Black hole geodesic parallel transport and the Marck reduction procedure

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The Wigner rotations arising from the combination of boosts along two different directions are rederived from a relative boost point of view and applied to study gyroscope spin precession along timelike geodesics in a Kerr spacetime. First this helps to clarify the geometrical properties of Marck's recipe for reducing the equations of parallel transport along such world lines expressed in terms of the constants of the motion to a single differential equation for the essential planar rotation. Second this shows how to bypass Marck's reduction procedure by direct boosting of orthonormal frames associated with natural observer families. Wigner rotations mediate the relationship between these two descriptions for reaching the same parallel transported frame along a geodesic. The comparison is particularly straightforward in the case of equatorial plane motion of a test gyroscope, where Marck's scalar angular velocity captures the essential cumulative spin precession relative to the spherical frame locked to spatial infinity. These cumulative precession effects are computed explicitly for both bound and unbound equatorial plane geodesic orbits. The latter case is of special interest in view of recent applications to the dynamics of a two-body system with spin. Our results are consistent with the point-particle limit of such two-body results and also pave the way for similar computations in the context of gravitational self-force.

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I. INTRODUCTION

Given any two forward pointing timelike unit vectors u_1 and u_2 in the tangent space to a Lorentzian spacetime (signature $+2$), which may be interpreted as the 4-velocities of a pair of test observers at that event, there is a unique active Lorentz transformation $B(u_2, u_1)$ which takes one (u_1) into the other (u_2), termed a relative observer boost, acting only in the plane of the two vectors as a hyperbolic rotation. This boost is most conveniently parametrized by the hyperbolic rotation angle, often called the rapidity $\beta = \operatorname{arccosh}(-u_1 \cdot u_2)$. For boosting between three such successive 4-velocities in the same plane, the rapidity parameters are additive, but when their 4-velocities are not coplanar, two successive relative observer boosts are equivalent to a single such boost followed by a rotation, called the Wigner rotation [1], due to the fact that the boost generators of the Lorentz group do not form a closed Lie subalgebra, but their commutators lead to rotation generators. All of the calculations in this case involve only special relativity. Furthermore, to combine boosts with the Lorentz group multiplication law, they should all be referred to a common time direction, say, u, i.e., $B(u_2, u)B(u_1, u)X$ when applied to some spacetime vector X [2–4].

The Wigner rotation is intimately connected with the Thomas precession effect in special relativity, most notably studied for a classical spinning electron in a circular orbit [5–14]. The Thomas precession is a dynamical expression of the instantaneous Wigner rotation effect, in the context of the succession of boosts from the laboratory frame to the particle rest frame along the changing direction of the particle trajectory. This takes a slightly different form in the general relativistic analysis of spin precession of a test gyroscope in a given curved spacetime through Fermi-Walker transport of the spin four-vector, but the Wigner rotation and a generalized Thomas precession remains important. Along geodesic world lines, Fermi-Walker transport reduces to parallel transport.

In the study of the precession of the parallel transported spin vector of a test gyroscope moving along a geodesic orbit in a rotating Kerr black hole spacetime, the key to geodesic motion and parallel transport is the Carter orthonormal frame [15] in Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$, which together with the Killing vectors ∂_t and ∂_ϕ, symmetric Killing 2-tensor $K_{\alpha\beta}$ and Killing-Yano 2-form $f_{\alpha\beta}$ leads to the separability of the geodesic equations and the reduction of the equations of parallel transport along geodesics expressible entirely in terms of the constants of the motion to a single first order differential equation for the essential planar rotation. The Carter orthonormal frame is boosted along the azimuthal direction

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associated with the rotational Killing vector field compared to the usual static observer frame, with the latter observers having world lines which are the time lines of the Boyer-Lindquist coordinate system. The Carter frame "corotates" with the black hole and is well defined everywhere outside its outer horizon. An orthonormal frame is said to be adapted to an observer if its timelike member is the four-velocity of that observer.

The Carter frame differs only by a boost from the more familiar slicing and threading "spherical" orthonormal frames associated with the Boyer-Lindquist coordinate system. The slicing frame or zero-angular-momentum-observer (ZAMO) frame is obtained by normalizing the orthogonal spatial coordinate frame to a spherical orthonormal triad and completing it by adding the unit normal n to the time coordinate hypersurfaces. These observers are also sometimes called the "locally nonrotating observers." The threading frame adapted to the so-called static observers following the time coordinate lines (when timelike, having four-velocity m) differs only by a boost in the 2-plane of the two Killing vector fields tangent to the cylinders of the t-ϕ coordinates—namely, a boost along the azimuthal direction. Similarly the Carter frame also differs only by a boost in that same 2-plane.

Each of these boosts just reflects how different observers in relative rotation around the azimuthal direction see the same orthonormal triad of vector fields rigidly attached to the coordinate system, which in turn is rigidly connected to the "distant stars" (spatial infinity) in the sense that the threading observers (sometimes called the "distantly nonrotating observers") see the incoming light rays from those distant stars forming a time-independent pattern on the celestial sphere. Since the spin vector along any timelike world line remains in the local rest space of that world line, one can also boost the spherical triad to that local rest space in order to measure the rotation of the spin with respect to the triad of vectors as seen by an observer following that world line. In a sense this subtracts the spherical aberration of the incoming light rays from spatial infinity on the local celestial sphere. Whether one boosts one of the other boosted spherical frames or performs a direct boost from the static frame to the local rest space of a gyro, the results differ only by orientation induced Wigner rotations reflecting the relative tilting of the local rest spaces and do not add to any accumulating angle of precession.

The local rest space of the gyro's geodesic world line is related to the Carter frame by two successive relative observer boosts at right angles, first along the radial direction which preserves the radial alignment of the electric and magnetic parts of the Killing-Yano 2-form while eliminating the radial relative velocity of the gyro, and then along an orthogonal angular direction to reach the local rest space of the geodesic by eliminating its angular velocity. Thus a sequence of three successive relative observer boosts, each at right angles to each other, takes the static observer frame to a frame aligned with the geodesic four-velocity. This geodesic frame is rotated with respect to a direct relative observer boost of the static observer axes to the gyro local rest space, which are an important comparison reference frame with respect to which the spin angular velocity precession has the stellar aberration effect subtracted away. Each pair of successive boosts leads to a Wigner rotation with respect to the equivalent direct boost. The result of the three successive boosts is then rotated with respect to the direct boost. One can evaluate each of these rotations.

Starting from the Carter frame, Marck's construction [16,17] of a parallelly propagated frame along a timelike geodesic with four-velocity U utilizes the electric part $f \llcorner U$ of the Killing-Yano 2-form as seen by the test observer following the gyro. This vector is parallel transported along the world line and defines the parallel transported normal direction to the 2-plane of the parallel transport rotation within the local rest space LRS_U of the gyro [18]. The rest of his construction uses a sequence of boosts along the radial and angular directions to determine a natural pair of orthonormal vectors in this plane to express the angular velocity of the parallel transport rotation within it. Ultimately it is the comparison of this angular velocity of rotation with the local static Cartesian frame associated with the spherical frame which allows one to extract the spin precession with respect to the distant stars. Absent the condition of stationarity which allows a connection between the "local sky" and the "distant sky," one only has spin transport via local Fermi-Walker transport without the possibility of comparing it to a preferred local reference frame.

We begin by rederiving the Wigner rotation associated with two successive boosts. We then use this formalism to relate various adapted orthonormal frames aligned with and defined along a given world line by boosting those associated with special observer frame fields existing in any stationary axisymmetric spacetime. Such adapted frames can be conveniently rotated within the local rest space of the world line in order to undergo special transport laws along the orbit, e.g., parallel transport for geodesics or Fermi-Walker transport for accelerated world lines.

We illustrate our approach by analyzing the well-known case of parallel transport along geodesic orbits in the Kerr spacetime, where the integrability of the geodesic equations further allows results to be expressed in terms of the constants of the motion. For simplicity we analyze in detail the simpler case of equatorial plane geodesic motion of a test gyroscope, both for parallel transport and its relation to the cumulative spin precession, a topic which has not been given much attention in the literature. In fact, in the latter case this total spin precession is a possible observational signature of hyperbolic encounters between two black holes, which can have a counterpart in the detection of gravitational wave signals and which necessitates further study. This case also has some relevance to recent work on the dynamics of a extended two-body systems with spin (see, e.g., Ref. [19]), whose point-particle limit is consistent with our results, which can also serve as a preliminary model for similar computations in the context of gravitational self-force.

The same manipulations described here to identify an adapted parallelly transported frame along timelike geodesics would apply to any spacetime given a preferred family of observers with an associated preferred adapted orthonormal frame, but without the special simplifications which occur due to the high symmetry of the Kerr spacetime. This approach gives a prescription for rewriting the equations of parallel transport along a timelike geodesic in terms of an angular velocity in the associated rest space with respect to axes which are locked onto the preferred observer adapted frame by removing the relative motion between the geodesic and the observer. The Gödel, Kasner, and de Sitter spacetimes are briefly discussed in Appendix B to illustrate this point. For a stationary asymptotically flat spacetime where the symmetry allows a connection between the local celestial sphere and a global such sphere at spatial infinity, one can then go further and interpret the results in terms of gyro spin precession with respect to distant observers, as developed in a previous series of articles [20–22].

The real significance of the Marck work is that, like the reducibility of the geodesic equations of motion to first order equations parametrized by the constants of the motion, the equations of parallel transport can be reduced to a similarly expressed single differential equation for an angle of rotation about an axis determined by the symmetry rather than a general rotation, once expressed in terms of an adapted frame along the geodesic obtained by boosting a frame adapted to a preferred family of test observers. In the special case of equatorial plane motion, this allows evaluation of the cumulative precession with respect to nonrotating observers at spatial infinity.

We use the signature $-+++$ and Greek and Latin index conventions $\alpha, \beta, \gamma = 0, 1, 2, 3$ and $i, j, k = 1, 2, 3$.

II. A RELATIVE OBSERVER BOOST

Consider two 4-velocities in the same tangent space, u and U. The orthogonal decomposition of U with respect to u and its local rest space LRS_u defines a relative velocity $\nu(U,u)$ and unit direction vector $\hat\nu(U,u) = \nu(U,u)/\|\nu(U,u)\|$

$$U = \gamma(U,u)[u + \nu(U,u)] \\ = \cosh\alpha\, u + \sinh\alpha\, \hat\nu(U,u), \tag{1}$$

where the associated gamma factor has the usual expression $\gamma(U,u) = (1 - \|\nu(U,u)\|^2)^{-1/2} = \cosh\alpha$ defining the rapidity $\alpha \geq 0$, in terms of which the relative speed is $\|\nu(U,u)\| = \tanh\alpha$. An active relative observer boost $B(U,u)$ of the tangent space takes u onto $U = B(U,u)u$ and acts as the identity orthogonal to the plane of u and U, mapping the local rest space LRS_u onto LRS_U [18,23,24]. Note that $B(U,u)\nu(U,u) = -\nu(u,U)$.

Let $P(u) = Id + u \otimes u^\flat$, $P(U) = Id + U \otimes U^\flat$ be the mixed tensors representing projections onto these respective local rest spaces, with Id denoting the identity tensor. Then restricting this boost to a map $B_{(\mathrm{lrs})}(U,u) = B(U,u)P(u)$ from LRS_u onto LRS_U [see Eq. (4.22) of Ref. [18] for additional details], one finds that

$$B_{(\mathrm{lrs})}(U,u)S = S + \frac{(U\cdot S)}{\gamma(U,u)+1}(U+u) \tag{2}$$

for any vector S belonging to the local rest space of u ($S\cdot u = 0 = U\cdot[B_{(\mathrm{lrs})}(U,u)S]$). Adjacent $\binom{1}{1}$ tensors are understood to be contracted on their adjacent indices, reflecting the composition of the corresponding linear maps of the tangent space. To express the full boost in this notation, let

$$X = X^{(\|)}u + X^{(\perp)}, \tag{3}$$

with

$$X^{(\|)} = -u\cdot X, \qquad X^{(\perp)} = X + (u\cdot X)u = P(u)X, \tag{4}$$

be a generic spacetime vector orthogonally decomposed with respect to u. Then one finds that

$$\begin{aligned} B(U,u)X &= B(U,u)(X^{(\|)}u + X^{(\perp)}), \\ &= X^{(\|)}U + X^{(\perp)} + \frac{X^{(\perp)}\cdot U}{\gamma(U,u)+1}(u+U) \\ &= \Big[Id - \frac{1}{\gamma+1}[(2\gamma+1)U - u]\otimes u^\flat \\ &\quad + \frac{1}{\gamma+1}(U+u)\otimes U^\flat\Big]X \end{aligned} \tag{5}$$

with the shorthand abbreviation $\gamma = \gamma(U,u)$.

Replacing U with Eq. (1) and then using the identity $Id = P(u) - u\otimes u^\flat$, one finds that

$$\begin{aligned} B(U,u) &= Id - (\gamma-1)[u\otimes u^\flat - \hat\nu\otimes\hat\nu^\flat] \\ &\quad - \gamma\nu[\hat\nu\otimes u^\flat - u\otimes\hat\nu^\flat] \\ &= B_{(\mathrm{lrs})u}(U,u) - \gamma\nu(U,u)\otimes u^\flat \\ &\quad + \gamma u\otimes\nu(U,u)^\flat - \gamma u\otimes u^\flat, \end{aligned} \tag{6}$$

where

$$\begin{aligned} B_{(\mathrm{lrs})u}(U,u) &= P(u)B(U,u)P(u) \\ &= P(u) + \frac{\gamma^2}{\gamma+1}\nu(U,u)\otimes\nu(U,u)^\flat \\ &= P(u) + (\gamma-1)\hat\nu(U,u)\otimes\hat\nu(U,u)^\flat. \end{aligned} \tag{7}$$

Its inverse map is

$$\begin{aligned} [B_{(\mathrm{lrs})u}(U,u)]^{-1} &= B_{(\mathrm{lrs})u}(u,U) \\ &= P(u)B(u,U)P(u) \\ &= P(u) - (\gamma-1)\hat\nu(U,u)\otimes\hat\nu(U,u)^\flat. \end{aligned} \tag{8}$$

Let $e(u)_a$ be an orthonormal spatial triad adapted to $u = e(u)_0$, and let $\omega(u)^\alpha$ be the dual frame with $\omega(u)^0 = -u^\flat$. The frame components of $B(U,u)$ with respect to this frame,

$$B(U,u)^\alpha{}_\beta = \omega^\alpha(B(U,u)e_\beta), \tag{9}$$

are then explicitly

$$\begin{aligned} &B^0{}_0 = \gamma, \qquad B^0{}_a = \gamma\nu(U,u)_a, \qquad B^a{}_0 = \gamma\nu(U,u)^a, \\ &B^a{}_b = P(u)^a{}_b + (\gamma-1)\hat\nu(U,u)^a\hat\nu(U,u)_b. \end{aligned} \tag{10}$$

However, using only orthonormal frames to express the relative boost matrices, the latter are a function only of the orthonormal components $\boldsymbol{\nu} = \langle \nu^1, \nu^2, \nu^3\rangle$ of the relative velocity $\nu(U,u)$ relative to the starting frame, which makes composing the boosts easier to manage,

$$\mathcal{B}(\boldsymbol{\nu}) = (B(U,u)^\alpha{}_\beta), \tag{11}$$

dropping for the moment the functional dependence (U,u).

Defining the generator $K(u)_i = e_i(u)\otimes u^\flat - u\otimes e_i^\flat$ of boosts in the direction $e(u)_i$, one has the following representation for the generator of a boost in the direction $\hat\nu(U,u)$:

$$K(u)_i\hat\nu(U,u)^i = \hat\nu\otimes u^\flat - u\otimes\hat\nu^\flat. \tag{12}$$

Then

$$\begin{aligned} (K(u)_i\hat\nu(U,u)^i)^2 &= (K(u)_i\hat\nu(U,u)^i)(K(u)_j\hat\nu(U,u)^j) \\ &= -u\otimes u^\flat + \hat\nu\otimes\hat\nu^\flat, \end{aligned} \tag{13}$$

and our previous expression (6) becomes

$$\begin{aligned} B(U,u) &= Id + (\gamma-1)(\hat\nu\cdot K(u))^2 + \gamma\nu(\hat\nu\cdot K(u)) \\ &= Id + (\cosh\alpha - 1)(K(u)_i\hat\nu(U,u)^i)^2 \\ &\quad + \sinh\alpha(K(u)_i\hat\nu(U,u)^i) \\ &= e^{\alpha K(u)_i\hat\nu(U,u)^i}. \end{aligned} \tag{14}$$

This corresponds directly to a matrix relation [see Eq. (A5) of Appendix A] when expressed in an adapted orthonormal frame.

For example, for a boost relative velocity in the 1-2 plane with direction $\hat\nu(U,u)^a = (\cos\delta, \sin\delta, 0)$ and speed $||\nu(U,u)|| = \tanh\alpha > 0$, we have $\boldsymbol{\nu} = \langle\nu^1,\nu^2,0\rangle = \langle\tanh\alpha\cos\delta, \tanh\alpha\sin\delta, 0\rangle$ and

$$\mathcal{B}(\boldsymbol{\nu}) = \begin{pmatrix} \cosh\alpha & \sinh\alpha\,\cos\delta & \sinh\alpha\,\sin\delta & 0 \\ \sinh\alpha\,\cos\delta & \cosh\alpha\cos^2\delta + \sin^2\delta & \frac{\sinh^2\alpha\,\cos\delta\sin\delta}{1+\cosh\alpha} & 0 \\ \sinh\alpha\,\sin\delta & \frac{\sinh^2\alpha\,\cos\delta\sin\delta}{1+\cosh\alpha} & \cosh\alpha\sin^2\delta + \cos^2\delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

with permutations of this holding for the remaining cases.

A. Composition of velocities

Consider expressing a single four-velocity U in terms of two distinct observer 4-velocities u and u',

$$U = \underbrace{\gamma(U,u)[u+\nu(U,u)]}_{U\,\mathrm{vs}\,u} = \underbrace{\gamma(U,u')[u'+\nu(U,u')]}_{U\,\mathrm{vs}\,u'}, \tag{16}$$

with

$$\begin{aligned} u &= \underbrace{\gamma(u,u')[u'+\nu(u,u')]}_{u\,\mathrm{vs}\,u'}, \\ u' &= \underbrace{\gamma(u,u')[u+\nu(u',u)]}_{u'\,\mathrm{vs}\,u}. \end{aligned} \tag{17}$$

A straightforward calculation shows that the gamma factors are related by

$$\frac{\gamma(U,u')}{\gamma(U,u)} = \gamma(u,u')[1-\nu(U,u)\cdot\nu(u',u)], \tag{18}$$

and the "velocity subtraction" formula is

$$P(u',u)\left[\frac{\nu(U,u)-\nu(u',u)}{1-\nu(U,u)\cdot\nu(u',u)}\right] = \gamma(u,u')\nu(U,u'). \tag{19}$$

Reexpressing this four-vector formula in terms of the boost rather than the projection for relative velocities in opposite directions leads to a more familiar formula written in terms of orthonormal components. In fact, it is enough to note that in the plane of u and u', the projection is $\gamma(u, u')$ times the boost but acts as the identity orthogonal to that plane.

III. COMBINATION OF BOOSTS

Boosts along a fixed spatial direction form a subgroup of the Lorentz group, but the subset of all boosts do not form a subgroup: the product of two boosts along distinct spatial directions is no longer a boost along any direction but instead the product of a single boost with a rotation, the Wigner rotation. To discuss this phenomenon, it is essential to introduce the matrix representation of boosts and rotations to divorce them from the various orthonormal frames in which our tensor representations can be expressed. The notation $B(u_2, u_1)$ facilitates expressing boosts from one local rest space to the next but is simple only when expressed in terms of an orthonormal frame adapted to the initial local rest space LRS_{u_1} so when composed with a second boost $B(u_3, u_2)$, that matrix representation of the successive boost is complicated when expressed in terms of the intermediate four-velocity u_2 instead of the original four-velocity u_1. Fortunately these boosts depend only on the orthonormal components of the relative velocity, so the alternative notation $\mathcal{B}(\boldsymbol{\nu})$ for a given relative velocity $\nu(u_2, u_1)$ simplifies the discussion.

Consider the combinations of two boosts in different directions. To this end let u_1, u_2, and u_3 be three 4-velocities, with the last one determining the final local rest space of the combination of Lorentz boosts associated with u_1 and u_2 with respect to the common time direction u,

$$\begin{aligned} u_3 &= B(u_1, u)u_2 \\ &= B(u_1, u)B(u_2, u)u, \end{aligned} \tag{20}$$

and define the relative gamma factors and relative velocities by

$$u_i = \gamma_i(u + \boldsymbol{\nu}_i), \tag{21}$$

where it is convenient to use the abbreviated notation $(\gamma_i, \boldsymbol{\nu}_i) = (\gamma(u_i, u), \nu(u_i, u))$.

Explicit calculation of u_3 using Eq. (6) (or by multiplication of the corresponding matrices using a computer algebra system) gives the relativistic addition of velocities formula in terms of the relative velocities

$$\gamma_3 = \gamma_1\gamma_2(1 + \boldsymbol{\nu}_1 \cdot \boldsymbol{\nu}_2) \tag{22}$$

and

$$\boldsymbol{\nu}_3 = \frac{1}{1 + \boldsymbol{\nu}_1 \cdot \boldsymbol{\nu}_2}\left[\left(1 + \frac{\gamma_1 \boldsymbol{\nu}_1 \cdot \boldsymbol{\nu}_2}{1 + \gamma_1}\right)\boldsymbol{\nu}_1 + \frac{1}{\gamma_1}\boldsymbol{\nu}_2\right]; \tag{23}$$

namely, $\boldsymbol{\nu}_3$ belongs to the 2-plane spanned by $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. If the two relative velocities are collinear, this reduces to the familiar scalar formula for velocity addition along one common direction.

Consider two successive active boosts of a vector $X = X^a e_a \in LRS_u$ $(a = 1, 2, 3)$—namely,

$$B(u_1, u)B(u_2, u)X = e_\alpha \mathcal{B}(\boldsymbol{\nu}_1)^\alpha{}_\gamma \mathcal{B}(\boldsymbol{\nu}_2)^\gamma{}_b X^b, \tag{24}$$

where the labeling from left to right of the boosts is appropriate for the group composition law, even though 2 acts first followed by 1 in this active point transformation. The first boost, $B(u_2, u){:}\, LRS_u \to LRS_{u_2}$, takes u to u_2, and the next boost, $B(u_1, u){:}\, LRS_{u_1} \to LRS_{u_3}$, takes u_2 to u_3, the time direction of the final local rest space of the combined boosts. One can also directly boost X from LRS_u to $B(u_3, u)X \in LRS_{u_3}$. These two vectors both belong to the final local rest space but differ by a Wigner rotation defined by

$$\begin{aligned} B(u_1, u)B(u_2, u)X &= R_{\mathrm{L}}^{(\mathrm{W})}(u_1, u_2, u)B(u_3, u)X \\ &= B(u_3, u)R_{\mathrm{R}}^{(\mathrm{W})}(u_1, u_2, u)X; \end{aligned} \tag{25}$$

namely,

$$\begin{aligned} R_{\mathrm{R}}^{(\mathrm{W})}(u_1, u_2, u) &= B(u_3, u)^{-1}B(u_1, u)B(u_2, u) \\ R_{\mathrm{L}}^{(\mathrm{W})}(u_1, u_2, u) &= B(u_1, u)B(u_2, u)B(u_3, u)^{-1} \\ &= B(u_3, u)R_{\mathrm{R}}^{(\mathrm{W})}(u_1, u_2, u)B(u_3, u)^{-1}, \end{aligned} \tag{26}$$

with the last line being a similarity transformation. The right Wigner rotation is a rotation in LRS_u, simply expressed in the adapted frame e_α, while the left Wigner matrix is the boosted rotation in LRS_{u_3}, not so simply expressed. However, the latter can always be written in the canonical form (see Appendix A) once the matrix components are computed with respect to the frame $e(u_3)_\alpha$.

Like the relative boost matrix, the Wigner rotation matrix depends only on the orthonormal components of the two successive boost relative velocities

$$\mathcal{R}_{\mathrm{R}}^{(\mathrm{W})}(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)^\alpha{}_\beta = [R_{\mathrm{R}}^{(\mathrm{W})}(u_1, u_2, u)]^\alpha{}_\beta \tag{27}$$

or

$$\mathcal{R}_{\mathrm{R}}^{(\mathrm{W})}(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) = \mathcal{B}(\boldsymbol{\nu}_3)^{-1}\mathcal{B}(\boldsymbol{\nu}_1)\mathcal{B}(\boldsymbol{\nu}_2). \tag{28}$$

We need the passive right action of the Lorentz group transforming orthonormal frames in succession, in contrast with the left action actively transforming the points of the tangent space. We successively boost the original frame e_α,

$$e(u_1)_\alpha = e_\gamma B(u_1, u)^\gamma{}_\alpha, \tag{29}$$

and then

$$
\begin{aligned}
e(u_3)_\alpha &= e(u_1)_\beta B(u_2,u)^\beta{}_\alpha = e_\gamma B(u_1,u)^\gamma{}_\beta B(u_2,u)^\beta{}_\alpha \\
&= e_\gamma [B(u_1,u)B(u_2,u)]^\gamma{}_\alpha \\
&= e_\gamma [B(u_3,u)R_{\rm R}^{(\rm W)}(u_1,u_2,u)]^\gamma{}_\alpha \\
&= e_\gamma [\mathcal{B}(\boldsymbol{\nu}_3)\mathcal{R}_{\rm R}^{(\rm W)}(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)]^\gamma{}_\alpha . \qquad (30)
\end{aligned}
$$

We consider below some explicit examples corresponding to special choices of $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. Without any loss of generality one can align $\boldsymbol{\nu}_1$ with the axis $e(u)_1$ and let the axis $e(u)_3$ be orthogonal to the 2-plane spanned by $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$; namely,

$$
\begin{aligned}
\boldsymbol{\nu}_1 &= \tanh\alpha_1 e(u)_1, \\
\boldsymbol{\nu}_2 &= \tanh\alpha_2[\cos\beta_2 e(u)_1 + \sin\beta_2 e(u)_2], \qquad (31)
\end{aligned}
$$

where $\alpha_1 > 0, \alpha_2 > 0, -\pi < \beta_2 \le \pi$ so that the speeds are $(\nu_1,\nu_2) = (\tanh\alpha_1, \tanh\alpha_2)$ and $\sin\beta_2 = e_3\cdot(\hat{\boldsymbol{\nu}}_1\times\hat{\boldsymbol{\nu}}_2)$. In this case we find

$$
\begin{aligned}
\boldsymbol{\nu}_1\cdot\boldsymbol{\nu}_2 &= \tanh\alpha_1\tanh\alpha_2\cos\beta_2, \\
\boldsymbol{\nu}_3 &= \tanh\alpha_3[\cos\beta_3 e(u)_1 + \sin\beta_3 e(u)_2], \qquad (32)
\end{aligned}
$$

with

$$
\cosh\alpha_3 = \cosh\alpha_1\cosh\alpha_1(1+\tanh\alpha_1\tanh\alpha_2\cos\beta_2) \qquad (33)
$$

and

$$
\tan\beta_3 = \frac{\nu_2\sin\beta_2}{\gamma_1(\nu_1+\nu_2\cos\beta_2)}. \qquad (34)
$$

The left Wigner rotation, $R_{\rm L}^{(\rm W)}(u_1,u_2,u)$, takes place in LRS_{u_3} in the plane of the two relative velocities, while the right one, $R_{\rm R}^{(\rm W)}(u_1,u_2,u)$, takes place in LRS_u, so its matrix is simple,

$$
\mathcal{R}_{\rm R}^{(\rm W)}(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) = R_3(-\theta^{(\rm W)}), \qquad (35)
$$

where the explicit expression $R_3(\theta)$ for a counterclockwise rotation in the 1-2 plane by an angle θ is given in Appendix A, and the Wigner angle here is

$$
\sin\theta^{(\rm W)} = \left(1+\frac{\gamma_1+\gamma_2}{1+\gamma_2\gamma_1(1+\nu_1\nu_2\cos\beta_2)}\right) \times\frac{\gamma_1\gamma_2\nu_1\nu_2}{(1+\gamma_1)(1+\gamma_2)}\sin\beta_2, \qquad (36)
$$

$$
\cos\theta^{(\rm W)} = 1-\frac{(1-\gamma_1)(1-\gamma_2)}{1+\gamma_2\gamma_1(1+\nu_1\nu_2\cos\beta_2)}\sin^2\beta_2. \qquad (37)
$$

Since the two speeds are nonnegative, $\theta^{(\rm W)}$ has the same sign as β_2, so a clockwise rotation results when $\beta_2>0$; namely, as long as the angle from the first to the second relative velocity in this plane is a positive counterclockwise angle between 0 and π, the Wigner rotation is clockwise, and vice versa for $\beta_2<0$. For boosts in the same direction ($\beta_2=0$), the rapidity is simply additive, $\alpha_3=\alpha_1+\alpha_2$, reflecting the fact that boosts along a fixed direction form a subgroup, and the Wigner rotation reduces to the identity.

For the complementary case $\boldsymbol{\nu}_1\cdot\boldsymbol{\nu}_2=0$ of boosts in orthogonal directions—say, for definiteness, $\beta_2=\pi/2$—the rotation angle is maximized for constant values of the two speeds, and these formulas reduce to

$$
\gamma_3 = \gamma_1\gamma_2, \qquad \boldsymbol{\nu}_3 = \boldsymbol{\nu}_1 + \gamma_1{}^{-1}\boldsymbol{\nu}_2, \qquad (38)
$$

since the first applied boost velocity $\boldsymbol{\nu}_2$ must be adjusted to the proper time of the second applied boost (ordered right to left). The Wigner angle in this case is given by

$$
\cos\theta^{(\rm W)} = \frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}, \qquad \sin\theta^{(\rm W)} = \frac{\gamma_1\gamma_2\nu_1\nu_2}{1+\gamma_1\gamma_2}. \qquad (39)
$$

Varying the speeds, one finds that the extreme values of the Wigner angle for orthogonal relative velocities occur when the speeds are equal, $\nu_1=\nu_2$ and $\beta_2=\pm\pi/2$; namely,

$$
\pm\sin\theta^{(\rm W)}_{(\rm ext)} = \frac{\gamma_1^2\nu_1^2}{1+\gamma_1^2} = \frac{\sinh^2\alpha_1}{1+\cosh^2\alpha_1} \qquad (40)
$$

or

$$
\cos\theta^{(\rm W)}_{(\rm ext)} = \frac{2\gamma_1}{1+\gamma_1^2} = \frac{2\cosh\alpha_1}{1+\cosh^2\alpha_1}, \qquad (41)
$$

which confines the Wigner angle to the interval $-\frac{\pi}{2}<\theta^{(\rm W)}_{(\rm ext)}<\frac{\pi}{2}$, approaching the end points at high speeds. The composite boost itself in this special case actually has a manageable expression. For $\beta_2=\pi/2$ it is

$$
\mathcal{B}(\boldsymbol{\nu}_3) = \begin{pmatrix}
\cosh\alpha_1\cosh\alpha_2 & \sinh\alpha_1\cosh\alpha_2 & \sinh\alpha_2 & 0 \\
\sinh\alpha_1\cosh\alpha_2 & 1+\frac{\sinh^2\alpha_1\cosh^2\alpha_2}{1+\cosh\alpha_1\cosh\alpha_2} & \frac{\cosh\alpha_2\sinh\alpha_1\sinh\alpha_2}{1+\cosh\alpha_1\cosh\alpha_2} & 0 \\
\sinh\alpha_2 & \frac{\cosh\alpha_2\sinh\alpha_1\sinh\alpha_2}{1+\cosh\alpha_1\cosh\alpha_2} & \frac{\cosh\alpha_2(\cosh\alpha_1+\cosh\alpha_2)}{1+\cosh\alpha_1\cosh\alpha_2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \qquad (42)
$$

A. Boosted observer adapted frames

A direct application of the above formalism within the tangent spaces to a spacetime arises when constructing an orthonormal frame defined along a given world line with four-velocity U by using relative observer boost to boost into LRS_U an orthonormal spatial frame field $e(u)_a$ ($a = 1, 2, 3$) in LRS_u adapted to an observer family with four-velocity $u = e(u)_0$ which exists in some open tube around the world line,

$$e(U,u)_a = B(U,u)e(u)_a = e(u)_\beta \mathcal{B}(\boldsymbol{\nu})^\beta{}_a. \tag{43}$$

This is a boost of the three vectors $e(u)_a$ along the direction of the relative motion of U, and u essentially shows how this frame would appear if it were not in relative motion.

However, if one is interested in a spatial frame belonging to LRS_U along this world line which undergoes a particular transport law which respects orthonormality, the desired frame will be related to the boosted frame by a rotation of the local rest space at each point of the world line. This translates the equations of the transport into differential equations for the relative rotation between the two frames. Both parallel transport and Fermi-Walker transport are such transports which have a physical interpretation, and they coincide along geodesics. We will illustrate this for the Kerr spacetime, where Marck showed how a particular parallel transported frame along timelike geodesic could be defined in terms of the constants of the motion, modulo a final rotation in a 2-plane, leading to a single differential equation for the angle of rotation in that plane, whose orientation is determined by the electric part of the Killing-Yano 2-form as seen by U. Independent of Marck's construction, one can evaluate the effective rotation of a parallel transported frame relative to a boosted static frame, differing from Marck's parallel transported frame only by a Wigner rotation.

In the Kerr case one has at least three natural special observer families due to its stationary axisymmetry, all in relative azimuthal motion with 4-velocities in the same relative observer plane spanned by the two Killing vector fields: the static observers, the ZAMOs, and the Carter observers. Their associated natural adapted spherical orthonormal frames are each related to the others by azimuthal boosts of the normalized spatial coordinate frame associated with the Boyer-Lindquist coordinate system, so when boosted to the local rest space of U, they generate spatial frames which differ one from the other by Wigner rotations.

IV. SPECIAL OBSERVERS AND ADAPTED FRAMES IN THE KERR SPACETIME

Consider the Kerr spacetime with metric written in the Boyer-Lindquist coordinate system (t, r, θ, ϕ) [25],

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \\ &\quad + \frac{2Mr}{\Sigma}(dt - a\sin^2\theta d\phi)^2, \end{aligned} \tag{44}$$

where M and a are the mass and the specific angular momentum of the source, respectively, and

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{45}$$

The inner and outer horizons are located at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

In this spacetime there exist at least three families of fiducial/special observers who play a role from either a geometrical point of view or a physical point of view. They are ZAMOs (with four-velocity $u = n$), static observers (with four-velocity $u = m$), and Carter observers (with four-velocity $u = u_{(\mathrm{Car})}$).

The ZAMOs have world lines orthogonal to the Boyer-Lindquist $t =$ constant hypersurfaces, the static observers have world lines aligned with Boyer-Lindquist temporal lines, and the Carter observers have four-velocity belonging to the intersection of the two 2-planes: the one spanned by the temporal and azimuthal Killing vectors and the one spanned by the two repeated principal null directions of the Kerr (Petrov type D) spacetime, aligned with $u_{(\mathrm{Car})} \pm e_{\hat{r}}$. One may form adapted frames to any test particle world line, e.g., moving along a timelike geodesic, by conveniently boosting adapted frames to each of them.

A. The static observes and their relative adapted frame

The static observers, which exist only in the spacetime region outside the black hole ergosphere where $g_{tt} < 0$, form a congruence of accelerated, nonexpanding, and locally rotating world lines. They are, however, nonrotating with respect to observers at rest at spatial infinity and have four-velocity $u = m$ where

$$m = \frac{1}{\sqrt{-g_{tt}}}\partial_t = \left(1 - \frac{2Mr}{\Sigma}\right)^{-1/2}\partial_t. \tag{46}$$

An orthonormal frame adapted to m is

$$\begin{aligned} e(m)_1 &= \frac{1}{\sqrt{g_{rr}}}\partial_r = \sqrt{\frac{\Delta}{\Sigma}}\partial_r \equiv e_{\hat{r}}, \\ e(m)_2 &= \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta = \frac{1}{\sqrt{\Sigma}}\partial_\theta \equiv e_{\hat{\theta}}, \\ e(m)_3 &= \frac{1}{\sqrt{g_{\phi\phi} - g_{t\phi}^2/g_{tt}}}\left(\partial_\phi - \frac{g_{t\phi}}{g_{tt}}\partial_t\right) \\ &= \frac{\sqrt{\Delta - a^2\sin^2\theta}}{\sin\theta\sqrt{\Delta\Sigma}}\left(\partial_\phi - \frac{2Mar\sin^2\theta}{\Delta - a^2\sin^2\theta}\partial_t\right). \end{aligned} \tag{47}$$

B. The ZAMOs and their relative adapted frame

The ZAMOs are locally nonrotating (but locally rotating in the azimuthal direction in the same sense as the rotation of the black hole) and exist everywhere outside of the outer horizon, They have four-velocity $u = n$, where

$$n = \sqrt{-g^{tt}}\left(\partial_t + \frac{g^{t\phi}}{g^{tt}}\partial_\phi\right) = \sqrt{\frac{A}{\Delta\Sigma}}\left(\partial_t + \frac{2aMr}{A}\partial_\phi\right) \equiv N^{-1}(\partial_t - N^\phi\partial_\phi), \quad (48)$$

where N and N^ϕ denote the lapse and shift functions, respectively, and

$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \quad (49)$$

The normalized spatial coordinate frame vectors

$$e(n)_1 = e_{\hat r}, \qquad e(n)_2 = e_{\hat\theta},$$
$$e(n)_3 = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi = \frac{\sqrt{\Sigma}}{\sin\theta\sqrt{A}}\partial_\phi \equiv e_{\hat\phi} \quad (50)$$

together with n form an orthonormal adapted frame. A boost along $e_{\hat\phi}$ maps n onto m, i.e.,

$$m = \gamma(m, n)[n + \nu(m, n)], \quad (51)$$

with relative velocity in the opposite azimuthal direction as the rotation of the black hole associated with the sign of a (resisting the “dragging of inertial frames”)

$$\nu(m, n) = -\frac{2Mr}{\Sigma}\frac{a\sin\theta}{\sqrt{\Delta}}e_{\hat\phi}, \quad (52)$$

and associated Lorentz factor $\gamma(m, n)$ so that ZAMOs and static observers share the same r-θ 2-plane of their local rest spaces.

C. The Carter observers and their relative adapted frame

The Carter family of observers $u = u_{(\mathrm{Car})}$ are geometrically special because their four-velocity is aligned with the intersection of two geometrically special 2-planes: the one which is the span of the two Killing vectors ∂_t and ∂_ϕ, and the other spanned by the two principal null directions of the spacetime. This coincidence connects them to the separability of the geodesic equations as well as to the alignment of all relevant vectors and tensors in the Kerr spacetime. In particular their relation to the Killing-Yano 2-form allows the solution of the equations of parallel transport found by Marck through successive boosts which isolate the effective spin precession from the various possible boosts of the spherical frame linked to spatial infinity.

The Carter observers are boosted in the opposite azimuthal direction from the static observers compared to the ZAMOs in order to “comove” with the black hole, their angular velocity at the outer horizon being defined as that of the black hole itself. Their four-velocity $u_{(\mathrm{Car})}$ is given by

$$u_{(\mathrm{Car})} = \frac{r^2 + a^2}{\sqrt{\Delta\Sigma}}\left(\partial_t + \frac{a}{r^2 + a^2}\partial_\phi\right),$$
$$u^\flat_{(\mathrm{Car})} = -\sqrt{\frac{\Delta}{\Sigma}}(dt - a\sin^2\theta d\phi), \quad (53)$$

with the $\flat$ symbol denoting the fully covariant form of any tensor. Decomposing it with respect to the static observers,

$$u_{(\mathrm{Car})} = \gamma(u_{(\mathrm{Car})}, m)[m + \nu(u_{(\mathrm{Car})}, m)], \quad (54)$$

leads to the relative velocity

$$\nu(u_{(\mathrm{Car})}, m) = \frac{a\sin\theta}{\sqrt{\Delta}}e(m)_3. \quad (55)$$

A spherical orthonormal frame adapted to $u_{(\mathrm{Car})}$ is obtained by using the triad boosted from the either the ZAMO or the static observer spherical frame along the azimuthal direction, with

$$e_1(u_{(\mathrm{Car})}) = e_{\hat r}, \qquad e_2(u_{(\mathrm{Car})}) = e_{\hat\theta} \quad (56)$$

and

$$e_3(u_{(\mathrm{Car})}) = \frac{a\sin\theta}{\sqrt{\Sigma}}\left(\partial_t + \frac{1}{a\sin^2\theta}\partial_\phi\right),$$
$$e_3(u_{(\mathrm{Car})})^\flat = -\frac{a\sin\theta}{\sqrt{\Sigma}}\left(dt - \frac{r^2 + a^2}{a}d\phi\right). \quad (57)$$

In terms of boost map we have

$$e(u_{(\mathrm{Car})})_\alpha = e(m)_\beta B(u_{(\mathrm{Car})}, m)^\beta{}_\alpha. \quad (58)$$

V. TIMELIKE GEODESICS

A geodesic timelike world line has a four-velocity unit tangent vector $U = U^\alpha\partial_\alpha$ with coordinate components $U^\alpha = dx^\alpha/d\tau$, which can be expressed using the Killing symmetries [15,26] as a system of first order differential equations

$$\frac{dt}{d\tau} = \frac{1}{\Sigma}\left[aB + \frac{(r^2+a^2)}{\Delta}P\right],$$
$$\frac{dr}{d\tau} = \epsilon_r \frac{1}{\Sigma}\sqrt{R},$$
$$\frac{d\theta}{d\tau} = \epsilon_\theta \frac{1}{\Sigma}\sqrt{\Theta},$$
$$\frac{d\phi}{d\tau} = \frac{1}{\Sigma}\left[\frac{B}{\sin^2\theta} + \frac{a}{\Delta}P\right], \tag{59}$$

where τ is a proper time parameter along the geodesic, ϵ_r and ϵ_θ are sign indicators, and

$$P = E(r^2+a^2) - La = Er^2 - ax,$$
$$B = L - aE\sin^2\theta = x + aE\cos^2\theta,$$
$$R = P^2 - \Delta(r^2+K),$$
$$\Theta = K - a^2\cos^2\theta - \frac{B^2}{\sin^2\theta}. \tag{60}$$

Here E and L denote the conserved Killing energy and angular momentum per unit mass and K is a separation constant, usually called the Carter constant, while the combination $x = L - aE$ proves to be useful. For example, in place of K one often uses

$$Q = K - (L-aE)^2 = K - x^2, \tag{61}$$

which vanishes for equatorial plane orbits. Corresponding to the four-velocity vector field U is the index-lowered 1-form

$$U^\flat = -Edt + \frac{\Sigma}{\Delta}\dot{r}dr + \Sigma\dot{\theta}d\theta + Ld\phi$$
$$= -Edt + \frac{\epsilon_r\sqrt{R(r)}}{\Delta}dr + \epsilon_\theta\sqrt{\Theta(\theta)}d\theta + Ld\phi. \tag{62}$$

Here we use the overdot notation $\dot{f} = df/d\tau$ for the proper time derivative along the geodesic. A remarkable (but not very familiar) property of the geodesic family of world lines is that they form an irrotational congruence, $dU^\flat = 0$, since each of the covariant component of U depends on the coordinates in a separated form. Consequently, there exists a new temporal parameter, say, T, such that

$$U^\flat = -dT. \tag{63}$$

The relation of T with the Boyer-Lindquist coordinates follows immediately,

$$T = Et - L\phi - \epsilon_r\int^r \frac{\sqrt{R(r)}}{\Delta}dr - \epsilon_\theta\int^\theta \sqrt{\Theta(\theta)}d\theta, \tag{64}$$

and can be further expressed in terms of elliptic functions [27].

For later use we decompose the geodesic four-velocity U with respect to the Carter observers,

$$U = \gamma(U, u_{(\mathrm{Car})})[u_{(\mathrm{Car})} + \nu(U, u_{(\mathrm{Car})})^a e(u_{(\mathrm{Car})})_a], \tag{65}$$

with

$$[\gamma(U, u_{(\mathrm{Car})})\nu(U, u_{(\mathrm{Car})})^a] = \left[\sqrt{\frac{\Sigma}{\Delta}}\dot{r}, \sqrt{\Sigma}\dot{\theta}, \frac{B}{\sqrt{\Sigma}\sin\theta}\right] \tag{66}$$

and

$$\gamma(U, u_{(\mathrm{Car})}) = \frac{P}{\sqrt{\Delta\Sigma}}. \tag{67}$$

A. Parallel transported frame along a geodesic bypassing Marck's approach

The construction of the various natural adapted frames along timelike geodesics is facilitated by using the boost maps introduced above [see, e.g., Eq. (6)] applied to the various spherical frames associated with the Boyer-Lindquist coordinates; namely,

$$e(U,u)_a = B(U,u)e(u)_a \tag{68}$$

for any observer family $u \in \{m, n, u_{(\mathrm{Car})}\}$, i.e.,

$$e(U,m)_a = B(U,m)e(m)_a,$$
$$e(U,n)_a = B(U,n)e(n)_a,$$
$$e(U,u_{(\mathrm{Car})})_a = B(U,u_{(\mathrm{Car})})e(u_{(\mathrm{Car})})_a. \tag{69}$$

The boosted spatial frames on the left are related to each other by Wigner rotations in LRS_U since the unboosted frames on the right are themselves related to each other by boosts. One can study their parallel transport along U, defining the corresponding angular velocities by

$$\frac{D}{d\tau_U}e(U,m)_a = \Omega(U,m)\times_U e(U,m)_a, \tag{70}$$

and similarly for $\Omega(U,n)$ and $\Omega(U,u_{(\mathrm{Car})})$.

One may then evaluate the angular velocity $\Omega(U,u) = \Omega(U,u)^a e(U,u)_a$ in terms of the relative motion of the geodesic U and these three observer families as a sum of the following three terms as given in Ref. [18],

$$\Omega(U,u) = -\gamma(U,u)B(U,u)[\omega_{(\mathrm{fw},u)} + \omega_{(\mathrm{sc},U,u)} + \omega_{(\mathrm{geo},U,u)}], \tag{71}$$

using its notation for the Fermi-Walker and the spatial curvature angular rotation vectors which characterize the covariant derivatives of the orthonormal frame along the orbit

$$P(u)\nabla_U e(u)_a = -\gamma(U,u)[\omega_{(\mathrm{fw},u)} + \omega_{(\mathrm{sc},U,u)}] \times_u e(u)_a,$$

as well as the geodetic precession term in the gyroscope precession formula [see Eq. (9.10) of Ref. [18]]

$$\omega_{(\mathrm{geo},U,u)} = \frac{1}{1+\gamma(U,u)}\nu(U,u) \times_u F^{(G)}_{(\mathrm{fw},U,u)}, \quad (72)$$

defined in terms of the spatial gravitational force $F^{(G)}_{(\mathrm{fw},U,u)} = -\nabla_U u$. Additional details, including notation, can be found in Ref. [18] and will not be repeated here.

Among the family of all frames adapted to U there is one which is geometrically special, corresponding to a parallel propagated frame along U. This frame was explicitly found by Marck long ago [16,17], and his elegant construction uses the properties of the Killing-Yano tensor of the Kerr spacetime to pin down the orientation of the 2-plane of the relative rotation. We have already discussed the geometrical meaning of this frame in previous articles [20–22], summarized in the next section. The question thus naturally arises whether it is possible to bypass Marck's approach and provide a prescription which can be used for other stationary axisymmetric spacetimes which do not admit a Killing-Yano tensor. The answer is yes, and the main computation needed is to determine the rotation matrix mapping the axes $e(U,u)_a$ onto parallel transported axes $e_{(\mathrm{par})}(U,u)_a$. This is a cumbersome task in practice, although not in principle. In fact, it is enough to arbitrarily rotate the frame $\{e(U,u)_a\}$ parametrizing the rotation by three Euler angles and solve the parallel transport equations for the new triad $\{e_{(\mathrm{par})}(U,u)_a\}$ for these angles, instead of for a single angle needed to complete Marck's construction. Let us describe this procedure in more detail.

Given any initial frame $e(U,u)_a$ along a geodesic, one can perform a generic rotation on it:

$$e_{(\mathrm{par})}(U,u)_a = e(U,u)_b R(U,u)^b{}_a. \quad (73)$$

The rotated frame can then be required to be parallel transported along U. The two conditions

$$\frac{D}{d\tau_U} e(U,u)_a = \Omega(U,u) \times_U e(U,u)_a,$$
$$\frac{D}{d\tau_U} e_{(\mathrm{par})}(U,u)_a = 0 \quad (74)$$

then determine the frame components $R(U,u)^b{}_a$ of the rotation matrix, obeying the equations

$$\left(\frac{d}{d\tau_U} R(U,u)^b{}_d\right)(R(U,u)^{-1})^d{}_a = \epsilon_{bac}\Omega(U,u)^c. \quad (75)$$

Therefore, going back to the local rest space of the chosen observer u by applying the boost operation yields

$$e_{(\mathrm{par})}(U,u)_a = e(U,u)_b R(U,u)^b{}_a = e(u)_c B(U,u)^c{}_b R(U,u)^b{}_a. \quad (76)$$

If the adapted triad $e(u)_a$ is in turn obtained by boosting to u the triad $e(u')_a$ adapted to another observer u', then

$$e(u,u')_c = B(u,u')e(u')_c = e(u')_d B(u,u')^d{}_c, \quad (77)$$

so

$$e_{(\mathrm{par})}(U,u)_a = e(u')_d B(u,u')^d{}_c B(U,u)^c{}_b R(U,u)^b{}_a, \quad (78)$$

where the combination of the two successive boosts leads to an additional Wigner rotation.

This procedure is general but may become very complicated computationally. In Appendix B we show how to construct a parallel transported frame along geodesics in some simple spacetimes, where Marck's recipe cannot be applied.

VI. FRAMING MARCK'S RESULT WITHIN THE PREVIOUS APPROACH

Whatever family of observers is chosen to start the procedure described above and leading to the construction of a parallelly propagated frame along U, all of the various steps can be easily performed, but this does not avoid long formulas. As always happens, some specific choice can be preferable if computational simplifications may arise. Knowing the result in advance (Marck's result), one can identify a suitable, special family of observers and elucidate this approach when going in reverse. This will be done "in steps" in the next subsections, revisiting recently obtained results.

A. Decomposing U in Carter's frame: A new family of radially moving observers $u_{(\mathrm{rad})}$

Consider a generic timelike geodesic with unit tangent vector (62) decomposed relative to the Carter observers [see Eqs. (65) and (66)]. To make the notation less cumbersome below, we introduce the abbreviations $\gamma(U,u_{(\mathrm{Car})}) = \gamma_{\mathrm{c}}$, $\nu(U,u_{(\mathrm{Car})}) = \nu_{\mathrm{c}}$. Let us introduce the angular part $\nu^\top$ of the Carter relative velocity and an orthogonal vector $\nu^\perp = e(u_{(\mathrm{Car})})_1 \times_{u_{(\mathrm{Car})}} \nu^\top$ of the same magnitude in the angular subspace

$$\nu^\top = \nu_{\mathrm{c}}^2 e(u_{(\mathrm{Car})})_2 + \nu_{\mathrm{c}}^3 e(u_{(\mathrm{Car})})_3 \equiv \|\nu^\top\|\hat{\nu}^\top,$$
$$\nu^\perp = -\nu_{\mathrm{c}}^3 e(u_{(\mathrm{Car})})_2 + \nu_{\mathrm{c}}^2 e(u_{(\mathrm{Car})})_3 \equiv \|\nu^\perp\|\hat{\nu}^\perp, \quad (79)$$

where $\|\nu^\top\| = \|\nu^\perp\| = \sqrt{(\nu_{\mathrm{c}}^2)^2 + (\nu_{\mathrm{c}}^3)^2}$, with

$$\nu_{\rm c}^1 = \frac{\Sigma\dot{r}}{P}, \qquad \nu_{\rm c}^2 = \frac{\Sigma\sqrt{\Delta}\dot{\theta}}{P}, \qquad \nu_{\rm c}^3 = \frac{B\sqrt{\Delta}}{P\sin\theta}. \tag{80}$$

We will use the notation

$$\begin{aligned}\hat{\nu}^{\perp} &= \cos\Upsilon e(u_{(\rm Car)})_2 + \sin\Upsilon e(u_{(\rm Car)})_3\\ &\equiv e(u_{(\rm rad)})_2,\\ \hat{\nu}^{\top} &= -\sin\Upsilon e(u_{(\rm Car)})_2 + \cos\Upsilon e(u_{(\rm Car)})_3\\ &\equiv e(u_{(\rm rad)})_3\end{aligned} \tag{81}$$

for a clockwise rotation by angle Υ in the 2-3 plane given by

$$\tan\Upsilon = -\frac{\nu_{\rm c}^2}{\nu_{\rm c}^3} = -\frac{\Sigma\dot{\theta}}{B}. \tag{82}$$

In the case of equatorial plane motion, one has $\Upsilon = 0$, while in the general case this rotates the azimuthal direction into the angular direction of relative motion.

Completing these new angular basis definitions to a new frame $\{u_{(\rm rad)}, e(u_{(\rm rad)})_a\}$ obtained by a boost along the radial direction to comove radially with the four-velocity U defines a key transitional frame in Marck's approach to reducing the equations of parallel transport along a geodesic

$$\begin{aligned}u_{(\rm rad)} &= \gamma^{\parallel}[u_{(\rm Car)} + \nu_{\rm c}^1 e(u_{(\rm Car)})_1],\\ &= \cosh\alpha u_{(\rm Car)} + \sinh\alpha e(u_{(\rm Car)})_1,\\ e(u_{(\rm rad)})_1 &= \gamma^{\parallel}[\nu_{\rm c}^1 u_{(\rm Car)} + e(u_{(\rm Car)})_1]\\ &= \sinh\alpha u_{(\rm Car)} + \cosh\alpha e(u_{(\rm Car)})_1,\end{aligned} \tag{83}$$

where the boost rapidity α is given by

$$\nu_{\rm c}^1 = \tanh\alpha, \qquad \gamma^{\parallel} = \cosh\alpha = \frac{P}{\sqrt{\Delta(r^2+K)}}, \tag{84}$$

and $e(u_{(\rm rad)})_2$ and $e(u_{(\rm rad)})_3$ are defined above in Eq. (81). In compact form we have

$$e(u_{(\rm rad)})_a = e(u_{(\rm Car)})_\beta [R_1(\Upsilon)\mathcal{B}(\boldsymbol{\nu}_1)]^\beta{}_a, \tag{85}$$

where with $\tanh\alpha = \nu_1^1 = \nu(u_{(\rm rad)}, u_{(\rm Car)})^1$ we have

$$\mathcal{B}(\boldsymbol{\nu}_1) = B_1(\alpha). \tag{86}$$

The notation and expressions for rotations and boosts along an axis are given in Appendix A.

B. Marck's frame

The (timelike) geodesic four-velocity U is boosted from $u_{(\rm rad)}$ in the direction $e(u_{(\rm rad)})_3$ of the angular motion

$$U = \cosh\beta u_{(\rm rad)} + \sinh\beta e(u_{(\rm rad)})_3, \tag{87}$$

where

$$\cosh\beta = \sqrt{\frac{K+r^2}{\Sigma}}, \qquad \sinh\beta = \sqrt{\frac{K-a^2\cos^2\theta}{\Sigma}}. \tag{88}$$

From this relation one can easily identify the orthogonal (spatial, azimuthal) direction in this plane,

$$e_{(\rm Mar)}(U)_3 = \sinh\beta u_{(\rm rad)} + \cosh\beta e(u_{(\rm rad)})_3. \tag{89}$$

Marck showed that a unit vector $e_{(\rm Mar)}(U)_2$ orthogonal to both U and $e_{(\rm Mar)}(U)_3$, which is also parallel propagated along U, arises naturally by normalizing the electric part $f_{\alpha\beta}U^\beta$ of the Killing-Yano 2-form f of the Kerr spacetime with respect to U. Because this 2-form is so simply expressed in both the Carter and intermediate frames (see Appendix A of Ref. [20]), the resulting vector frame components are obtained by a simple anisotropic rescaling of the two vector components of U expressed in the form (87). The normalized electric part of the Killing-Yano 2-form is then

$$e_{(\rm Mar)}(U)_2 = -\sin\Xi e(u_{(\rm rad)})_1 + \cos\Xi e(u_{(\rm rad)})_2, \tag{90}$$

where

$$\cos\Xi = \frac{r}{\sqrt{K}}\sinh\beta, \qquad \sin\Xi = -\frac{a\cos\theta}{\sqrt{K}}\cosh\beta. \tag{91}$$

The last frame vector is then

$$\begin{aligned}e_{(\rm Mar)}(U)_1 &= e_{(\rm Mar)}(U)_2 \times_U e_{(\rm Mar)}(U)_3\\ &= \cos\Xi e(u_{(\rm rad)})_1 + \sin\Xi e(u_{(\rm rad)})_2.\end{aligned} \tag{92}$$

This rotation in the 2-plane orthogonal to the final boost in the angular motion direction simply realigns what was originally the θ direction to be the normal to the plane of the instantaneous parallel transport rotation while rotating the radial direction to remain orthogonal to it and within the plane of the parallel rotation containing the angular direction of motion.

In compact form we have

$$\begin{aligned}e_{(\rm Mar)}(U)_a &= e(u_{(\rm rad)})_\beta[\mathcal{B}(\boldsymbol{\nu}_2)R_3(\Xi)]^\beta{}_a\\ &= e(u_{(\rm Mar)})_\beta[R_3(\Upsilon)\mathcal{B}(\boldsymbol{\nu}_1)\mathcal{B}(\boldsymbol{\nu}_2)R_3(\Xi)]^\beta{}_a,\end{aligned} \tag{93}$$

where

$$\mathcal{B}(\boldsymbol{\nu}_2) = B_3(\beta), \tag{94}$$

with $\tanh\beta = \nu_2{}^3 = \nu(U, u_{(\mathrm{rad})})^3$. The successive boosts can then be replaced by the direct boost and a Wigner rotation.

This boosted and then rotated adapted frame $\{U, e_{(\mathrm{Mar})}(U)_a\}$ is a degenerate Frenet-Serret frame along U; namely,

$$\frac{DU}{d\tau} = 0, \qquad \frac{De_{(\mathrm{Mar})}(U)_2}{d\tau} = 0,$$
$$\frac{De_{(\mathrm{Mar})}(U)_1}{d\tau} = \mathcal{T} e_{(\mathrm{Mar})}(U)_3,$$
$$\frac{De_{(\mathrm{Mar})}(U)_3}{d\tau} = -\mathcal{T} e_{(\mathrm{Mar})}(U)_1, \tag{95}$$

with

$$\mathcal{T} = \frac{\sqrt{K}}{\Sigma}\left[\frac{P}{r^2+K} + \frac{aB}{K - a^2\cos^2\theta}\right] = \frac{\sqrt{K}}{(r^2+K)(K-a^2\cos^2\theta)}(KE + ax), \tag{96}$$

the only surviving (spacetime) torsion of the world line, which reversed in sign describes the scalar angular velocity of two frame vectors with respect to parallel transport.

The vector angular velocity of the frame is aligned with the parallel transported third spatial direction in this frame, i.e.,

$$\Omega_{(\mathrm{par})} = -\mathcal{T} e_{(\mathrm{Mar})}(U)_2. \tag{97}$$

Thus a parallel transported frame is obtained by further rotating this pair of frame vectors along the world line at the opposite angular velocity $d\Psi/d\tau = \mathcal{T}$,

$$e_{(\mathrm{par})}(U)_a = e_{(\mathrm{Mar})}(U)_\beta R_2(-\Psi)^\beta{}_a. \tag{98}$$

Introducing the Wigner rotation, this finally becomes

$$e_{(\mathrm{par})}(U)_a = e(u_{(\mathrm{Car})})_\beta L^\beta{}_a, \tag{99}$$

where

$$L = R_3(\Upsilon)\mathcal{B}(\boldsymbol{\nu}_1)\mathcal{B}(\boldsymbol{\nu}_2)R_3(\Xi)R_2(-\Psi) = R_3(\Upsilon)\mathcal{B}(\boldsymbol{\nu}_3)\mathcal{R}_{\mathrm{R}}^{(\mathrm{W})}(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)R_3(\Xi)R_2(-\Psi). \tag{100}$$

Figure 1 illustrates the succession of three rotations in Eq. (100) modulo the relative boosts in the sequence transforming from the Carter frame to the final parallel transported frame, identifying the three distinct local rest spaces in this sequence for comparison purposes.

Note that in the case of equatorial plane motion $\theta = \pi/2$ where $U^\theta = 0$, the angles Υ and Ξ vanish so that the intermediate Marck frame remains aligned with the spherical frame (modulo boosts), and only the final rotation Ψ remains. From Eq. (61) it follows that $K = x^2$, and in turn from Eq. (84) we have

$$\cosh\alpha = \frac{P}{\sqrt{\Delta(r^2+x^2)}}. \tag{101}$$

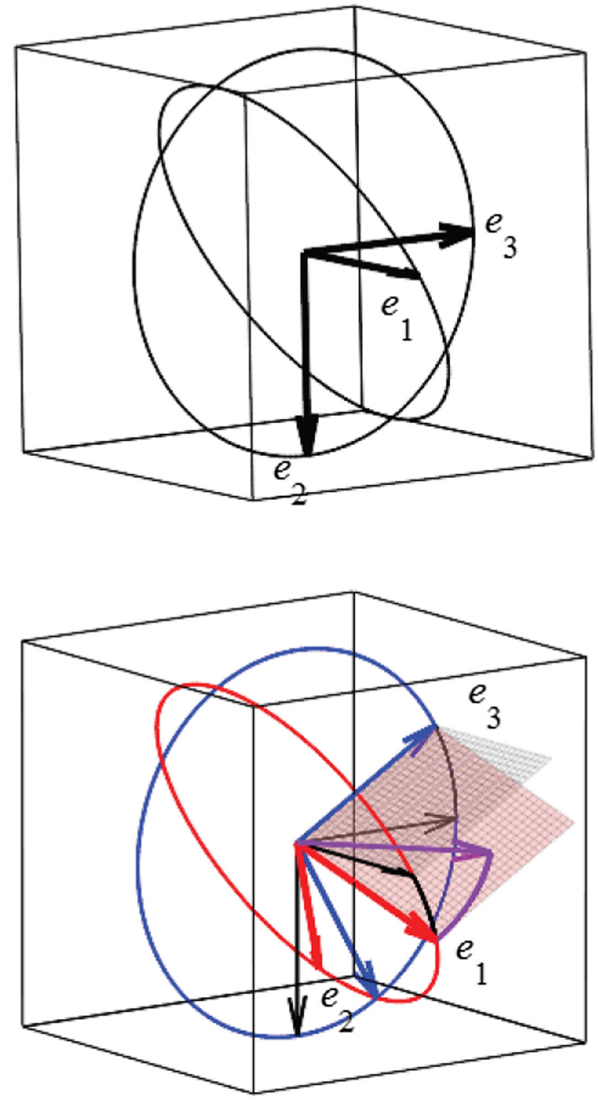

FIG. 1. The initial and final rotations required to orient the spherical frame with the final Marck frame adapted to the plane of the parallel transport rotation, where the final rotation (Ψ) takes place. The top panel shows the initial orientation of a spherical frame in the equatorial plane. The azimuthal vector e_3 is first rotated about the radial vector e_1 by the angle Υ to align it with the direction of the angular relative velocity. The two successive boosts to the intermediate Marck frame comoving with the geodesic have relative velocities spanning the upper plane, where the resulting Wigner rotation takes place. Then e_2 is rotated about the new e_3 by the angle Ξ to align it with the electric part of the Killing-Yano 2-form, which is the normal to the (lower) parallel rotation plane, leading to the bottom panel, with the final triad in bold.

Finally, from Eq. (88) we find that

$$\sinh\beta = \frac{|x|}{r}. \tag{102}$$

The torsion simplifies to

$$\mathcal{T} = \frac{|x|}{r^2 + x^2}\left(E + \frac{a}{x}\right). \tag{103}$$

Note that the sign of $x = L - aE$ depends on the relative signs of L and a, i.e., if the orbit is either prograde (same sign) or retrograde (opposite sign). For example, if $a > 0$, then $x > 0$ ($x < 0$) for prograde (retrograde) orbits.

We have shown in the previous section how to resolve the parallel transport equations relative to a boosted observer frame along the world line in a general spacetime using a "brute force" approach without relying on the Killing-Yano 2-form essential for Marck's elegant construction for the Kerr spacetime. The complication is that in general one cannot confine the essential rotation to a parallel transported 2-plane but would require a general rotation parametrized by three angles and a vector angular velocity rather than a single scalar one. For the Kerr spacetime one simply boosts the static spherical frame into LRS_U and evaluates the parallel transport angular velocity of the boosted axes. Such a direct calculation was done for general motion in Ref. [22] and is not so complicated, but it lacks the geometrical interpretation of Marck's construction. Marck's scalar angular velocity captures the essential precession of a gyro spin vector with respect to the celestial sphere at spatial infinity, i.e., the rotation of the spin vector which accumulates to give a net change in its orientation over a finite interval of time, disregarding wandering Wigner rotations in the local rest space of the gyro.

C. Relationship of the Killing-Yano 2-form to the axis of rotation

Neglecting the details of the Kerr metric components which deform them from their flat spacetime values, we can get a sense of how the 2-plane of the parallel transport rotation along a timelike geodesic depends on its location, i.e., how the orientation of the axis of rotation in the comoving local rest space depends on position. The electric part of this 2-form as seen in that local rest space defines the axis of the parallel transport rotation—namely, the direction of the associated angular velocity.

In the spherical coordinate frame the Killing-Yano 2-form

$$\begin{aligned} F = {}& dt \wedge [-ad(r\cos\theta)] \\ & + [(r^2 + a^2)rd\theta - a^2\sin\theta\cos\theta dr] \wedge \sin\theta d\phi \end{aligned} \tag{104}$$

has a single electric part term wedged into dt and a magnetic part consisting of two angular terms. The nonvanishing magnetic part when $a = 0$ (coming from the $d\theta \wedge d\phi$ term corresponding to a radial vector) leads to a comoving electric part vector in LRS_U (boosted from that radial magnetic part vector term) which is perpendicular to the plane of the planar geodesic motion in the Schwarzschild case, leading to parallel transport rotation only in that plane in the comoving local rest space of the geodesic.

On the other hand, the electric part linear in a represents a constant vertical vector field in the Cartesian coordinates associated with the spherical coordinate system in the usual way ($z = r\cos\theta$), thus aligned with the axis of rotation of the black hole and contributing a smaller parallel transport angular velocity about that vertical direction. The last term quadratic in a is a higher order angular correction which grows with angular distance from the equatorial plane, where it vanishes.

VII. SPIN-PRECESSION FRAMES

In a stationary spacetime one can formulate a precise way of measuring locally the precession of the spin of a test gyroscope in geodesic motion with respect to the distant stars; namely, the celestial sphere at spatial infinity [20–22]. The static observers determine a local reference frame rigidly linked to the distant celestial sphere in the sense that these observers see an unchanging distant sky pattern of incoming photons from spatial infinity. Thus one can establish a Cartesian frame in the local rest space along each static observer world line which establishes the local celestial sky, and the Killing symmetry links that frame to a single Cartesian frame at spatial infinity. With the additional axial symmetry and reflection symmetry across the equatorial plane, the natural Cartesian frame linked to the Boyer-Lindquist spherical frame is uniquely a candidate for this link, but off the equatorial plane of the Kerr spacetime, though not unique (in the case of nonspherical symmetry), it is the simplest frame to serve this purpose. Spheroidal spatial coordinates would provide another spherical orthonormal triad which could be used to introduce a corresponding Cartesian frame, for example, that would differ from the Boyer-Lindquist frame except on the equatorial plane.

For a gyro at rest with respect to the static observer grid, the precession is unambiguous and straightforward to measure, but for relative motion one has the additional complication that in the local rest space of the gyro, the local axes linked to spatial infinity by the incoming null geodesics from spatial infinity in the static observer local rest space are distorted by spatial aberration. Boosting those axes into the local rest space of the gyro enables one to measure the relative rotation of the spin vector, or equivalently, boosting the spin vector back to the static observer rest frame gives the same result (because of the isometric property of the boost). Of course one can compare the projection of the spin vector into the static observer rest space, the spin vector as seen by that observer, but even for circular motion in flat spacetime, this projected spin vector does not undergo a uniform precession, instead undergoing a periodic tilting (in spacetime) effect to maintain orthogonality with the four-velocity of the gyro resulting in a changing spin vector magnitude and orbital-dependent additional rotation. The traditional Thomas precession formula for circular orbits, for example, describes this boosted spin vector precession [28,29].

The Carter spatial frame is azimuthally boosted with respect to the static observer spherical frame, which in turn is azimuthally boosted with respect to the Boyer-Lindquist spherical coordinate normalized frame. The Carter frame is associated with the Killing tensor and Killing 2-form that in turn are associated with the separability of the geodesic equations, and the solution of the equations for Fermi-Walker transport along those geodesics, the latter of which describes free fall gyro spin behavior in the timelike case. The Marck procedure for solving the differential equations for a parallelly transported frame along geodesics (which is also a Fermi-Walker frame in this case) relies on a two step procedure for transforming to the local rest frame of the geodesic, first boosting the Carter frame along the radial direction common to all three frames (ZAMO, static, Carter) to comove radially with the gyro, preserving the Killing 2-form in this step, then followed by a boost in the remaining angular direction to comove with the gyro. This defines his preliminary frame $e_{(\mathrm{Mar})}(U)_\alpha$, which isolates the remaining parallel transport rotation to a 2-plane of the first and last spatial frame vectors, leaving the remaining frame vector aligned with its spatial normal direction invariant under parallel transport.

The spin vector of a test gyro moving along such a geodesic simply has constant components in this final parallel transported frame. However, to compare this evolution with the (nonrotating) celestial sphere at spatial infinity, one needs the Marck frame which is anchored to the local spherical frame modulo boosts. The Marck frame vector $e_{(\mathrm{Mar})}(U)_1$ is locked to the radial direction $e_{\hat r}$ in the spherical grid of the static observers following the time lines, differing only by a boost due to the radial motion of the gyro alone until the final rotational realignment into the plane of the parallel transport rotation. Along the gyro geodesic world line, the spherical axes rotate with an orbital angular velocity with respect to spatial infinity, so one must subtract this more complicated rotation from the simpler parallel transport rotation to obtain the net rotation of the axes with respect to spatial infinity. For equatorial plane motion this is simple since both the orbital and parallel transport rotations lie in a 2-plane, and it is a matter of subtracting the two scalar angular velocities to get the net angular velocity of precession in the Marck frame. These matters are discussed in Refs. [20–22].

VIII. ISOLATING CUMULATIVE PRECESSION EFFECTS

We now discuss the cumulative precession effects on a test gyroscope moving along an equatorial plane geodesic orbit. While the bound case has been given much more attention in the literature and explicit expressions already exist describing the precession [30–33], the unbound case has been studied only within certain approximation schemes; namely, in a post-Newtonian expansion (weak field and slow motion, in a power series in the reciprocal of the speed of light, $1/c$) and in a post-Minkowskian expansion (weak field, power series in the gravitational constant G) [19,34–39].

We evaluate exactly the total spin-precession angle Ψ, the accumulated azimuthal phase Φ, and the associated spin-precession invariant [40]

$$\psi = 1 - \frac{\Psi}{\Phi}. \tag{105}$$

For bound orbits these quantities are evaluated between two successive passages at periastron corresponding to one period of the radial motion, whereas for unbound (hyperboliclike) orbits they are evaluated for the entire scattering process (from the two asymptotic states at spatial infinity). We provide in both cases closed form analytical expressions in terms of elliptic functions as well as approximate expressions which facilitate comparison with known results. The combined quantity ψ is a sort of average azimuthal precession rate since the full spin precession with respect to spatial infinity measured from the perihelion is just $\psi\Phi$.

A. Gyroscope moving along a bound equatorial orbit

In the case of bound equatorial orbits not captured by the black hole, the radial motion is periodic and confined between a minimum radius r_{per} (periastron) and a maximum radius r_{apo} (apastron). It is convenient to introduce the relativistic anomaly $\chi \in [0, 2\pi]$ such that

$$r = \frac{Mp}{1 + e\cos\chi}, \tag{106}$$

with dimensionless semilatus rectum $p = 1/u_p$ and eccentricity $0 \le e < 1$ (see, e.g., Ref. [20] for additional details). The parameters (u_p, e) are related in turn to the energy and angular momentum (per unit mass) (E, L) entering the geodesic equation (59), with $E < 1$. This relation (106) with these parameters represents a classical Newtonian orbit in the Boyer-Lindquist polar coordinates in the equatorial plane which precesses due to general relativity according to a function χ rather than the azimuthal angle ϕ itself.

In terms of χ the rate of gyroscope precession and azimuthal change are given by

$$\frac{d\phi}{d\chi} = u_p^{1/2} \frac{\hat{x} + \hat{a}E - 2u_p\hat{x}(1 + e\cos\chi)}{[1 + u_p^2\hat{x}^2(e^2 - 2e\cos\chi - 3)]^{1/2}[1 - 2u_p(1 + e\cos\chi) + \hat{a}^2u_p^2(1 + e\cos\chi)^2]},$$
$$\frac{d\Psi}{d\chi} = u_p^{1/2} \frac{\hat{a} + E\hat{x}}{[1 - u_p^2\hat{x}^2(3 - e^2 + 2e\cos\chi)]^{1/2}[1 + u_p^2\hat{x}^2(1 + e\cos\chi)^2]}, \tag{107}$$

where $d\Psi/d\tau = \mathcal{T}$ is defined in Eq. (103) and

$$M\frac{d\chi}{d\tau} = u_p^{3/2}(1 + e\cos\chi)^2[1 + u_p^2\hat{x}^2(e^2 - 2e\cos\chi - 3)]^{1/2}. \tag{108}$$

Equation (107), once integrated over a radial period, i.e.,

$$\Phi = \int_0^{2\pi} \frac{d\phi}{d\chi} d\chi, \qquad \Psi = \int_0^{2\pi} \frac{d\Psi}{d\chi} d\chi, \tag{109}$$

then leads to Φ, Ψ, and ψ being expressible in terms of elliptic functions. We find that

$$\Phi = -\frac{\kappa}{\hat{a}^2e^2u_p^2\sqrt{eu_p\hat{x}^2}(b_+ - b_-)}\{[\hat{L} - 2u_p\hat{x}(1 + eb_+)]k_+\Pi(k_+, \kappa) - [\hat{L} - 2u_p\hat{x}(1 + eb_-)]k_-\Pi(k_-, \kappa)\},$$
$$\Psi = -\frac{i}{2}\frac{\kappa(\hat{a} + E\hat{x})}{\hat{x}^2(eu_p)^{3/2}}[k\Pi(k, \kappa) - \bar{k}\Pi(\bar{k}, \kappa)], \tag{110}$$

where $\hat{L} = L/M$,

$$\kappa^2 = \frac{4eu_p^2\hat{x}^2}{1 - (1 - e)(3 + e)u_p^2\hat{x}^2}, \qquad k_\pm = \frac{2}{1 + b_\pm}, \qquad b_\pm = \frac{1 - \hat{a}^2u_p \pm \sqrt{1 - \hat{a}^2}}{\hat{a}^2eu_p}, \qquad k = 2ieu_p\hat{x}\frac{1 + i(1 - e)u_p\hat{x}}{1 + (1 - e)^2u_p^2\hat{x}^2}, \tag{111}$$

with the overbar denoting complex conjugation, and where

$$\Pi(n, m) = \int_0^{\frac{\pi}{2}} \frac{dz}{(1 - n\sin^2 z)\sqrt{1 - m^2\sin^2 z}} \tag{112}$$

is the complete elliptic integral of the third kind [41].

Expanding these expressions in terms of the eccentricity e so that $\Phi = \Phi_0 + e^2\Phi_{e^2} + O(e^4)$ and similarly for Ψ and ψ, one finds that

$$\frac{\Phi}{2\pi} = \frac{1}{\sqrt{1 - 6u_p + 8\hat{a}u_p^{3/2} - 3\hat{a}^2u_p^2}}$$
$$+ e^2\frac{3u_p^2(-1 + 2u_p + (-3 + 22u_p)\sqrt{u_p}\hat{a} - 33u_p^2\hat{a}^2 + 13u_p^{5/2}\hat{a}^3)(\hat{a}\sqrt{u_p} - 1)^3}{4(1 - 2u_p + \hat{a}^2u_p^2)(1 - 6u_p + 8\hat{a}u_p^{3/2} - 3\hat{a}^2u_p^2)^{5/2}} + O(e^4),$$
$$\frac{\Psi}{2\pi} = \frac{\sqrt{1 - 3u_p + 2\hat{a}u_p^{3/2}}}{\sqrt{1 - 6u_p + 8\hat{a}u_p^{3/2} - 3\hat{a}^2u_p^2}} + e^2\frac{3P(u_p, \hat{a})(\hat{a}\sqrt{u_p} - 1)^2u_p^2}{4(1 - 2u_p + \hat{a}^2u_p^2)(1 - 6u_p + 8\hat{a}u_p^{3/2} - 3\hat{a}^2u_p^2)^{5/2}} + O(e^4), \tag{113}$$

where

$$P(u_p, \hat{a}) = 14\hat{a}^5u_p^{9/2} + (-81u_p + 13)u_p^3\hat{a}^4 + 4u_p^{5/2}(47u_p - 12)\hat{a}^3 + (-225u_p^2 - 3 + 68u_p)u_p\hat{a}^2$$
$$+ 2u_p^{1/2}(72u_p^2 - 22u_p + 1)\hat{a} - 1 - 42u_p^2 + 15u_p. \tag{114}$$

Finally,

$$\psi = 1 - \sqrt{1 - 3u_p + 2\hat{a}u_p^{3/2}} - \frac{3}{2}e^2 \frac{(2\hat{a}^3 u_p^{5/2} - 3\hat{a}^2 u_p^2 - 2\hat{a}u_p^{3/2} + 4u_p - 1)u_p^2(\hat{a}\sqrt{u_p} - 1)^2}{(1 - 6u_p + 8\hat{a}u_p^{3/2} - 3\hat{a}^2 u_p^2)(\hat{a}^2 u_p^2 - 2u_p + 1)(1 - 3u_p + 2\hat{a}u_p^{3/2})^{1/2}} + O(e^4), \tag{115}$$

which correctly goes to zero far from the black hole where $u_p \to 0$.

B. Gyroscope moving along an unbound equatorial orbit

In the case of unbound orbits not captured by the black hole, Eq. (106) represents a classical Newtonian parabolic ($e = 1$) or hyperbolic ($e > 1$) orbit which precesses due to general relativity, with a minimal radius $r_{\rm per}$ of closest approach. We consider only the hyperboliclike orbits of the latter type resembling a classical scattering process in which the geodesic path does not circle the black hole more than once, which occurs as long as the periastron is not too close to the black hole. Mathematically this corresponds to $\phi(\chi_{(\max)}) < \pi$. We compare the direction of the spin of the gyroscope before starting its gravitational interaction with the black hole (i.e., at $\tau \to -\infty$) with that after their interaction (i.e., at $\tau \to \infty$). The relativistic anomaly now varies in the range $\chi \in [-\chi_{(\max)}, \chi_{(\max)}]$, where $\chi_{(\max)} = \arccos(-1/e)$.

For computational purposes it is convenient to parametrize the orbit instead in terms of the dimensionless inverse radial variable $u = M/r$ such that

$$\left(\frac{du}{d\tau}\right)^2 = \frac{2\hat{x}^2}{M^2} u^4 (u - u_1)(u - u_2)(u - u_3),$$
$$\frac{d\phi}{d\tau} = \frac{2\hat{x}}{M\hat{a}^2} u^2 \frac{u_4 - u}{(u - u_+)(u - u_-)}. \tag{116}$$

Here $u_1 < u_2 < u_3$ are the ordered roots of the equation

$$u^3 - (\hat{x}^2 + 2\hat{a}\,\hat{E}\,\hat{x} + \hat{a}^2)\frac{u^2}{2\hat{x}^2} + \frac{u}{\hat{x}^2} + \frac{\hat{E}^2 - 1}{2\hat{x}^2} = 0, \tag{117}$$

whereas

$$u_\pm = \frac{M}{r_\pm}, \qquad u_4 = \frac{L}{2x}, \tag{118}$$

for hyperbolic orbits $u_1 < 0 < u \le u_2 < u_3$, with u_2 corresponding to the distance of closest approach.

The rates of gyroscope precession and azimuthal change are given by

$$\frac{d\phi}{du} = \pm \frac{\sqrt{2}}{\hat{a}^2} \frac{u_4 - u}{(u - u_+)(u - u_-)} \times \frac{1}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}},$$
$$\frac{d\Psi}{du} = \pm \frac{\hat{a} + E\hat{x}}{\sqrt{2}\hat{x}(1 + \hat{x}^2 u^2)} \times \frac{1}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}}, \tag{119}$$

which can be integrated in terms of elliptic functions. Since the scattering process is symmetric with respect to the closest approach distance, the total change results from twice the integration between 0 and u_2, i.e.,

$$\Phi = 2\int_0^{u_2} \frac{d\phi}{du} du, \qquad \Psi = 2\int_0^{u_2} \frac{d\Psi}{du} du, \tag{120}$$

where we have assumed that $\Phi(u_2) = 0 = \Psi(u_2)$ and that the plus sign must be selected in Eq. (119). The following explicit expressions in terms of elliptic functions hold,

$$\Phi = -\frac{4\sqrt{2}}{\hat{a}^2(u_+ - u_-)\sqrt{u_3 - u_1}} \left[\frac{u_4 - u_+}{u_1 - u_+}(\Pi(\alpha, \beta_+, m) - \Pi(\beta_+, m)) - \frac{u_4 - u_-}{u_1 - u_-}(\Pi(\alpha, \beta_-, m) - \Pi(\beta_-, m))\right],$$
$$\Psi = -i\frac{\sqrt{2}m(\hat{a} + E\hat{x})}{\hat{x}^2(u_2 - u_1)^{3/2}}[\beta(\Pi(\alpha, \beta, m) - \Pi(\beta, m)) - \bar{\beta}(\Pi(\alpha, \bar{\beta}, m) - \Pi(\bar{\beta}, m))], \tag{121}$$

where

$$m = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}, \qquad \alpha = \sqrt{\frac{-u_1}{u_2 - u_1}}, \qquad \beta_\pm = \frac{u_2 - u_1}{u_\pm - u_1}, \qquad \beta = \frac{i\hat{x}(u_2 - u_1)}{1 - i\hat{x}u_1}, \tag{122}$$

and

$$\Pi(\varphi, n, k) = \int_0^{\varphi} \frac{dz}{(1 - n\sin^2 z)\sqrt{1 - k^2 \sin^2 z}}, \qquad (123)$$

with $\Pi(\pi/2, n, k) = \Pi(n, k)$, is the incomplete elliptic integral of the third kind [41].

For the case of "simple" scattering orbits under consideration here, the total change in the azimuthal angle is less than 2π (less than a single revolution about the black hole). The scattering angle of the whole process can then be defined as

$$\frac{1}{2}\chi_{\rm scat} = \frac{1}{2}\Phi - \frac{\pi}{2} \qquad (124)$$

with $\chi_{\rm scat} < \pi$.

To compare with the existing literature (see, e.g., Ref. [35]), we define an energy-related variable $\bar{E}$

$$\bar{E} = \frac{1}{2}\sqrt{E^2 - 1} \qquad (125)$$

in place of the energy per unit mass E and a combined (dimensionless) variable

$$\alpha = \frac{1}{\sqrt{2\bar{E}j^2}}, \qquad (126)$$

defined through $\bar{E}$ and the (dimensionless) angular momentum per unit mass $j \equiv \hat{L} = L/M$ and used in turn in place of $\bar{E}$. Once all factors of c are restored,

$$\bar{E} \to \bar{E}c^2, \qquad j \to \frac{j}{c}, \qquad (127)$$

one can perform the post-Newtonian expansion of Eq. (119) and integrate order by order by taking the Hadamard's *partie finie*, following the prescriptions of Ref. [35] (see Sec. III B there). The final result for $\Psi = \Psi_0 + \hat{a}\Psi_{\hat{a}} + \hat{a}^2\Psi_{\hat{a}^2} + O(\hat{a}^3)$ then has the form of a power series in $1/j$, that is,

$$\begin{aligned}
\frac{1}{2}\Psi_0 &= B(\alpha) + \frac{1}{j^2}\left[\frac{3}{2}B(\alpha) + \frac{1}{2}\frac{(1+3\alpha^2)}{\alpha(\alpha^2+1)}\right] + \frac{1}{j^4}\left[\frac{3(2+35\alpha^2)}{8\alpha^2}B(\alpha) + \frac{(-1+67\alpha^2+181\alpha^4+105\alpha^6)}{8\alpha^3(\alpha^2+1)^2}\right] \\
&\quad + \frac{1}{j^6}\left[\frac{3(-1+140\alpha^2+770\alpha^4)}{16\alpha^4}B(\alpha) + \frac{3+193\alpha^2+5913\alpha^4+18597\alpha^6+19740\alpha^8+6930\alpha^{10}}{48\alpha^5(\alpha^2+1)^3}\right] + O(1/j^8), \\
\frac{1}{2}\Psi_{\hat{a}} &= -\frac{1}{j^3}\left[3B(\alpha) + \frac{(1+3\alpha^2)}{\alpha(\alpha^2+1)}\right] - \frac{1}{j^5}\left[\frac{3(3+35\alpha^2)}{2\alpha^2}B(\alpha) + \frac{(71+184\alpha^2+105\alpha^4)}{2\alpha(\alpha^2+1)^2}\right] + O(1/j^7), \\
\frac{1}{2}\Psi_{\hat{a}^2} &= \frac{1}{j^4}\left[\frac{3}{2}B(\alpha) + \frac{(2+3\alpha^2)}{2\alpha(\alpha^2+1)}\right] + \frac{1}{j^6}\left[\frac{9(4+35\alpha^2)}{4\alpha^2}B(\alpha) + \frac{2+228\alpha^2+561\alpha^4+315\alpha^6}{4\alpha^3(\alpha^2+1)^2}\right] + O(1/j^8), \qquad (128)
\end{aligned}$$

where

$$B(\alpha) = \arctan(\alpha) + \frac{\pi}{2}. \qquad (129)$$

Similarly

$$\begin{aligned}
\frac{1}{2}\Phi_0 &= B(\alpha) + \frac{1}{j^2}\left[3B(\alpha) + \frac{(2+3\alpha^2)}{(\alpha^2+1)\alpha}\right] + \frac{1}{j^4}\left[\frac{15(1+7\alpha^2)}{4\alpha^2}B(\alpha) + \frac{81+190\alpha^2+105\alpha^4}{4\alpha(\alpha^2+1)^2}\right] \\
&\quad + \frac{1}{j^6}\left[\frac{105(3+11\alpha^2)}{4\alpha^2}B(\alpha) + \frac{256+3663\alpha^2+10143\alpha^4+10185\alpha^6+3465\alpha^8}{12\alpha^3(\alpha^2+1)^3}\right] + O(1/j^8), \\
\frac{1}{2}\Phi_{\hat{a}} &= -\frac{1}{j^3}\left[4B(\alpha) + 2\frac{(1+2\alpha^2)}{(\alpha^2+1)\alpha}\right] - \frac{1}{j^5}\left[12\frac{(1+7\alpha^2)}{\alpha^2}B(\alpha) + \frac{(1+65\alpha^2+152\alpha^4+84\alpha^6)}{\alpha^3(\alpha^2+1)^2}\right] + O(1/j^7), \\
\frac{1}{2}\Phi_{\hat{a}^2} &= \frac{1}{j^4}\left[\frac{3}{2}B(\alpha) + \frac{(2+3\alpha^2)}{2(\alpha^2+1)\alpha}\right] + \frac{1}{j^6}\left[\frac{3(11+70\alpha^2)}{2\alpha^2}B(\alpha) + \frac{4+167\alpha^2+383\alpha^4+210\alpha^6}{2\alpha^3(\alpha^2+1)^2}\right] + O(1/j^8). \qquad (130)
\end{aligned}$$

The above expression for Φ_0 agrees with Eqs. (45) and (46) (in the point-particle limit $\nu = 0$) of Ref. [35].

IX. CONCLUDING REMARKS

All of the relevant observer-adapted frames needed to construct geometrically motivated frames along a general timelike geodesic in a Kerr black hole spacetime are described in terms of combinations of boost operations applied to the natural orthonormal frames associated with Boyer-Lindquist coordinates. Thus Marck's seemingly arbitrary recipe for obtaining a parallel transported frame along timelike geodesics acquires a nice interpretation in terms of identifying a parallel transported axis of rotation and an angular velocity of rotation in the 2-plane orthogonal to it, anchored radially to the usual coordinate grid of the spacetime. The sequence of boosts and rotations needed to interpret Marck's frame has now been made explicit. Furthermore, we have shown how to bypass Marck's procedure using a suitable combination of boosts and (associated) Wigner rotations starting from any given family of fundamental observers. We have explicitly given the connection between the ZAMO and Carter observer-adapted frames and the parallel transported one.

Finally, we have discussed the cumulative precession effects of a test gyroscope moving along both bound and unbound equatorial plane geodesic orbits by evaluating the total spin-precession angle and the cumulative azimuthal phase as well as the associated spin-precession invariant. The latter is a new addition to the current literature, which also plays an important role in the physics of a two-body system with spin. Indeed, in this case one has a deflection of the orbit (by the orbital" scattering angle χ) and a rotation of the spin vector (by the spin" rotation angle ψ) in the full scattering process.

APPENDIX A: BOOST AND ROTATION MATRICES

The six-dimensional Lie algebra of the Lorentz matrix group is generated by 4×4 matrices corresponding to the components of mixed second rank tensors which are antisymmetric when the flat Minkowski metric [component matrix $(\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$] is used to lower or raise indices to a completely covariant or covariant form. Six linearly independent matrices from this set,

$$L_{\alpha\beta} = ([L_{\alpha\beta}]^{\gamma}{}_{\delta}), \qquad [L_{\alpha\beta}]^{\gamma\delta} = \delta^{\gamma\delta}_{\alpha\beta} = 2\delta^{[\gamma}{}_{\alpha}\delta^{\delta]}{}_{\beta}, \quad \text{(A1)}$$

form a basis of the matrix Lie algebra. Three rotation generators are

$$J_1 = (L_{23}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = (L_{31}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = (L_{12}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(A2)}$$

and three boost generators are

$$K_1 = (L_{01}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = (L_{02}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = (L_{03}{}^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad \text{(A3)}$$

The rotation and boost matrices are obtained exponentiating linear combinations of these two subsets of the Lie algebra, collapsing the exponential series to three terms from the cubic identities

$$(n^i J_i)^3 = -n^i J_i, \qquad (n^i K_i)^3 = n^i K_i \qquad \text{(A4)}$$

satisfied for a unit vector $\delta_{ij}n^i n^j = 1$. One finds that

$$\begin{aligned} R(\theta, n^i) &= e^{\theta n^i J_i} \\ &= Id + \sin\theta (n^i J_i) - (\cos\theta - 1)(n^i J_i)^2, \\ B(\alpha, n^i) &= e^{\alpha n^i K_i} \\ &= Id + \sinh\alpha (n^i K_i) + (\cosh\alpha - 1)(n^i K_i)^2. \end{aligned} \qquad \text{(A5)}$$

These represent an active rotation by an angle θ of the 2-plane orthogonal to n^i in space (in the direction related to n^i by the right-hand rule), and an active boost by the rapidity α along the spatial velocity $\nu^i = \tanh\alpha n^i$. The boost may instead be parametrized by the relative velocity components ν^i themselves,

$$\mathcal{B}(\boldsymbol{\nu}) = B(\alpha, n^i). \qquad \text{(A6)}$$

The special rotations and boosts along an axis are particularly useful. Letting $R_i(\theta) = \exp(\theta J_i)$, $B_i(\alpha) = \exp(\alpha K_i)$, one has explicitly

$$R_3(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{(A7)}$$

and permutations thereof, and similarly

$$B_1(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(A8)}$$

which equals $\mathcal{B}(\langle\nu^1, 0, 0\rangle)$ with $\nu^1 = \tanh\alpha$. Note that $R_3(\theta)$ is an active counterclockwise rotation of the 1-2 plane; namely, rotating the first frame vector towards the second.

APPENDIX B: PARALLEL TRANSPORTED FRAMES ALONG GEODESICS IN ABSENCE OF KILLING-YANO SYMMETRY

In absence of a Killing-Yano tensor, Marck's recipe cannot be applied, and one has to rely to the general procedure outlined in Sec. VA. We consider below some simple examples for which the solutions to the timelike geodesic equations are known analytically.

1. The Gödel spacetime

Gödel spacetime [42,43] is a stationary axisymmetric solution of Einstein's equations with separable geodesics. One can start from a natural observer family with its adapted orthonormal frame, then boost it along a geodesic and rotate it so that it is parallel transported.

The spacetime metric in cylindrical-like coordinates (t, r, ϕ, z) adapted to the stationary axisymmetry about any point in this homogeneous spacetime is given by

$$ds^2 = \frac{2}{\omega^2}[-dt^2 + dr^2 + \sinh^2 r(1 - \sinh^2 r)d\phi^2 + 2\sqrt{2}\sinh^2 r dt d\phi + dz^2]. \quad \text{(B1)}$$

This nonvacuum spacetime has a dust fluid source with energy-momentum tensor $T = \rho u \otimes u$, with constant energy density ρ proportional to the cosmological constant $\Lambda = -\omega^2 = -4\pi\rho$ and unit timelike four-velocity $u = (\omega/\sqrt{2})\partial_t$ aligned with the time coordinate lines. We assume that $\omega > 0$ to describe an intrinsic counterclockwise rotation of the spacetime around the z axis.

The geodesic equations are separable and the covariant four-velocity 1-form of a general timelike geodesic has the following separated form:

$$U^\flat = -Edt + Ld\phi + bdz + U_r dr, \quad \text{(B2)}$$

where E, L, and b are Killing constants and

$$U_r^2 = \frac{1}{\omega^2\sinh^2 r}\left[\mathcal{A}\cosh^2 r + \mathcal{B} + \frac{\mathcal{C}}{\cosh^2 r}\right], \quad \text{(B3)}$$

while the constants $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by

$$\mathcal{A} = -[\omega^2(b^2 + E^2) + 2],$$
$$\mathcal{B} = -\mathcal{A} + 2\omega^2 E(E + \sqrt{2}L),$$
$$\mathcal{C} = -\omega^2(\sqrt{2}E + L)^2, \quad \text{(B4)}$$

with $\mathcal{A} + \mathcal{B} + \mathcal{C} = -\omega^2 L^2$. Turning points for radial motions are the roots of the equation $U_r = 0$, i.e., $r = r_\pm$ such that

$$\cosh^2 r_\pm = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} \quad \text{(B5)}$$

with $r_- \leq r \leq r_+$. Circular geodesics have $r_- = r_+$.

We start with an orthonormal frame adapted to the static observers with four-velocity $u = e_0 = (\omega/\sqrt{2})\partial_t$,

$$e_1 = \frac{\omega}{\sqrt{2}}\partial_r, \qquad e_3 = \frac{\omega}{\sqrt{2}}\partial_z,$$
$$e_2 = \omega\tanh r\left(\partial_t + \frac{1}{\sqrt{2}\sinh^2 r}\partial_\phi\right), \quad \text{(B6)}$$

with respect to which the orthonormal components of U are

$$U = \frac{\omega E}{\sqrt{2}}e_0 + \frac{\omega U_r}{\sqrt{2}}e_1 + \frac{\omega[L - \sqrt{2}E\sinh^2(r)]}{\sqrt{2}\sinh(r)\cosh(r)}e_2 + \frac{b\omega}{\sqrt{2}}e_3. \quad \text{(B7)}$$

Setting $b = 0$ confines the geodesic motion to a plane orthogonal to the axis of cylindrical symmetry, allowing the vector e_3 orthogonal to these planes to be parallel transported along U so that the situation is exactly analogous to the case of equatorial plane motion in Kerr, where parallel transport rotation is confined to the equatorial plane directions. Boosting the frame (B6) into LRS_U leads to

$$E_a = e_a + \frac{U + e_0}{1 - U \cdot e_0}(U \cdot e_a), \quad \text{(B8)}$$

where $E_3 = e_3$ is invariant. Rotating this frame around E_3 by an angle $\beta(r)$ using the radial coordinate as a parameter along the geodesic leads to a parallel transported frame $\{F_a\}$,

$$F_1 = \cos\beta E_1 + \sin\beta E_2,$$
$$F_2 = -\sin\beta E_1 + \cos\beta E_2,$$
$$F_3 = E_3. \quad \text{(B9)}$$

For noncircular orbits where r can be used to parametrize the orbit, one finds that the parallel transport angle satisfies

$$\frac{d\beta}{dr} = \frac{1}{\omega\sinh^2 r U_r}\left[2\sinh^2 r - \sqrt{2}\omega(E + \sqrt{2}L) + \frac{\omega}{\cosh^2 r}(L + \sqrt{2}E)\right], \quad \text{(B10)}$$

which requires piecing together continuously the solutions for the half of the orbit where r is increasing with those for the second half of the orbit where r is decreasing, during one oscillation in the radial coordinate from its minimum to maximum value and back.

Remarkably this can be integrated to find, up to an additive constant,

$$\beta = \frac{1}{\sqrt{\omega^2 E^2 + 2}} \arcsin\left[\frac{\mathcal{U}(r, r_+) + \mathcal{U}(r, r_-)}{\mathcal{U}(r_+, r_-)}\right] - \arctan\left(\frac{\sinh r_+}{\sinh r_-}\frac{\sqrt{\mathcal{U}(r, r_-)}}{\sqrt{\mathcal{U}(r_+, r)}}\right) + \frac{1}{2}\arctan\left(\frac{\cosh^2 r_+ \mathcal{U}(r_-, r) + \cosh^2 r_- \mathcal{U}(r_+, r)}{2\cosh r_+ \cosh r_- \sqrt{\mathcal{U}(r, r_-)\mathcal{U}(r_+, r)}}\right), \tag{B11}$$

where

$$\mathcal{U}(r_1, r_2) = \cosh^2 r_1 - \cosh^2 r_2. \tag{B12}$$

In the limit $r \to r_+$ we find that

$$\beta(r_+) = \frac{\pi}{4}\left[\frac{2}{\sqrt{\omega^2 E^2 + 2}} - 3\right], \tag{B13}$$

while in the limit $r \to r_-$

$$\beta(r_-) = -\frac{\pi}{4}\left[\frac{2}{\sqrt{\omega^2 E^2 + 2}} - 1\right], \tag{B14}$$

so

$$\beta(r_+) + \beta(r_-) = -\frac{\pi}{2} \tag{B15}$$

and

$$\beta(r_+) - \beta(r_-) = \frac{\pi}{\sqrt{\omega^2 E^2 + 2}} - \pi. \tag{B16}$$

However, for general motion $b \neq 0$, the absence of a Killing-Yano tensor leaves only the brute force approach to finding a parallel transported frame. The boosted frame along the geodesic is still given by Eq. (B8), but with the additional term $U \cdot e_3 = \omega b/\sqrt{2} \neq 0$. First one needs to rotate the frame so that E_3 becomes the parallel transported vector and then perform an additional rotation about that direction to fix the remaining two vectors. Starting from a triad of orthonormal vectors parametrized by three Euler angles $\theta_i(r)$,

$$(H_1 H_2 H_3) = (E_1 E_2 E_3) R(\theta_1, \theta_2, \theta_3),$$
$$R(\theta_1, \theta_2, \theta_3) = R_3(\theta_1) R_1(\theta_2) R_3(\theta_3), \tag{B17}$$

where here $R_i(\theta)$ refer to the obvious three-dimensional submatrices of those of Appendix A. The first two angles (θ_1, θ_2) determine the orientation of the axis of parallel transport rotation, while the last angle describes the rotation in the orthogonal 2-plane. The ordinary differential equations along the geodesic for these angles set the Lie algebra derivative of the rotation matrix equal to the parallel transport angular velocity in the comoving frame [see Eq. (75)], where the proper time derivative is converted into a derivative with respect to r using the chain rule $d/d\tau = U^r d/dr = (\omega^2/2) U_r d/dr$

$$(dR/d\tau R^{-1})_{ab} = \epsilon_{abc}\Omega^c. \tag{B18}$$

Evaluating the explicit frame components of the angular velocity (B18) with respect to the frame (B8), one finds the system

$$\begin{aligned}
\Omega^1 &= -\frac{\omega^3 b U_r}{\sqrt{2}(\omega E + \sqrt{2})} \\
&= \sin\theta_2 \sin\theta_3 \frac{d\theta_1}{d\tau} + \cos\theta_3 \frac{d\theta_2}{d\tau}, \\
\Omega^2 &= -\frac{\omega^3 b}{(\omega E + \sqrt{2})\sinh 2r}[\sqrt{2}L - E(\cosh 2r - 1)] \\
&= \sin\theta_2 \cos\theta_3 \frac{d\theta_1}{d\tau} - \sin\theta_3 \frac{d\theta_2}{d\tau}, \\
\Omega^3 &= \frac{\omega^2 L}{\cosh 2r - 1} + \frac{\omega^2(L + \sqrt{2}E)}{\cosh 2r + 1} \\
&\quad - \omega\left[1 + \frac{\omega^2 b^2}{\sqrt{2}(\omega E + \sqrt{2})}\right] \\
&= \frac{d\theta_3}{d\tau} + \cos\theta_2 \frac{d\theta_1}{d\tau}.
\end{aligned} \tag{B19}$$

This procedure could have been used for the Kerr spacetime, but the required equations for the three Euler angles are more complicated by the fact that they depend explicitly on the two nonignorable coordinates (r, θ) rather than just r alone. Unfortunately, in either case they are still too complicated to be solved analytically.

2. The Kasner spacetime

Another illustrative example is the Kasner vacuum spacetime [44], with metric written in Cartesian-like coordinates (t, x, y, z)

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{B20}$$

where the constant parameters p_i ($i = 1, 2, 3$) satisfy the relations

$$p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2. \tag{B21}$$

Here the natural "fiducial observers" are the comoving observers following the time coordinate lines (at fixed values of the spatial coordinates) with four-velocity $e_0 = \partial_t$ and their associated orthonormal triad is

$$e_a = t^{-p_a}\partial_a, \qquad (a = 1, 2, 3). \tag{B22}$$

This frame is parallel transported along the temporal geodesic world lines

$$\nabla_{e_0} e_a = 0. \tag{B23}$$

The general timelike geodesics are characterized by the four-velocity U given by

$$U = U_t dt + p_x dx + p_y dy + p_z dz, \tag{B24}$$

where

$$U_t = -U^t = -\sqrt{1 + t^{-2p_1} p_x^2 + t^{-2p_2} p_y^2 + t^{-2p_3} p_z^2}, \tag{B25}$$

and with p_x, p_y, p_z (Killing) constants. As above, one can form a frame adapted to U by boosting the comoving observer adapted frame along U,

$$E_a = e_a + \frac{U + e_0}{1 - U \cdot e_0}(U \cdot e_a), \tag{B26}$$

where now

$$U \cdot e_0 = -U_t, \qquad U \cdot e_a = t^{-p_a} p_{x^a}. \tag{B27}$$

The same Euler angle approach as in the previous case can be taken, except here the explicit equations are much more complicated because of the time dependence of the spacetime metric.

3. de Sitter spacetime

Finally, consider the de Sitter spacetime with metric written in Cartesian-like coordinates (t, x, y, z)

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij} dx^i dx^j, \tag{B28}$$

satisfying the Einstein's equations with cosmological constant $\Lambda = 3H^2$. The above line element is actually associated with the part of de Sitter's spacetime with constant positive curvature only.

The static observers have four-velocity $e_0 = \partial_t$ and associated orthonormal frame

$$e_a = e^{-Ht}\partial_a, \quad (a = 1, 2, 3). \tag{B29}$$

It is easy to show that this frame is parallel propagated along the geodesics

$$U = U^t \partial_t + e^{-2Ht} C^i \partial_i, \tag{B30}$$

where C^i are Killing constants and

$$U^t = \sqrt{1 + e^{-2Ht}\delta_{ij} C^i C^j}. \tag{B31}$$

In fact, it is enough to boost the frame (B29) along U, i.e.,

$$E_a = e_a + \frac{U + e_0}{1 - U \cdot e_0}(U \cdot e_a) \tag{B32}$$

with

$$U \cdot e_0 = -U^t, \qquad U \cdot e_a = e^{-Ht} C^a. \tag{B33}$$

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