Basis of surface curvature-dependent terms in $6D$

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Total derivative terms play an important role in the integration of the conformal anomaly. In four dimensional space $4D$ there is only one such term, namely $\Box R$. In the case of six dimensions $6D$ the structure of surface terms is more complicated, and it is useful to construct a basis of linear independent total derivative terms. We briefly review the general scheme of integrating the anomaly and present the reduction of the minimal set of the surface terms in $6D$ from eight to seven. Furthermore, we discuss the comparison with the previously known equivalent reduction based on the general covariance and obtain it also from the conformal symmetry. Our results confirm that the anomaly induced effective action in $6D$ really has a qualitatively new (compared to previously elaborated $2D$ and $4D$ cases) ambiguity, which is parametrized by the two parameters ξ_1 and ξ_2.

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I. INTRODUCTION

The integration of trace anomaly is the simplest way to derive the effective action (EA) of vacuum. The anomaly-induced action proved being a powerful tool due to the compact and useful form of the result, which finds many applications (see, e.g., [1] for the review). The integration of the anomaly was originally done in the two dimensional space $2D$ in the important work of Polyakov [2]. The generalization for $4D$ was done by Riegert [3] and Fradkin and Tseytlin [4]. There are interesting general features of anomaly, which can not be seen in $2D$ and can be merely noticed in $4D$. The reason is that in $4D$ there is only one possible surface term $\Box R$ in the anomaly, while in $2D$ there are no such terms at all.

Things change dramatically in $6D$, where we meet a bunch of the surface terms, which make integration of the anomaly quite a challenging task. As we have discussed in the previous papers [5,6], the number of possible covariant surface terms with the proper dimension coming from the derivatives of the metric may be larger [7], but it can be reduced to eight [6]. In the present contribution we show how this number can be reduced further to seven terms and discuss the relation of the corresponding identity to the diffeomorphism invariance from one side and with the conformal property of the Gauss-Bonnet term in $6D$ from another side. Indeed, the identity itself has been known previously [8], but in what follows we present its direct derivation and also show the relation to conformal transformation of the metric.

The paper is organized as follows. In Sec. II we briefly review the general scheme of integrating the anomaly in even dimension and present the result of anomaly integration in $6D$. In Sec. III we describe the reduction of the basis of surface terms. Section IV describes how the main reduction formula is related to the general covariance and conformal invariance of the term which is topological invariant in $6D$, and why this identity is valid beyond this particular dimension. Finally, in Sec. V we draw our conclusions and discuss the implications of this work for the integration of the anomaly.

II. ANOMALY INDUCED EFFECTIVE ACTION

The general structure of conformal anomaly in an arbitrary even dimension $D = 2n$ includes the following three types of terms:

(i) Conformal invariant structures, such as $C^2_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ in $4D$. In the simplest case of $2D$ there are no conformal terms, while in higher dimensions there may be much more such terms $\sum c_r W^r_D$, with the sum over r. For instance, there are three of them in $6D$ [9].

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(ii) The topological invariant

$$E_{(2n)} = \frac{1}{2^n}\varepsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\varepsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1\delta_1}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}. \tag{1}$$

(iii) The set of surface terms $\Xi_D = \sum \gamma_k \chi_k$. In $4D$ there is only one surface term $\Box R$, and in higher dimensions there are always much more such terms. The main subject of the present communication is the reduction of the minimal set of surface terms in $6D$.

The reason for the described classification of constituents of the trace anomaly (see, e.g., [10–12]) is that the anomaly reflects the form of the one-loop divergences in the vacuum sector, and the last satisfy conformal Noether identity in case when quantized matter fields are conformal. Thus the anomaly can be presented in the universal form

$$T = \langle T^\mu_\mu \rangle = c_r W^r_D + aE_D + \Xi_D. \tag{2}$$

For the integration of anomaly it proves useful to start from the conformal transformation of the metric tensor $g_{\mu\nu}$,

$$g_{\mu\nu} = e^{2\sigma(x)}\bar{g}_{\mu\nu}. \tag{3}$$

The main ingredient of the scheme described in [3,4] is the transformation rule for the corrected topological invariant,

$$\sqrt{-g}\tilde{E}_D = \sqrt{-\bar{g}}(\bar{\tilde{E}}_D + \kappa\bar{\Delta}_D\sigma), \tag{4}$$

where $D = 2, 4, 6, \ldots$ and

$$\tilde{E}_D = E_D + \sum_i \alpha_i \Xi_i \tag{5}$$

is the modified Euler density which is a sum of the original topological term E_D and a special linear combination of the total derivatives of the curvature-dependent terms Ξ_i with the coefficients α_i. Finally, Δ_D is the conformal operator, which is a D-dimensional generalization of the Paneitz operator in $4D$, [13,14]. The $6D$ solution for α_i is [6]

$$\alpha_1 = \frac{3}{5}, \quad \alpha_2 = -\frac{9}{10} - \frac{5}{4}\xi_1 + \frac{3}{8}\xi_2, \quad \alpha_3 = \xi_1, \quad \alpha_4 = 0,$$
$$\alpha_5 = \frac{84}{5} + 3\xi_1 + \frac{11}{2}\xi_2, \quad \alpha_6 = -\frac{36}{5} - 2\xi_1 - 5\xi_2$$
$$\alpha_7 = -\frac{18}{5} - \xi_1 - \frac{7}{2}\xi_2, \quad \alpha_8 = \xi_2. \tag{6}$$

Here ξ_1 and ξ_2 are two arbitrary parameters which can be fixed only if we find more than one identically vanishing linear combination of the surface terms. Indeed, the two-parameter ambiguity in the conformal operator $\bar{\Delta}_D$ has been found in the paper [15].

As far as the coefficients α_i in Eq. (5) are established, the integration of conformal anomaly becomes a relatively simple exercise, and the general answer can be written in the form [6]

$$\Gamma_{\rm ind} = S_c + \iint_{xy}\left\{\frac{1}{4}c_r W^r_D + \frac{a}{8}\tilde{E}_D(x)\right\}G(x,y)\tilde{E}_D(y) + \sum_k(\gamma_k - \alpha_k)\sum_i c_{ik}\int_x \mathcal{L}_i. \tag{7}$$

Here $S_c = S_c[g_{\mu\nu}]$ is an arbitrary conformal invariant functional, $\int_x \equiv \int d^Dx\sqrt{-g}$, $G(x,y)$ is the Green function of the conformal operator Δ_D and, finally, $\mathcal{L}_i$ are local Lagrangians which generate the surface terms in the anomaly through the relations,

$$-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\sum_i c_{ik}\int_x \mathcal{L}_i = \chi_k. \tag{8}$$

One can see that surface terms χ_k play a decisive role in the integration of anomaly. Therefore it is very desirable to establish a minimal set of linear independent surface terms. According to the previous publications, e.g., [7] or [6] there are eight such terms. In the next section we show how this number can be reduced to seven.

III. REDUCTION OF SIX DERIVATIVE SURFACE TERMS

The set of six derivative surface terms which was used in [6] looks as follows:

$$\Xi_1 = \Box^2 R, \quad \Xi_2 = \Box R^2_{\mu\nu\alpha\beta}, \quad \Xi_3 = \Box R^2_{\mu\nu}, \quad \Xi_4 = \Box R^2,$$
$$\Xi_5 = \nabla_\mu\nabla_\nu(R^\mu{}_{\lambda\alpha\beta}R^{\nu\lambda\alpha\beta}), \quad \Xi_6 = \nabla_\mu\nabla_\nu(R_{\alpha\beta}R^{\mu\alpha\nu\beta})$$
$$\Xi_7 = \nabla_\mu\nabla_\nu(R^\mu_\alpha R^{\nu\alpha}), \quad \Xi_8 = \nabla_\mu\nabla_\nu(RR^{\mu\nu}). \tag{9}$$

Let us start with the following statement which can be obtained by direct calculation. Performing the conformal transformations of the structures Ξ_k one can prove, with the help of the software *Mathematica* [16], that the following linear combination of surface terms is conformal invariant:

$$\sqrt{-g}(\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8) = \sqrt{-\bar{g}}(\bar{\Xi}_2 - 4\bar{\Xi}_3 + \bar{\Xi}_4 - 4\bar{\Xi}_5 + 8\bar{\Xi}_6 + 8\bar{\Xi}_7 - 4\bar{\Xi}_8). \tag{10}$$

Could the above combination be identically vanishing, indicating linear dependence of the set (9)? The answer to this question is positive, and we demonstrate this in what follows. For the sake of this proof, we introduce the following notations:

$$\begin{aligned}
&\Sigma_1 = \Box^2 && R\Sigma_2 = (\nabla_\lambda R_{\mu\nu\alpha\beta})^2 && \Sigma_3 = R_{\mu\alpha\nu\beta}\nabla^\mu\nabla^\nu R^{\alpha\beta}\\
&\Sigma_4 = R_{\mu\nu}R^{\mu\lambda\alpha\beta}R^\nu{}_{\lambda\alpha\beta} && \Sigma_5 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\mu\nu} && \Sigma_6 = R^\mu{}_\alpha{}^\nu{}_\beta R^\alpha{}_\lambda{}^\beta{}_\tau R^\lambda{}_\mu{}^\tau{}_\nu\\
&\Sigma_7 = (\nabla_\lambda R_{\mu\nu})^2 && \Sigma_8 = R_{\mu\nu}\Box R^{\mu\nu} && \Sigma_9 = (\nabla_\mu R)^2\\
&\Sigma_{10} = R\Box R && \Sigma_{11} = (\nabla_\alpha R_{\mu\nu})\nabla^\mu R^{\nu\alpha} && \Sigma_{12} = R^{\mu\nu}\nabla_\mu\nabla_\nu R\\
&\Sigma_{13} = R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\nu\beta} && \Sigma_{14} = R_{\mu\nu}R^{\mu\alpha}R^\nu_\alpha. &&
\end{aligned} \tag{11}$$

Let us elaborate each of the terms Ξ_i using notations (11). We start from the trivial simplest case and then go the more complicated part.

$$\Box^2 R = \Sigma_1. \tag{12}$$

$$\Box R^2_{\mu\nu\alpha\beta} = 2(\nabla_\lambda R_{\mu\nu\alpha\beta})^2 + 2R_{\mu\nu\alpha\beta}\Box R^{\mu\nu\alpha\beta}. \tag{13}$$

Using the properties of the Riemann tensor and Bianchi identities, the second term in the last relation can be rewritten as [17]

$$R_{\mu\nu\alpha\beta}\Box R^{\mu\nu\alpha\beta} = 4R_{\mu\alpha\nu\beta}\nabla^\mu\nabla^\nu R^{\alpha\beta} + 2R_{\mu\nu}R^{\mu\lambda\alpha\beta}R^\nu{}_{\lambda\alpha\beta} - R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\mu\nu} - 4R^\mu{}_\alpha{}^\nu{}_\beta R^\alpha{}_\lambda{}^\beta{}_\tau R^\lambda{}_\mu{}^\tau{}_\nu. \tag{14}$$

Thus we arrive at the first relations

$$\Box R^2_{\mu\nu\alpha\beta} = 2\Sigma_2 + 8\Sigma_3 + 4\Sigma_4 - 2\Sigma_5 - 8\Sigma_6, \tag{15}$$

$$\Box R^2_{\mu\nu} = 2(\nabla_\lambda R_{\mu\nu})^2 + 2R_{\mu\nu}\Box R^{\mu\nu} = 2\Sigma_7 + 2\Sigma_8, \tag{16}$$

$$\Box R^2 = 2(\nabla_\lambda R)^2 + 2R\Box R = 2\Sigma_9 + 2\Sigma_{10}. \tag{17}$$

Furthermore,

$$\begin{aligned}
\nabla_\mu\nabla_\nu(R^\mu{}_{\lambda\alpha\beta}R^{\nu\lambda\alpha\beta})
&= \nabla_\mu[(\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} + R^\mu{}_{\lambda\alpha\beta}\nabla_\nu R^{\nu\lambda\alpha\beta}]\\
&= \nabla_\mu[(\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} + R^\mu{}_{\lambda\alpha\beta}\nabla^\alpha R^{\lambda\beta} - R^\mu{}_{\lambda\alpha\beta}\nabla^\beta R^{\lambda\alpha}]\\
&= (\nabla_\mu\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} + (\nabla_\nu R^\mu{}_{\lambda\alpha\beta})\nabla_\mu R^{\nu\lambda\alpha\beta}\\
&\quad + (\nabla_\mu R^\mu{}_{\lambda\alpha\beta})\nabla^\alpha R^{\lambda\beta} + R^\mu{}_{\lambda\alpha\beta}\nabla_\mu\nabla^\alpha R^{\lambda\beta}\\
&\quad - (\nabla_\mu R^\mu{}_{\lambda\alpha\beta})\nabla^\beta R^{\lambda\alpha} - R^\mu{}_{\lambda\alpha\beta}\nabla_\mu\nabla^\beta R^{\lambda\alpha}\\
&= (\nabla_\mu\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} + (\nabla_\nu R^\mu{}_{\lambda\alpha\beta})\nabla_\mu R^{\nu\lambda\alpha\beta}\\
&\quad + 2(\nabla_\mu R^\mu{}_{\lambda\alpha\beta})\nabla^\alpha R^{\lambda\beta} + 2\Sigma_3.
\end{aligned} \tag{18}$$

The first term in the expression (18) can be transformed as follows:

$$\begin{aligned}
(\nabla_\mu\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta}
&= (\nabla_\nu\nabla_\mu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} + R^{\nu\lambda\alpha\beta}[\nabla_\mu,\nabla_\nu]R^\mu{}_{\lambda\alpha\beta}\\
&= (\nabla_\nu\nabla_\alpha R_{\lambda\beta})R^{\nu\lambda\alpha\beta} - (\nabla_\nu\nabla_\beta R_{\lambda\alpha})R^{\nu\lambda\alpha\beta} + R_{\kappa\nu}R^\kappa{}_{\lambda\alpha\beta}R^{\nu\lambda\alpha\beta}\\
&\quad - R^\kappa{}_{\lambda\mu\nu}R^\mu{}_{\kappa\alpha\beta}R^{\nu\lambda\alpha\beta} - R^\kappa{}_{\alpha\mu\nu}R^\mu{}_{\lambda\kappa\beta}R^{\nu\lambda\alpha\beta} - R^\kappa{}_{\beta\mu\nu}R^\mu{}_{\lambda\alpha\kappa}R^{\nu\lambda\alpha\beta}\\
&= 2\Sigma_3 + \Sigma_4 - R^\kappa{}_{\lambda\mu\nu}R^\mu{}_{\kappa\alpha\beta}R^{\nu\lambda\alpha\beta} - 2R^\kappa{}_{\alpha\mu\nu}R^\mu{}_{\lambda\kappa\beta}R^{\nu\lambda\alpha\beta}.
\end{aligned} \tag{19}$$

At this moment we remember that

$$\begin{aligned}
R^\kappa{}_{\lambda\mu\nu}R^\mu{}_{\kappa\alpha\beta}R^{\nu\lambda\alpha\beta} &= R_{\kappa\lambda\mu\nu}R^{\mu\kappa}{}_{\alpha\beta}R^{\alpha\beta\nu\lambda}\\
&= -R_{\kappa\nu\lambda\mu}R^{\mu\kappa}{}_{\alpha\beta}R^{\alpha\beta\nu\lambda} - R_{\kappa\mu\nu\lambda}R^{\mu\kappa}{}_{\alpha\beta}R^{\alpha\beta\nu\lambda}.
\end{aligned} \tag{20}$$

By making change of indices $\nu \leftrightarrow \lambda$ in the first of these expressions, we arrive at

$$R^\kappa{}_{\lambda\mu\nu}R^\mu{}_{\kappa\alpha\beta}R^{\nu\lambda\alpha\beta} = -R_{\kappa\lambda\nu\mu}R^{\mu\kappa}{}_{\alpha\beta}R^{\alpha\beta\lambda\nu} + \Sigma_5 = \frac{1}{2}\Sigma_5. \tag{21}$$

Next, the last term of (19) can be transformed as

$$\begin{aligned}
R^\kappa{}_{\alpha\mu\nu}R^\mu{}_{\lambda\kappa\beta}R^{\nu\lambda\alpha\beta} &= R^\kappa{}_\alpha{}^\mu{}_\nu R_\mu{}^\lambda{}_\kappa{}^\beta R^\nu{}_\lambda{}^\alpha{}_\beta\\
&= R^\mu{}_\nu{}^\kappa{}_\alpha R^\nu{}_\lambda{}^\alpha{}_\beta R^\lambda{}_\mu{}^\beta{}_\kappa = \Sigma_6
\end{aligned} \tag{22}$$

thus we get

$$(\nabla_\mu\nabla_\nu R^\mu{}_{\lambda\alpha\beta})R^{\nu\lambda\alpha\beta} = 2\Sigma_3 + \Sigma_4 - \frac{1}{2}\Sigma_5 - 2\Sigma_6. \tag{23}$$

The second term of the second expression of (18) can be developed as

$$(\nabla_\nu R_{\mu\lambda\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta} = -(\nabla_\lambda R_{\nu\mu\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta} - (\nabla_\mu R_{\lambda\nu\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta}. \tag{24}$$

By using the index exchange $\nu \leftrightarrow \lambda$ in the last formula we get

$$\begin{aligned}
(\nabla_\nu R_{\mu\lambda\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta}
&= -(\nabla_\nu R_{\lambda\mu\alpha\beta})\nabla^\mu R^{\lambda\nu\alpha\beta} - (\nabla_\mu R_{\lambda\nu\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta}\\
&= -(\nabla_\nu R_{\mu\lambda\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta} + \Sigma_2,
\end{aligned} \tag{25}$$

hence

$$(\nabla_\nu R_{\mu\lambda\alpha\beta})\nabla^\mu R^{\nu\lambda\alpha\beta} = \frac{1}{2}\Sigma_2. \tag{26}$$

Using the first reduced Bianchi identity, we develop the third term of (18) such that

$$(\nabla_\mu R^\mu{}_{\lambda\alpha\beta})\nabla^\alpha R^{\lambda\beta} = (\nabla_\alpha R_{\lambda\beta})\nabla^\alpha R^{\lambda\beta} - (\nabla_\beta R_{\lambda\alpha})\nabla^\alpha R^{\lambda\beta} = \Sigma_7 - \Sigma_{11}. \tag{27}$$

Replacing (23), (26), and (27) into (18) we obtain

$$\nabla_\mu\nabla_\nu(R^\mu{}_{\lambda\alpha\beta}R^{\nu\lambda\alpha\beta}) = \frac{1}{2}\Sigma_2 + 4\Sigma_3 + \Sigma_4 - \frac{1}{2}\Sigma_5 - 2\Sigma_6 + 2\Sigma_7 - 2\Sigma_{11}. \tag{28}$$

The next step is to consider

$$\begin{aligned}
&\nabla_\mu\nabla_\nu(R_{\alpha\beta}R^{\mu\alpha\nu\beta})\\
&= \nabla_\mu[(\nabla_\nu R_{\alpha\beta})R^{\mu\alpha\nu\beta} + R_{\alpha\beta}\nabla_\nu R^{\nu\beta\mu\alpha}]\\
&= \nabla_\mu[(\nabla_\nu R_{\alpha\beta})R^{\mu\alpha\nu\beta} + R_{\alpha\beta}\nabla^\mu R^{\beta\alpha} - R_{\alpha\beta}\nabla^\alpha R^{\beta\mu}]\\
&= (\nabla_\mu\nabla_\nu R_{\alpha\beta})R^{\mu\alpha\nu\beta} + (\nabla_\nu R_{\alpha\beta})\nabla_\mu R^{\mu\alpha\nu\beta} + (\nabla_\mu R_{\alpha\beta})^2\\
&\quad + R_{\alpha\beta}\Box R^{\alpha\beta} - (\nabla_\mu R_{\alpha\beta})\nabla^\alpha R^{\beta\mu} - R^\alpha_\beta\nabla_\mu\nabla_\alpha R^{\beta\mu}\\
&= \Sigma_3 + (\nabla_\nu R_{\alpha\beta})^2 - (\nabla_\nu R_{\alpha\beta})\nabla^\beta R^{\alpha\nu} + \Sigma_7 + \Sigma_8 - \Sigma_{11}\\
&\quad - \frac{1}{2}R^{\alpha\beta}\nabla_\alpha\nabla_\beta R - R^\alpha_\beta[\nabla_\mu, \nabla_\alpha]R^{\beta\mu}\\
&= \Sigma_3 + 2\Sigma_7 + \Sigma_8 - 2\Sigma_{11} - \frac{1}{2}\Sigma_{12}\\
&\quad - R^\alpha_\beta R^\beta{}_{\kappa\mu\alpha}R^{\kappa\mu} - R^\alpha_\beta R_{\kappa\alpha}R^{\beta\kappa},
\end{aligned} \tag{29}$$

that means

$$\nabla_\mu\nabla_\nu(R_{\alpha\beta}R^{\mu\alpha\nu\beta}) = \Sigma_3 + 2\Sigma_7 + \Sigma_8 - 2\Sigma_{11} - \frac{1}{2}\Sigma_{12} + \Sigma_{13} - \Sigma_{14}. \tag{30}$$

Next,

$$\begin{aligned}
&\nabla_\mu\nabla_\nu(R^\mu_\alpha R^{\nu\alpha})\\
&= \nabla_\mu\left[(\nabla_\nu R^\mu_\alpha)R^{\nu\alpha} + \frac{1}{2}R^\mu_\alpha\nabla^\alpha R\right]\\
&= (\nabla_\mu\nabla_\nu R^\mu_\alpha)R^{\nu\alpha} + (\nabla_\nu R^\mu_\alpha)\nabla_\mu R^{\nu\alpha} + \frac{1}{4}(\nabla_\alpha R)^2 + \frac{1}{2}R^\mu_\alpha\nabla_\mu\nabla^\alpha R\\
&= \frac{1}{2}(\nabla_\nu\nabla_\alpha R)R^{\nu\alpha} + ([\nabla_\mu,\nabla_\nu]R^\mu_\alpha)R^{\nu\alpha} + \frac{1}{4}\Sigma_9 + \Sigma_{11} + \frac{1}{2}\Sigma_{12}\\
&= \frac{1}{4}\Sigma_9 + \Sigma_{11} + \Sigma_{12} + R_{\kappa\nu}R^{\kappa\alpha}R^\nu_\alpha + R^\alpha{}_{\kappa\mu\nu}R^{\mu\kappa}R^\nu_\alpha.\\
&= \frac{1}{4}\Sigma_9 + \Sigma_{11} + \Sigma_{12} - \Sigma_{13} + \Sigma_{14}.
\end{aligned} \tag{31}$$

Finally, even simpler operations provide the last ingredients,

$$\begin{aligned}
\nabla_\mu\nabla_\nu(RR^{\mu\nu}) &= \frac{1}{2}R\Box R + (\nabla_\mu R)^2 + R^{\mu\nu}\nabla_\mu\nabla_\nu R,\\
\nabla_\mu\nabla_\nu(RR^{\mu\nu}) &= \Sigma_9 + \frac{1}{2}\Sigma_{10} + \Sigma_{12}.
\end{aligned} \tag{32}$$

Now we possess all what is needed to solve the equation of our interest,

$$a\Xi_1 + b\Xi_2 + c\Xi_3 + d\Xi_4 + e\Xi_5 + f\Xi_6 + g\Xi_7 + h\Xi_8 \equiv 0. \tag{33}$$

The solution for the coefficients of this equation is as follows:

$$\begin{aligned}
&a = 0 \quad b = \beta \quad c = -4\beta \quad d = \beta\\
&e = -4\beta \quad f = 8\beta \quad g = 8\beta \quad h = -4\beta,
\end{aligned} \tag{34}$$

where β is an arbitrary number which can be equal to one. Therefore, we have proved the identity

$$\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8 = 0. \tag{35}$$

Equation (35) resolves the main problem which we posed at the beginning of this contribution. Namely, it reduces the number of linearly independent six-derivative surface terms from eight to seven. Still this is not a complete solution of all relevant issues which one meets in the part of surface terms, and one can find the description of the remaining problems in the next section.

IV. TWO SIDES OF THE IDENTITY (35)

After sending the first version of this manuscript to arXiv we learned about the well-known paper [8], where the identity (35) has been used for deriving other relations between the equations of motion of the six-derivative actions in $6D$. The way this identity has been obtained in the mentioned work came from the similar consideration in [18] for the Gauss-Bonnet invariant in $4D$. The relation can be obtained as a Noether identity for the diffeomorphism invariance of the corresponding topological action (in some formulas we avoid using condensed notation for a D dimensional integral, just to stress its dimension),

$$S^{(D)}_{GB} = \int d^Dx\sqrt{-g}E_D. \tag{36}$$

The Gauss-Bonnet term (1) in $6D$ can be cast into the form

$$E_6 = \frac{1}{8}\varepsilon^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\varepsilon^{\gamma_1\delta_1\ldots\gamma_3\delta_3}R_{\alpha_1\beta_1\gamma_1\delta_1}\ldots R_{\alpha_3\beta_3\gamma_3\delta_3} = -8\mathcal{L}_1 + 4\mathcal{L}_2 - 24\mathcal{L}_3 + 24\mathcal{L}_4 + 16\mathcal{L}_5 + 3\mathcal{L}_6 - 12\mathcal{L}_7 + \mathcal{L}_8. \tag{37}$$

It is interesting that the first of these presentations does not admit simple generalization to an arbitrary dimension D, while for the second one it is not an obstacle. In what follows we will assume that E_6 means the expression in the right-hand side (r.h.s.) when it is considered in $D \neq 6$.

The Noether identity for the general covariance of the action has the form

$$\nabla_\mu\left[\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}S^{(D)}_{GB}\right] = 0. \tag{38}$$

The last identity reflects only the covariance of the action (36) and does not use the topological nature of this expression. Therefore this identity is going to hold even for the dimension D where this action is not topological. At the same time, since in the "proper" dimension the equation of motion for the topological action is supposed to vanish (see [19] and the book [20] for detailed discussion), its trace is also vanishing [5]. One can anticipate that in the case of the action (36) this can produce another identity, which can be related to (38) due to the topological nature of the action in the "proper" dimension. Let us check the situation in the case of $6D$.

In what follows we will need the list of the six-derivative actions which are not full derivatives. One can define these actions in the form $I_n = \int_x \mathcal{L}_n$, where

$$\begin{aligned}
&\mathcal{L}_1 = R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\tau}R^{\lambda}{}_{\rho}{}^{\tau}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta},\\
&\mathcal{L}_2 = R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta},\\
&\mathcal{L}_3 = R_{\alpha\beta}R^{\alpha}{}_{\gamma\lambda\tau}R^{\beta\gamma\lambda\tau},\\
&\mathcal{L}_4 = R^{\alpha\beta}R^{\lambda\tau}R_{\alpha\lambda\beta\tau}, \qquad \mathcal{L}_5 = R^{\alpha}_{\lambda}R^{\beta}_{\alpha}R^{\lambda}_{\beta},\\
&\mathcal{L}_6 = RR^2_{\alpha\beta\lambda\tau}, \qquad \mathcal{L}_7 = RR^2_{\alpha\beta},\\
&\mathcal{L}_8 = R^3, \qquad \mathcal{L}_9 = R^{\alpha\beta}\Box R_{\alpha\beta}, \qquad \mathcal{L}_{10} = R\Box R.
\end{aligned}\tag{39}$$

Furthermore, let us give the list of the corresponding equations of motion [17] (see also [8,21])

$$\Phi^{\mu\nu}_n = \frac{1}{\sqrt{-g}}\frac{\delta I_n}{\delta g_{\mu\nu}} \quad \text{and their traces} \quad \Phi_n = g_{\mu\nu}\Phi^{\mu\nu}_n, \tag{40}$$

which have the following form:

$$\begin{aligned}
\Phi^{\mu\nu}_1 &= \frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\tau}R^{\lambda}{}_{\rho}{}^{\tau}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} - 3R^{\alpha}{}_{\lambda}{}^{(\mu}{}_{\tau}R_{\rho\alpha\sigma}{}^{\nu)}R^{\lambda\rho\tau\sigma} + 3\nabla_\lambda\nabla_\beta(R^{\lambda}{}_{\rho}{}^{(\mu}{}_{\sigma}R^{\nu)\rho\beta\sigma}) - 3\nabla^\lambda\nabla^\tau(R^{(\mu}{}_{\rho}{}^{\nu)}{}_{\sigma}R^{\rho}{}_{\lambda}{}^{\sigma}{}_{\tau}),\\
\Phi_1 &= \frac{D-6}{2}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\tau}R^{\lambda}{}_{\rho}{}^{\tau}{}_{\sigma}R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} + \frac{3}{2}\Xi_5 - 3\Xi_6,
\end{aligned}\tag{41}$$

$$\begin{aligned}
\Phi^{\mu\nu}_2 &= \frac{1}{2}g^{\mu\nu}R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} - 3R^{\alpha(\mu}{}_{\lambda\tau}R_{\rho\sigma\alpha}{}^{\nu)}R^{\lambda\tau\rho\sigma} - 6\nabla^\beta\nabla_\tau(R^{\tau(\mu}{}_{\rho\sigma}R_{\beta}{}^{\nu)\rho\sigma}),\\
\Phi_2 &= \frac{D-6}{2}R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} - 6\Xi_5,
\end{aligned}\tag{42}$$

$$\begin{aligned}
\Phi^{\mu\nu}_3 &= \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha}{}_{\gamma\lambda\tau}R^{\beta\gamma\lambda\tau} - R^{(\mu}_{\alpha}R^{\nu)\gamma\lambda\tau}R^{\alpha}{}_{\gamma\lambda\tau} - 2R_{\alpha\beta}R^{\alpha}{}_{\gamma}{}^{(\mu}{}_{\tau}R^{\nu)\tau\beta\gamma} - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\alpha\gamma\lambda\tau}R^{\beta}{}_{\gamma\lambda\tau}) + \nabla_\alpha\nabla^{(\mu}(R^{\nu)\gamma\lambda\tau}R^{\alpha}{}_{\gamma\lambda\tau})\\
&\quad - \frac{1}{2}\Box(R^{\mu}{}_{\gamma\lambda\tau}R^{\nu\gamma\lambda\tau}) - 2\nabla_\gamma\nabla_\lambda(R^{(\mu}_{\alpha}R^{\nu)\lambda\alpha\gamma}) - 2\nabla_\alpha\nabla^\tau(R^{\alpha}_{\beta}R^{\beta(\mu}{}_{\tau}{}^{\nu)}),\\
\Phi_3 &= \frac{D-6}{2}R_{\alpha\beta}R^{\alpha}{}_{\gamma\lambda\tau}R^{\beta\gamma\lambda\tau} - \frac{1}{2}\Xi_2 - \frac{D-2}{2}\Xi_5 - 2\Xi_6 - 2\Xi_7,
\end{aligned}\tag{43}$$

$$\begin{aligned}
\Phi^{\mu\nu}_4 &= \frac{1}{2}g^{\mu\nu}R^{\alpha\beta}R^{\lambda\tau}R_{\alpha\lambda\beta\tau} - 3(R_{\lambda\alpha}{}^{(\mu}{}_{\beta}R^{\nu)\lambda}R^{\alpha\beta}) - \Box(R^{\alpha\mu\beta\nu}R_{\alpha\beta}) - \nabla_\alpha\nabla_\beta(R^{\mu\nu}R^{\alpha\beta})\\
&\quad - g^{\mu\nu}\nabla^\alpha\nabla^\beta(R^{\lambda\tau}R_{\lambda\alpha\tau\beta}) + 2\nabla_\lambda\nabla^{(\mu}(R^{\nu)\alpha\lambda\beta}R_{\alpha\beta}) + \nabla_\alpha\nabla_\beta(R^{\alpha(\mu}R^{\nu)\beta}),\\
\Phi_4 &= \frac{D-6}{2}R^{\alpha\beta}R^{\lambda\tau}R_{\alpha\lambda\beta\tau} - \Xi_3 - (D-2)\Xi_6 + \Xi_7 - \Xi_8,
\end{aligned}\tag{44}$$

$$\begin{aligned}
\Phi^{\mu\nu}_5 &= \frac{1}{2}g^{\mu\nu}R^{\alpha}_{\lambda}R^{\beta}_{\alpha}R^{\lambda}_{\beta} - 3R^{\mu}_{\beta}R^{\nu\lambda}R^{\beta}_{\lambda} + 3\nabla^\alpha\nabla^{(\mu}(R^{\nu)}_{\lambda}R^{\lambda}_{\alpha}) - \frac{3}{2}g^{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\lambda\alpha}R^{\beta}_{\lambda}) - \frac{3}{2}\Box(R^{\mu}_{\lambda}R^{\nu\lambda}),\\
\Phi_5 &= \frac{D-6}{2}R^{\alpha}_{\lambda}R^{\beta}_{\alpha}R^{\lambda}_{\beta} - \frac{3}{2}\Xi_3 - 3\frac{D-2}{2}\Xi_7,
\end{aligned}\tag{45}$$

$$\Phi_6^{\mu\nu} = \frac{1}{2}g^{\mu\nu}RR^2_{\alpha\beta\lambda\tau} - R^{\mu\nu}R^2_{\alpha\beta\lambda\tau} - 2RR_{\lambda\tau}{}^{\alpha(\mu}R_\alpha{}^{\nu)\lambda\tau} + \nabla^\mu\nabla^\nu R^2_{\alpha\beta\lambda\tau} - g^{\mu\nu}\Box R^2_{\alpha\beta\lambda\tau} - 4\nabla^\beta\nabla^\lambda(RR_\beta{}^{(\mu}{}_\lambda{}^{\nu)}),$$
$$\Phi_6 = \frac{D-6}{2}RR^2_{\alpha\beta\lambda\tau} - (D-1)\Xi_2 - 4\Xi_8, \tag{46}$$

$$\Phi_7^{\mu\nu} = \frac{1}{2}g^{\mu\nu}RR^2_{\alpha\beta} - R^{\mu\nu}R^2_{\alpha\beta} - 2RR^\mu_\lambda R^{\nu\lambda} + \nabla^\mu\nabla^\nu(R^2_{\alpha\beta}) - g^{\mu\nu}\Box R^2_{\alpha\beta} + 2\nabla_\alpha\nabla^{(\mu}(R^{\nu)\alpha}R) - g^{\mu\nu}\nabla_\alpha\nabla_\beta(RR^{\alpha\beta}) - \Box(RR^{\mu\nu}),$$
$$\Phi_7 = \frac{D-6}{2}RR^2_{\alpha\beta} - (D-1)\Xi_3 - \Xi_4 - (D-2)\Xi_8, \tag{47}$$

$$\Phi_8^{\mu\nu} = \frac{1}{2}g^{\mu\nu}R^3 - 3R^{\mu\nu}R^2 + 3\nabla^\mu\nabla^\nu R^2 - 3g^{\mu\nu}\Box R^2,$$
$$\Phi_8 = \frac{D-6}{2}R^3 - 3(D-1)\Xi_4, \tag{48}$$

$$\Phi_9^{\mu\nu} = \frac{1}{2}g^{\mu\nu}R^{\alpha\beta}\Box R_{\alpha\beta} - R^{\alpha\beta}\nabla^{(\mu}\nabla^{\nu)}R_{\alpha\beta} - 2R_\alpha^{(\mu}\Box R^{\nu)\alpha} + 2\nabla^\alpha\nabla^{(\mu}\Box R_\alpha^{\nu)} - g^{\mu\nu}\nabla_\alpha\nabla_\beta\Box R^{\alpha\beta} - \Box^2R^{\mu\nu} + 2\nabla^\alpha(R_{\alpha\beta}\nabla^{(\mu}R^{\nu)\beta})$$
$$- 2\nabla^\alpha(R^{\beta(\mu}\nabla^{\nu)}R_{\alpha\beta}) + \nabla^{(\mu}(R_{\alpha\beta}\nabla^{\nu)}R^{\alpha\beta}) - \frac{1}{2}g^{\mu\nu}\nabla^\lambda(R_{\alpha\beta}\nabla_\lambda R^{\alpha\beta}),$$
$$\Phi_9 = \frac{D-6}{2}R^{\alpha\beta}\Box R_{\alpha\beta} - \frac{D}{2}\Xi_1 - \frac{D}{2}\Xi_3 + \frac{D-4}{4}\Xi_4 + 2(D-2)\Xi_6 - 2\Xi_7 - (D-4)\Xi_8, \tag{49}$$

$$\Phi_{10}^{\mu\nu} = -\frac{1}{2}g^{\mu\nu}(\nabla_\alpha R)^2 + (\nabla^\mu R)(\nabla^\nu R) + 2\nabla^\mu\nabla^\nu\Box R - 2g^{\mu\nu}\Box^2R - 2R^{\mu\nu}\Box R,$$
$$\Phi_{10} = -\frac{D-6}{2}R\Box R - 2(D-1)\Xi_1 - \frac{D-2}{4}\Xi_4. \tag{50}$$

In the derivation of traces we used expressions given in Appendix A.

Taking the last observation and new notation into account, by combining Eqs. (41)–(48) we arrive at the following relation:

$$\frac{1}{\sqrt{-g}}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int_x E_6 = \frac{D-6}{2}E_6 - 3(D-5)(\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8). \tag{51}$$

The first term in the r.h.s. of the Eq. (51) obviously vanish in $D = 6$. At the same time, the left-hand side (l.h.s.) also vanish in $D = 6$, because in this specific dimension it is the trace of the variational derivative of the topological term.[1] In this sense the relation (51) proves that the remaining term in the r.h.s. also vanish in $D = 6$. However, this term is exactly an identity (35) which we proved directly in the previous section. It is worth noticing that the proof which we presented there does not depend on the dimension.

[1]One can prove this even without taking trace (see, e.g., [20]), but such a proof requires choosing a special coordinate system. In general coordinates this equation does not look trivial, as it was discussed in [19,22].

Taking the identity (35) into account, we arrive at the simple rule of conformal shift of the term under consideration, namely

$$\frac{g_{\mu\nu}}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_x E_6 = \frac{D-6}{2}E_6, \tag{52}$$

which is perfectly consistent to the main relation of integrating anomaly (5).

To close the story, let us mention that there is yet another equivalent form of our identity (35)

$$\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8 = \frac{1}{4}\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}\nabla_\mu\nabla^\nu(R^{\xi\eta}{}_{\alpha\beta}R^{\kappa\chi}{}_{\lambda\tau}), \tag{53}$$

where (in Euclidean signature)

$$\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi} = \epsilon^{\rho\mu\alpha\beta\lambda\tau}\epsilon_{\rho\nu\xi\eta\kappa\chi} = 5!\delta^{[\mu}_\nu\delta^\alpha_\nu\delta^\beta_\eta\delta^\lambda_\kappa\delta^{\tau]}_\chi. \tag{54}$$

The proof of the relation

$$\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}\nabla_\mu\nabla^\nu(R^{\xi\eta}{}_{\alpha\beta}R^{\kappa\chi}{}_{\lambda\tau}) \equiv 0 \tag{55}$$

can be found in Appendix B.

V. CONCLUSIONS AND DISCUSSIONS

As we have just mentioned above, Eq. (35) reduces the number of surface terms which is needed to construct the full basis of such terms in $6D$. Let us discuss the importance of this formula in the general context.

Consistent and complete integration of trace anomaly in $6D$ requires several mathematical results which are all not easy to accomplish, mainly because the practical calculations in $6D$ are essentially more involved than the ones in $4D$. In the first place one needs the main formula (5) which immediately produces the nonlocal part of the anomaly-induced action [6,10]. The general formal expression for this action (7) for an arbitrary even dimension has been constructed in Ref. [6], where we also reported on the explicit realization of the key formula (5) in the case of $6D$. Then, looking at the general expression (7) we can see that the remaining part of the effective action is related to the integration of total derivative terms.

Usually the importance of total derivative terms in the anomaly is underestimated, since it is supposed that they can be modified or eliminated by adding finite local counterterms. Such an addition is a mathematically legal procedure, because the gravitational vacuum action is not quantized in the framework of semiclassical theory. Therefore, even if the local nonconformal terms are not needed for renormalization, one can add them without changing the general structure of quantum theory in curved space.

In some cases such an addition can be pretty well justified. The main example of this sort is the Starobinsky inflation [23,24] where the R^2 term with a very large coefficient is required to provide the control over perturbations and, in general, correspondence with the existing observational data. The attempts to explain the magnitude of this coefficient from quantum field theory arguments are currently at the rudimentary level (see, e.g., [25]) and hence the introduction of the large coefficient of R^2 is a phenomenological operation. In general, and especially in $6D$, there are no observational evidence which can be used to fix the coefficients of the total derivative terms. Therefore for us the importance of these terms is certain and without doubts.[2]

In this situation the formula defining the part of effective action which comes from the total derivative terms in the anomaly is (8). Then the reduction of the number of the total derivatives χ_k in the r.h.s. of this equation from eight to seven increases our chances to find the solution. And, from the general perspective, it would be interesting to have an independent, new and nontrivial confirmation of the possibility to integrate total derivatives with the local gravitational terms, according to Eq. (5).

Two concluding observations are in order. First of all, since in $6D$ the structure (36) is topological, its variational derivative is zero. At the same time, in $6D$ even the identity for the trace is not easy to prove explicitly, as the reader could ensure from Sec. III. The second aspect is that the topological term (1) is unique and, therefore, the vanishing linear combination (35) is also unique[3] The important consequence of this uniqueness is that further reduction of the solution (6) is impossible, because (1) was already taken into account in [6]. Thus the main result of the present work is that now we can affirm that the fundamental difference between the $2D$ and $4D$ formulas (4) from one side and similar formula in $6D$ from another side is that in the last case this important formula has two-parameter ambiguity. The changes of ξ_1 or ξ_2 do not produce a change of conformal functional S_c, which is the unique ambiguous part of the effective action in $2D$ and $4D$ cases. Therefore, now we can claim that in $6D$ we meet a qualitatively new kind of ambiguity, that is something which does not take place in $2D$ and $4D$ cases.

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APPENDIX A: USEFUL RELATIONS FOR TOTAL DERIVATIVES

Let us give a useful list of relations for total derivatives,

$$\nabla_\mu\nabla_\nu(R^\mu{}_{\alpha\beta\lambda}R^{\nu\lambda\beta\alpha}) = \frac{1}{2}\Xi_5, \tag{A1}$$

$$\nabla_\mu(R^{\mu\nu}\nabla_\nu R) = -\frac{1}{4}\Xi_4 + \Xi_8, \tag{A2}$$

$$\nabla_\alpha(R_{\mu\nu}\nabla^\alpha R^{\mu\nu}) = \frac{1}{2}\Xi_3, \tag{A3}$$

$$\nabla_\mu(R^\nu_\lambda\nabla_\nu R^{\mu\lambda}) = \frac{1}{8}\Xi_4 + \Xi_7 - \frac{1}{2}\Xi_8, \tag{A4}$$

$$\nabla_\mu(R^{\mu\alpha\nu\beta}\nabla_\nu R_{\alpha\beta}) = -\frac{1}{2}\Xi_3 + \frac{1}{8}\Xi_4 + \Xi_6 + \Xi_7 - \frac{1}{2}\Xi_8, \tag{A5}$$

$$\nabla_\mu\nabla_\nu\Box R^{\mu\nu} = \frac{1}{2}\Xi_1 + \frac{1}{2}\Xi_3 - \frac{1}{4}\Xi_4 - 2\Xi_6 + \Xi_8. \tag{A6}$$

[2]Further arguments concerning the ambiguities related to local terms in the induced action can be found in Ref. [26], where one can see also the relation to the nonlocal structures in the case of almost vanishing masses of quantum fields.

[3]We are grateful to Dr. Sourya Ray for stressing this point to us.

APPENDIX B: PROOF OF THE RELATION (55)

Let us denote the object of our interest Ω and take one of the derivatives,

$$\Omega = \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}\nabla_\mu\nabla^\nu(R^{\xi\eta}{}_{\alpha\beta}R^{\kappa\chi}{}_{\lambda\tau}) \\ = \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}\nabla_\mu[R^{\kappa\chi}{}_{\lambda\tau}\nabla^\nu R^{\xi\eta}{}_{\alpha\beta} + R^{\xi\eta}{}_{\alpha\beta}\nabla^\nu R^{\kappa\chi}{}_{\lambda\tau}]. \quad \text{(B1)}$$

Using antisymmetry of the object (54) and the Bianchi identity, the last expression transforms into

$$\Omega = 2\nabla_\mu(\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}R^{\kappa\chi}{}_{\lambda\tau}\nabla^\nu R^{\xi\eta}{}_{\alpha\beta}) \\ = -2\nabla_\mu[\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}(R^{\kappa\chi}{}_{\lambda\tau}\nabla^\xi R^{\eta\nu}{}_{\alpha\beta} + R^{\kappa\chi}{}_{\lambda\tau}\nabla^\eta R^{\nu\xi}{}_{\alpha\beta})]. \quad \text{(B2)}$$

Once again using antisymmetry of (54) we arrive at

$$\Omega = -4\nabla_\mu(\delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\kappa\chi}R^{\kappa\chi}{}_{\lambda\tau}\nabla^\nu R^{\xi\eta}{}_{\alpha\beta}). \quad \text{(B3)}$$

Comparing (B2) and (B3) one can check that

$$\Omega = -2\Omega, \quad \text{(B4)}$$

which is equivalent to Eq. (55).

Let us stress that the analog of this result can be found in [18] for $4D$ and can be also found in [8] for $6D$. The derivation of this identity in both cases was based on the relation (38) which reflects diffeomorphism invariance of the action (36) with $D = 6$ and E_6 defined as in the r.h.s. of Eq. (37). For this reason the identity is valid in any dimension D. At the same time the same identity can be also obtained in exactly $D = 6$ as a Noether identity of the conformal symmetry (52).

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