At $N_f = 0$, it is a pure $SU(N_c)$ Yang-Mills gauge theory [say SU(2)], potentially with a θ-term equation (6). At $\theta = 0$, the phase is a trivially gapped confined phase (IR confinement) with no SPT state.

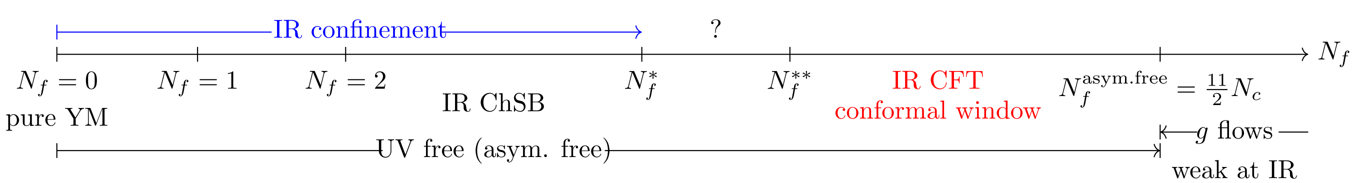

FIG. 1. Candidate phases of fundamental QCD_4 and their possible dynamical fates. "ChSB" means the "chiral symmetry-breaking phase." "Pure YM" means the pure Yang-Mills gauge theory with a $SU(N_c)$ gauge group. "CFT" means conformal field theory. "UV free" or "asym. free" means the asymptotic free. The question mark "?" means the detailed structure of the phase boundaries requires further studies.

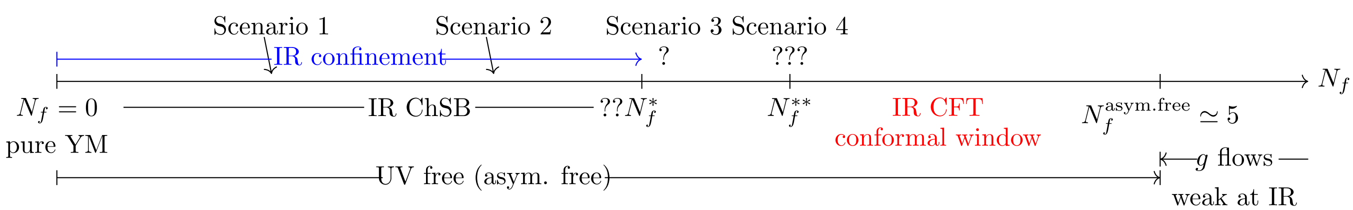

FIG. 2. Candidate phases of adjoint QCD_4 with an SU(2) gauge group ($N_c = 2$) and their possible dynamical fates. Scenarios 1, 2, 3, and 4 are from the list summarized in Sec. IVA 2. The question marks "?, ??, and ???" mean the detailed structure of the phase boundaries requires further studies.

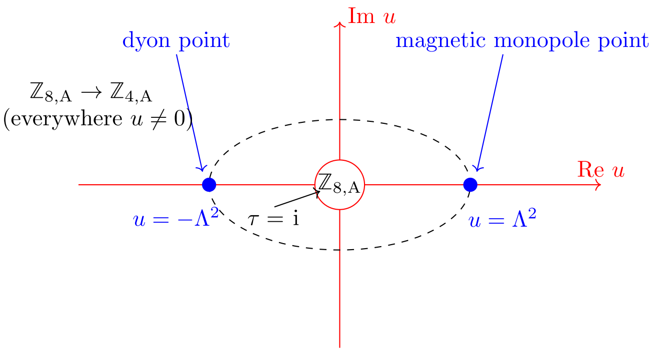

FIG. 3. For the $\mathcal{N} = 2$ SYM or Seiberg-Witten theory [13], there is a moduli space of the supersymmetric vacua, labeled by the expectation value of the complex $u = \mathrm{Re}u + \mathrm{iIm}u \equiv \langle \mathrm{Tr}(\phi^2) \rangle \in \mathbb{C}$ for the $\mathcal{N} = 2$ chiral operator. The $\tau \equiv \tau_{\mathrm{U}(1)} \equiv \frac{\theta}{2\pi} + \frac{2\pi \mathrm{i}}{e^2} = \mathrm{i}$ is the special point at the self-dual value of the coupling τ. The coupling τ is for the Coulomb phase of the U(1) gauge theory (the Coulomb branch for the moduli space of vacua). Even though the SYM is supersymmetric, Ref. [2] enumerates the possible vacua by supersymmetry-breaking deformations. Let us relate the supersymmetric vacua to the adjoint QCD_4 vacua listed in the scenarios [2]: ◇ The vacua of the magnetic monopole point ($u = \Lambda^2$) and dyon point ($u = -\Lambda^2$) are related by the broken symmetry $\mathbb{Z}_{8,A}$ generator, which gives rise to scenario 1. ◇ Generic vacua of $u \neq 0$, and $u \neq \pm\Lambda^2$ on the plane are related to scenario 2. ◇ The vacua of the $u = 0$ with a self-dual coupling τ are related to scenario 4.

However, at $\theta = \pi$, the phase has mixed higher anomalies [35] and potentially newly found higher 't Hooft anomalies [12].

(ii) At $N_f = 1$, it is a pure $\mathcal{N} = 1$ supersymmetric Yang-Mills gauge theory (SYM) [36]. Moreover, there are N_c supersymmetric breaking vacua due to gaugino condensation [37], which breaks $\mathbb{Z}_{2N_c}$ down to $\mathbb{Z}_2$ (simply $\mathbb{Z}_2^F$). This $\mathcal{N} = 1$ SYM phase is also known to be confined through monopole condensation, by embedding into a $\mathcal{N} = 2$ SYM theory with $N_c = 2$ [13].

(iii) At lower N_f, there should be a confinement (IR confinement) and chiral symmetry breaking (IR ChSB).

(iv) At larger N_f, one expects again a range of N_f with a range of *conformal window* phenomena of Bank-Zaks [32].

To proceed further, we recall that the UV internal global symmetry is $\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,A}}{\mathbb{Z}_2^F}\right) \times \mathbb{Z}^e_{2,[1]}$. Now we organize a list of possible fates of the dynamics of adjoint QCD_4 with N_f Weyl fermions proposed from [2,3]. There are four scenarios summarized in Table I and below:

(1) The N_c copies of (or more specifically here $N_c = 2$) of the 4D $\mathbb{CP}^1$ sigma model at low energy with spontaneous symmetry-breaking (SSB) Goldstone modes, proposed by [2]. Its global symmetry is as follows:

$$\mathrm{O}(2) \times \mathbb{Z}^e_{2,[1]}. \tag{31}$$

In summary, scenario 1 has

$$\text{“chiral symmetry breaking, and confinement.”} \tag{32}$$

To understand better the target space of the $\mathbb{CP}^1$ sigma model, here we can consider the breaking of the 0-form symmetry group G as the total space E breaking to a smaller fiber F (a subgroup or a normal subgroup, as the fiber or the stabilizer), where the order parameter parametrizes the base manifold B (the base space or the orbit). In short, we formally and mathematically write

$$\begin{array}{ccc} F \hookrightarrow E & \text{stabilizer} \hookrightarrow & \text{total space} \\ \downarrow, & & \downarrow \\ B & & \text{orbit} \end{array}. \tag{33}$$

Then we obtain a relation for scenario 1:

$$\begin{array}{ccc} S^1 = \mathrm{U}(1)_R & \hookrightarrow & S^3 = \mathrm{SU}(2)_R \\ & & \downarrow \\ & & S^2 = \mathbb{CP}^1 \end{array}, \tag{34}$$

or more precisely a relation:

$$\begin{array}{ccc} \mathrm{O}(2)_R = \mathrm{U}(1)\rtimes\mathbb{Z}_2 & \hookrightarrow & \left(\frac{\mathrm{SU}(2)_R\times\mathbb{Z}_{8,A}}{\mathbb{Z}_2^F}\right) \\ & & \downarrow \\ & & \mathbb{CP}^1\rtimes\frac{\mathbb{Z}_{8,A}}{\mathbb{Z}_2\times\mathbb{Z}_2^F}. \end{array} \tag{35}$$

The $\mathbb{CP}^1\rtimes\frac{\mathbb{Z}_{8,A}}{\mathbb{Z}_2\times\mathbb{Z}_2^F}$ has two copies of $\mathbb{CP}^1$ as the target space, parametrizing the order parameter of the base manifold B (the base space or the orbit).

(2) A free massless Dirac fermion (equivalently, two massless Weyl fermions, or two massless Majorana fermions) and a $\mathbb{Z}_2$ discrete gauge theory have a 4D TQFT with a $\mathbb{Z}_4$ symmetry (spontaneously broken from the $\mathbb{Z}_8$ symmetry) proposed by [2]. The IR symmetry is

$$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{4,A}}{\mathbb{Z}_2^F}\right) \times \mathbb{Z}^e_{2,[1]}. \tag{36}$$

In summary, scenario 2 has

$$\text{“chiral symmetry breaking } \mathbb{Z}_8 \to \mathbb{Z}_4, \text{ and confinement.”} \tag{37}$$

TABLE I. Scenarios 1, 2, 3, and 4 are from the list summarized in Sec. IV A 2. The $\cancel{(\ldots)}$ means that symmetry $(\ldots)$ leads to SSB. We find an obstruction for scenario 3 based on the higher symmetry-extension construction of Ref. [7]. We should note that, educated by Ref. [2] and summarized in Fig. 3, scenario 1 is consistent with the supersymmetry (SUSY) breaking of $\mathcal{N}=2$ SYM from the magnetic monopole point and dyon point (as 2 copies of $\mathbb{CP}^1$ model). Scenario 2 is consistent with the SUSY breaking of $\mathcal{N}=2$ SYM from the generic point from $u \neq 0$, and $u \neq \pm\Lambda^2$. Scenario 4 is consistent with the SUSY breaking of $\mathcal{N}=2$ SYM from the $u=0$ with a self-dual coupling τ.

Scenario	Internal Global Symmetry $\mathbb{G}$	Chiral Symmetry	1-form $\mathbb{Z}^e_{2,[1]}$ Symmetry, De-/Confinement	Anomaly Matched with UV	Plausible Candidates
1. Ref. [2]	$\mathrm{O}(2)\times \mathrm{U}(1)_{[1]}$	SSB	Enhanced and preserved; confined.	Yes	Yes (favored by eneretic?)
2. Ref. [2]	$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{4,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\mathbb{Z}^e_{2,[1]}\times\ldots$	SSB $\mathbb{Z}_{8,\mathrm{A}}\to\mathbb{Z}_{4,\mathrm{A}}$	Preserved; confined. But + new deconfined $\mathbb{Z}_2$-TQFT with emergent new $\mathbb{Z}_{2,[1]}$ SSB.	yes	Yes
3. Ref. [3]	$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\mathbb{Z}^e_{2,[1]}$	Preserved	Preserved; confined.	Obstruction of symmetric TQFT	Obstruction. Not compatible w/symmetry extension [7]
4. Ref. [2]	$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\cancel{\mathrm{U}(1)^e_{[1]}}\times\cancel{\mathrm{U}(1)^m_{[1]}}$	Preserved	Enhanced but SSB; deconfined.	Yes	Yes

However, as explained in [2], there is an additional emergent new deconfined $\mathbb{Z}_2$ TQFT with emergent new $\mathbb{Z}_{2,[1]}$ symmetries spontaneously broken.

(3) A free massless Dirac fermion (equivalently, two massless Weyl fermions, or two massless Majorana fermions) and a 4D TQFT preserving the full $\mathbb{Z}_8$ symmetry, proposed by [3]. The two massless Weyl fermions actually have a U(2) continuous global symmetry. The IR symmetry we focus is

$$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\mathbb{Z}^e_{2,[1]}. \tag{38}$$

In summary, scenario 3 proposed that

$$\text{"chiral symmetry fully preserved, and confinement."} \tag{39}$$

(4) A 4D U(1) gauge theory in Coloumb phase with a $\mathbb{Z}_{2N_cN_f}=\mathbb{Z}_8$ symmetry, proposed by [2]. The IR symmetry we focus is

$$\left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right)\times\cancel{\mathrm{U}(1)^e_{[1]}}\times\cancel{\mathrm{U}(1)^m_{[1]}} \tag{40}$$

The $\cancel{(\ldots)}$ means a spontaneous symmetry breaking of …; thus, for 1-form symmetry breaking here, it leads to a deconfinement of U(1) gauge theory. In summary, scenario 4 proposed that

$$\text{"chiral symmetry preserved, and deconfinement."} \tag{41}$$

(5) Note that there is another scenario from Ref. [1] proposing only a free massless Dirac fermion at IR (equivalently, two massless Weyl fermions, or two massless Majorana fermions), and two vacua (two degenerate ground states) due to $\mathbb{Z}_{8,\mathrm{A}}\to\mathbb{Z}_{4,\mathrm{A}}$, without any 1-form symmetry. This scenario is certainly incomplete due to the lack of matching the higher 't Hooft anomalies of 1-form symmetry. As Ref. [1] also notices later, the more complete scenario is adding a TQFT sector, following scenario 2.

B. Deconfined quantum criticality in 3 + 1 dimensions and more comments

In this work, we obtain a higher symmetry-extension generalization of Ref. [7]'s method to construct symmetric anomalous TQFT saturating higher 't Hooft anomalies. We have obtained a symmetric anomalous TQFT, valid for scenario 2 from Cordova-Dumitrescu (Ref. [2]), see Eqs. (29) and (30). However, we are unable to obtain a symmetric anomalous TQFT proposed by scenario 3 motivated by Bi-Senthil (Ref. [3]) based on a symmetry-extension construction.

It is worthwhile to understand the exotic and interesting physics of scenario 3 better. Scenario 3 is motivated by the deconfined quantum criticality in $3+1$ dimensions. It is proposed that a critical theory can be realized as a phase transition between two conventional Landau-Ginzburg symmetry-breaking orders [31], or a phase transition between two different SPT orders (see [3] and references therein). The adjoint QCD_4 is a UV description [UV side of Eq. (1)] of the phase transition, while the IR description is currently unclear [IR side of Eq. (1)].

The novelty of scenario 3 is that the gapless sector is a free CFT as two free Weyl fermions (a single free Dirac fermion). So, the hope is that the possible UV-IR duality equation (1) in $3+1$D is between a strongly coupled and interacting UV gauge theory and a free non-interacting massless IR theory, up to a gapped fully symmetric TQFT sector to saturate the higher 't Hooft anomalies.

Our present work shows an obstruction for scenario 3 from a symmetry-extension construction alone. The implications of our finding are as follows:

(I) We should remind the readers that the symmetry-extension construction is fairly general enough to saturate a large class of higher 't Hooft anomalies of bosonic systems. Although the adjoint QCD_4 is a fermionic system (the UV completion requires fermionic degrees of freedom, where there are gauge-invariant fermionic operators), the type I and II anomalies, Eqs. (2) and (3), are bosonic anomalies in nature.

(II) Despite the fact that fully symmetric TQFT under scenario 3 cannot be obtained via our symmetry-extension construction, we may still be able to use the symmetry-extension construction to derive other symmetric anomalous TQFTs, suitable to propose new candidate phases of other *deconfined quantum criticality* (dQCP), in $3+1$ and other dimensions.

We should also notice that the recent numerical attempts [38,39] suggest that the adjoint QCD_4 with the SU(2) gauge group and N_f number of adjoint Weyl fermions may have IR dynamics as follows:

(i) At $N_f = 2$, (as 1 adjoint Dirac fermion), according to [39], the IR theory may be very close to the onset of the conformal window, instead of the conventional confining behavior. In addition, the anomalous dimension of the fermionic condensate is reported to be close to 1. The numerical data seems to suggest the IR theory can be an interacting CFT (more exotic), instead of a free CFT (all the proposed scenarios so far, discussed in Table I).

(ii) At $N_f = 4$, (as 2 adjoint Dirac fermion), Ref. [38] discusses the candidate IR theory. Reference [38] points out the theory is gapless (or massless), while a future endeavor is required to distinguish whether it shows the confinement or the conformal behavior.

To unambiguously determine the IR dynamics, apart from the given numerical inputs [38,39], we note that further lattice studies are still necessary.

Finally, we remark that many anomalies discussed in Sec. II B, following [2], are nonperturbative global anomalies instead of perturbative anomalies. The nonperturbative anomalies have classifications from finite groups (e.g., $\mathbb{Z}_n$ classes), instead of a $\mathbb{Z}$ classification. Examples include the old and the new SU(2) anomalies [15,26], and also the recent higher 't Hooft anomalies of SU(N) YM gauge theory; see [12,35], and references therein. For these nonperturbative global anomalies, we can saturate certain 't Hooft anomalies of ordinary or higher global symmetries by symmetry-preserving TQFTs or the so-called long-ranged entangled topological order sectors, via our higher symmetry-extension approach; see a companion work along this direction [12].

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APPENDIX A: COBORDISM THEORY AND HIGHER SYMMETRY EXTENSION: CONSTRUCTION OF SYMMETRIC TQFTS

By the result from Ref. [40] (p. 251), the cohomology ring of the infinite lens space $\mathrm{B}\mathbb{Z}_{2^n} = S^\infty/\mathbb{Z}_{2^n}$ with coefficients $\mathbb{Z}_{2^n}$ is the polynomial ring generated by a and b over the $\mathbb{Z}_{2^n}$ quotient by the relation $a^2 = 2^{n-1}b$:

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n}) = \mathbb{Z}_{2^n}[a,b]/(a^2 = 2^{n-1}b) \quad \text{for } n \geq 2, \qquad (A1)$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \quad \text{for } n \geq 2, \qquad (A2)$$

where $\tilde{a} = a \bmod 2$, $\tilde{b} = b \bmod 2$, and there is a $(2,2^n)$-Bockstein $\beta_{(2,2^n)}$ with $\beta_{(2,2^n)}(a) = \tilde{b}$. Here H^* is the cohomology ring, $\Lambda_{\mathbb{Z}_2}$ denotes the exterior algebra over $\mathbb{Z}_2$, and $\otimes$ is the tensor product. The $(2,2^n)$-Bockstein homomorphism $\beta_{(2,2^n)}\colon \mathrm{H}^*(-,\mathbb{Z}_{2^n}) \to \mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ is associated to the extension $\mathbb{Z}_2 \to \mathbb{Z}_{2^{n+1}} \to \mathbb{Z}_{2^n}$.

Notice that the notations $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$ and $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$ will be used later, since we will encounter cases $n=2$ and $n=3$. We will use the uniform notation and explain wherever they appear.

1. Pullback trivialization of $A\mathcal{P}_2(B_2)$ in $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$

Following the mathematical conventional notation, we will also denote the 5D topological term

$$A \cup \mathcal{P}_2(B_2) \quad \text{as } a \cup \mathcal{P}_2(x_2) \tag{A3}$$

in Appendix A and after. The a here is a background probed field, which should not be confused with the SU(2) dynamical gauge field.

a. Computation

$\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)\equiv(\mathrm{Spin}\times(\mathrm{SU}(2)\times\mathbb{Z}_8)/\mathbb{Z}_2)/\mathbb{Z}_2$ where the quotient is with respect to the diagonal center $\mathbb{Z}_2$ subgroup.

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)),\mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2). \tag{A4}$$

Here Ext is the Ext functor, $\mathcal{A}_2$ is the mod 2 Steenrod algebra; more precisely, $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}$ is the internal degree t part of the sth derived functor of $\mathrm{Hom}_{\mathcal{A}_2}^*$. $MT(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))$ is the Madsen-Tillmann spectrum of the group $\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)$ and the bordism group $\Omega_d^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)=\pi_d(MT(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\wedge(\mathrm{B}^2\mathbb{Z}_2)_+)$ is the stable homotopy group of the spectrum $MT(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\wedge(\mathrm{B}^2\mathbb{Z}_2)_+$. Here $\wedge$ is the smash product, X_+ is the disjoint union of the space X and a point. "$\Rightarrow$" means "convergent to". For more detail, see [9].

Similarly, as the discussion in [11,41], we know

$$MT(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)) = M\mathrm{Spin}\wedge\Sigma^{-3}M\mathrm{SO}(3)\wedge(\mathrm{B}\mathbb{Z}_4)^{2\xi} \tag{A5}$$

where 2ξ is twice the sign representation, $(\mathrm{B}\mathbb{Z}_4)^{2\xi}$ is the Thom space $\mathrm{Thom}(\mathrm{B}\mathbb{Z}_4;2\xi)$, $M\mathrm{Spin}$ is the Thom spectrum of the group Spin, $M\mathrm{SO}(3)$ is the Thom spectrum of the group SO(3), and Σ is the suspension.

Note that $(\mathrm{B}\mathbb{Z}_4)^{2\xi}=\Sigma^{-2}M\mathbb{Z}_4$.

We have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)) & \longrightarrow & \mathrm{BSO}(3)\times\mathrm{B}\mathbb{Z}_4 \\ \downarrow & & \downarrow{\scriptstyle w_2'+\tilde{b}} \\ \mathrm{BSO} & \xrightarrow{\ w_2\ } & \mathrm{B}^2\mathbb{Z}_2 \end{array} \tag{A6}$$

where $\tilde{b}$ is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_2)$, $w_2=w_2(TM)$ is the Stiefel-Whitney class of the tangent bundle TM, and $w_2'=w_2'(\mathrm{SO}(3))$ is the Stiefel-Whitney class of the universal SO(3) bundle.

Hence we have the constraint

$$w_2(TM)=w_2'(\mathrm{SO}(3))+\tilde{b}. \tag{A7}$$

Since $\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2)=\mathcal{A}_2\otimes_{\mathcal{A}_2(1)}\{\mathbb{Z}_2\oplus M\}$ where $\mathcal{A}_2(1)$ is the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2 and M is a M is a graded $\mathcal{A}_2(1)$ module with the degree i homogeneous part, $M_i=0$ for $i<8$.

For $t-s<8$, we can identify the E_2 page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)\otimes\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2) \otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2). \tag{A8}$$

$\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)=\mathbb{Z}_2[w_2',w_3']U$ where U is the Thom class and w_i' is the Stiefel-Whitney class of the universal SO(3) bundle.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)$ is shown in Fig. 4.

$\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2)=(\mathbb{Z}_2[\tilde{b}]\otimes\Lambda_{\mathbb{Z}_2}(\tilde{a}))U$ where U is the Thom class, $\tilde{a}$ is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_2)$, and $\tilde{b}$ is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_2)$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2)$ is shown in Fig. 5.

$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[x_2,x_3,x_5,\ldots]$ where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, $x_3=\mathrm{Sq}^1x_2$, $x_5=\mathrm{Sq}^2x_3$, etc.

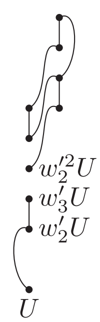

FIG. 4. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)$. Each dot indicates a $\mathbb{Z}_2$, the short straight line indicates a Sq^1 and the curved line indicates a Sq^2.

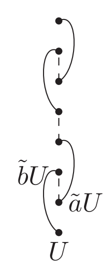

FIG. 5. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2)$. The dashed lines indicate a (2, 4)-Bockstein. Each dot indicates $\mathbb{Z}_2$, the short straight line indicates a Sq^1, and the curved line indicates a Sq^2.

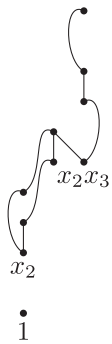

FIG. 6. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$. Each dot indicates $\mathbb{Z}_2$, the short straight line indicates a Sq^1, and the curved line indicates a Sq^2.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ is shown in Fig. 6.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3), \mathbb{Z}_2) \otimes \mathrm{H}^{*+2}(M\mathbb{Z}_4, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ is shown in Fig. 7.

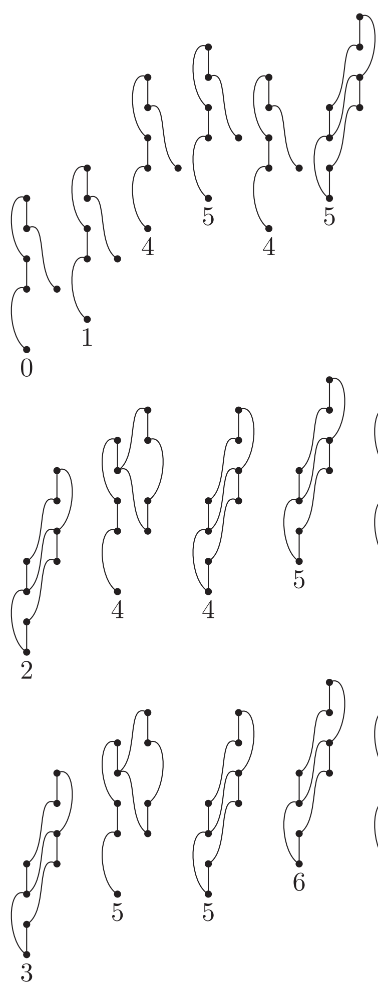

FIG. 7. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3), \mathbb{Z}_2) \otimes \mathrm{H}^{*+2}(M\mathbb{Z}_4, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$. Each dot indicates $\mathbb{Z}_2$, the short straight line indicates a Sq^1, and the curved line indicates a Sq^2. Each label indicates its degree.

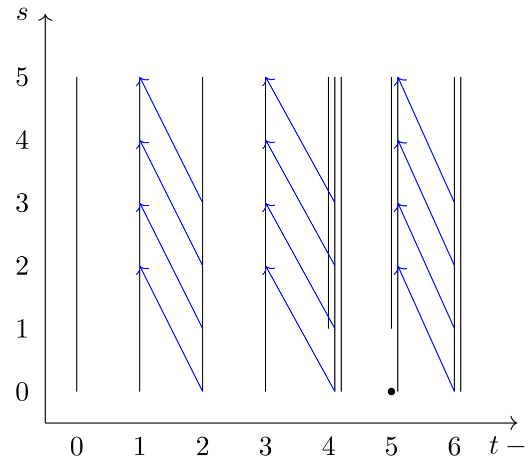

FIG. 8. $\Omega_*^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}$. The arrows indicate differentials.

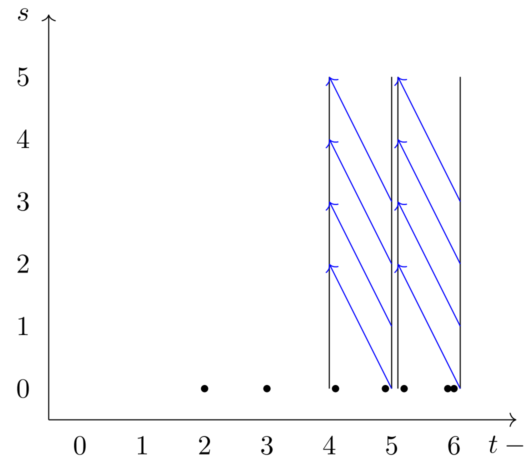

FIG. 9. $(\Omega_*^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2))/(\Omega_*^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)})$. The arrows indicate differentials.

There is a differential d_2 corresponding to the (2, 4)-Bockstein [42] as indicated in Fig. 5. There is also a differential d_2 maps $x_2x_3 + x_5$ to $x_2^2h_0^2$ since $\beta_{(2,4)}(\mathcal{P}_2(x_2)) = x_2x_3 + x_5$ [9]. Since $\beta_{(2,4)}(a\mathcal{P}_2(x_2)) = \tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5)$, there is a differential d_2 maps $\tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5)$ to $\tilde{a}x_2^2h_0^2$.

TABLE II. Bordism group $\Omega_i^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}$ in dimensions i. Here a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$ and b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$. w_i' is the Stiefel-Whitney class of the universal SO(3) bundle.

i	$\Omega_i^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}$	Cobordism Invariants
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	a
2	0	
3	$\mathbb{Z}_4$	ab
4	$\mathbb{Z}^2$	
5	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$w_2'w_3'$, ab^2

TABLE III. Bordism group $\Omega_i^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$ in dimensions i. Here a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_4)$ and b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4,\mathbb{Z}_4)$. $\tilde{a}=a \mod 2$ and $\tilde{b}=b \mod 2$. $w_2=w_2(TM)$ is the Stiefel-Whitney class of the tangent bundle. Note that $w_2x_3=w_3x_2$ (see [9]). w_i' is the Stiefel-Whitney class of the universal SO(3) bundle.

i	$\Omega_i^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$	Cobordism Invariants
0	$\mathbb{Z}$	
1	$\mathbb{Z}_4$	a
2	$\mathbb{Z}_2$	x_2
3	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\tilde{a}x_2$, ab
4	$\mathbb{Z}^2\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\tilde{b}x_2$, $\mathcal{P}_2(x_2)$
5	$\mathbb{Z}\times\mathbb{Z}_2^3\times\mathbb{Z}_4^2$	$w_2'w_3'$, $\tilde{a}\,\tilde{b}\,x_2$, w_2x_3, ab^2, $a\mathcal{P}_2(x_2)$

Note that the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)\otimes\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2)$ is contained in that of $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)\otimes\mathrm{H}^{*+2}(M\mathbb{Z}_4,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, and we draw the E_2 page for it individually in Fig. 8. The remaining part is shown in Fig. 9.

See Table II for the bordism group data.

See Table III for the bordism group data.

b. Manifold generator

Now we determine the manifold generator of the $\mathbb{Z}_4$-valued invariant $a\cup\mathcal{P}_2(x_2)$.

$$\begin{aligned}\Omega_5^{\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)=\{&\text{5-manifolds } M \text{ with maps}\\ &f\colon M\to\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\\ &\text{and } g\colon M\to\mathrm{B}^2\mathbb{Z}_2\}/\text{bordism}.\end{aligned}\tag{A9}$$

Here bordism is an equivalence relation. (M,f,g) and (M',f',g') are bordant if there exists a 6-manifold $\mathcal{M}$ and maps $F\colon\mathcal{M}\to\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))$, $G\colon\mathcal{M}\to\mathrm{B}^2\mathbb{Z}_2$ such that the boundary of $\mathcal{M}$ is the disjoint union of M and M' and the induced $\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)$ structures on M and M' from that determined by F on $\mathcal{M}$ coincide with those determined by f and f', respectively, and $G|_M=g$, $G|_{M'}=g'$.

We have the homotopy pullback square (A6).

In order to give a map $f\colon M\to\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))$, we need only give maps $f_1\colon M\to\mathrm{BSO}$, $f_2\colon M\to\mathrm{BSO}(3)$ and $f_3\colon M\to\mathrm{B}\mathbb{Z}_4$ with $f_1^*(w_2)=f_2^*(w_2')+f_3^*(\tilde{b})$.

The bordism invariant $a\cup\mathcal{P}_2(x_2)$ is actually $f_3^*(a)\cup\mathcal{P}_2(g^*(x_2))=f_3\cup\mathcal{P}_2(g)$.

Now let M be the lens space $S^5/\mathbb{Z}_4$; M is orientable but not spin.

Take $f_1=TM$ (since M is orientable, the tangent bundle TM determines a map $M\to\mathrm{BSO}$), $f_2=0$, and f_3 is the generator of $\mathrm{H}^1(M,\mathbb{Z}_4)$.

By the cell structure of the lens space, f_3 induces a chain map between the cellular chain complexes of M and $\mathrm{B}\mathbb{Z}_4$, and we draw the chain map below degree 2:

$$\begin{array}{ccccc}\mathbb{Z} & \xrightarrow{4} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}\\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow\\ \mathbb{Z} & \xrightarrow{4} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}.\end{array}\tag{A10}$$

So $f_3^*(\tilde{b})$ is nonzero, since $f_1^*(w_2)$ is also nonzero, the cohomology group $\mathrm{H}^2(M,\mathbb{Z}_2)$ is $\mathbb{Z}_2$, and we have a commutative diagram

$$\begin{array}{ccc}M & \xrightarrow{f_3} & \mathrm{B}\mathbb{Z}_4\\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \tilde{b}}\\ \mathrm{BSO} & \xrightarrow[w_2]{} & \mathrm{B}^2\mathbb{Z}_2.\end{array}\tag{A11}$$

So we get a map $f\colon M\to\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))$.

Take $g=w_2(TM)$.

$$\int_M f_3\cup\mathcal{P}_2(g)=1 \mod 4.\tag{A12}$$

The partition function

$$Z(M)=\mathrm{i}^{\int_M f_3\cup\mathcal{P}_2(g)}=\mathrm{i}.\tag{A13}$$

So (M,f,g) is the manifold generator of the $\mathbb{Z}_4$-valued invariant $f_3\cup\mathcal{P}_2(g)$.

2. Pullback trivialization

Consider the pullback of $\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))$ to $\mathrm{BSpin}\times\mathrm{B}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)$:

$$\begin{aligned}\mathrm{B}\mathbb{Z}_2&\to\mathrm{BSpin}\times\mathrm{B}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)\\ &\to\mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)).\end{aligned}\tag{A14}$$

Since $w_2=0$ in Spin, so $w_2x_3=w_3x_2$ is trivialized.

Furthermore, consider the pullback of $\mathrm{BSpin}\times\mathrm{B}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)$ to $\mathrm{BSpin}\times\mathrm{BSU}(2)\times\mathrm{B}\mathbb{Z}_8$:

$$\begin{aligned}\mathrm{B}\mathbb{Z}_2&\to\mathrm{BSpin}\times\mathrm{BSU}(2)\times\mathrm{B}\mathbb{Z}_8\\ &\to\mathrm{BSpin}\times\mathrm{B}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8).\end{aligned}\tag{A15}$$

To simplify the computation, we only compute $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$ which is a subgroup of $\Omega_5^{\mathrm{Spin}}(\mathrm{BSU}(2)\times\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$.

Note that $\mathcal{P}_2(x_2)=x_2^2=\mathrm{Sq}^2(x_2)=(w_2(TM)+w_1(TM)^2)x_2=0 \mod 2$ on spin manifolds where we have used the Wu formula, so $\mathcal{P}_2(x_2)$ can be divided by 2.

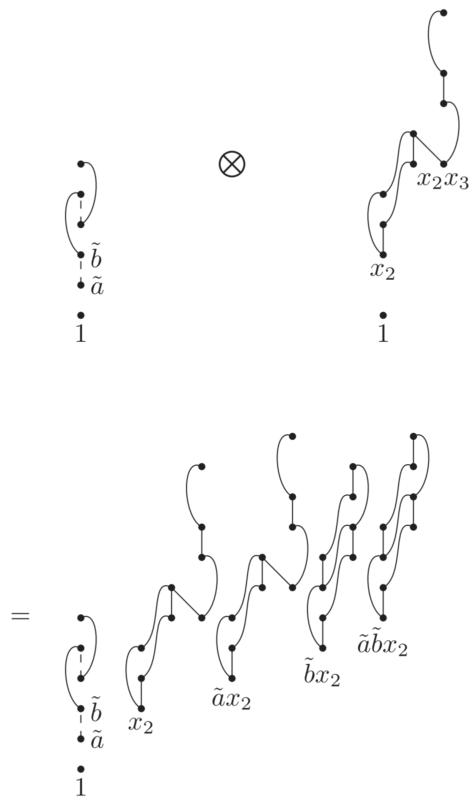

FIG. 10. The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$. The dashed lines indicate a (2, 8)-Bockstein. Each dot indicates $\mathbb{Z}_2$, the short straight line indicates a Sq^1, and the curved line indicates a Sq^2.

a. Computation

We have the Adams spectral sequence

$$E_2^{s,t}=\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2). \quad \text{(A16)}$$

For $t-s<8$,

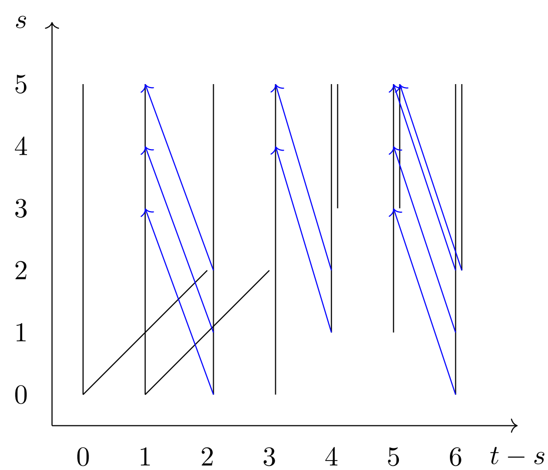

FIG. 11. $\Omega_*^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8)$. The arrows indicate differentials.

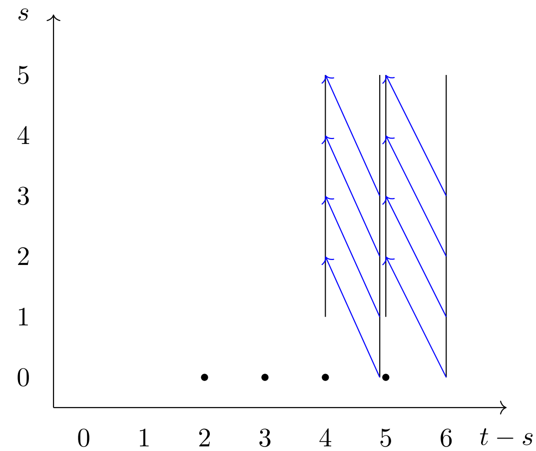

FIG. 12. $(\Omega_*^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2))/(\Omega_*^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8))$. The arrows indicate differentials.

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2). \quad \text{(A17)}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ is shown in Fig. 10.

Note that the $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)$ is contained in that of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, and we draw the E_2 page for it individually in Fig. 11. The remaining part is shown in Fig. 12.

There is a differential d_3 corresponding to the (2, 8)-Bockstein [42] as indicated in Fig. 10 and a differential d_2 corresponding to the (2, 4)-Bockstein $\beta_{(2,4)}(\mathcal{P}_2(x_2))=x_2x_3+x_5$.

See Table IV for the bordism group data.

See Table V for the bordism group data.

One $\mathbb{Z}_2$-valued bordism invariant of $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$ is $\tilde{a}\cup\frac{\mathcal{P}_2(x_2)}{2}$. Here $\tilde{a}$ is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_2)$ and x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$.

TABLE IV. Bordism group $\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8)$ in dimensions i. Here $\tilde{\eta}$ is the 1D eta invariant, Arf is the Arf invariant, and $\mathfrak{P}$ is the Postnikov square. a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_8)$, and b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_8,\mathbb{Z}_8)$. $\tilde{a}=a \bmod 2$ and $\tilde{b}=b \bmod 2$.

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8)$	Cobordism Invariants
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2\times\mathbb{Z}_8$	$\tilde{\eta}$, a
2	$\mathbb{Z}_2^2$	$\tilde{a}\tilde{\eta}$, Arf
3	$\mathbb{Z}_2\times\mathbb{Z}_{16}$	$\tilde{a}$Arf, $\mathfrak{P}(a)$
4	$\mathbb{Z}$	
5	$\mathbb{Z}_{16}$	$\mathfrak{P}(b)$[a]

[a]Here we proposed that the 5D cobordism invariant of $\mathbb{Z}_{16}$ is the Postnikov square. A caveat is that we have not yet been able to fully verify that the Postnikov square is the only possibility, although partial evidences suggest this to be true. The readers need to be cautious using this particular result. The verification is left for future work.

TABLE V. Bordism group $\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$ in dimensions i. Here $\tilde{\eta}$ is the 1D eta invariant, Arf is the Arf invariant, and $\mathfrak{P}$ is the Postnikov square. a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$ and b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$. $\tilde{a} = a \bmod 2$ and $\tilde{b} = b \bmod 2$.

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$	Cobordism Invariants
0	$\mathbb{Z}$	
1	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\tilde{\eta}$, a
2	$\mathbb{Z}_2^3$	$\tilde{a}\,\tilde{\eta}$, Arf, x_2
3	$\mathbb{Z}_2^2 \times \mathbb{Z}_{16}$	$\tilde{a}$Arf, x_3, $\mathfrak{P}(a)$
4	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\tilde{b}x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$
5	$\mathbb{Z}_2^2 \times \mathbb{Z}_{16}$	$\tilde{a}\,\tilde{b}\,x_2$, $\tilde{a}\frac{\mathcal{P}_2(x_2)}{2}$, $\mathfrak{P}(b)$[a]

[a]Here we proposed that this 5D cobordism invariant of $\mathbb{Z}_{16}$ is the Postnikov square. A caveat is that we have not yet been able to fully verify that the Postnikov square is the only possibility, although partial evidences suggest this to be true. The readers need to be cautious using this particular result. The verification is left for future work.

b. Further trivialization: First approach

Define $\mathbb{G}$ to be a group which sits in a homotopy pullback square

$$\begin{array}{ccccc} \mathrm{B}\mathbb{G} & & \longrightarrow & & \mathrm{B}^2\mathbb{Z}_2 \\ \downarrow & & \mathrm{BSO}(3)\times\mathrm{B}\mathbb{Z}_4 & & \downarrow x_2 \\ & \overset{j_2}{\nearrow} & & \overset{w_2'+\tilde{b}}{\searrow} & \\ \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)) & & & & \mathrm{B}^2\mathbb{Z}_2 \\ & \underset{j_1}{\searrow} & & \underset{w_2}{\nearrow} & \\ & & \mathrm{BSO} & & \end{array} \tag{A18}$$

where $j_1^*(w_2) = j_2^*(w_2' + \tilde{b})$, $\tilde{b}$ is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_2)$, $w_2 = w_2(TM)$ is the Stiefel-Whitney class of the tangent bundle TM, and $w_2' = w_2'(\mathrm{SO}(3))$ is the Stiefel-Whitney class of the universal SO(3) bundle.

In general, if we have a homotopy pullback square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z, \end{array} \tag{A19}$$

then there is a fiber sequence

$$\Omega Z \to P \to X \times Y \to Z \tag{A20}$$

where ΩZ is the loop space of Z.

So there is a fiber sequence

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{B}\mathbb{G} \to \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) \times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_2 \tag{A21}$$

where the last map is $(u, v) \mapsto j_1^*(w_2)(u) - x_2(v) = j_2^*(w_2' + \tilde{b})(u) - x_2(v)$.

Then we define $\mathbb{G}'$ to be a group which sits in a homotopy pullback square

$$\begin{array}{ccccc} \mathrm{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{B}\mathbb{G}' & \longrightarrow & \mathrm{BSpin}\times\mathrm{B}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8)\times\mathrm{B}^2\mathbb{Z}_2 \\ & & \downarrow & & \downarrow \\ \mathrm{B}\mathbb{Z}_2 & \longrightarrow & \mathrm{B}\mathbb{G} & \longrightarrow & \mathrm{B}(\mathrm{Spin}\times_{\mathbb{Z}_2}(\mathrm{SU}(2)\times_{\mathbb{Z}_2}\mathbb{Z}_8))\times\mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{A22}$$

Since w_2 is identified with x_2 in $\mathrm{B}\mathbb{G}$, it is trivialized in $\mathrm{B}\mathbb{G}'$ because $x_2 = w_2 = 0$ due to the spin structure, so $a \cup \mathcal{P}_2(x_2)$ is clearly trivialized by being pulled back to $\Omega_5^{\mathbb{G}'}$.

Although our starting point was the symmetry extension, this is a symmetry-breaking case in disguise.

c. Further trivialization: Second approach

Define $\mathbb{G}$ to be a group which sits in a homotopy pullback square

$$\begin{array}{ccccc} \mathrm{B}\mathbb{G} & & \longrightarrow & & \mathrm{B}^2\mathbb{Z}_2 \\ \downarrow & & & & \downarrow x_2 \\ \mathrm{B}(\mathrm{Spin}\times\mathrm{SU}(2)\times\mathbb{Z}_8) & \xrightarrow{j_3} & \mathrm{B}\mathbb{Z}_8 & \xrightarrow{\tilde{b}} & \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{A23}$$

In general, if we have a homotopy pullback square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z, \end{array} \tag{A24}$$

then there is a fiber sequence

$$\Omega Z \to P \to X \times Y \to Z \tag{A25}$$

where ΩZ is the loop space of Z.

So there is a fiber sequence

$$\mathrm{B}\mathbb{Z}_2 \to \mathrm{B}\mathbb{G} \to \mathrm{B}(\mathrm{Spin} \times \mathrm{SU}(2) \times \mathbb{Z}_8) \times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_2 \tag{A26}$$

where the last map is $(u, v) \mapsto j_3^*(\tilde{b})(u) - x_2(v)$.

Since $\mathcal{P}_2(x_2) = x_2 \underset{0}{\cup} x_2 + x_2 \underset{1}{\cup} \delta x_2$, $\delta x_2 = 2\mathrm{Sq}^1 x_2$, $x_2 \underset{0}{\cup} x_2 = x_2^2$, so $\frac{\mathcal{P}_2(x_2)}{2} = \frac{x_2^2}{2} + x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2$.

Since x_2 is identified with $\tilde{b} = \beta_{(2,8)}a$ in $\mathrm{B}\mathbb{G}$ where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$ and $\mathrm{Sq}^1\beta_{(2,8)} = 0$ [9], so $\tilde{a} \cup (x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2)$ is trivialized in $\Omega_5^{\mathbb{G}}$.

Note that $\tilde{a} \cup \frac{x_2^2}{2}$ is still not trivialized.

This is also a symmetry-breaking case, since x_2 is locked with a. In physics, the locking between two probed background fields means that the global symmetry between two sectors are locked together, which results in global symmetry breaking.

d. Further trivialization: Third approach

Consider the pullback of $\mathrm{B}^2\mathbb{Z}_2$ to $\mathrm{B}^2\mathbb{Z}_4$:

$$\mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_4 \to \mathrm{B}^2\mathbb{Z}_2. \tag{A27}$$

Since $\frac{\mathcal{P}_2(x_2)}{2} = \frac{x_2^2}{2} + x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2$, $x_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ is pulled back to $\tilde{x}_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2)$ and the following diagram

$$\begin{array}{ccc} \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^1} & \mathrm{H}^3(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) \\ \downarrow \cdot 2 & & \downarrow \mathrm{id} \\ \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4) & \xrightarrow{\beta_{(2,4)}} & \mathrm{H}^3(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) \end{array} \tag{A28}$$

is commutative by the naturality of the Bockstein homomorphism, we have that $\mathrm{Sq}^1\tilde{x}_2 = 0$, so $\tilde{a} \cup (x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2)$ is trivialized in $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_4)$.

Note that $\tilde{a} \cup \frac{x_2^2}{2}$ is still not trivialized.

e. Summary

The term $\tilde{a} \cup \frac{x_2^2}{2}$ cannot be trivialized.

Consider $M = S^1 \times S^2 \times S^2$, the partition function

$$Z(M) = (-1)^k \int_M \tilde{a} \cup \frac{x_2^2}{2} = (-1)^k \int_{S^2 \times S^2} \frac{x_2^2}{2}. \tag{A29}$$

Since

$$\mathrm{H}^n(S^2 \times S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ \mathbb{Z} & n = 0, 4 \\ 0 & n = 1, 3, n \geq 4 \end{cases} \tag{A30}$$

where the two generators of $\mathrm{H}^2(S^2 \times S^2, \mathbb{Z})$ are a, b, the generator of $\mathrm{H}^4(S^2 \times S^2, \mathbb{Z})$ is ab.

No matter how to pullback, when $x_2 = a + b \bmod 2$, $(-1)^k \int_{S^2 \times S^2} \frac{x_2^2}{2} = (-1)^k$ can be nontrivial.

This conclusion will be stated more formally and proved in the next Appendix.

In this Appendix, we compute the bordism group $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$ and find a bordism invariant $a \cup \mathcal{P}_2(x_2)$ of it. Then we find the manifold generator of $a \cup \mathcal{P}_2(x_2)$ and consider the pullback trivialization problem of $a \cup \mathcal{P}_2(x_2)$. We first compute the bordism group $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$ and find that $a \cup \mathcal{P}_2(x_2)$ becomes $\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}$ in $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$. Moreover, we find that the summand $\tilde{a} \cup (x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2)$ of $\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}$ can be trivialized $[\mathcal{P}_2(x_2) = x_2^2 + 2x_2 \underset{1}{\cup} \mathrm{Sq}^1 x_2]$, but $\tilde{a} \cup \frac{x_2^2}{2}$ cannot be trivialized. We conclude that $a \cup \mathcal{P}_2(x_2)$ cannot be trivialized via extending the global symmetry by 0-form symmetry and 1-form symmetry.

APPENDIX B: PROOF: A COUNTEREXAMPLE

By direct computation, we find that $a \cup \mathcal{P}_2(x_2)$ is a bordism invariant of $\Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$.

We consider the trivialization problem: Can we trivialize the topological term $a \cup \mathcal{P}_2(x_2)$ via extending the global symmetry by 0-form $K_{[0]}$ symmetry and 1-form $K_{[1]}$ symmetry?

We can reformulate it mathematically: Can we find finite Abelian groups $K_{[0]}$ and $K_{[1]}$ such that

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G} \xrightarrow{f} \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) \times \mathrm{B}^2\mathbb{Z}_2 \tag{B1}$$

is a fibration and $(fg)^*(a \cup \mathcal{P}_2(x_2)) = 0$ for any 5-manifold M and any map $g\colon M \to \mathrm{B}\mathbb{G}$?

There is a group homomorphism:

$$\Omega_5^{\mathbb{G}} \xrightarrow{\phi} \Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)$$
$$(M, g) \mapsto (M, fg). \tag{B2}$$

So the trivialization problem is asking whether we can find $\mathbb{G}$ and f such that $\phi^*(a \cup \mathcal{P}_2(x_2)) = 0$ for any $(M, g) \in \Omega_5^{\mathbb{G}}$. By direct computation, we find that $a \cup \mathcal{P}_2(x_2)$ becomes $\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}$ in $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$.

Our main result is

Claim 1: We cannot find finite Abelian groups $K_{[0]}$ and $K_{[1]}$ such that

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G} \xrightarrow{f} \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) \times \mathrm{B}^2\mathbb{Z}_2 \tag{B3}$$

is a fibration and $(fg)^*(a \cup \mathcal{P}_2(x_2)) = 0$ for any 5-manifold M and any map $g\colon M \to \mathrm{B}\mathbb{G}$.

Claim 2: We cannot find finite Abelian groups $K_{[0]}$ and $K_{[1]}$ such that

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G} \xrightarrow{f} \mathrm{BSpin} \times \mathrm{BSU}(2) \times \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \tag{B4}$$

is a fibration and $(fg)^*(\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}) = 0$ for any 5-manifold M and any map $g\colon M \to \mathrm{B}\mathbb{G}$.

Clearly claim 2 implies claim 1 since if we can find Abelian groups $K_{[0]}$ and $K_{[1]}$ such that

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G} \xrightarrow{f} \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) \times \mathrm{B}^2\mathbb{Z}_2 \tag{B5}$$

is a fibration and $(fg)^*(a \cup \mathcal{P}_2(x_2)) = 0$ for any 5-manifold M and any map $g\colon M \to \mathrm{B}\mathbb{G}$, then we can define $\mathbb{G}'$ which sits in a homotopy pullback square

$$\begin{array}{ccccc} \mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} & \longrightarrow & \mathrm{B}\mathbb{G}' & \xrightarrow{f'} & \mathrm{BSpin} \times \mathrm{BSU}(2) \times \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \\ & & \downarrow & & \downarrow \\ \mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} & \longrightarrow & \mathrm{B}\mathbb{G} & \xrightarrow{f} & \mathrm{B}(\mathrm{Spin} \times_{\mathbb{Z}_2} (\mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) \times \mathrm{B}^2\mathbb{Z}_2. \end{array} \tag{B6}$$

Then

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G}' \xrightarrow{f'} \mathrm{BSpin} \times \mathrm{BSU}(2) \times \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \tag{B7}$$

is a fibration and $(f'g')^*(\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}) = 0$ for any 5-manifold M and any map $g'\colon M \to \mathrm{B}\mathbb{G}'$.

Since $\mathrm{H}_i(\mathrm{BSpin}, \mathbb{Z}) = 0$ for $i = 1, 2, 3$, $\mathrm{H}^2(\mathrm{BSpin}, K_{[0]}) = \mathrm{H}^3(\mathrm{BSpin}, K_{[1]}) = 0$ by the universal coefficient theorem, similarly we have $\mathrm{H}^2(\mathrm{BSU}(2), K_{[0]}) = \mathrm{H}^3(\mathrm{BSU}(2), K_{[1]}) = 0$. So in order to prove claim 2, we need only prove the following:

Claim 3: We cannot find finite Abelian groups $K_{[0]}$ and $K_{[1]}$ such that

$$\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \to \mathrm{B}\mathbb{G} \xrightarrow{f} \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \tag{B8}$$

is a fibration and $(fg)^*(\tilde{a} \cup \frac{\mathcal{P}_2(x_2)}{2}) = 0$ for any spin 5-manifold M and any map $g\colon M \to \mathrm{B}\mathbb{G}$.

We prove claim 3 by finding a counterexample.

For $M = S^1 \times S^2 \times S^2$, let a, b be the generators of $\mathrm{H}^2(S^2 \times S^2, \mathbb{Z}_2)$, c be the generator of $\mathrm{H}^1(S^1, \mathbb{Z}_8)$, and let $h\colon M \to \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$ be given by $(c, a+b)$. The lifting problem

$$\begin{array}{ccc} & & \mathrm{B}\mathbb{G} \\ & \overset{g}{\nearrow} & \downarrow f \\ S^1 \times S^2 \times S^2 & \xrightarrow{h} & \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \end{array} \tag{B9}$$

has a solution, but $(c \bmod 2) \cup \frac{\mathcal{P}_2(a+b)}{2} \neq 0$.

In general, if $F \to E \xrightarrow{p} B \xrightarrow{q} \Sigma F$ is a fiber sequence, then $[M, F] \to [M, E] \xrightarrow{p_*} [M, B] \xrightarrow{q_*} [M, \Sigma F]$ is an exact sequence of Abelian groups, so the lifting problem has a solution if and only if $q_*(h) = 0$ where $q\colon \mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2K_{[0]} \ltimes \mathrm{B}^3K_{[1]}$. So we need to prove that $q \circ h = 0$.

Again apply the exact sequence $[M, \Sigma F] \to [M, \Sigma E] \to [M, \Sigma B]$ to $M = S^1 \times S^2 \times S^2$ and the fibration

$$\begin{array}{ccc} \mathrm{B}^2K_{[1]} & \longrightarrow & \mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]} \\ & & \downarrow \\ & & \mathrm{B}K_{[0]}, \end{array} \tag{B10}$$

we get that if the image of $q \circ h$ in $[M, \Sigma B]$ is zero, then $q \circ h$ is the image of some map in $[M, \Sigma F]$.

We can write the composition q' of the map $\mathrm{B}^2K_{[0]} \ltimes \mathrm{B}^3K_{[1]} \to \mathrm{B}^2K_{[0]}$ and q as

$$q' = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{B11}$$

where $q_1 \in H^2(\mathrm{B}\mathbb{Z}_8, K_{[0]})$, $q_2 \in H^2(\mathrm{B}^2\mathbb{Z}_2, K_{[0]})$. We assume that $q_2 = 0$ to ensure that the 1-form symmetry (here the 1-form $\mathbb{Z}_2^e$-symmetry) is not broken.

$q' \circ h = q_1 \circ c + q_2 \circ (a + b)$, since $q_1 \circ c = 0$ and $q_2 \circ (a + b) = 0$ for $M = S^1 \times S^2 \times S^2$, so $q \circ h$ is the image of some map in $[M, \mathrm{B}^3K_{[1]}]$. So $q \circ h = q_3 \circ c + q_4 \circ (a + b)$ where $q_3 \in H^3(\mathrm{B}\mathbb{Z}_8, K_{[1]})$, $q_4 \in H^3(\mathrm{B}^2\mathbb{Z}_2, K_{[1]})$. Since $q_3 \circ c = 0$ and $q_4 \circ (a + b) = 0$ for $M = S^1 \times S^2 \times S^2$, we have proven that $q \circ h = 0$.

In this Appendix, we give a proof of the conclusion in the previous Appendix. This answers the first question (Question 1) in Sec. I.

APPENDIX C: PULLBACK TRIVIALIZATION OF $\mathcal{P}_2(B_2)$ IN $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

There is a group homomorphism:

$$\begin{aligned} \Omega_4^{\mathrm{SO}}(X) &\xrightarrow{\rho} \Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2) \\ (M, g) &\mapsto (M, fg). \end{aligned} \tag{C1}$$

We want to extend the 1-form $\mathbb{Z}_2$ symmetry by the 0-form $K_{[0]}$ symmetry and 1-form $K_{[1]}$ symmetry such that $\rho^* \mathcal{P}_2(\tilde{g}) = \mathcal{P}_2(fg) = 0$ for any $(M, g) \in \Omega_4^{\mathrm{SO}}(X)$ where $(M, \tilde{g}) \in \Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$.

We consider the trivialization problem: Does there exist a fibration $f\colon X \to \mathrm{B}^2\mathbb{Z}_2$ with fiber $\mathrm{B}K_{[0]} \ltimes \mathrm{B}^2K_{[1]}$ where $K_{[0]}$ and $K_{[1]}$ are finite Abelian groups such that $\mathcal{P}_2(fg) = 0$ for any oriented 4-manifold M and any map $g\colon M \to X$?

The answer to this problem is negative. For $M = S^2 \times S^2$, let a, b be the generators of $\mathrm{H}^2(S^2 \times S^2, \mathbb{Z}_2)$. The lifting problem

$$\begin{array}{ccc} & & X \\ & \overset{g}{\nearrow} & \downarrow f \\ S^2 \times S^2 & \xrightarrow{a+b} & \mathrm{B}^2\mathbb{Z}_2 \end{array} \tag{C2}$$

always has a solution, but $\mathcal{P}_2(fg) = \mathcal{P}_2(a+b)$ is nontrivial. Similarly as before, we need only prove that the composition map $S^2 \times S^2 \xrightarrow{a+b} \mathrm{B}^2\mathbb{Z}_2 \xrightarrow{q} \mathrm{B}^2K_{[0]} \ltimes \mathrm{B}^3K_{[1]}$ is zero.

This can be proven similarly as before.

So $\mathcal{P}_2(x_2)$ cannot be trivialized.

In this Appendix, we consider the pullback trivialization problem of $\mathcal{P}_2(x_2)$, and we give a similar proof that $\mathcal{P}_2(x_2)$ also cannot be trivialized via extending the global symmetry by the 0-form symmetry and 1-form symmetry. This answers the second question (question 2) in Sec. I.

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Correction: The end of the heading for Appendix A was inadvertently deleted during the production cycle and has been restored.