Remarks on the Novikov-Shifman-Vainshtein-Zakahrov β functions in two-dimensional $\mathcal{N}=(0,2)$ supersymmetric models

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The Novikov-Shifman-Vainshtein-Zakahrov β functions in two-dimensional $\mathcal{N}=(0,2)$ supersymmetric models are revisited. We construct and discuss a broad class of such models using the gauge formulation. All of them represent direct analogs of four-dimensional $\mathcal{N}=1$ Yang-Mills theories and are free of anomalies. Following the same line of reasoning as in four dimensions we distinguish between the holomorphic and canonical coupling constants. This allows us to derive the *exact* two-dimensional β functions in all models from the above class. We then compare our results with a few examples that have been studied previously.

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I. INTRODUCTION AND CONCLUSION

The $2d/4d$ parallels are known and have been used since the time of Polyakov who found asymptotic freedom (AF) in $2d$ nonlinear sigma models [1], in analogy with AF in $4d$ Yang-Mills theories [2,3]. In the past three decades, $2d/4d$ correspondence acquired a much deeper meaning by virtue of supersymmetry. Much of nonperturbative dynamics in both $2d/4d$ supersymmetric gauge theories has been thoroughly understood and found to correspond to each other. By the "$2d/4d$ correspondence" we mean here the cases in which either some of the $2d/4d$ physics contents are exactly the same, e.g., the Alday-Gaiotto-Tachikawa correspondence [4], or the dynamical behaviors in $2d$ and $4d$ coincide; for instance, the BPS[1] spectra, certain correlation functions, dualities, etc., are identical [5–10]. Among these phenomena, an instructive example is provided by non-Abelian BPS vortex strings [11–13], both in $4d$ $\mathcal{N}=2$ and $\mathcal{N}=1$ gauge theories, whose low-energy dynamics are captured by $2d$ $\mathcal{N}=(2,2)$ and heterotic $\mathcal{N}=(0,2)$ sigma models, respectively [14–18]. The above vortex strings present a "bridge" between $4d$ and $2d$ physics providing a quantitative explanation why the $2d$ dynamics are in correspondence with the dynamics in its $4d$ progenitor. This correspondence was established in a wide class of theories from both $2d$ and $4d$ directions, perturbatively and nonperturbatively [6,19–24].

The goal of this paper is to derive Novikov-Shifman-Vainshtein-Zakahrov (NSVZ)–like β functions [25–29] in general two-dimensional $\mathcal{N}=(0,2)$ supersymmetric gauge theories adding new evidence for the $2d/4d$ correspondence. A number of $2d$ analogs of the NSVZ β functions were obtained in the past via both perturbative methods and instanton calculus in the $\mathcal{N}=(0,2)\mathbb{CP}^1$ model [24] and in a large class of heterotically deformed nonlinear sigma models (NLSMs) that are deformations of their $\mathcal{N}=(2,2)$ cousins [6]. Here we focus on another general class of $\mathcal{N}=(0,2)$ gauged linear sigma models (GLSMs) and obtain the general form of the corresponding β functions. They have the same structure as the NSVZ β function in $4d$. In those cases where comparison with the previous results is possible our newly derived GLSM β functions are identical to those of NLSMs. This is not surprising since the NLSMs studied previously can be embedded in GLSMs.

We want to emphasize not only the ubiquity of $2d/4d$ correspondence but also the conspiracy of methodologies applicable to both $2d$ and $4d$ theories. Historically, $2d$ sigma models were considered as simplified toy models useful for understanding real world physics in $4d$. Instead, in this paper, we follow the opposite direction, from $4d$ to $2d$, establishing and using the $2d$ analog of the Konishi anomaly [30] and scaling anomalies in $2d$ $\mathcal{N}=(0,2)$ gauge theories, *à la* Arkani-Hamed and Murayama in the $4d$ $\mathcal{N}=1$ case [31]. This observation helps us relate

[1]BPS spectra here are meant to those physical states in the short supermultiplets of given supersymmetries.

holomorphic coupling constants to canonic ones in $2d$ GLSMs, thus trivializing derivation of their β functions. The general master formula obtained in this paper is

$$\beta(g^2) = -\frac{g^4}{4\pi}\frac{\sum_i q_i + \frac{1}{2}\sum_a \tilde{q}_a\gamma_a}{1-\frac{\sum_i q_i}{8\pi}g^2}, \tag{1.1}$$

in the case of $2d$ $\mathcal{N}=(0,2)$ gauge theories with a single Fayet-Iliopoulos (FI) coupling

$$\xi \equiv \frac{2}{g^2},$$

where q_i's are the $U(1)$ gauge charges of the bosonic matter fields and $\tilde{q}_a$ and γ_a are the $U(1)$ gauge charges and anomalous dimensions of the fermionic matter fields.

The paper is organized as follows: We will briefly review the building blocks of $2d$ $\mathcal{N}=(0,2)$ supersymmetric GLSMs in Sec. II and a nonrenormalization theorem for the FI coupling constants in Sec. III. We then explain the difference between holomorphic and canonical coupling constants from both the perspectives of the Konishi anomaly and the scaling anomalies of matter fields, and we derive the master equation (1.1) in Sec. IV. Finally, we apply the formula in several examples.

II. TWO-DIMENSIONAL $\mathcal{N}=(0,2)$ GLSMs

The $\mathcal{N}=(0,2)$ superspace is parametrized by $2d$ bosonic spacetime

$$x^{\pm\pm} \equiv x^0 \pm x^1$$

and their $\mathcal{N}=(0,2)$ fermionic partners θ^+ and $\bar{\theta}^+$. The supercharges are defined in terms of these coordinates as follows:

$$Q_+ \equiv \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\partial_{++},$$
$$\bar{Q}_+ \equiv -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\partial_{++}, \tag{2.1}$$

where

$$\partial_{++} \equiv 2\partial_{x^{++}}, \qquad \partial_{--} \equiv 2\partial_{x^{--}}.$$

Accordingly, the superderivatives are given by

$$D_+ \equiv \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\partial_{++},$$
$$\bar{D}_+ \equiv -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\partial_{++}, \tag{2.2}$$

which satisfy the conditions

$$D_+^2 = \bar{D}_+^2 = 0, \qquad \{D_+, \bar{D}_+\} = 2i\partial_{++}.$$

With this notation, it is not difficult to build three types of supermultiplets to construct $\mathcal{N}=(0,2)$ GLSMs [14,32].

A. Gauge multiplets

The $\mathcal{N}=(0,2)$ gauge multiplet $U_{--} = (A_{--}, \lambda_-, \bar{\lambda}_-, D)$ is real and adjoint-valued

$$U_{--} = A_{--} - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D \tag{2.3}$$

in superfield formalism. Here

$$A_{--} \equiv A_0 - A_1, \qquad A_{++} \equiv A_0 + A_1$$

are the $2d$ gauge fields, λ_- and $\bar{\lambda}_-$ are the gaugino fields, and the real field D is auxiliary. The field A_{++} is an $\mathcal{N}=(0,2)$ singlet.

Next, we can promote superderivatives to be covariant, namely

$$\mathcal{D}_+ \equiv \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\nabla_{++} \equiv \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\partial_{++} - iA_{++}),$$
$$\bar{\mathcal{D}}_+ \equiv -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\nabla_{++} \equiv -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\partial_{++} - iA_{++}),$$
$$D_{--} \equiv \partial_{--} - iU_{--} = \nabla_{--} - 2\theta^+\bar{\lambda}_- - 2\bar{\theta}^+\lambda_- - 2i\theta^+\bar{\theta}^+ D. \tag{2.4}$$

The superfield strength of the gauge multiplet is given by

$$\Upsilon_- = [\bar{\mathcal{D}}_+, \mathcal{D}_{--}] = -2(\lambda_- - i\theta^+(D - iB) - i\theta^+\bar{\theta}^+\mathcal{D}_{++}\lambda_-), \tag{2.5}$$

where

$$B = \partial_0 A_1 - \partial_1 A_0 - i[A_0, A_1] \tag{2.6}$$

is the field strength of the A_μ field. The conjugated superfield $\bar{\Upsilon}_-$ is defined accordingly. The action of the gauge multiplet is as follows:

$$S_{\text{gauge}} = \frac{1}{8e^2}\text{Tr}\int d^2x d\theta^+ d\bar{\theta}^+ \bar{\Upsilon}_-\Upsilon_-$$
$$= \frac{1}{e^2}\text{Tr}\int d^2x\left(\frac{1}{2}B^2 + i\bar{\lambda}_-\nabla_{++}\lambda_- + \frac{1}{2}D^2\right). \tag{2.7}$$

Here e^2 is the gauge coupling. The corresponding NLSM can be obtained in the limit $e^2 \to \infty$.

B Chiral multiplets

The $\mathcal{N}=(0,2)$ chiral multiplet $\Phi^i = (\phi^i, \psi_+^i)$ satisfies the usual chiral constraint

$$\bar{\mathcal{D}}_+\Phi^i = 0. \tag{2.8}$$

In the superfield formalism it is written as

$$\Phi^i = \phi^i + \sqrt{2}\theta^+\psi^i_+ - i\theta^+\bar{\theta}^+\nabla_{++}\phi^i, \tag{2.9}$$

where

$$\nabla_\mu\phi^i = (\partial_\mu - iq_iA_\mu)\phi^i.$$

Moreover, q_i is the charge of the field Φ^i with respect to the $U(1)$ gauge field. The action of the chiral multiplets can be written as

$$\begin{aligned} S_{\text{chiral}} &= -\frac{i}{2}\int d^2x d\theta^+ d\bar{\theta}^+ \sum_i \bar{\Phi}_i \mathcal{D}_{--}\Phi^i \\ &= \int d^2x \sum_i (-|\nabla_\mu\phi^i|^2 + i\bar{\psi}_{+i}\nabla_{--}\psi^i_+ \\ &\quad - \sqrt{2}iq_i\bar{\phi}_i\lambda_-\psi^i_+ + \sqrt{2}q_i\bar{\psi}_{+i}\bar{\lambda}_-\phi^i + q_i\bar{\phi}_iD\phi^i). \end{aligned} \tag{2.10}$$

C Fermi multiplets

Another important matter superfield consists of a fermion χ^a_- and an auxiliary field G^a,

$$(\chi^a_-, G^a) \in \Gamma^a_-. \tag{2.11}$$

It is not necessarily chiral, but, instead, satisfies the constraint

$$\bar{\mathcal{D}}_+\Gamma^a_- = \sqrt{2}E^a(\Phi), \tag{2.12}$$

where $E(\Phi)$ is an arbitrary holomorphic function with respect to chiral boson fields Φ's. In the superfield formalism, it can be expanded as

$$\Gamma^a_- = \chi^a_- - \sqrt{2}\theta^+G^a - i\theta^+\bar{\theta}^+\nabla_{++}\chi_- - \sqrt{2}\bar{\theta}^+E^a(\Phi). \tag{2.13}$$

The action for the Fermi multiplet reduces to

$$\begin{aligned} S_{\text{Fermi}} &= -\frac{1}{2}\int d^2x d\theta^+ d\bar{\theta}^+ \sum_a \bar{\Gamma}_{-a}\Gamma^a_- \\ &= \int d^2x \sum_{a,i}\bigg(i\bar{\chi}_{-a}\nabla_{++}\chi^a_- + |G^a|^2 - |E^a(\phi)|^2 \\ &\quad - \bar{\chi}_{-a}\frac{\partial E^a}{\partial\phi^i}\psi^i_+ + \text{H.c.}\bigg). \end{aligned} \tag{2.14}$$

Note that the gauge field strength Υ_- is a particular case of the Fermi multiplets in the adjoint representation of the gauge group, satisfying

$$\bar{\mathcal{D}}_+\Upsilon_- = 0. \tag{2.15}$$

D Superpotentials

Last but not least, we need to introduce superpotentials $J_a(\Phi)$ as holomorphic functions of chiral superfields, whose action reduces to half of the superspace (accompanied by Fermi multiplets Γ^a_-),

$$\begin{aligned} S_J &= -\frac{1}{\sqrt{2}}\sum_a \int d^2x d\theta^+ \Gamma^a_- J_a + \text{H.c.} \\ &= \sum_a \int d^2x G^a J^a(\phi) + \sum_i \chi_{-a}\frac{\partial J^a}{\partial\phi^i}\psi^i_+ + \text{H.c.} \end{aligned} \tag{2.16}$$

Of the utmost interest is the FI term as a superpotential given by the gauge field strength, if it admits $U(1)$ factors,

$$\begin{aligned} S_\tau &= \frac{1}{4}\text{Tr}\int d^2x d\theta^+ \tau\Upsilon_-|_{\bar{\theta}^+=0} + \text{H.c.} \\ &= \text{Tr}\int d^2x\left(-\xi D + \frac{\theta}{2\pi}B\right), \end{aligned} \tag{2.17}$$

where for simplicity we only consider theories with a single FI term, and

$$\tau = \frac{\theta}{2\pi} + i\xi \tag{2.18}$$

is the complex FI coupling constant.

E GLSM action

Overall we assemble all the above ingredients and arrive at the action of $\mathcal{N} = (0,2)$ supersymmetric GLSM,

$$S = S_{\text{gauge}} + S_{\text{chiral}} + S_{\text{Fermi}} + S_\tau + \text{H.c.} \tag{2.19}$$

Here and below, without loss of generality, we will consider theories in which the superpotentials are limited to FI terms. Importantly, for such theories to be consistent at the quantum level (i.e., free of internal anomalies), we need to impose constraints on the representations of the chiral and Fermi multiplets to get rid of the gauge anomalies (see also in [33]),

$$\begin{aligned} U(1)\,\text{gauge}&: \sum_i q_i^2 = \sum_a \tilde{q}_a^2, \\ \text{non}-\text{Abelian gauge}&: \sum_i t_2(i) = t_2(A) + \sum_a t_2(a), \end{aligned} \tag{2.20}$$

where q_i and $\tilde{q}_a$ are $U(1)$ gauge charges of chiral and Fermi multiplets, t_2 is the dual Coxeter number, and i, a, and A denote the representations of chiral, Fermi, and gauge multiplets.

III. A NONRENORMALIZATION THEOREM FOR THE HOLOMORPHIC COUPLING τ

In $2d$ gauge theories, the gauge coupling e has a dimension of mass and is thus superrenormalizable. For energy scale $\mu \ll e$, the gauge multiplets will be nondynamical, and we arrive at NLSMs. Therefore the only sensible parameter in the theory is its FI coupling constant τ, which is marginal and runs at the quantum level. In much the same way as with the gauge couplings in $4d$ $\mathcal{N}=1$ gauge theories, the $2d$ FI parameter τ, as the coupling of the $\mathcal{N}=(0,2)$ superpotential, is subject to a nonrenormalization theorem and receives *at most a one-loop correction* (see, e.g., [28]). We will follow [28,31] in reviewing the relevant argument.

From Eq. (2.19), we see that the action S depends on τ holomorphically. It is convenient to use the notation

$$2\pi i\tau = -2\pi\xi + i\theta \equiv -\frac{4\pi}{g^2} + i\theta. \tag{3.1}$$

Let us ask ourselves: When we change the cutoff from M_0 to μ, how does the coupling $2\pi i\tau(\mu)$ (in the Wilsonian sense) change to keep the low-energy physics intact? To answer this question, let us examine an ansatz

$$2\pi i\tau(\mu) = 2\pi i\tau(M_0) + f\left(2\pi i\tau(M_0), \log\frac{M_0}{\mu}\right). \tag{3.2}$$

It is worth noting that a 2π shift of the θ angle leads to no change of physics, and therefore at most,

$$f\left(2\pi i\tau(M_0), \log\frac{M_0}{\mu}\right) \to f\left(2\pi i\tau(M_0), \log\frac{M_0}{\mu}\right) + 2\pi i F\left(\log\frac{M_0}{\mu}\right), \quad \text{for } \theta \to \theta + 2\pi, \tag{3.3}$$

where function $F(\log\frac{M_0}{\mu})$ can only take integer values. Furthermore, because $F(0)=0$, by continuity we conclude that function f is periodic with respect to the θ angle. Therefore, the β function for $2\pi i\tau$,

$$\beta(2\pi i\tau) = \mu\frac{\partial}{\partial\mu}(2\pi i\tau(\mu)) = \mu\frac{\partial f}{\partial\mu}, \tag{3.4}$$

is periodic with respect to θ and admits a Fourier expansion,

$$\beta(2\pi i\tau) = \sum_{n\geq 0} b_n e^{2\pi i n\tau}. \tag{3.5}$$

It is clear that in perturbation theory we can only have nonnegative integer values of n appearing in the expansion (3.5). Also, in the perturbative regime we have at most b_0 nonzero, i.e.,

$$\beta(2\pi i\tau) = b_0. \tag{3.6}$$

In perturbation theory it is obvious that all b_n's with $n=1,2,3,\ldots$, vanish. Hence the nonrenormalization theorem of the absence of higher loops is proven for the holomorphic coupling.

Nonperturbatively, one needs to apply the anomalous R symmetry of $\mathcal{N}=(0,2)$, which guarantees that the θ angle receives no quantum corrections at all. Consequently $\beta(2\pi i\tau)$ is independent of $\mathrm{Im}(2\pi i\tau)$, *and, simultaneously is holomorphic* in $2\pi i\tau$. It implies that $\beta(2\pi i\tau)$ can only be a constant; i.e., Eq. (3.6) holds both perturbatively and nonperturbatively.

Before proceeding to the discussion of the canonical coupling τ_c in next sections, let us first calculate b_0 that would be used later. It can easily be obtained by inspecting the D term of the action (2.19),

$$S_D = \int d^2x\left(\frac{1}{2e^2}D^2 - \xi D + \sum_i q_i\bar{\phi}_i D\phi^i\right). \tag{3.7}$$

From (3.7) we see that the real part of τ receives a tadpole one-loop correction.[2] The tadpole graph emerges through contracting ϕ and $\bar{\phi}$. As a result,

$$\xi(\mu) = \xi(M_0) - \frac{\sum_i q_i}{2\pi}\log\left(\frac{M_0}{\mu}\right), \tag{3.8}$$

which implies, in turn, that

$$\beta(\xi) = \frac{\sum_i q_i}{2\pi}, \text{or, say}, \beta(g^2) = -\frac{\sum_i q_i}{4\pi}, \tag{3.9}$$

and

$$b_0 = -\sum_i q_i.$$

IV. FROM THE HOLOMORPHIC TO CANONIC COUPLING

As is known from [28], all higher order loops in the gauge coupling renormalization appear in passing from the holomorphic to canonic coupling from the Z factors of the matter fields (which are converted into the anomalous dimensions in the β functions). To see how this happens we must convert the kinetic terms of the matter fields into (2.17) by virtue of anomalies. In other words, we must take into account a subtle difference between the Wilsonian Lagrangian and the one partical irreducible functional (see [25–28]).

Below we will discuss two alternative (but related) derivations, through the Konishi anomaly [30] and through the scale anomaly [31].

[2] As in the $4d$ case, the tadpole correction appears if and only if $\sum_i q_i \neq 0$.

A. The Konishi anomaly in $\mathcal{N}=(0,2)$ GLSM

It is not difficult to establish the $2d$ analog of the Konishi anomaly. To this end, as an example, we will consider the operator $\sum_a \bar{\Gamma}_{-a}\Gamma_-^a$ appearing in (2.14) (assuming that $E^a=0$). Classically, the equation of motion for this operator is

$$\mathcal{D}_+\left(\sum_a \bar{\Gamma}_{-a}\Gamma_-^a\right)=0. \tag{4.1}$$

This follows, e.g., from inspection of the $\bar{\theta}^+$ component. However, at the quantum level this particular component contains a well-known anomaly in the derivative of the χ_- current (see more details in Appendix B and also [34]), analogous to the triangle anomaly in the axial current in $4d$,[3]

$$\partial_{++}\left(\sum_a \bar{\chi}_{-a}\chi_-^a\right)=\sum_a \frac{\tilde{q}_a}{2\pi}B\Big|_{U(1)}, \tag{4.2}$$

where B is defined in (2.5). Note that the relative coefficient between D and B in (2.5) is rigidly fixed by $\mathcal{N}=(0,2)$ supersymmetries. Needless to say, the full derivative in the $U(1)$ part does not appear in the action classically (it can be dropped). However, at the quantum level we can establish the following relations (after evolving the action from M_0 down to μ):

$$\begin{aligned}
&\Delta\mathcal{L}_\Gamma(\mu)-\frac{1}{2}Z_{\text{Fermi}}\int d\bar{\theta}^+d\theta^+(\bar{\Gamma}_{-a}\Gamma_-^a)\\
&=-\frac{Z_{\text{Fermi}}}{2}\int d\bar{\theta}^+\mathcal{D}_+(\bar{\Gamma}_{-a}\Gamma_-^a)\\
&=i\frac{Z_{\text{Fermi}}}{2}\partial_{++}\left(\sum_a \bar{\chi}_{-a}\chi_-^a\right)=iZ_{\text{Fermi}}\sum_a\frac{\tilde{q}_a}{4\pi}B\Big|_{U(1)}\\
&=iZ_{\text{Fermi}}\sum_a\frac{\tilde{q}_a}{8\pi}\left(\int d\theta^+\Upsilon_- +\int d\bar{\theta}^+\bar{\Upsilon}_-\right)\Big|_{U(1)},
\end{aligned} \tag{4.3}$$

where in the last step, we uplifted the equation to the level of superspace; cf. (2.17). The Υ_- part gives the evolution of the wave function renormalization of fermion Γ_-^a to the FI-coupling constant τ; see also Eq. (4.7). Adding the one-loop tadpole graph and differentiating over $\mu/\partial\mu$ we arrive at the $\tilde{q}_a\gamma_a$ term in (1.1).

B. Scaling anomalies

Now we would like to discuss the $2d$ $\mathcal{N}=(0,2)$ β function along the lines of [31]. It is true that the holomorphic τ only receives a one-loop correction; however, because of the normalization point running down from M_0 to μ, the kinetic terms of the matter fields will receive a wave function renormalization,

$$\begin{aligned}
&\sum_i \bar{\Phi}_i\mathcal{D}_{--}\Phi^i\to\sum_i Z_i(\mu)\bar{\Phi}_i\mathcal{D}_{--}\Phi^i,\\
&\sum_a \bar{\Gamma}_{-a}\Gamma_-^a\to\sum_a Z_a(\mu)\bar{\Gamma}_{-a}\Gamma_-^a;
\end{aligned} \tag{4.4}$$

see Sec. IV A for $\bar{\Gamma}_{-a}\Gamma_-^a$.

To keep all matter fields canonically normalized, we need to change field variables, i.e., redefine

$$\Phi^i\equiv\frac{1}{\sqrt{Z_i(\mu)}}\Phi^{i\prime},\qquad \Gamma_-^a\equiv\frac{1}{\sqrt{Z_a(\mu)}}\Gamma_-^{a\prime}. \tag{4.5}$$

However, such rescaling will result in anomalous Jacobians from the functional measure. Formally we have

$$\begin{aligned}
[d\Phi^i]&=\left[d\left(\frac{1}{\sqrt{Z_i(\mu)}}\Phi^{i\prime}\right)\right]=\text{sDet}\left(\frac{1}{\sqrt{Z_i(\mu)}}\right)[d\Phi^{i\prime}]\\
&=[d\Phi^{i\prime}]e^{-\frac{1}{2}\log Z_i(\mu)\text{sTr}_{\Phi^i}\mathbb{1}},\\
[d\Gamma_-^a]&=\left[d\left(\frac{1}{\sqrt{Z_a(\mu)}}\Gamma_-^{a\prime}\right)\right]=\text{sDet}\left(\frac{1}{\sqrt{Z_a(\mu)}}\right)[d\Gamma_-^{a\prime}]\\
&=[d\Gamma_-^{a\prime}]e^{-\frac{1}{2}\log Z_a(\mu)\text{sTr}_{\Gamma_-^a}\mathbb{1}},
\end{aligned} \tag{4.6}$$

where "sDet" and "sTr" denote the superdeterminant and supertrace, respectively. The supertrace is superficially vanishing due to supersymmetries. Nevertheless, in a nontrivial gauge field background, we can show that they give rise to terms proportional to the $U(1)$ field strength Υ_-. More specifically,

$$\begin{aligned}
&\text{sTr}_{\Phi^i}\mathbb{1}=-i\frac{q_i}{8\pi}\int d^2x d\theta^+\Upsilon_-|_{\bar{\theta}^+=0},\quad\text{and}\\
&\text{sTr}_{\Gamma_-^a}\mathbb{1}=i\frac{\tilde{q}_a}{8\pi}\int d^2x d\theta^+\Upsilon_-|_{\bar{\theta}^+=0}.
\end{aligned} \tag{4.7}$$

The derivation of this formula is presented in Appendix B. Therefore, the holomorphic τ will receive *nonholomorphic* corrections from wave function renormalizations,

$$\tau\to\tau_c=\tau+\sum_i i\frac{q_i}{4\pi}\log Z_i(\mu)-\sum_a i\frac{\tilde{q}_a}{4\pi}\log Z_a(\mu). \tag{4.8}$$

The anomalous dimensions of Φ^i and Γ_-^a are given by

$$\gamma_i=-\mu\frac{\partial}{\partial\mu}\log Z_i(\mu)\quad\text{and}\quad\gamma_a=-\mu\frac{\partial}{\partial\mu}\log Z_a(\mu), \tag{4.9}$$

and they are nonholomorphic. This statement is in one-to-one correspondence with the NSVZ β function in four dimensions.

Differentiating $\log\mu$ on both sides of Eq. (4.8) and using Eq. (3.9), we have

[3] The triangle anomalous graph in four dimensions is replaced in two dimensions by a diangle graph. That is why the right-hand side in (4.2) is linear in $\tilde{q}_a$.

$$\beta(\tau_c) = i\left(\frac{\sum_i q_i}{2\pi} - \sum_i \frac{q_i}{4\pi}\gamma_i + \sum_a \frac{q_a}{4\pi}\gamma_a\right). \tag{4.10}$$

In terms of coupling constant

$$\mathrm{Im}(\tau_c) = \xi_c \equiv \frac{2}{g_c^2} \tag{4.11}$$

we have

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi}\left(\sum_i q_i - \frac{1}{2}\sum_i q_i\gamma_i + \frac{1}{2}\sum_a \tilde{q}_a\gamma_a\right). \tag{4.12}$$

Furthermore, from Eq. (3.7), the β function of g_c^2, or say, ξ, is nothing other than the wave function renormalization of chiral multiplets, i.e.,

$$\gamma_i = \frac{\beta(g_c^2)}{g_c^2}. \tag{4.13}$$

Using it, we arrive at the master formula

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi}\frac{\sum_i q_i + \frac{1}{2}\sum_a \tilde{q}_a\gamma_a}{1 - \frac{\sum_i q_i}{8\pi} g_c^2}. \tag{4.14}$$

Remark: The gauge multiplets have no contribution to the β function, because τ_c is associated with the $U(1)$ factor gauge group, with respect to which the gauge multiplet is $U(1)$ neutral.

V. EXAMPLES

In this section, we will apply Eq. (1.1) in various examples.

A. $\mathcal{N} = (2,2)\mathbb{CP}^{N-1}$ model

For $\mathcal{N} = (2,2)$ supersymmetries, the $\mathcal{N} = (0,2)$ chiral and Fermi multiplets are combined to an $\mathcal{N} = (2,2)$ chiral multiplet. We have

$$q_i = \tilde{q}_a \quad \text{and} \quad Z_i = Z_a, \quad \text{for } i = a = 1, 2, \ldots. \tag{5.1}$$

Therefore the holomorphic τ and canonical τ_c coincide, and the β-function terminates at one-loop, in terms of g_c^2,[4]

$$\beta(g_c^2) = -\frac{\sum_i q_i}{4\pi} g_c^4. \tag{5.2}$$

Especially for a $U(1)$ gauge theory with all $q_i = 1$, we have the standard $\mathcal{N} = (2,2)\mathbb{CP}^{N-1}$ sigma model, and its β function is

$$\beta(g_c^2) = -\frac{N}{4\pi} g_c^4. \tag{5.3}$$

[4]Exactly the same occurs in $4d$ Yang-Mills [28,29].

B. $\mathcal{N} = (0,2)\mathbb{CP}^{N-1}$ model

We can deform the previous $\mathcal{N} = (2,2)\mathbb{CP}^{N-1}$ model by deleting part of $\mathcal{N} = (2,2)$ $U(1)$ field strength, considered in [16]. In the language $\mathcal{N} = (0,2)$ supersymmetries, the $\mathcal{N} = (2,2)$ $U(1)$ field strength $\Sigma_{(2,2)}$ can be decomposed as

$$\Sigma_{(2,2)} = \Sigma_{(0,2)} \oplus \Upsilon_-, \tag{5.4}$$

where $\Sigma_{(0,2)}$ is a $\mathcal{N} = (0,2)$ chiral superfield and Υ_- is the $\mathcal{N} = (0,2)$ Fermi multiplet as the field strength of the $U(1)$ gauge multiplet. $\mathcal{N} = (2,2)$ chiral multiplet $\Phi^i_{(2,2)}$ also admits a decomposition as

$$\Phi^i_{(2,2)} = \Phi^i \oplus \Gamma^i_-, \tag{5.5}$$

and the $\mathcal{N} = (0,2)$ Fermi multiplet Γ^i_- satisfies the constraint

$$\bar{\mathcal{D}}_+\Gamma^i_- \propto \Sigma_{(0,2)}\Phi^i. \tag{5.6}$$

Now, if we delete $\Sigma_{(0,2)}$, the deformed theory will have only $\mathcal{N} = (0,2)$ supersymmetry, and the Fermi multiplets satisfy

$$\bar{\mathcal{D}}_+\Gamma^i_- = 0. \tag{5.7}$$

Its β function turns out to be

$$\beta(g_c^2) = -\frac{Ng_c^4}{4\pi}\frac{1 + \frac{1}{2}\gamma}{1 - \frac{N}{8\pi} g_c^2}, \tag{5.8}$$

where γ denotes the anomalous dimension of Fermi multiplet Γ^i_-. We want to further comment that, in [24], the authors also considered a type of deformed $\mathcal{N} = (0,2)\mathbb{CP}^1$ model at the level of NLSM, which is different from ours. However, we do see that the β functions of the two models are similar. To compare the difference between our model and that in [24], we discuss its nonlinear formalism in Appendix A.

C. Heterotically deformed $\mathcal{N} = (0,2)\mathbb{CP}^{N-1}$ model

We can also consider a further deformation from the $\mathcal{N} = (0,2)\mathbb{CP}^{N-1}$ model discussed above, by adding an additional *gauge singlet* $\mathcal{N} = (0,2)$ Fermi multiplet,

$$\Omega_- = \eta_- - \sqrt{2}\theta^+ H - i\theta^+\bar{\theta}^+\partial_{++}\eta_-, \tag{5.9}$$

to the $\mathcal{N} = (0,2)$ model, with the corresponding deformed term in the action,

$$\mathcal{S}_\Omega = \int d^2x d\theta^+ d\bar{\theta}^+ \left(-\frac{1}{2}\bar{\Omega}_-\Omega_- + \frac{\kappa}{2}\bar{\Phi}_i \Gamma^i_- \Omega_- + \text{H.c.} \right), \tag{5.10}$$

where κ is an additional coupling. It is crucial to note that, since we start from the $\mathcal{N}=(0,2)$ model, all Fermi multiplets satisfy

$$\bar{\mathcal{D}}_+ \Gamma^i_- = \bar{\mathcal{D}}_+ \Omega_- = 0. \tag{5.11}$$

This constraint turns out to be important because it guarantees that the interaction term can be recast in half superspace as

$$\frac{\kappa}{2}\int d^2x d\theta^+ d\bar{\theta}^+ \bar{\Phi}_i \Gamma^i_- \Omega_- = \frac{\kappa}{2}\int d^2x d\theta^+ \bar{\mathcal{D}}_+ \bar{\Phi}_i \Gamma^i_- \Omega_-. \tag{5.12}$$

It was argued in [23] that this type of interaction is subject to a "D-term" nonrenormalization theorem in $2d$; see also [6]. Therefore, the holomorphic coupling constant κ is *not* renormalized. Here we pause and remark that, if one tries to perform the heterotic deformation from $\mathcal{N}=(2,2)\mathbb{CP}^{N-1}$ GLSM, there would be no nonrenormalization theorem to protect the coupling κ, because in the $\mathcal{N}=(2,2)$ case, $\bar{\mathcal{D}}_+\Gamma^i_- \propto \Sigma_{(0,2)}\Phi^i$; see Eq. (5.6). This differs from the situation in [6], where the heterotic deformation is indeed performed on $\mathcal{N}=(2,2)\mathbb{CP}^{N-1}$ NLSM, because the superderivative acting on the Fermi multiplet in NLSM automatically vanishes.

Since the coupling κ receives no renormalization, we thereby will focus on the β function of ξ, or say g_c^{-2}, in the presence of the coupling constant κ. Let us first write down the action in components:

$$\begin{aligned}\mathcal{S}_\Omega &= \int d^2x (i\bar{\eta}_-\partial_{++}\eta_- + \bar{H}H) \\ &\quad + \kappa \int d^2x (i\nabla_{++}\bar{\phi}_i \chi^i_- \eta_- + G^i \bar{\psi}_{+i}\eta_- - H\bar{\psi}_{+i}\chi^i_-) \\ &\quad + \text{H.c.}\end{aligned} \tag{5.13}$$

The key observation (see also [6]) is that the evolution of the interaction term $i\kappa\nabla_{++}\bar{\phi}_i\chi^i_-\eta_-$ and its Hermitian conjugate will give a *finite* shift to the kinetic term of ϕ^i, i.e.,

$$\begin{aligned}&\left\langle \kappa \int d^2x (i\nabla_{++}\bar{\phi}_i\chi^i_-\eta_-), \bar{\kappa}\int d^2y (i\nabla_{++}\phi^i\bar{\chi}_{-i}\bar{\eta}_-) \right\rangle \\ &\quad = -\frac{|\kappa|^2}{4\pi Z_\chi Z_\eta}\int d^2x |\nabla_\mu \phi^i|^2,\end{aligned} \tag{5.14}$$

where we take fermions as quantum fluctuations and bosons as a background. We write the wave function renormalizations of χ_- and η_- explicitly. It was argued in [6] that this $|\kappa|^2$ iteration is limited to one-loop in the computation of the quantum correction in the instanton background. Here we have a similar situation—our $2d$ GLSM admits an (anti)vortex background, say,

$$\nabla_z \bar{\phi}_i = 0 \quad \text{or} \quad \nabla_z \phi^i = 0, \tag{5.15}$$

where ∇_z is the Euclidean continuation of ∇_{++}. In this background, the iteration of $|\kappa|^2$ will not enter higher loops. Nevertheless, the wave function renormalization of the fields ψ^i_- and η_- will still enter higher loops evaluation. Therefore, we define a new coupling,

$$h^2 \equiv \frac{|\kappa|^2}{Z_\chi Z_\eta}, \tag{5.16}$$

whose β function is given by

$$\beta(h^2) = \mu\frac{\partial}{\partial\mu}h^2 = h^2(\gamma_\chi + \gamma_\eta), \tag{5.17}$$

where

$$\gamma_\chi = -\mu\frac{\partial}{\partial\mu}\log Z_\chi(\mu) \quad \text{and} \quad \gamma_\eta = -\mu\frac{\partial}{\partial\mu}\log Z_\eta(\mu) \tag{5.18}$$

are the anomalous dimensions of the fields χ^i_- and η_-.

Now we assemble this additional contribution to the one-loop correction of the holomorphic coupling ξ. The imaginary part of Eq. (4.8) is thus modified as

$$\frac{2}{g_c^2} = \frac{2}{g^2} - \frac{h^2}{4\pi} + \frac{N}{4\pi}\log Z_\phi(\mu) - \frac{N}{4\pi}\log Z_\chi(\mu). \tag{5.19}$$

Differentiating with respect to the running scale μ, and using Eqs. (4.13) and (5.17), we arrive at the β function for g_c^2 in the heterotically deformed $\mathcal{N}=(0,2)\mathbb{CP}^{N-1}$ GLSM,

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi}\frac{N(1+\frac{\gamma_\chi}{2}) - h^2(\gamma_\chi+\gamma_\eta)}{1-\frac{N}{8\pi}g_c^2}. \tag{5.20}$$

Finally, we can compare Eq. (5.23) to the master formula in [6]. In [6], the kinetic term of the fermion χ^i_- (in their notation, it was ψ^i_R) is nonlinearly coupled to the bosonic field ϕ^i. It makes the definition of the wave function renormalizations of the two theories different up to a scale factor g_c^2, i.e.,

$$Z_{\chi\text{here}} = g_c^2 Z_{\chi\text{there}}. \tag{5.21}$$

Therefore it leads us to define

$$h'^2 = h^2 g_c^2 \quad \text{and} \quad \gamma'_\chi = \gamma_\chi - \frac{\beta(g_c^2)}{g_c^2}. \tag{5.22}$$

Under these new definitions, we exactly reproduce the master formula in [6],

$$\beta(g_c^2) = -\frac{g_c^2}{4\pi} \frac{N g_c^2 (1 + \frac{\gamma'_\chi}{2}) - h'^2(\gamma'_\chi + \gamma_\eta)}{1 - \frac{h'^2}{4\pi}}. \tag{5.23}$$

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APPENDIX A: NLSM OF $\mathcal{N} = (0,2)\mathbb{CP}^{N-1}$ MODEL

In this Appendix, we transform the action of the deformed $\mathcal{N} = (0,2)\mathbb{CP}^{N-1}$ model of Sec. V B into the corresponding NLSM version. The NLSM can be obtained by integrating out the gauge multiplet of its GLSM cousin at the energy scale $\mu \ll e$. Then, one can study the model in the geometric formalism. First, by integrating the D term, Eq. (3.7), one finds the potential

$$V_D = \left(\sum_i \bar{\phi}_i \phi^i - \xi\right)^2. \tag{A1}$$

On the level of NLSM, it constrains all bosonic fields on $\mathbb{S}^{2N-1}$; i.e., ϕ^i must satisfy the equation

$$\sum_i \bar{\phi}_i \phi^i - \xi = 0. \tag{A2}$$

On the other hand, integrating the gaugino fields λ_- and $\bar{\lambda}_-$ in Eq. (2.10), we see that the fermion fields ψ_+^i are subject to constraints

$$\sum_i \bar{\phi}_i \psi_+^i = 0, \tag{A3}$$

implying that ψ_+^i's live on the tangent bundle of the manifold. In fact, we can rewrite Eqs. (A2) and (A3) together in terms of superfields,

$$\sum_i \bar{\Phi}_i \Phi^i - \xi = 0. \tag{A4}$$

To obtain the $\mathbb{CP}^{N-1}$ model, we need to also take account of the $U(1)$ gauge imposed on Φ^i's. We can use this gauge to fix one of the chiral multiplet, say, the Nth field Φ^N, to have its bosonic field *real*,

$$\Phi^N = \varphi + \sqrt{2}\theta^+ \kappa_+ + \cdots, \tag{A5}$$

where φ now is a real boson and κ_+ is its superpartner that is still a complex Weyl fermion. Further, we define the gauge invariant coordinates,

$$Z^i = z^i + \sqrt{2}\theta^+ \zeta_+^i \equiv \frac{\Phi^i}{\Phi^N}, \quad \text{for } i = 1, 2, \ldots, \quad N-1, \tag{A6}$$

from which we find

$$z^i = \frac{\phi^i}{\varphi} \quad \text{and} \quad \zeta_+^i = \frac{1}{\varphi}\left(\psi_+^i - \frac{\phi^i}{\varphi}\kappa_+\right). \tag{A7}$$

Now, we can solve for Φ^i in terms of Z^i. From Eq. (A4), we express Φ^N as

$$|\Phi^N|^2 = \frac{\xi}{1 + \bar{Z}_i Z^i}, \tag{A8}$$

or, in components,

$$\varphi = \frac{\sqrt{\xi}}{\sqrt{1 + \bar{z}_i z^i}} \equiv \frac{\sqrt{\xi}}{\rho} \quad \text{and} \quad \kappa_+ = -\frac{\sqrt{\xi}}{\rho^3} \bar{z}_i \zeta_+^i. \tag{A9}$$

We then solve

$$\phi^i = \frac{\sqrt{\xi}}{\rho} z^i \quad \text{and} \quad \psi_+^i = \frac{\sqrt{\xi}}{\rho}\left(\delta_j^i - \frac{1}{\rho^2} z^i \bar{z}_j\right)\zeta_+^j, \quad \text{for } i = 1, 2, \ldots, N-1. \tag{A10}$$

Next, we integrate out the gauge fields A_μ in Eqs. (2.10) and (2.14), and we find

$$A_{++} = \frac{i\xi}{2\rho^2}(\partial_{++}\bar{z}_i z^i - \bar{z}_i \partial_{++} z^i) + i g_{i\bar{j}} \bar{\zeta}_+^{\bar{j}} \zeta_+^i,$$
$$A_{--} = \frac{i\xi}{2\rho^2}(\partial_{--}\bar{z}_i z^i - \bar{z}_i \partial_{--} z^i) + i \bar{\chi}_{-a} \chi_-^a, \tag{A11}$$

where, to distinguish the Fermi multiplet Γ^a from the bosonic one Φ^i, we use the Latin letter a to label them with

$$i = 1, 2, \ldots, N-1 \quad \text{and} \quad a = 1, 2, \ldots, N.$$

Moreover,

$$g_{i\bar{j}} = \frac{\xi}{\rho^2}\left(\delta_{i\bar{j}} - \frac{1}{\rho^2}\bar{z}_i z_{\bar{j}}\right) \tag{A12}$$

is the standard Fubini-Study metric on $\mathbb{CP}^{N-1}$. The bosonic part of the gauge field is, in fact, the $U(1)$ piece of the holonomy group $U(N-1)$ of $\mathbb{CP}^{N-1}$ [34] and couples to the left moving fermion χ_-^a. It implies that the left mover lives on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{CP}^{N-1}$.

Using Eqs. (A9)–(A11), we can recast Eqs. (2.10), (2.14), and (2.17) to obtain the NLSM action

$$\mathcal{S}_{\rm NLSM} = \int d^2x (g_{i\bar{j}}\partial_\mu \bar{z}^{\bar{j}}\partial^\mu z^i + i g_{i\bar{j}}\bar{\zeta}_+^{\bar{j}} \nabla_{--}^{U(N-1)}\zeta_+^i + i\bar{\chi}_{-a}\nabla_{++}^{U(1)}\chi_-^a) + 2(g_{i\bar{j}}\bar{\zeta}_+^{\bar{j}}\zeta_+^i)(\bar{\chi}_{-a}\chi_-^a)), \tag{A13}$$

where

$$\nabla^{U(N-1)}\zeta_+^i \equiv d\zeta_+^i + \Gamma^i_{jk}dz^j\psi_+^k, \quad \text{with} \quad \Gamma^i_{jk} = g^{\bar{l}i}\partial_k g_{j\bar{l}},$$
$$\nabla^{U(1)}\chi_-^a \equiv d\chi_-^a - i\omega\chi_-^a, \quad \text{with} \quad \omega = \frac{i\xi}{2\rho^2}(d\bar{z}_i z^i - \bar{z}_i dz^i). \tag{A14}$$

One can clearly see that unlike the $\mathcal{N} = (2,2)\mathbb{CP}^{N-1}$ case, the deformed model has all its left movers living on $\mathcal{O}(-1)^{\oplus N}$. We remark here that at the level of NLSM, the study of isometry/holonomy anomalies is easy. The $N-1$ right movers ζ_+^i living on a tangent bundle of $\mathbb{CP}^{N-1}$ contribute to the anomaly proportional to the first Chern class of $T\mathbb{CP}^{N-1}$,

$$\mathcal{A}_{\zeta_+} = c_1(T\mathbb{CP}^{N-1}) = \frac{N}{4\pi}d\omega. \tag{A15}$$

On the other hand, the N left movers χ_-^a on $\mathcal{O}(-1)^{\oplus N}$ contribute

$$\mathcal{A}_{\chi_-} = -\frac{N}{4\pi}d\omega. \tag{A16}$$

Therefore, the deformed model is anomaly-free as its GLSM cousin; for more details see [34].

APPENDIX B: SCALING ANOMALIES: TECHNICALITIES

In this Appendix we explain the technique to compute the anomalous Jacobian in Sec. IV B, say, $\mathrm{sTr}_{\Phi^i}\mathbb{1}$ and $\mathrm{sTr}_{\Gamma_-^a}\mathbb{1}$ in Eq. (4.6). A careless treatment of the chiral multiplet $\Phi^i = (\phi^i, \psi_+^i)$ seemingly tells us that

$$\mathrm{sTr}_{\Phi^i}\mathbb{1} = \mathrm{Tr}_{\phi^i}\mathbb{1} - \mathrm{Tr}_{\psi_+^i}\mathbb{1} = 0. \tag{B1}$$

One has to regularize the above supertrace by introducing regulators. To find a proper regulator, it is sufficient to look at the equation of motion of the superfield Φ^i which enters the action $S_{\rm chiral}$ [see Eq. (2.10)],

$$\mathcal{D}_+\mathcal{D}_{--}\Phi^i = \cdots. \tag{B2}$$

We need to further act by $\bar{\mathcal{D}}_+$ to project the operator equation into the half chiral superspace, i.e.,

$$\bar{\mathcal{D}}_+\mathcal{D}_+\mathcal{D}_{--}\Phi^i = \bar{\mathcal{D}}_+(\cdots). \tag{B3}$$

After some algebra, we find

$$\bar{\mathcal{D}}_+\mathcal{D}_+\mathcal{D}_{--}\Phi^i \propto (\nabla_\mu^2 + q_i D)\phi^i + \sqrt{2}\theta^+(\nabla_\mu^2 + iq_i B)\psi_+^i + \cdots. \tag{B4}$$

Therefore, the supertrace Eq. (B1) is regularized as

$$\mathrm{sTr}_{\Phi^i}\mathbb{1} = \lim_{M^2\to\infty}(\mathrm{Tr}_{\phi^i} e^{\frac{1}{M^2}(\nabla_\mu^2 + q_i D)} - \mathrm{Tr}_{\psi_+^i} e^{\frac{1}{M^2}(\nabla_\mu^2 + iq_i B)}). \tag{B5}$$

For trivial fields D and B, the above trace is surely zero. But now let us turn on nonzero but constant D and B backgrounds. We have

$$\begin{aligned}
&\mathrm{Tr}_{\phi^i} e^{\frac{1}{M^2}(\nabla_\mu^2 + q_i D)} \\
&= \int d^2x \left\langle x \middle| e^{\frac{\partial_\mu^2}{M^2}}\left(1 + \frac{1}{M^2}(q_i D + \mathcal{O}(A_\mu)) + \mathcal{O}\left(\frac{1}{M^4}\right)\right) \middle| x \right\rangle \\
&= \frac{1}{4\pi}\int d^2x \left(M^2 + (q_i D + \mathcal{O}(A_\mu)) + \mathcal{O}\left(\frac{1}{M^2}\right)\right), \\
&\mathrm{Tr}_{\psi_+^i} e^{\frac{1}{M^2}(\nabla_\mu^2 + iq_i B)} \\
&= \int d^2x \left\langle x \middle| e^{\frac{\partial_\mu^2}{M^2}}\left(1 + \frac{1}{M^2}(iq_i B + \mathcal{O}(A_\mu)) + \mathcal{O}\left(\frac{1}{M^4}\right)\right) \middle| x \right\rangle \\
&= \frac{1}{4\pi}\int d^2x \left(M^2 + (iq_i B + \mathcal{O}(A_\mu)) + \mathcal{O}\left(\frac{1}{M^2}\right)\right).
\end{aligned} \tag{B6}$$

Therefore, putting $M^2 \to \infty$, we arrive at

$$\mathrm{sTr}_{\Phi^i}\mathbb{1} = \frac{q_i}{4\pi}\int d^2x (D - iB), \tag{B7}$$

or, in superspace,

$$\mathrm{sTr}_{\Phi^i}\mathbb{1} = -i\frac{q_i}{8\pi}\int d^2x d\theta^+ \Upsilon_-|_{\bar{\theta}^+=0}. \tag{B8}$$

Similarly, for Fermi multiplet Γ_-^a, we also impose $\bar{\mathcal{D}}_+\mathcal{D}_+\mathcal{D}_{--}$ upon Γ_-^a and find

$$\begin{aligned}
\bar{\mathcal{D}}_+\mathcal{D}_+\mathcal{D}_{--}\Gamma_-^a &= \bar{\mathcal{D}}_+\mathcal{D}_{--}\mathcal{D}_+\Gamma_-^a + \bar{\mathcal{D}}_+(\bar{\Upsilon}_-\Gamma_-^a) \\
&\propto (\nabla_\mu^2 - i\tilde{q}_a B)\chi_-^a - \sqrt{2}\theta^+(\nabla_\mu^2 - \tilde{q}_a D)G^a \\
&\quad + \bar{\mathcal{D}}_+(\bar{\Upsilon}_-\Gamma_-^a) + \cdots
\end{aligned} \tag{B9}$$

Thus, we regularize the supertrace of the Fermi multiplet as

$$\begin{aligned}
\mathrm{sTr}_{\Gamma_-^a}\mathbb{1} &= \lim_{M^2\to\infty}\left(-\mathrm{Tr}_{\chi_-^a} e^{\frac{1}{M^2}(\nabla_\mu^2 - i\tilde{q}_a B)} + \mathrm{Tr}_{G^a} e^{\frac{1}{M^2}(\nabla_\mu^2 - \tilde{q}_a D)}\right) \\
&= -\frac{\tilde{q}_a}{4\pi}\int d^2x (D - iB) = i\frac{\tilde{q}_a}{8\pi}\int d^2x d\theta^+ \Upsilon_-|_{\bar{\theta}^+=0};
\end{aligned} \tag{B10}$$

cf. Sec. IV A. From Eqs. (B8) and (B10), we establish the relation between canonical coupling τ_c and holomorphic τ in Eq. (4.8), i.e.,

$$\tau_c = \tau + \sum_i i\frac{q_i}{4\pi}\log Z_i(\mu) - \sum_a i\frac{\tilde{q}_a}{4\pi}\log Z_a(\mu). \quad \text{(B11)}$$

We further remark that, as a consistency check, given a complexified $U(1)$ rotation of the chiral or Fermi matter, e.g.,

$$\Phi^i \to e^{\alpha}\Phi^i, \quad \text{(B12)}$$

the anomalous Jacobian takes the form

$$\mathcal{J}(\alpha) = e^{\alpha(\mathrm{sTr}_{\Phi^i}\mathbb{1})} = e^{\alpha\frac{q_i}{4\pi}\int d^2x(D-iB)}. \quad \text{(B13)}$$

For real α, such as the wave function renormalization or a scale transformation, the anomalous Jacobian only gives a correction to the D term, because $\mathrm{Im}\mathcal{J}(\alpha)$ cancels with the contribution from $\bar{\Phi}^i$. It simply signals that fermions do not contribute to the one-loop β function. On the other hand, for imaginary α, it is equivalent to a chiral rotation. We see that $\mathcal{J}(\alpha)$ and its conjugation only contribute to the flux B term, which gives us the correct chiral anomaly from the chiral fermions ψ^i_+ (Sec. IV A).

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