Relaxations of perturbations of spacetimes in general relativity coupled to nonlinear electrodynamics

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Three well-known exact regular solutions of general relativity coupled to nonlinear electrodynamics (NED), namely the Maxwellian, Bardeen, and Hayward regular spacetimes, which can describe either a regular black hole or a geometry without horizons, have been considered. Relaxation times for the scalar, electromagnetic (EM) and gravitational perturbations of black holes and no-horizon spacetimes have been estimated in comparison with the ones of the Schwarzschild and Reissner-Nordström spacetimes. It has been shown that the considered geometries in general relativity coupled to the NED have never-vanishing circular photon orbits, and on account of this fact, these spacetimes always oscillate the EM perturbations with quasinormal frequencies. Moreover, we have shown that the EM perturbations in the eikonal regime can be a powerful tool to confirm (i) that the light rays do not follow null geodesics in the NED by the relaxation rates and (ii) if the underlying solution has a correct weak field limit to the Maxwell electrodynamics by the angular velocity of the circular photon orbit.

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I. INTRODUCTION

On September 14, 2015, the first ever detection of gravitational waves (GWs) from the coalescence of two stellar mass black holes (BHs) by the LIGO Scientific Collaboration led to birth of an entirely new field of astronomy—GW astronomy [1]. Afterward, LIGO and VIRGO scientific collaborations announced the detection of several GWs from the merger of stellar mass BHs [2–5] and neutron stars (NSs) [6]. Ground-based GW detectors such as LIGO and VIRGO have a few-kilometer-long arms and can only observe the GW sources of which the radiation is emitted at frequencies in the deca- and hectahertz band. Therefore, the ground-based GW detectors are sensitive to the coalescence of NSs and stellar mass BHs. On the other hand, GWs at very low frequencies have wavelength larger than the Earth's size as the frequency of the GW is proportional to the inverse of the mass of the source. In this case, these GWs cannot be detected by the ground-based detectors. To detect them, large enough antennas, so-called space-based antennas, away from the Earth's surface are required. Space-based GW detectors, such as Laser Interferometer Space Antenna (LISA) can have million-kilometer-long arms, and they are atmosphere, turbulence, and seismic noise free and can therefore be sensitive in the millihertz band. The space-based GW detectors are expected to be sensitive to coalescences of supermassive BH-BH and supermassive BH-NS [7]. The coalescence of the BHs (NSs) in binary occurs into three phases: inspiral, merger, and ringdown—each of which is calculated by the different methods. The inspiral represents the early evolution of the close binary system, and since the binary components are far enough away from each other, it can be treated by the post-Newtonian approximation by expanding expressions in powers of small relative velocity v/c. In the phase of merger, strong and highly dynamical gravitational fields develop, which can be treated only via numerical relativity simulations. Finally, in the last, merger phase, the final object relaxes to its equilibrium state by radiating GWs of which the frequencies are called quasinormal (QN), since they are complex and subject to decay through the imaginary part. The merger is calculable via perturbation theory (semi)analytically. In the BH perturbations theory, one obtains the wave equation by introducing the linear small perturbation to a fixed BH background spacetime and solving the Einstein equations in the linear order of perturbations. A perturbed BH in its queue goes through the following three stages: transient, quasinormal mode (QNM) ringdown, and power law tail. Where the transient phase strongly depends on the initial perturbations,

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while the ringdown is independent of the initial perturbations. Moreover, the ringdown is characterized by the QNMs which encode information about the BH [8–11].

The above discussion regards only the gravitational perturbations. However, for scalar and electromagnetic (EM) perturbations, also through the typical standard analysis, similar wave equations can be obtained, despite the different underlying physics. The scalar [12–15], gravitational [16,17], and EM [18,19] perturbations of regular BHs have been studied. One of the important properties of the perturbations is the relaxation time, which is defined by the inverse of the imaginary part of the QNMs, $\tau = 1/\omega_i$. In this paper, we aim to study in the eikonal regime the relaxation times of the scalar, electromagnetic, and gravitational perturbations of the Maxwellian, Bardeen, and Hayward regular BHs in general relativity coupled to nonlinear electrodynamics (NED). In the eikonal regime, scalar and gravitational perturbations behave similarly, following the null geodesics of spacetime [20]; however, the EM perturbations of spacetimes in the NED behave differently [18,19], due the fact that in the NED light rays do not follow null geodesics of the spacetime, instead following null geodesics of the optical metric [21–27].

The paper is organized as follows. In Sec. II, we present the regular BH solutions in general relativity coupled to the NED and study the main properties of the spacetime. In Sec. III, the scalar, EM, and gravitational perturbations of regular spacetimes in general relativity coupled to the NED are described, and in the eikonal regime, their propagations and relaxation times are studied in comparison with the ones of the Schwarzschild and Reissner-Nordström (RN) spacetimes. Finally, we summarize our results in Sec. IV. In this paper, we mainly use the geometric units $c = G = 1$ and adopt the $(-,+,+,+)$ convention for the signature of the metric.

II. BACKGROUND

The action of a system of general relativity coupled to nonlinear electrodynamics is given as

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - \mathcal{L}(F)), \tag{1}$$

where g is the determinant of the metric tensor, R is the Ricci scalar, and $\mathcal{L}$ is the Lagrangian density describing the NED theory. $F \equiv F_{\alpha\beta}F^{\alpha\beta}$, with the EM field tensor being $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, with A^α the 4-potential. Since $F_{\alpha\beta}$ is antisymmetric, it has only six nonzero components.

The covariant equations of motion are written in the form

$$G_{\alpha\beta} = T_{\alpha\beta}, \tag{2}$$

$$\nabla_\beta(\mathcal{L}_F F^{\alpha\beta}) = 0, \tag{3}$$

where $T_{\alpha\beta}$ and $G_{\alpha\beta} = R_{\alpha\beta} - Rg_{\alpha\beta}/2$ are the energy-momentum tensor of the NED field and the Einstein tensor, respectively. The energy-momentum tensor of the NED is determined by the relation

$$T_{\alpha\beta} = 2\left(\mathcal{L}_F F^\gamma_\alpha F_{\beta\gamma} - \frac{1}{4}g_{\alpha\beta}\mathcal{L}\right), \tag{4}$$

where $\mathcal{L}_F = \partial_F\mathcal{L}$.

Let us consider the line element of the static, spherically symmetric BH given in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where general relativity (GR) and NED evaluate the lapse function $f(r)$.

In general, the EM 4-potential can be written in the following form,

$$\bar{A}_\alpha = \varphi(r)\delta^t_\alpha - Q_m\cos\theta\delta^\phi_\alpha, \tag{6}$$

where $\varphi(r)$ and Q_m are the electric potential and the total magnetic charge, respectively. Since the construction of the electrically and magnetically charged spacetime solutions have been shown in Refs. [28–30], we do not report the derivation of the solution here; instead, we specify the model of NED and perform the further calculations. In the following, we consider a generic class of magnetically charged regular BH solutions, which is given by the function [28,30]

$$f(r) = 1 - \frac{2Mr^{\mu-1}}{(r^\nu + q^\nu)^{\frac{\mu}{\nu}}}, \tag{7}$$

corresponding to the Lagrangian density

$$\mathcal{L} = \frac{4\mu}{\alpha}\frac{(\alpha F)^{\frac{\nu+3}{4}}}{[1 + (\alpha F)^{\frac{\nu}{4}}]^{1+\frac{\mu}{\nu}}}, \tag{8}$$

where q is the magnetic charge parameter. Here, $\mu \geq 0$ and $\nu > 0$ are dimensionless constants, and the value of μ characterizes the strength of nonlinearity of the EM field. Notice that $\mu = 0$ corresponds to the absence of NED, which reduces the spacetime to the Schwarzschild solution. Also taking $\mu \geq 3$ ensures the regularity of the spacetime everywhere [28]. Finally, M is the gravitational mass. In this framework, several classes of well-known regular BH solutions can be obtained such as (i) $\nu = 1$—Maxwellian solutions that corresponds to the Maxwell field in weak EM field regime, (ii) $\nu = 2$—Bardeen-like solutions, and (iii) $\nu = 3$—Hayward-like solutions. Hereafter, we perform calculations in these three types of regular spacetimes and compare their behavior relative to each other and to the Schwarzschild BH.

The main properties of these spacetimes have been studied in Refs. [28,31], so here we shall mention the most crucial points, such as horizons of the spacetimes, since these are important for our further calculations. The coordinate singularity so-called event horizon of the spacetime is defined by the divergence of the spacetime metric through the g_{rr} component of the spacetime metric, that corresponds to $f(r) = 0$ in our case. When $q = 0$, one recovers the Schwarzschild spacetime in Schwarzschild coordinates, with the coordinate singularity at radius $r = 2M$ and the curvature singularity at $r = 0$. In the NED solutions, the presence of the charge parameter decreases the radius of event horizon and determines the existence of an inner horizon close to the center. With the increasing value of the charge parameter, the outer horizon's location decreases, while the inner horizon's location increases. For a specific value of q, we obtain an extreme value for the horizon radius where the two horizons coincide. This corresponds to the solution of equations $f(r) = 0$ and $f'(r) = 0$. For values of the charge parameter above the extremal one, both horizons disappear, and the spacetime no longer represents a BH; instead, it represents no-horizon spacetime. By solving $f = 0 = f'$ for $\mu = 3$, we find two equations,

$$(r^\nu + q^\nu)^{3/\nu} - 2Mr^2 = 0, \qquad r^\nu - 2q^\nu = 0. \tag{9}$$

By solving them simultaneously, we find the values $r_{\rm ext}$ and $q_{\rm ext}$ that denote the boundary of the BH case with the horizonless case. These values are:

(i) Maxwellian BH: $(16/27 \approx 0.5926, 8/27 \approx 0.2963)$;
(ii) Bardeen BH: $(4\sqrt{2}/\sqrt{27} \approx 1.0887, 4/\sqrt{27} \approx 0.7698)$;
(iii) Hayward BH: $(4/3 \approx 1.3333, 4/(3\times 2^{1/3}) \approx 1.0583)$,

In Fig. 1, the boundaries of BH and no-horizon Maxwellian, Bardeen, and Hayward regular spacetimes are presented for the case of $\mu = 3$.[1]

Thus, possible values of the charge parameter for the spacetimes (7) with $\mu = 3$ to represent the BHs lay in the following ranges: for a Maxwellian BH, $q/M \in [0, 0.2963]$; for a Bardeen BH, $q/M \in [0, 0.7698]$; and for a Hayward BH, $q/M \in [0, 1.0583]$. Since these ranges are different, to facilitate the comparison, we normalize the charge parameter as $Q_n \equiv q/q_{\rm ext}$, and for the BH regime of the spacetime, Q_n lies in the range $Q_n \in [0, 1]$. $Q_n \in [1, \infty)$ corresponds to the no-horizon spacetimes.

One of the astrophysically important orbits around BHs is light rings (photon spheres). It is well-known fact that in the LED and other NED nonrelated spacetimes light ray always follows the null geodesics and the radius, r_c, of circular null geodesics (CNG) of the spacetime (5) is determined by solving equation $\mathrm{CNG} \equiv 2f(r_c) - r_c f'(r_c) = 0$. In our case, it takes the form

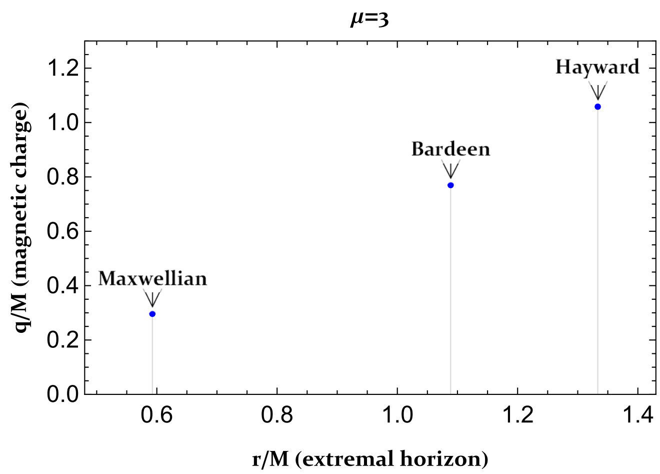

FIG. 1. Boundaries between the Maxwellian, Bardeen, and Hayward regular BHs and the no-horizon spacetimes in the parametric space. The blue points correspond to the extremal BHs $(r_{\rm ext}/M,\ q_{\rm ext}/M)$. Shaded regions represent the possible values of the magnetic charge parameter for the spacetime to represent BHs.

$$Mr_c^{\mu-1}(q_c^\nu + r_c^\nu)^{-\frac{\mu}{\nu}-1}[(\mu-3)q_c^\nu - 3r_c^\nu] + 1 = 0. \tag{10}$$

The radius r_c can be considered as the one of the circular massless neutrino orbits [32]. As in the case of the event horizon, the presence of the charge parameter evaluates the inner and outer CNG orbits in regular spacetimes, and with increasing the value of charge parameter the inner and outer CNGs approach each other and before disappearing they coincide at the extremal CNG. The extremal CNG is determined by solving the equations $\mathrm{CNG} = 0$ and $\mathrm{CNG}_{,r} = 0$, simultaneously. The extremal CNG is located at the following coordinates of parametric space $[r_{c/{\rm ext}}/M, q_{c/{\rm ext}}/M(Q_n)]$,

(i) Maxwellian spacetime: (0.9492,0.3164(1.0679));
(ii) Bardeen spacetime: (1.7173,0.8586(1.1154));
(iii) Hayward spacetime: (2.0833,1.2183(1.1513)),

and they follow the relation

$$r_{c/{\rm ext}} = \left(\frac{\mu(\nu+4) + \sqrt{\mu}\sqrt{\mu(\nu+2)^2 + 4\nu(\mu-3)} - 6}{6}\right)^{1/\nu} \times q_{c/{\rm ext}}. \tag{11}$$

Since we are mainly focusing our attention on the minimal value $\mu = 3$ that makes the spacetimes regular then, expression (11) takes a more compact form as

$$r_{c/{\rm ext}} = (\nu+2)^{1/\nu} q_{c/{\rm ext}}. \tag{12}$$

By comparing the values given above with the ones of the extremal horizons, or seeing Fig. 2, one can make sure that even no-horizon spacetime can possess the circular null geodesics, for a limited range of values of the charge parameters as:

[1]Boundaries of the Maxwellian regular BHs and no-horizon spacetimes for different values of μ were studied in Ref. [19].

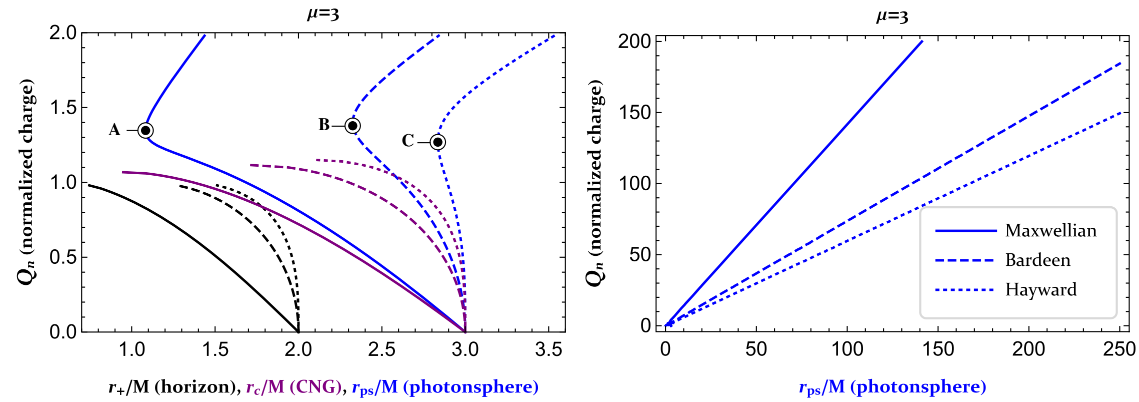

FIG. 2. Left panel: dependence of radii of event horizon (black), circular null geodesics (purple), and light ring (blue) of the Maxwellian ($\nu = 1$, solid), Bardeen ($\nu = 2$, dashed), and Hayward ($\nu = 3$, dotted) regular BHs on the normalized charge parameters. Points A, B, and C correspond to minimal radii of photon spheres in corresponding spacetimes. Right panel: radii of the light rings in the Maxwellian, Bardeen, and Hayward spacetimes for the large value of the normalized charge parameter.

(i) Maxwellian no-horizon spacetime: $Q_n \in (1, 1.0679]$ (or $q/M \in (0.2963, 0.3164])$ at $r_c/M \in [0.9492, 1.3731)$;
(ii) Bardeen no-horizon spacetime: $Q_n \in (1, 1.1154]$ (or $q/M \in (0.7698, 0.8586])$ at $r_c/M \in [1.7173, 2.3012)$;
(iii) Hayward no-horizon spacetime: $Q_n \in (1, 1.1513]$ (or $q/M \in (1.0583, 1.2183])$ at $r_c/M \in [2.0833, 2.6524)$;
(iv) RN naked singularity spacetime: $Q_n \in (1, 1.0607]$ (or $q/M \in (1, 1.0607])$ at $r_c/M \in [1.5, 2)$.

However, in the NED, light rays do not follow the null geodesics of the original metric; instead, they propagate along the null geodesics of the effective (or optical) metric, which is given by [21,22,33]

$$ds^2 = -\frac{1}{\mathcal{L}_F}\left[f(r)dt^2 - \frac{dr^2}{f(r)}\right] + \frac{r^2}{\Phi}d\Omega^2, \qquad (13)$$

where $\Phi = \mathcal{L}_F + 2F\mathcal{L}_{FF}$. Thus, the photon sphere of spacetime (5) is located at the unstable circular null geodesics of the metric (13) that is determined by solving equation

$$\left(\frac{r^2}{\Phi}\right)'_{ps}\frac{f_{ps}}{\mathcal{L}_{Fps}} - \frac{r^2_{ps}}{\Phi_{ps}}\left(\frac{f}{\mathcal{L}_F}\right)'_{ps} = 0. \qquad (14)$$

Let us write Eq. (14) for considered spacetimes. For the Maxwellian spacetime with $\mu = 3$, $\nu = 1$,

$$12M - \frac{28Mq}{r} + \frac{(6q^2 + 7qr - 4r^2)(q+r)^3}{r^4} = 0. \qquad (15)$$

For the Bardeen spacetime $\mu = 3$, $\nu = 2$,

$$18M - \frac{52q^2}{r^2} + \sqrt{\frac{q^2}{r^2}+1}\left(\frac{8q^6}{r^5} + \frac{24q^4}{r^3} + \frac{10q^2}{r} - 6r\right) = 0. \qquad (16)$$

For the Hayward spacetime $\mu = 3$, $\nu = 3$,

$$24M - 8r + \frac{10q^9}{r^8} + \frac{39q^6}{r^5} + \frac{21q^3(r-4M)}{r^3} = 0. \qquad (17)$$

Despite that Eqs. (15)–(17) are analytically not solvable, it is not difficult to check that for all values of q they always have at least one real zero (see the right panel of Fig. 2). In Fig. 2, the loci of the characteristic orbits are depicted. One can see that, unlike the case of standard LED or other NED not related spacetimes, in the considered regular spacetimes given by the line element (5) with metric function (7) and $\mu = 3$, $\nu = 1$, 2, 3, the GR coupled to the NED theory (8) always gives a nonvanishing circular photon orbit. It must be noted that different radial coordinates denote different radial distances in different spacetimes. Therefore, in principle, we cannot directly compare the values of CNGs for different metrics. In the cases under consideration here, radial distances are determined by the line elements (5) and (13), which depend upon the parameters q and ν and reduce to known metrics in the limits of vanishing q. By evaluating the area of the surfaces of revolution for $t = \text{cont}$ and $r = r_{0ps}$, it is easy to verify that for nonvanishing values of q the radii of CNGs identify indeed spherical 2-surfaces for every constant t slice. Also, since such areas monotonically increase with q, we know that the behavior shown in Fig. 2 is qualitatively valid. The existence of circular photon orbits in all the Bardeen spacetimes was first demonstrated in Ref. [32]. Here, we have shown that the Maxwellian, Bardeen, and Hayward regular BH and no-horizon spacetimes in the NED model (8) have this property. One can see from Fig. 2 that with increasing the values of the charge parameters, the radii of the photon spheres decrease until the values that are given by the points A, B, C, then they start to increase again. Therefore, these points correspond to minimal radii of photon spheres in corresponding spacetimes. Thus, the minimal photon spheres of the regular Maxwellian, Bardeen, and Hayward spacetimes are located at

(i) $A(r_{0ps}/M, Q_n) = (1.0834, 1.3468)$,
(ii) $B(r_{0ps}/M, Q_n) = (2.3251, 1.3986)$,
(iii) $C(r_{0ps}/M, Q_n) = (2.8355, 1.2690)$,

and they correspond to the no-horizon spacetimes. Interestingly, we see that for a range of coordinate radii $r \in (r_{0ps}, 3M]$, there may exist two photon spheres with the same radius for different values of the charge parameter.

III. PERTURBATIONS OF SPACETIMES IN GENERAL RELATIVITY COUPLED TO NED

It is known that most of the problems concerning the perturbations of BHs can be reduced to a second order partial differential equation after decoupling of angular variables and considering the perturbations as harmonically time dependent, in the following form:

$$\left(\frac{\partial^2}{\partial x^2} + \omega_j^2 - V_j(r)\right)\Psi_j(r) = 0. \tag{18}$$

Where j stands for sc (scalar), em (EM) and gr (gravitational) perturbations, and x is the tortoise coordinate that is defined as $dx = dr/f$. Let us present the explicit forms of potentials of scalar V_{sc}, electromagnetic V_{em}, and gravitational V_{gr} perturbations of the BHs in the NED, which are given separately with brief explanations in Refs. [14,34].

A. Scalar perturbations

Since the scattering potential of the test scalar field in the field of the spherically symmetric BHs is presented in Ref. [14], we will only provide the potential:

$$V_{sc} = f\left[\frac{\ell(\ell+1)}{r^2} + \frac{f'}{r}\right]. \tag{19}$$

B. Gravitational and EM perturbations

Both the axial and polar EM perturbations of the BHs in the NED that have been studied in our preceding papers [18,19] are just special cases of the gravitational perturbation due to the fact that the EM one was neglected. Here, we briefly give the general case where both perturbations are taken into account.

The EM perturbation of the magnetically charged (with the 4-potential $\bar{A}_\phi = -Q_m\cos\theta$) BH in the NED is given as $A_\phi = \bar{A}_\phi + \delta A_\phi$, where

$$\delta A_\phi = \psi(r)e^{-i\sigma t}\sin\theta\partial_\theta P_k(\cos\theta), \tag{20}$$

with $\sigma = \omega_{em}$ and k is a multipole number of the EM perturbations, which is restricted by the condition $k \geq 1$. The gravitational perturbation in the "Regge-Wheeler" gauge is introduced as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ * & 0 & 0 & h_1(r) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} e^{-i\omega t}\sin\theta\partial_\theta P_\ell(\cos\theta). \tag{21}$$

We insert the perturbed metric and EM 4-potential from Eqs. (20)–(21) into the Einstein and Maxwell equations in the equations of motion (2) and (3) and expand to first order in the perturbations. Thus, for the gravitational perturbations, we obtain the following equations,

$$h_0'' + i\omega h_1' + i\omega\frac{2h_1}{r} - \frac{h_0[\lambda + 2f + (r^2f')' + \bar{\mathcal{L}}r^2]}{r^2 f} = 0, \tag{22}$$

$$i\omega h_0' - i\omega\frac{2h_0}{r} - \omega^2 h_1 + h_1\frac{f[\lambda + (r^2f')' + \bar{\mathcal{L}}r^2]}{r^2} = 0, \tag{23}$$

$$i\omega h_0 = -f(h_1 f)', \tag{24}$$

with

$$\lambda = (\ell+2)(\ell-1). \tag{25}$$

By eliminating h_0 from Eq. (23) by using Eq. (24), we arrive at the master equation (18) for the gravitational perturbations with the potential

$$V_{gr} = f\left[\frac{\ell(\ell+1)}{r^2} + \frac{r(rf')' + 2(f-1)}{r^2} + \bar{\mathcal{L}}\right], \tag{26}$$

by introducing the following notation:

$$\Psi_{gr} = \frac{f}{r}h_1. \tag{27}$$

Let us now consider propagation of the EM perturbation (20) in the perturbed spacetime (21). The equation that governs the EM perturbation (3) with both perturbations (20) and (21) appears to be independent of the gravitational perturbations in the linear order expansion. The dynamics of the EM perturbation of the BHs in general relativity coupled to the NED without gravitational perturbations was studied in our previous papers [18,19]. Therefore, we report only the potential without giving the details of derivation as[2]

[2]Since the EM perturbation is independent of the gravitational one, in the linear order expansion, one can replace the multipole number of the EM perturbation k with ℓ.

$$V_{em} = f\left[\frac{\ell(\ell+1)}{r^2}\left(1+\frac{4Q_m^2\bar{\mathcal{L}}_{\bar{F}\bar{F}}}{r^4\bar{\mathcal{L}}_{\bar{F}}}\right) - \frac{f\bar{\mathcal{L}}_{\bar{F}}'^2 - 2\bar{\mathcal{L}}_{\bar{F}}(f\bar{\mathcal{L}}_{\bar{F}}')'}{4\bar{\mathcal{L}}_{\bar{F}}^2}\right], \tag{28}$$

where

$$\Psi_{em} = \sqrt{\mathcal{L}_F}\psi_1. \tag{29}$$

C. Eikonal regime

In the large multipole numbers limit, the potentials (19), (26), and (28) take the following forms:

$$V_{gr} = V_{sc} = f\frac{\ell^2}{r^2} + O(\ell), \tag{30}$$

$$V_{em} = f\frac{\ell^2}{r^2}\left(1+\frac{4Q_m^2\bar{\mathcal{L}}_{\bar{F}\bar{F}}}{r^4\bar{\mathcal{L}}_{\bar{F}}}\right) + O(\ell). \tag{31}$$

It is known that in the eikonal (large multipole number) regime the QNMs of all perturbations of any stationary, spherically symmetric, and asymptotically flat black holes in any dimensions are characterized by the parameters of the circular null geodesics [20]; namely, the real part of the QNMs is determined by the angular velocity of the unstable null geodesics Ω_c, while the imaginary part of the QNMs is determined by the instability timescale of the orbit, the so-called Lyapunov exponent, λ, as

$$\omega = \Omega_c\ell - i\left(n+\frac{1}{2}\right)|\lambda|, \tag{32}$$

where Ω_c and λ are determined by the spacetime metric (5) as

$$\Omega_c = \sqrt{\frac{f_c}{r_c^2}}, \tag{33}$$

$$\lambda = \sqrt{-\frac{r_c^2}{2f_c}\left(\frac{d^2}{dx^2}\frac{f}{r^2}\right)\bigg|_{r=r_c}}, \tag{34}$$

where r_c is the radius of the unstable null circular orbit which is determined by solving the equation $r_c f_c' - 2f_c = 0$. However, as was mentioned in Refs. [18,19], the relation (32) is not a universal feature of all stationary, spherically symmetric, and asymptotically flat black holes in any dimensions, as it is not satisfied in several cases such as in the EM perturbations of BHs in NED [18,19] and the gravitational perturbations of BHs in the Einstein-Lovelock theory [35,36].

Thus, from the form of the potential (30), one can deduce that in the eikonal regime the scalar and gravitational perturbations of BHs and no-horizon spacetimes in general relativity coupled to NED behave similarly and propagate along null geodesics. Their oscillations and damping rates (or relaxation time) are characterized by the unstable circular null geodesics. At the same time, as has already been pointed out, the EM perturbations follow the trajectory of light rays or null geodesics of the effective metric (13). Since in the Maxwellian, Bardeen, and Hayward spacetimes of GR with NED there are always nonvanishing circular photon orbits irrespective of the value of the mass and charge parameters, in the eikonal regime, BHs and no-horizon spacetimes always oscillate EM perturbations with QN frequencies—see Figs. 3 and 4.

Since the Maxwellian, Bardeen, and Hayward solutions of (8) reduce to the Schwarzschild one when the charge parameter is set to zero, as $f_{\rm NED}(Q\to 0) = f_{\rm Schw} \equiv 1\text{–}2M/r$, all further calculations performed on these

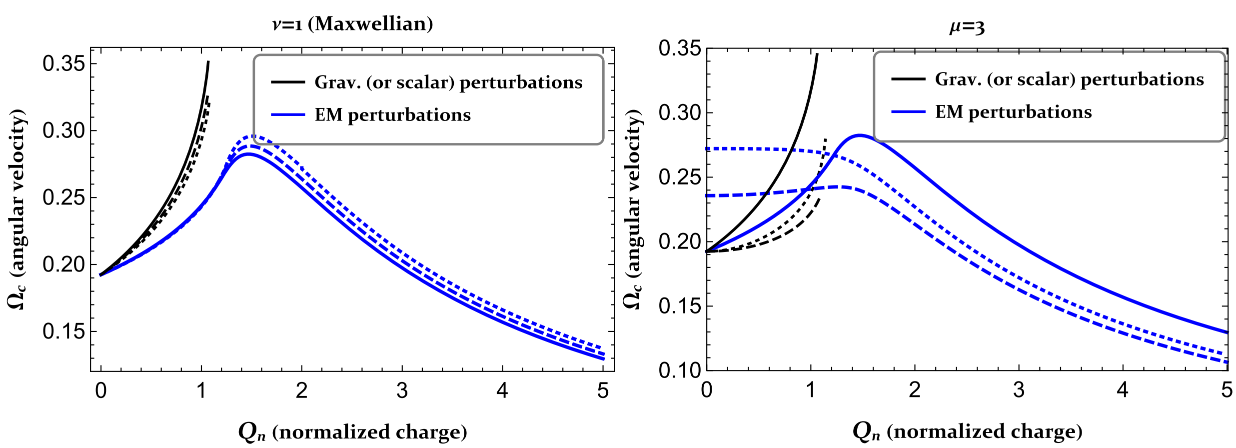

FIG. 3. Dependence of angular velocities of the circular null geodesics (black) and photon orbit (blue, thick) in the generic class of spacetimes (5) with metric function (7) in general relativity coupled to the NED from normalized charge parameters. Left panel: the Maxwellian regular spacetimes ($\nu = 1$) with different values of μ as $\mu = 3$—solid, $\mu = 5$—dashed, $\mu = 12$—dotted curves. Right panel: the Maxwellian regular spacetimes ($\mu = 3$, $\nu = 1$), solid; the Bardeen regular spacetimes ($\mu = 3$, $\nu = 2$), dashed, and the Hayward regular spacetimes ($\mu = 3$, $\nu = 3$), dotted curves.

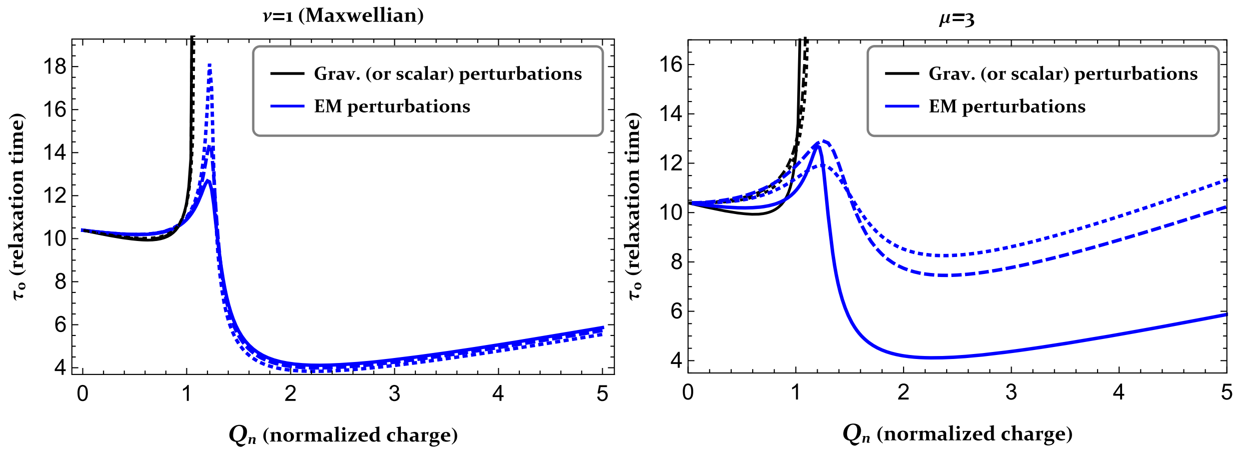

FIG. 4. Relaxation times of the gravitational (and scalar) (black) and EM (blue, thick) perturbations of a generic class of spacetimes (5) with metric function (7) in general relativity coupled to the NED on normalized charge parameters in the large multipole numbers limit. Left panel: the Maxwellian regular spacetimes ($\nu = 1$) with different values of μ as $\mu = 3$, solid; $\mu = 5$, dashed; and $\mu = 12$, dotted curves. Right panel: the Maxwellian regular spacetimes ($\mu = 3$, $\nu = 1$), solid; the Bardeen regular spacetimes ($\mu = 3$, $\nu = 2$), dashed; and the Hayward regular spacetimes ($\mu = 3$, $\nu = 3$), dotted curves.

spacetimes must coincide with the one that corresponds to the Schwarzschild spacetime in that limit, i.e.,

$$A_{\mathrm{NED}}(Q_n \to 0) = A_{\mathrm{Schw}}, \tag{35}$$

where A is any physical quantity. However, at $Q_n = 0$, the angular velocities of the circular photon orbits in the Bardeen and Hayward spacetimes are different, and they do not match the ones of the Maxwellian BH, which coincides with the one of the Schwarzschild BH (see the right panel of Fig. 3), as they have the following limits:

$$\Omega_{\mathrm{NED}}(Q_n \to 0) = \sqrt{\frac{\nu+1}{2}}\Omega_{\mathrm{Schw}}. \tag{36}$$

By comparing (35) and (36), one easily realizes that only the Maxwellian spacetimes $\nu = 1$ have the correct Schwarzschild limit. This confirms once more the fact that the Bardeen and Hayward solutions have the wrong limit at the weak field limit [37,38]. It was shown in our previous papers [18,19] that from the imaginary part of the eikonal QNMs of the EM perturbations of BHs in NED one can verify that light rays do not follow null geodesics of the spacetime. Here, we have shown that from the real part of the eikonal QNMs of the EM perturbations of spacetimes in NED one can verify if the solution (or equivalently the NED model) has the correct behavior in the weak EM field limit.

Let us analyze the eikonal QNMs of the Maxwellian, Bardeen, and Hayward spacetimes which are presented in Figs. 3 and 4. In the eikonal regime, the spacetimes (7) always oscillate the gravitational (scalar) perturbations with bigger real frequency of QNMs than the Schwarzschild one $[\omega_r(Q_n \neq 0) > \omega_r(Q_n = 0)]$. The Maxwellian spacetime is the most favored to oscillate gravitational (scalar) perturbation with bigger real frequency, rather than the Hayward spacetime, while the Bardeen one is the least favored. Moreover, the Maxwellian spacetimes with smaller μ ($\mu \geq 3$) are always better oscillators than the ones with bigger μ.

We consider now the relaxation times of the perturbations in the eikonal regime. In Fig. 4, the relaxation times of the fundamental (the least damped) mode of the gravitational (and scalar) and EM perturbations are presented. In the left panel, the Maxwellian spacetimes with different μ are shown, while in the right panel, the Maxwellian, Bardeen, and Hayward spacetimes with $\mu = 3$ have been plotted. One can see from the figures that the relaxation times of perturbations of the Maxwellian spacetimes do not depend strongly on the parameter μ. Moreover, the relaxation times of the gravitational (and scalar) perturbations of the regular Maxwellian, Bardeen, and Hayward BHs are in the similar intermediate ranges, while the ones of the no-horizon spacetimes diverge to infinity at the values which correspond to extreme values of the circular null geodesics. On the other hand, the relaxation times of the EM perturbations of these spacetimes qualitatively behave similarly, but quantitatively, their differences are significant in the no-horizon spacetimes; i.e., the Hayward no-horizon spacetime oscillates the EM perturbations with the least damping, while the Maxwellian no-horizon spacetime has the fastest relaxation rate.

In Table I, we present the above-discussed features of the relaxation times of the perturbations of the regular spacetimes in unit of *seconds* in comparison with the ones of the Schwarzschild and RN spacetimes. To write the dimensionful relaxation time in Table I from the dimensionless one in Fig. 4, one uses the following relation:

TABLE I. Relaxation times of the gravitational (and scalar) and EM perturbations of the regular spacetimes in comparison with the ones of the RN and Schwarzschild spacetimes in units of $[(M/M_\odot)]$ microseconds. **CC** stands for the corresponding charge. Note that * indicates that in the paper [39] by Hod it was shown that in the eikonal regime the RN BH with $Q_n \approx 0.73$ has the fastest relaxation rate.

Spacetimes	Shortest Gravitational	CC	Longest Gravitational	CC	Shortest EM	CC	Longest (local) EM	CC
Schwarzschild BH	51.0262	0	51.0262	0	51.0262	0	51.0262	0
RN BH*	50.0378	0.73	55.5503	1	50.0378	0.73	55.5503	1
RN naked singularity	$\gtrsim$55.5503	$\gtrsim$1	∞	1.06	$\gtrsim$55.5503	$\gtrsim$1	∞	1.06
Maxwellian BH	48.7728	0.62	57.5190	1	50.0385	0.52	53.4715	1
Maxwellian no horizon	$\gtrsim$57.5190	$\gtrsim$1	∞	1.07	20.1975	2.25	62.3036	1.2
Bardeen BH	$\gtrsim$51.0262	$\gtrsim$0	63.0313	1	$\gtrsim$51.0262	$\gtrsim$0	58.3588	1
Bardeen no horizon	$\gtrsim$63.0313	$\gtrsim$1	∞	1.11	36.6051	2.4	63.1749	1.24
Hayward BH	$\gtrsim$51.0262	$\gtrsim$0	61.0489	1	$\gtrsim$51.0262	$\gtrsim$0	55.8758	1
Hayward no horizon	$\gtrsim$61.0489	$\gtrsim$1	∞	1.15	40.5379	2.35	58.4751	1.24

$$\tau_{\mathrm{ful}} = \frac{GM}{c^3}\tau_{\mathrm{less}} \approx 4.92\times 10^{-6}\tau_{\mathrm{less}}\left(\frac{M}{M_\odot}\right)\ \mathrm{sec}. \tag{37}$$

Since the RN spacetimes are a solution of general relativity coupled with linear electrodynamics (LED), the EM perturbations follow null geodesics in the same manner as the scalar and gravitational ones. Therefore, in Table I, relaxation times of the scalar and gravitational perturbations are identical.

The relaxation times of the nonfundamental modes are easily determined from the relation [40]

$$\tau_n = \frac{\tau_0}{2n+1}. \tag{38}$$

IV. CONCLUSION

In this paper, we have studied scalar, electromagnetic, and gravitational perturbations of spacetimes in general relativity coupled to the NED. Specifically, we have chosen a generic model of NED from which the Maxwellian (i.e., corresponding to the Maxwell field in the weak field limit), Bardeen, and Hayward solutions can be obtained as special cases. In NED, light rays do not follow null geodesics of the given spacetime; instead, they follow null geodesics of the optical metric. We have shown for the first time that in the Maxwellian, Bardeen, and Hayward spacetimes there is always at least one nonvanishing radius for the circular photon orbit around a central gravitating object, while the existence of the circular null geodesics of the spacetime is restricted by the spacetime parameters. These play a fundamental role in the propagation and relaxation periods of the scalar, EM, and gravitational perturbations in the eikonal regime. To be more precise, since in the large multipole numbers limit scalar and gravitational perturbations follow the null geodesics of the spacetimes, they behave similarly; therefore, they are indistinguishable from the characteristic frequencies of the perturbations. On the other hand, the EM perturbations follow the light ray trajectory, and due to the fact that in the Maxwellian, Bardeen, and Hayward spacetimes there is always a nonvanishing circular photon orbit, even no-horizon spacetimes always oscillate the EM perturbations with QNMs.

Moreover, we have shown that the EM perturbations in the eikonal regime can be a powerful tool to confirm (i) (ii) if the underlying solution has a correct weak field limit to the Maxwell electrodynamics by the angular velocity of the circular photon orbit.

We have shown that the relaxation times of gravitational (and scalar) and EM perturbations of the regular Maxwellian, Bardeen, and Hayward BHs are very similar. However, in the horizonless case, they behave differently. Interestingly, the RN naked singularity and regular no-horizon spacetimes with the extreme circular null geodesics oscillate the gravitational (and scalar) perturbations with normal modes without damping; i.e., the scalar and gravitational perturbations of these spacetimes never come back to equilibrium. However, the EM perturbations always have damping, and they come to relaxation faster than the gravitational ones. Moreover, the relaxation times of the EM perturbations of these spacetimes show qualitatively similar behavior, but quantitatively, their differences become significant in the horizonless spacetimes. In other words, the Hayward no-horizon spacetime oscillates the EM perturbations with the least damping, while the Maxwellian no-horizon spacetime is the most favored in terms of fastness of the relaxation rate.

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