Two new approaches to the anomalous limit of Brans-Dicke theory to Einstein gravity

Valerio Faraoni[1,*] and Jeremy Côté[2,†]

[1]*Department of Physics and Astronomy and STAR Research Cluster, Bishop's University, 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*

[2]*Department of Physics and Astronomy, Bishop's University, 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*

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Contrary to common belief, (electro)vacuum Brans-Dicke gravity does not reduce to general relativity for large Brans-Dicke coupling ω, a problem which has never been fully solved. Two new approaches, independent from each other, shed light on this issue producing the same result: in the limit $\omega \to \infty$ an (electro)vacuum Brans-Dicke spacetime reduces to a solution of the Einstein equations sourced, not by (electro)vacuum, but by a minimally coupled scalar field. The latter is shown to coincide with the Einstein frame scalar field. The first method employs a direct analysis of the Einstein frame, while the second (complementary and independent) method uses an imperfect fluid representation of Brans-Dicke gravity together with a little known 1-parameter symmetry group of this theory.

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I. INTRODUCTION

There is currently a major theoretical and experimental effort to detect, or constrain, deviations of gravity from Einstein's general relativity (GR), which span many areas of research and many spatial scales [1–3], including cosmology [4,5], supermassive black holes [6], stellar mass black hole binaries emitting gravitational waves [7], and Solar System tests [1–3,8–12]. The prototypical alternative to GR is scalar-tensor gravity [13–15], which is motivated by fundamental theoretical considerations and by observational cosmology. On the one hand, every attempt to quantize gravity produces deviations from GR in the form of higher order equations of motion, extra gravitational degrees of freedom (d.o.f.), or curvature corrections to the Einstein-Hilbert action (for example, the low-energy limit of bosonic string theory, the simplest string theory, yields a Brans-Dicke gravity with coupling parameter $\omega = -1$ [16]). On the other hand, from the cosmological point of view, explaining the present acceleration of the Universe without invoking an *ad hoc* dark energy [17] has led to the very popular $f(R)$ class of theories [18] (where R is the Ricci scalar of spacetime). This is nothing but a subclass of Brans-Dicke theories in disguise, with coupling parameter $\omega = 0$ and a complicated scalar field potential (see Ref. [19] for reviews). Also the most successful model of inflation in the early Universe, Starobinsky inflation, is based on quadratic curvature corrections to Einstein gravity, embodied in the $f(R) = R + \alpha R^2$ Lagrangian [20].

It is assumed that viable modified theories of gravity have some limit to GR and observational tests then tell us that gravity must indeed be close to GR on the scales at which such tests are available (which, admittingly, do not span a vast range [1]). Possessing a GR limit seems to be an essential ingredient of the required "closeness to GR," but there are still gaps in our theoretical understanding of this limit even for the simplest alternative to GR and the simplest incarnation of scalar-tensor gravity, which is the original Brans-Dicke theory [13]. Here we clarify a puzzle in the limit of vacuum Brans-Dicke gravity (which is relevant for Solar System tests) to GR.

The scalar-tensor action is (we follow the notation of Ref. [21], using units in which Newton's constant G and the speed of light c are unity)

$$S_{\mathrm{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)}, \tag{1.1}$$

where $\phi > 0$ is the Brans-Dicke scalar field, the function $\omega(\phi)$ (a constant parameter in the original Brans-Dicke theory [13]) is the "Brans-Dicke coupling," $V(\phi)$ is a scalar field potential, and $S^{(m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(m)}$ is the matter part of the action.

The variation of the action (1.1) with respect to the inverse metric g^{ab} and to the Brans-Dicke scalar ϕ yields the (Jordan frame) field equations [13,14]

*vfaraoni@ubishops.ca
†jcote16@ubishops.ca

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi}{\phi}T^{(m)}_{ab} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \quad (1.2)$$

$$\Box\phi = \frac{1}{2\omega+3}\left(\frac{8\pi T^{(m)}}{\phi} + \phi\frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi}\nabla^c\phi\nabla_c\phi\right) \quad (1.3)$$

where R_{ab} is the Ricci scalar and ∇_a is the covariant derivative of the spacetime metric g_{ab}, while $T^{(m)} \equiv g^{cd}T^{(m)}_{cd}$ is the trace of the matter energy-momentum tensor $T^{(m)}_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{(m)}}{\delta g^{ab}}$. The matter stress-energy tensor and the effective stress-energy tensor of the scalar ϕ are covariantly conserved separately.

Let us briefly review the limit to GR of Brans-Dicke theory and its anomaly. GR is reproduced from Jordan frame Brans-Dicke gravity when ϕ becomes constant: then the effective gravitational coupling $G_{\mathrm{eff}} \simeq \phi^{-1}$ also becomes constant (for more general scalar-tensor theories in which $\omega = \omega(\phi)$, the limit to GR is $\omega \to \infty$ in conjunction with $\omega^{-3}d\omega/d\phi \to 0$ [15,22]). The contentious issue is how fast ϕ approaches a constant. It is commonly believed (e.g., [23]) that (Jordan frame) Brans-Dicke gravity reduces to GR as the Brans-Dicke parameter $\omega \to \infty$, with the Brans-Dicke scalar ϕ following the asymptotics

$$\phi = \phi_\infty + \mathcal{O}\left(\frac{1}{\omega}\right), \quad (1.4)$$

where $\phi_\infty > 0$ is a constant. However, a number of analytic solutions of the Brans-Dicke field equations have been reported which fail to reduce to the corresponding solutions of GR as $\omega \to \infty$ [24,25] (including anomalies in matter different from (electro)vacuum [26–28], to which we will instead restrict). In these situations, the asymptotic behavior of the Brans-Dicke scalar is not the one expected [Eq. (1.4)] but[1]

$$\phi = \phi_\infty + \mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right). \quad (1.5)$$

It has also been realized that the asymptotic behavior (1.5) of the Brans-Dicke field is usually accompanied by a vanishing trace $T^{(m)}$ of the matter energy-momentum tensor [25] (this condition is trivially satisfied *in vacuo*). This coincidence hints to some degree of conformal invariance, which has motivated an explanation of why the $\omega \to \infty$ limit of Brans-Dicke theory fails to reproduce GR *in vacuo* or in electrovacuo [30,31]. We summarize this explanation in Sec. IV.

The limit to GR is important for three reasons. First, it is related to the weak-field limit of gravity, in which the relativistic corrections to Newtonian gravity are parametrized by the so-called parametrized post-Newtonian (PPN) formalism [10]. This formalism is the basis for constraining ω with Solar System experiments [9,10]. Second, various authors have studied attractor mechanisms in which scalar-tensor gravity converges toward GR, such as during the early evolution of the Universe [32,33]. Third, scalar-tensor gravity could be an "excitation" of GR in the context of the thermodynamics of spacetime, in which the Einstein equations are derived as a sort of macroscopic equation of state [34]. Then, GR would represent a "state of equilibrium" while deviations from it (for example, through the excitation of other gravitational scalar d.o.f. such as the Brans-Dicke scalar ϕ) could be nonequilibrium states [35,36].

Here we revisit the anomaly in the $\omega \to \infty$ limit of Brans-Dicke theory with two new approaches. The first one consists of using the Einstein conformal frame, while the second describes the Brans-Dicke field equations as effective Einstein equations with an effective imperfect fluid made of terms including derivatives of the scalar field ϕ. Both approaches produce the same result.

II. EINSTEIN FRAME APPROACH

The first approach relies on the fact that scalar-tensor gravity has another close relation with GR, besides the $\omega \to \infty$ limit. Let us restrict, for simplicity, to *vacuum* Brans-Dicke theory: the Einstein frame formulation of this theory is formally GR with a metric $\tilde{g}_{ab}$ and a (minimally coupled and canonical) scalar field $\tilde{\phi}$. So the question arises naturally: what is the relation between the $\omega \to \infty$ limit $(g^{(\infty)}_{ab}, \phi^{(\infty)})$ of a Brans-Dicke spacetime (g_{ab}, ϕ) and its Einstein frame version $(\tilde{g}_{ab}, \tilde{\phi})$? One could expect these two to coincide, but they do not, as is elucidated in the following.

A. Einstein frame

In addition to the Jordan frame (g_{ab}, ϕ), another representation of scalar-tensor gravity, the Einstein frame $(\tilde{g}_{ab}, \tilde{\phi})$, is used [37]. The metric tensors in the Einstein and in the Jordan frames are related by the conformal transformation

$$g_{ab} \to \tilde{g}_{ab} \equiv \phi g_{ab}, \quad (2.1)$$

while, for the two scalar fields, we have

$$d\tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi}}\frac{d\phi}{\phi}. \quad (2.2)$$

Restricting ourselves to Brans-Dicke theory with constant ω, the scalar field is redefined nonlinearly as

[1] See also the discussion of Ref. [29].

$$\phi \to \tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi}} \ln\left(\frac{\phi}{\phi_0}\right), \tag{2.3}$$

where ϕ_0 is an integration constant and $\omega \neq -3/2$. Since both ϕ and g_{ab} depend on the parameter ω, barring miraculous cancellations, in general the Einstein frame metric $\tilde{g}_{ab}$ given by Eq. (2.1) depends on ω. Similarly, the Einstein frame scalar $\tilde{\phi}$ given by Eq. (2.3) depends on ω.

In the Einstein frame, the Brans-Dicke action (i.e., (1.1) with $\omega = \text{const}$) assumes the form

$$S_{\rm BD} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(m)}}{\phi^2(\tilde{\phi})}\right], \tag{2.4}$$

where

$$U(\tilde{\phi}) = \frac{V(\phi)}{16\pi\phi^2}\bigg|_{\phi=\phi(\tilde{\phi})} \tag{2.5}$$

(Einstein frame quantities are denoted by a tilde). The action (2.4) is formally the Einstein-Hilbert action of GR with a matter scalar field which has canonical kinetic energy density, except that now this scalar couples nonminimally to matter. The Einstein frame field equations are

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = 8\pi\Bigg(e^{-\sqrt{\frac{64\pi}{|2\omega+3|}}\tilde{\phi}}T_{ab}^{(m)} + \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi} - U(\tilde{\phi})\tilde{g}_{ab}\Bigg), \tag{2.6}$$

$$\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} - \frac{dU}{d\tilde{\phi}} + 8\sqrt{\frac{\pi}{|2\omega+3|}}e^{-\sqrt{\frac{64\pi}{|2\omega+3|}}\tilde{\phi}}\mathcal{L}^{(m)} = 0. \tag{2.7}$$

Let us restrict, for simplicity, to vacuum Brans-Dicke theory by setting $T_{ab}^{(m)} = 0$. Then the explicit coupling between the Einstein frame scalar $\tilde{\phi}$ and matter disappears from the action and from the field equations and the Einstein frame action (2.4) formally reduces to the Einstein-Hilbert action of GR. Therefore, the Einstein frame pair $(\tilde{g}_{ab}, \tilde{\phi})$ is formally a scalar field solution of GR corresponding to the original Jordan frame spacetime (g_{ab}, ϕ) and it could appear as a natural candidate for a "GR limit" of the latter.

B. Comparison between the Einstein frame and the $\omega \to \infty$ limit of the Jordan frame

Since we are interested in the anomaly discussed in the literature for the limit $\omega \to \infty$ of Brans-Dicke theory, we are only interested in the vacuum case in the following.

As already seen, in the Einstein frame both the metric $\tilde{g}_{ab}$ and the scalar $\tilde{\phi}$ depend on the parameter ω, but the gravitational coupling is constant. One cannot regard the Einstein frame fields $(\tilde{g}_{ab}, \tilde{\phi})$ as a limit to GR of (g_{ab}, ϕ) because any dependence on the parameter ω (which is absent in GR) must disappear after taking the GR limit. Moreover, $\tilde{g}_{ab}$ is a solution of the coupled Einstein-Klein-Gordon equations with Klein-Gordon field $\tilde{\phi}$. There are different matter sources in the two conformal frames: vacuum in the Jordan frame and a Klein-Gordon field in the Einstein frame, and the GR limit of a Brans-Dicke solution must have the same matter source.[2] It is more meaningful to compare Einstein and Jordan frame in the $\omega \to \infty$ limit, as is done in the next subsection.

C. Comparison between the $\omega \to \infty$ limits of the Einstein and Jordan frames

The $\omega \to \infty$ limit of a Jordan frame spacetime which is a solution of the Brans-Dicke field equations (1.2) and (1.3) is expected to produce[3] $\phi_\infty = \text{const} > 0$ and the limit $g_{ab}^{(\infty)}$ of the metric, while the scalar field potential reduces to an effective cosmological constant $\Lambda \equiv V(\phi_\infty)/(2\phi_\infty)$. The conformal map relating the Jordan and Einstein frame metrics yields, in this limit,

$$\tilde{g}_{ab} = \phi g_{ab} \to \phi_\infty g_{ab}^{(\infty)} \quad \text{as } \omega \to \infty. \tag{2.8}$$

By dropping the irrelevant multiplicative constant ϕ_∞, which can always be absorbed by a coordinate redefinition, one has $\tilde{g}_{ab}(\omega) \to g_{ab}^{(\infty)}$ in this limit. Therefore, the Einstein frame *geometry* always coincides with the $\omega \to \infty$ limit of the Jordan frame *geometry*. The Einstein frame scalar field is given by Eq. (2.3). In the limit $\omega \to \infty$, the square root in the right-hand side of Eq. (2.3) diverges, then the entire right-hand side diverges, unless ϕ becomes a constant ϕ_∞ as $\omega \to \infty$, which is indeed what happens. This is not sufficient, however, to avoid the divergence of $\tilde{\phi}$: it must also be $\phi_0 = \phi_\infty$, which makes $\ln(\phi/\phi_\infty)$ vanish so that $\tilde{\phi}$ has a chance to remain finite in this limit. The divergence of $\tilde{\phi}$ as $\omega \to \infty$ would be unphysical since this scalar field must be well defined for all values of ω, including large ones (except, possibly, at spacetime singularities). Hence, we require $\phi_0 = \phi_\infty$ and we write

$$\tilde{\phi} = \sqrt{\frac{|2\omega+3|}{16\pi}} \ln\left(\frac{\phi}{\phi_\infty}\right). \tag{2.9}$$

[2] ϕ has a gravitational nature in the Jordan frame but it appears as a matter field in the Einstein frame, blurring the sharp distinction between geometry and matter present in Einstein theory. Ambiguities in the identification of matter and gravitational fields are an obstacle to creating a metatheory of gravitational theories [38].

[3] The constant ϕ_∞ is positive because it corresponds to the inverse of the gravitational coupling strength.

This argument fixes the arbitrary integration constant ϕ_0 appearing in Eq. (2.3).

If the Jordan frame scalar has asymptotics

$$\phi = \phi_\infty + \mathcal{O}\left(\frac{1}{\omega^p}\right) \tag{2.10}$$

for some[4] p, then the Einstein frame scalar (2.9) can be written as

$$\tilde{\phi} = \frac{1}{2\sqrt{2\pi}} \frac{\ln\left[1+\mathcal{O}(1/\omega^p)\right]}{\frac{1}{\sqrt{|\omega+3/2|}}} \approx \frac{1}{2\sqrt{2\pi}} \mathcal{O}\left(\frac{1}{\omega^{p-1/2}}\right) \tag{2.11}$$

using the linear expansion $\ln(1+x) \simeq x$ for $|x| \ll 1$. We do not want to prescribe the asymptotics of ϕ, and attempts to do so would require independent arguments but, *a priori*, there are only three possibilities:

(i) If $p > 1/2$, in particular in the "standard" situation $\phi = \phi_\infty + \mathcal{O}(1/\omega)$ corresponding to $p = 1$ [23,39], then $\tilde{\phi} \to 0$ as $\omega \to \infty$ and, *in vacuo*, no scalar field source is left in the limit of the Einstein frame field equations (except, possibly, for a cosmological constant if $V(\phi_\infty) \neq 0$).

(ii) If $p = 1/2$, which is the "anomalous" situation studied in the literature [24,25,30,31], $\tilde{\phi}$ does not vanish but reduces to a function $\tilde{\phi}_\infty$ which does not depend on ω. Then the limit of the Einstein frame equations contains a (canonical) scalar field source $\tilde{\phi}_\infty$, unless this function is constant. If $\tilde{\phi}_\infty$ is a constant instead of depending on the coordinates, this scalar field disappears from the field equations (except if there is a potential $V(\phi)$ with $V(\phi_\infty) \neq 0$, in which case a cosmological constant remains). It would be tempting to assume that $\tilde{\phi}_\infty$ is constant, but this assumption is unwarranted at this stage.

(iii) If $p < 1/2$ then $\tilde{\phi} \to \infty$ as $\omega \to +\infty$, which is unphysical. Therefore, this case is excluded and the asymptotics described by $p = 1/2$ are truly a borderline, extreme situation. This fact is not evident from the standard Jordan frame analysis: one needs to analyze the Einstein frame to reach this conclusion.

The analysis of the Einstein frame, and of Eq. (2.9) that accompanies it, selects the scaling $\sim\omega^{-1/2}$ as a critical behavior which is a boundary of the possible behaviors of ϕ *in the Jordan frame*. This conclusion about the Jordan frame was not expected to come from an analysis of the Einstein frame.

Let us compare now the previous considerations with the asymptotics of the Jordan frame Brans-Dicke scalar. By inverting Eq. (2.3), one writes the Jordan frame scalar as

$$\phi = \phi_0 \exp\left(\sqrt{\frac{16\pi}{|2\omega+3|}}\tilde{\phi}\right). \tag{2.12}$$

The Jordan frame scalar ϕ depends on ω, a property familiar from the study of exact Jordan frame solutions of Brans-Dicke theory, especially in cosmology [31,40] and in spherical symmetry [41–44]. In order for the $\omega \to \infty$ limit to not cause unphysical divergences in the Einstein frame, $\tilde{\phi}$ must go over to a scalar function $\tilde{\phi}_\infty(x)$ independent of ω in this limit, say

$$\tilde{\phi}(x) = \tilde{\phi}_\infty(x) + \mathcal{O}\left(\frac{1}{\omega^q}\right) \tag{2.13}$$

with $q > 0$. There are now two possibilities:

1. $\tilde{\phi}_\infty(x) \neq 0$.—If $\tilde{\phi}_\infty(x) \neq 0$, Eq. (2.12) implies that

$$\phi(x) = \phi_0 + \mathcal{O}\left(\frac{1}{\sqrt{|\omega|}}\right): \tag{2.14}$$

these are exactly the asymptotics (1.5) which are known to make the $\omega \to \infty$ limit fail to reproduce GR in the Jordan frame. Hence, a finite and nonzero Einstein frame $\tilde{\phi}$ in the $\omega \to \infty$ limit (i.e., a finite $\tilde{\phi}_\infty(x) \neq 0$) *always* implies an "anomalous" scaling of the Jordan frame ϕ of (electro) vacuum Brans-Dicke theory if $\tilde{\phi}_\infty \neq 0$. *Vice-versa*, using Eq. (2.9), one concludes that the "anomalous" Jordan frame scaling (2.14) always determines a finite nonzero $\tilde{\phi}_\infty$ in the Einstein frame, and there is a one-to-one correspondence between these two phenomena in the two frames.

2. $\tilde{\phi}_\infty(x) = 0$.—If instead $\tilde{\phi}_\infty(x) = 0$, say $\tilde{\phi}(x) = \mathcal{O}(1/\omega^q)$ with $q > 0$, then the comparison of Eqs. (2.11) and (2.13) gives $p = q + 1/2 > 1/2$ and it follows from Eq. (2.12) that

$$\phi = \phi_0 + \mathcal{O}\left(\frac{1}{\omega^{q+1/2}}\right) \quad (q > 0). \tag{2.15}$$

The "standard" behavior [23] of Eq. (1.4) is reproduced by the special value $q = 1/2$.

D. General conclusions

It is not clear from this analysis whether values of p other than $1/2$ and 1 are possible, but nothing seems to forbid them as long as one keeps $p > 1/2$. However, such values have not appeared in the literature thus far. All that one can say in the Einstein frame approach described here is that p must be larger than, or equal to, $1/2$ and that $p > 1/2$ corresponds to the vanishing of the function $\tilde{\phi}_\infty$ (which does not depend on ω) and of the Einstein frame scalar $\tilde{\phi}$ in the $\omega \to \infty$ limit.

The way ϕ approaches a constant is detailed by the decay of its gradient as $|\omega|$ becomes larger and larger,

[4]Here $\mathcal{O}(\omega^{-p})$ denotes a scalar function of the coordinates which is of order ω^{-p} as $\omega \to \infty$.

$$\nabla_a\phi = \phi_0\sqrt{\frac{16\pi}{|2\omega+3|}}\exp\left(\sqrt{\frac{16\pi}{|2\omega+3|}}\tilde{\phi}\right)\tilde{\nabla}_a\tilde{\phi}. \qquad (2.16)$$

Let us examine now the limit of the Jordan frame field equations (1.2) and, in particular, of the term in their right-hand side

$$A_{ab} \equiv \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right). \qquad (2.17)$$

In the Jordan frame analysis available in the literature, the failure of Brans-Dicke theory to reproduce the expected GR limit (which corresponds to $\phi = \text{const}$ and to the right-hand side of the vacuum field equations equal to zero) has been traced to the fact that, in conjunction with the asymptotics (1.5), the tensor A_{ab} does not vanish in the $\omega\to\infty$ limit but remains of order unity [24,25]. In fact, using Eq. (2.16), the tensor A_{ab} (before taking any limit) reads

$$A_{ab} = \frac{\omega}{\phi^2}\phi_0^2 e^{2\sqrt{\frac{16\pi}{|2\omega+3|}}\tilde{\phi}}\frac{16\pi}{|2\omega+3|} \times\left(\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - \frac{1}{2}g_{ab}g^{cd}\nabla_c\tilde{\phi}\nabla_d\tilde{\phi}\right) \qquad (2.18)$$

$$= 16\pi\,\text{sign}(\omega)\left|\frac{\omega}{2\omega+3}\right|\left(\frac{\phi_0}{\phi}\right)^2 e^{2\sqrt{\frac{16\pi}{|2\omega+3|}}} \times\left(\nabla_a\tilde{\phi}\nabla_b\tilde{\phi} - \frac{1}{2}\frac{\tilde{g}_{ab}}{\phi}\phi\tilde{g}^{cd}\nabla_c\tilde{\phi}\nabla_d\tilde{\phi}\right) \qquad (2.19)$$

and, taking the $\omega\to\infty$ limit in which $\phi\to\phi_0$,

$$A_{ab}\to A_{ab}^{(\infty)} = 8\pi\,\text{sign}(\omega)\left(\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi}\right). \qquad (2.20)$$

The Einstein frame metric $\tilde{g}_{ab}^{(\infty)}$ solves the Einstein equations which have as a matter source the scalar field $\tilde{\phi}$ with canonical stress-energy tensor $A_{ab}^{(\infty)}$, obtained as the limit of the *Jordan frame* stress-energy tensor:

$$8\pi\tilde{T}_{ab}[\tilde{\phi}]\Big|_{\text{Einstein frame}} = A_{ab}^{(\infty)}\Big|_{\text{Jordan frame limit}}. \qquad (2.21)$$

The Einstein frame scalar $\tilde{\phi}$ is minimally coupled to the curvature and has canonical kinetic energy density (if $\omega > 0$). If we focus only on the metric tensor and the spacetime geometry, the two candidates for a GR limit of Brans-Dicke theory, i) the $\omega\to\infty$ limit of the Einstein frame metric and ii) the $\omega\to\infty$ limit of the Jordan frame metric coincide (apart from the irrelevant positive multiplicative constant ϕ_∞ which can always by eliminated by rescaling the coordinates).

Since this GR limit $\tilde{g}_{ab}^{(\infty)} = g_{ab}^{(\infty)}$ obtained with these two different methods is not a solution of the vacuum Einstein equations (which would require $A_{ab}^{(\infty)}$ to vanish identically) but solves the coupled Einstein-Klein-Gordon equations, the limit to GR is regarded as an anomaly. The previous discussion using the Einstein frame deepens our understanding of the limit to GR.

Summary. The limit $\tilde{g}_{ab}^{(\infty)}$ of the Einstein frame metric coincides with the $\omega\to\infty$ limit $g_{ab}^{(\infty)}$ of the Jordan frame metric g_{ab}, but it does not solve the Einstein equations with the same matter source $-\frac{V(\phi_\infty)}{2\phi_\infty}g_{ab} \equiv -\Lambda g_{ab}$ and $\phi = \text{const}$ expected to be left over in the Jordan frame equations (1.2) after the $\omega\to\infty$ limit. The Einstein frame limiting metric $\tilde{g}_{ab}^{(\infty)}$ solves instead the Einstein equations with a canonical minimally coupled scalar field $\tilde{\phi}$ described by the stress-energy tensor

$$\tilde{T}_{ab} = \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}g_{ab}^{(\infty)}g_{(\infty)}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi} - U(\tilde{\phi})\tilde{g}_{ab}. \qquad (2.22)$$

III. EXAMPLES

All stationary, spherically symmetric, asymptotically flat black holes of Brans-Dicke theory without potential reduce to those of GR, a well-known no-hair theorem (which is generalized to axial symmetry, to more general scalar-tensor theories, and to potentials with zero minimum) [45]. For these black holes, the Brans-Dicke scalar ϕ becomes trivial (i.e., constant) outside the event horizon. Apart from this situation, the most general spherical, static, asymptotically flat solution of the vacuum Brans-Dicke equations is a 3-parameter family depending on parameters α_0, β_0, and γ. If $\gamma\neq 0$, this general solution is[5] [41,42,44,48]

$$ds^2 = -e^{(\alpha_0+\beta_0)/r}dt^2 + e^{(\beta_0-\alpha_0)/r}\left(\frac{\gamma/r}{\sinh(\gamma/r)}\right)^4 dr^2 + e^{(\beta_0-\alpha_0)/r}\left(\frac{\gamma/r}{\sinh(\gamma/r)}\right)^2 r^2 d\Omega_{(2)}^2, \qquad (3.1)$$

$$\phi(r) = \phi_0 e^{-\beta_0/r}, \qquad \beta_0 = \frac{\sigma}{\sqrt{|2\omega+3|}}, \qquad (3.2)$$

where $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere, and[6]

$$4\gamma^2 = \alpha_0^2 + 2\sigma^2, \qquad (3.3)$$

[5] As shown in Ref. [44], under certain conditions this solution can be recast in the less general Campanelli-Lousto form [46,47].

[6] This relation only holds for $\gamma > 0$ and there is no loss of generality in choosing positive γ when $\gamma\neq 0$ [49].

while σ is a scalar charge. If $\gamma = 0$, the general solution is instead [44] the Brans class IV spacetime [50]

$$ds^2 = -e^{-2B/r}dt^2 + e^{2B(C+1)/r}(dr^2 + r^2 d\Omega_{(2)}^2), \quad (3.4)$$

$$\phi(r) = \phi_0 e^{-BC/r}, \quad (3.5)$$

where

$$B = -\frac{(\alpha_0 + \beta_0)}{2}, \qquad C = -\frac{2\beta_0}{(\alpha_0 + \beta_0)}. \quad (3.6)$$

According to the results of the previous section, in the limit $\omega \to \infty$ this spacetime should reduce to a solution of GR with the same symmetries but sourced by a canonical, minimally coupled scalar field $\Phi(r)$ which must coincide with the Einstein frame scalar $\tilde{\phi}(r)$. But the general spherical, static, asymptotically flat solution of the Einstein equations sourced by a free scalar field is well known: it is the Fisher-Janis-Newman-Winicour-Buchdahl-Wyman (FJNWBW) solution hosting a central naked singularity [49,51]. Let us check if our statement based on Sec. II is true. By taking the limit $\omega \to \infty$, the Brans-Dicke line element (3.1) reduces to

$$ds_{(\infty)}^2 = -e^{\alpha_0/r}dt^2 + e^{-\alpha_0/r}\left(\frac{\gamma/r}{\sinh(\gamma/r)}\right)^4 dr^2 + e^{-\alpha_0/r}\left(\frac{\gamma/r}{\sinh(\gamma/r)}\right)^2 r^2 d\Omega_{(2)}^2. \quad (3.7)$$

This is indeed recognized as the FJNWBW geometry [49,51].

The Jordan frame scalar field $\phi(r) = \phi_0 e^{-\beta_0/r} \to \phi_0 =$ const as $\omega \to \infty$. However, the scalar field sourcing the FJNWBW geometry in GR is [49,51]

$$\Phi(r) = \frac{\Phi_*}{r}, \qquad \Phi_* = \frac{-\sigma}{4\sqrt{\pi}} \quad (3.8)$$

and is exactly the Einstein frame cousin of $\phi(r)$, i.e., $\Phi(r) = \tilde{\phi}(r)$ [44].

Let us consider now the case $\gamma = 0$, in which the Brans-Dicke solution is the Brans class IV geometry (3.4)–(3.6) [50]. It is well known that, in the $\omega \to \infty$ limit, this solution does not reduce to the corresponding vacuum solution of the Einstein equations with the same symmetries, which is the Schwarzschild solution, and it cannot describe a black hole but only a wormhole throat or a naked singularity [52,53] (indeed, the failure of the Brans solutions [50] to reproduce the Schwarzschild spacetime was one of the first occurrences prompting investigation of the anomaly in the GR limit of Brans-Dicke theory [25]). It is easy to see that the $\omega \to \infty$ limit of Eqs. (3.4)–(3.6) produces $B \to \alpha_0/2$, $C \to 0$, and the Yilmaz geometry [54,55]

$$ds_{(\infty)}^2 = -e^{\alpha_0/r}dt^2 + e^{-\alpha_0/r}(dr^2 + r^2 d\Omega_{(2)}^2), \quad (3.9)$$

which is indeed the Einstein frame counterpart of Brans class IV [44]. The Jordan frame scalar field of Brans class IV $\phi(r) = \phi_0 e^{-BC/r} \to \phi_0 =$ const but the scalar field sourcing the Yilmaz geometry is instead

$$\Phi(r) = \frac{\Phi_0}{r}, \quad (3.10)$$

which coincides with the Einstein frame counterpart of $\phi(r)$, i.e., $\Phi(r) = \tilde{\phi}(r)$ again.

IV. THE EFFECTIVE FLUID APPROACH

Let us discuss now an independent approach to the problem of the anomaly in the GR limit of BD theory. In this approach, the Brans-Dicke field equations are rewritten as effective Einstein equations with the terms dependent on ϕ and its derivatives moved to the right-hand side to provide an extra source (in addition to the real matter stress-energy tensor $T_{ab}^{(m)}$). In the rest of this article we consider only vacuum or conformally invariant matter with $T^{(m)} = 0$. Since we want to define an effective fluid with 4-velocity proportional to the gradient of ϕ, this 4-vector must necessarily be timelike and this section, unlike the previous one, is restricted to situations in which $\nabla^c\phi$ is always timelike.

A. Imperfect fluid equivalent of the Brans-Dicke field

When the gradient $\nabla^a\phi$ is timelike, one introduces the fluid 4-velocity

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}, \quad (4.1)$$

which is normalized, $u^a u_a = -1$. This 4-velocity provides a $3+1$ splitting of spacetime into the time direction u^a and the three-dimensional space Σ_t perceived by the comoving observers of the effective fluid, which has the Riemannian metric

$$h_{ab} \equiv g_{ab} + u_a u_b. \quad (4.2)$$

${h_a}^b$ is the usual projection operator which satisfies

$$h_{ab}u^a = h_{ab}u^b = 0, \quad (4.3)$$

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^a{}_a = 3. \quad (4.4)$$

The fluid 4-acceleration

$$\dot{u}^a \equiv u^b\nabla_b u^a \quad (4.5)$$

is purely spatial, $\dot{u}^c u_c = 0$. The projection of the velocity gradient onto the 3-space Σ_t is the purely spatial tensor

$$V_{ab} \equiv h_a{}^c h_b{}^d \nabla_d u_c, \quad (4.6)$$

which is decomposed according to

$$V_{ab} = \theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab}, \tag{4.7}$$

where $\theta_{ab} = V_{(ab)}$ (expansion tensor) is the symmetric part of V_{ab}, $\theta \equiv \theta^c{}_c = \nabla^c u_c$ is its trace,

$$\sigma_{ab} \equiv \theta_{ab} - \frac{\theta}{3} h_{ab} \tag{4.8}$$

(shear tensor) is the trace-free part of θ_{ab}, and the vorticity tensor $\omega_{ab} = V_{[ab]}$ vanishes identically because the fluid is generated by a scalar and u^a is irrotational. The tensors h_{ab}, V_{ab}, expansion θ_{ab}, and shear σ_{ab} are purely spatial,

$$\theta_{ab} u^a = \theta_{ab} u^b = \sigma_{ab} u^a = \sigma_{ab} u^b = 0, \tag{4.9}$$

and $\sigma^a{}_a = 0$. In general, it is [56]

$$\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b = V_{ab} - \dot{u}_a u_b. \tag{4.10}$$

The projection of this equation onto the time direction produces $\dot{u}_a$, while the projection onto the 3-space orthogonal to u^a gives V_{ab}. In our particular case (4.1), we have [57]

$$h_{ab} = g_{ab} - \frac{\nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi}, \tag{4.11}$$

$$\nabla_b u_a = \frac{1}{\sqrt{-\nabla^e \phi \nabla_e \phi}} \left(\nabla_a \nabla_b \phi - \frac{\nabla_a \phi \nabla^c \phi \nabla_b \nabla_c \phi}{\nabla^e \phi \nabla_e \phi} \right), \tag{4.12}$$

$$\dot{u}_a = (-\nabla^e \phi \nabla_e \phi)^{-2} \nabla^b \phi [(-\nabla^e \phi \nabla_e \phi) \nabla_a \nabla_b \phi + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi]. \tag{4.13}$$

The kinematic quantities of the effective fluid are given in Ref. [57], while the effective stress-energy tensor of ϕ is described by

$$8\pi T_{ab}^{(\phi)} = \frac{\omega}{\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} (\nabla_a \nabla_b \phi - g_{ab} \Box \phi) - \frac{V}{2\phi} g_{ab} \tag{4.14}$$

and it is covariantly conserved, together with that of ordinary matter

$$\nabla^b T_{ab}^{(m)} = 0, \qquad \nabla^b T_{ab}^{(\phi)} = 0. \tag{4.15}$$

The effective stress-energy tensor $T_{ab}^{(\phi)}$ can be written [57,58] in the imperfect fluid form

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}, \tag{4.16}$$

where

$$\rho = T_{ab} u^a u^b, \tag{4.17}$$

$$q_a = -T_{cd} u^c h_a{}^d, \tag{4.18}$$

$$\Pi_{ab} \equiv P h_{ab} + \pi_{ab} = T_{cd} h_a{}^c h_b{}^d, \tag{4.19}$$

$$P = \frac{1}{3} g^{ab} \Pi_{ab} = \frac{1}{3} h^{ab} T_{ab}, \tag{4.20}$$

$$\pi_{ab} = \Pi_{ab} - P h_{ab}, \tag{4.21}$$

are the effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses (the trace-free part π_{ab} of the stress tensor Π_{ab}) in the comoving frame. In this frame, we have

$$q_c u^c = \Pi_{ab} u^b = \pi_{ab} u^b = \Pi_{ab} u^a = \pi_{ab} u^a = 0, \quad \pi^a{}_a = 0. \tag{4.22}$$

These effective quantities were computed explicitly in Refs. [57,58], obtaining:

$$8\pi \rho^{(\phi)} = -\frac{\omega}{2\phi^2} \nabla^e \phi \nabla_e \phi + \frac{V}{2\phi} + \frac{1}{\phi} \left(\Box \phi - \frac{\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \right), \tag{4.23}$$

$$8\pi q_a^{(\phi)} = \frac{\nabla^c \phi \nabla^d \phi}{\phi (-\nabla^e \phi \nabla_e \phi)^{3/2}} (\nabla_d \phi \nabla_c \nabla_a \phi - \nabla_a \phi \nabla_c \nabla_d \phi) \tag{4.24}$$

$$= -\frac{\nabla^c \phi \nabla_a \nabla_c \phi}{\phi (-\nabla^e \phi \nabla_e \phi)^{1/2}} - \frac{\nabla^c \phi \nabla^d \phi \nabla_c \nabla_d \phi}{\phi (-\nabla^e \phi \nabla_e \phi)^{3/2}} \nabla_a \phi, \tag{4.25}$$

$$8\pi \Pi_{ab}^{(\phi)} = \left(-\nabla^e \phi \nabla_e \phi \right)^{-1} \left[\left(-\frac{\omega}{2\phi^2} \nabla^e \phi \nabla_e \phi - \frac{\Box \phi}{\phi} - \frac{V}{2\phi} \right) (\nabla_a \phi \nabla_b \phi - g_{ab} \nabla^e \phi \nabla_e \phi) - \frac{\nabla^d \phi}{\phi} \left(\nabla_d \phi \nabla_a \nabla_b \phi - \nabla_b \phi \nabla_a \nabla_d \phi - \nabla_a \phi \nabla_d \nabla_b \phi + \frac{\nabla_a \phi \nabla_b \phi \nabla^c \phi \nabla_c \nabla_d \phi}{\nabla^e \phi \nabla_e \phi} \right) \right] \tag{4.26}$$

$$= \left(-\frac{\omega}{2\phi^2} \nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi} \right) h_{ab} + \frac{1}{\phi} h_a{}^c h_b{}^d \nabla_c\nabla_d\phi, \tag{4.27}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi} \right), \tag{4.28}$$

$$8\pi\pi_{ab}^{(\phi)} = \frac{1}{\phi\nabla^e\phi\nabla_e\phi}\left[\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\left(\Box\phi - \frac{\nabla^c\phi\nabla^d\phi\nabla_d\nabla_c\phi}{\nabla^e\phi\nabla_e\phi} \right) \right.$$
$$\left. + \nabla^d\phi\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi} \right) \right]. \tag{4.29}$$

B. A symmetry of (electro)vacuum Brans-Dicke gravity and of its equivalent effective fluid

In (electro)vacuum, the Brans-Dicke action (1.1) is invariant under the 1-parameter group of symmetries [30]

$$g_{ab} \to \bar{g}_{ab} = \phi^{2\alpha} g_{ab}, \tag{4.30}$$

$$\phi \to \bar{\phi} = \phi^{1-2\alpha}, \qquad \alpha \neq 1/2, \tag{4.31}$$

provided that the Brans-Dicke parameter and the scalar field potential are changed to

$$\bar{\omega}(\omega, \alpha) = \frac{\omega + 6\alpha(1-\alpha)}{(1-2\alpha)^2}, \tag{4.32}$$

$$\bar{V}(\bar{\phi}) = \bar{\phi}^{\frac{-4\alpha}{1-2\alpha}} V(\bar{\phi}^{\frac{1}{1-2\alpha}}) \tag{4.33}$$

[because $\alpha \neq 1/2$, the conformal transformation (4.30) is completely different from the transformation leading to the Einstein frame metric (2.1)]. To understand the use of α, the idea is that one first discovers a 1-parameter symmetry group of (electro-)vacuum Brans-Dicke theory. This group is parametrized exactly by the parameter α appearing in the exponents of both metric and scalar field redefinitions (in different ways). This parameter is well defined independently of any limit of the theory. Then, we discover that the limit $\omega \to \infty$ of the theory corresponds to the limit $\alpha \to 1/2$ and we use this fact advantageously for the particular problem at hand. Is is shown in Ref. [30] that the operations (4.30), (4.31) constitute a 1-parameter group of symmetries which, thus far, has seen two uses in the literature: to generate new solutions of the field equations from known ones [59], and to study the anomaly in the $\omega \to \infty$ limit of Brans-Dicke gravity [30,31]. It is this second use that we are interested in here.

The relation (4.32) between the parameters $\bar{\omega}$ and α can be inverted, obtaining

$$\alpha = \frac{2\bar{\omega} + 3 \pm \sqrt{(2\bar{\omega}+3)(2\omega+3)}}{2(2\bar{\omega}+3)}, \tag{4.34}$$

and $\alpha \to 1/2$ as $\bar{\omega} \to \infty$, hence one can trade these two limits and think of obtaining larger and larger $\bar{\omega}$ by means of consecutive symmetry transformations (4.30), (4.31). In (electro)vacuo, this transformation connects theories within an equivalence class and a change of the ω-parameter $\omega \to \bar{\omega}$ simply moves a Brans-Dicke theory within this equivalence class [30,31]. This is true also for the $\bar{\omega} \to \infty$ limit, which can be seen as a transformation (4.30), (4.31) leading to larger and larger $\bar{\omega}$ and cannot break this restricted conformal invariance and move the theory outside of the equivalence class. GR, which is not conformally invariant, does not belong to this class and it cannot be reproduced[7] by the $\omega \to \infty$ limit under these circumstances—one needs to first break the conformal invariance and exit this class to be able to obtain GR as the limit [30,31].

Naturally, the symmetry (4.30), (4.31) corresponds to a symmetry of the effective fluid. Under (4.30), (4.31), the fluid 4-velocity is mapped to [57]

$$u_c \to \bar{u}_c \equiv \frac{\bar{\nabla}_c\bar{\phi}}{\sqrt{-\bar{g}^{cd}\bar{\nabla}_c\bar{\phi}\bar{\nabla}_d\bar{\phi}}} = \phi^\alpha u_c, \tag{4.35}$$

$$u^c \to \bar{u}^c = \phi^{-\alpha} u^c, \tag{4.36}$$

while $\bar{g}^{ab}\bar{u}_a\bar{u}_b = g^{ab}u_a u_b = -1$. The effective fluid quantities transform according to [57]

$$\bar{T}_{ab}^{(\bar{\phi})} = T_{ab}^{(\phi)} + \frac{\alpha}{4\pi\phi}\left[\frac{(1+\alpha)}{\phi}\nabla_a\phi\nabla_b\phi + \frac{(\alpha-2)}{2\phi}\nabla^e\phi\nabla_e\phi g_{ab} \right.$$
$$\left. - (\nabla_a\nabla_b\phi - g_{ab}\Box\phi) \right], \tag{4.37}$$

$$\bar{\rho}^{(\bar{\phi})} = \phi^{-2\alpha}\left[(1-2\alpha)\rho^{(\phi)} - \frac{\alpha(3\alpha+\omega)}{8\pi\phi^2}\nabla^e\phi\nabla_e\phi + \frac{\alpha V}{8\pi\phi} \right], \tag{4.38}$$

$$\bar{q}_a^{(\bar{\phi})} = (1-2\alpha)\phi^{-\alpha} q_a^{(\phi)}, \tag{4.39}$$

[7] Quantization breaks this symmetry [60].

$$\bar{P}^{(\bar{\phi})} = \phi^{-2\alpha}\left[(1-2\alpha)P^{(\phi)} + \frac{\alpha(\alpha-\omega-2)}{8\pi\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\alpha V}{8\pi\phi}\right], \tag{4.40}$$

$$\bar{\Pi}^{(\bar{\phi})}_{ab} = (1-2\alpha)\Pi^{(\phi)}_{ab} + \frac{\alpha}{8\pi\phi}\left[\frac{(\alpha-\omega-2)}{\phi}\nabla^e\phi\nabla_e\phi - V\right]h_{ab}, \tag{4.41}$$

$$\bar{\pi}^{(\bar{\phi})}_{ab} = (1-2\alpha)\pi^{(\phi)}_{ab}. \tag{4.42}$$

In the limit $\alpha \to 1/2$ (corresponding to $\bar{\omega} \to \infty$) the imperfect fluid quantities, i.e., the heat flux $\bar{q}^{(\bar{\phi})}_a$ and the anisotropic stresses $\bar{\pi}^{(\bar{\phi})}_{ab}$, which are proportional to $(1-2\alpha)$ vanish identically, but there remain nonvanishing contributions to the effective energy density and pressure:

$$\bar{T}^{(\bar{\phi})}_{ab} \to \bar{T}^{(\infty)}_{ab} = \frac{1}{8\pi}\left[\frac{(2\omega+3)}{2\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) - \frac{V(\phi)}{2\phi}g_{ab}\right], \tag{4.43}$$

$$\bar{\rho}^{(\bar{\phi})} \to \bar{\rho}_{(\infty)} = \frac{1}{\phi}\left[-\frac{(2\omega+3)}{32\pi\phi^2}\nabla^c\phi\nabla_c\phi + \frac{V}{16\pi\phi}\right], \tag{4.44}$$

$$\bar{P}^{(\bar{\phi})} \to \bar{P}_{(\infty)} = \frac{1}{\phi}\left[-\frac{(2\omega+3)}{32\pi\phi^2}\nabla^c\phi\nabla_c\phi - \frac{V}{16\pi\phi}\right]. \tag{4.45}$$

By introducing $\tilde{g}_{ab} \equiv \phi g_{ab}$, $\tilde{g}^{ab} \equiv \phi^{-1}g^{ab}$, $\Phi = \sqrt{\frac{|2\omega+3|}{16\pi}}\ln(\frac{\phi}{\phi_0})$ (i.e., the Einstein frame metric, inverse metric, and scalar field), these quantities are written as

$$\bar{T}^{(\infty)}_{ab} = \mathrm{sign}(2\omega+3)\left[\tilde{\nabla}_a\Phi\tilde{\nabla}_b\Phi - \frac{1}{2}\tilde{g}_{ab}(\tilde{g}^{cd}\tilde{\nabla}_c\Phi\tilde{\nabla}_d\Phi)\right] - U(\Phi)\tilde{g}_{ab}, \tag{4.46}$$

$$\bar{\rho}_{(\infty)} = -\frac{\mathrm{sign}(2\omega+3)}{2}\tilde{g}^{ab}\tilde{\nabla}_a\Phi\tilde{\nabla}_b\Phi + U(\Phi), \tag{4.47}$$

$$\bar{P}_{(\infty)} = -\frac{\mathrm{sign}(2\omega+3)}{2}\tilde{g}^{ab}\tilde{\nabla}_a\Phi\tilde{\nabla}_b\Phi - U(\Phi). \tag{4.48}$$

The stress-energy tensor reduces to that of a minimally coupled scalar field which has the perfect fluid form [58,61]. If $2\omega+3>0$, it is a canonical scalar, while if $2\omega+3<0$, it is a phantom field. The imperfect fluid terms (the heat flux density $\bar{q}^a$ and the anisotropic stresses $\bar{\pi}_{ab}$) vanish in the limit $\alpha \to 1/2$, while the second order derivatives of ϕ (which cause the effective stress-energy tensor of the Brans-Dicke field to deviate from the canonical form) are all contained in the terms $\rho^{(\phi)}$ and $P^{(\phi)}$ which, being weighted by a factor $(1-2\alpha)$, vanish in this limit. What is more, if $2\omega+3>0$, the canonical and minimally coupled scalar field Φ left in the limit is nothing but the Einstein frame scalar corresponding to the Jordan frame ϕ, i.e., $\Phi = \tilde{\phi}$. However, the Einstein frame was not used in any way in the considerations of this section. This result reproduces that of Sec. II using an independent and complementary method, but is limited by the requirement that the gradient $\nabla^c\phi$ be timelike.

As a final comment, consider the Brans-Dicke field equation (1.3) for the scalar field which, in the presence of conformal matter (including (electro)vacuum), and for $\omega = \mathrm{const}$ and $V(\phi) \equiv 0$, reduces to $\Box\phi = 0$. Since this equation does not contain explicitly the parameter ω, its form is not affected by the limit $\omega \to \infty$ and, indeed, the scalar field Φ left behind in the $\omega \to \infty$ limit solves the same formal equation $\Box\Phi = 0$ (although the metric and its covariant derivative operator change during the limit).

V. CONCLUSIONS

When considering particular solutions of Brans-Dicke theory, one should keep in mind that the limit of the metric tensor taken in a specific coordinate system may not be unique [62]. A possible, rigorous way to take the limit of a spacetime as a parameter varies consists of the geometric, coordinate-invariant method of Ref. [63]. The two methods employed in the present work are also covariant.

We have studied the limit of (electro)vacuum Brans-Dicke gravity with no scalar field potential as the Brans-Dicke coupling parameter ω becomes infinite. The first method, based on an analysis of Einstein frame quantities, elucidates the relation between the two GR relatives of Jordan frame spacetimes in Brans-Dicke theory, the Einstein frame formulation and the Jordan frame limit $\omega \to \infty$. The second method combines an effective fluid description of Brans-Dicke gravity (in which the Jordan frame scalar ϕ is equivalent to an imperfect fluid sourcing effective Einstein equations) with a 1-parameter symmetry group of the theory [30]. The two methods are independent of each other and complementary and produce the same result. As $\omega \to \infty$, the metric g_{ab} of an (electro)vacuum Brans-Dicke spacetime (M, g_{ab}, ϕ) (where M is the spacetime manifold) tends to a metric which does not solve the (electro)vacuum Einstein equations. It solves instead the Einstein equations sourced by a minimally coupled scalar field. Moreover, this scalar coincides with the Einstein frame scalar field $\tilde{\phi}$ corresponding to the Jordan frame scalar ϕ present before taking any limit. Contrary to much reasoning in the literature, the methods used do not rely crucially on assuming *a priori* the asymptotics of the Brans-Dicke scalar field as $\omega \to \infty$. The formal explanation of the failure of the $\omega \to \infty$ limit of (electro)vacuum Brans-Dicke theory to reproduce a GR solution with the same matter source, proposed long ago in Ref. [30], is put on a more physical basis by the present analysis. Essentially, the $\omega \to \infty$ limit does not freeze the scalar gravitational d.o.f. of Brans-Dicke theory but demotes it to the role of an "ordinary" minimally coupled scalar field with stress-energy tensor quadratic in the first derivatives of this field and no contribution linear in the second derivatives. This is still a nontrivial dynamical field, but it has changed its gravitational nature to that of a matter field.

As an example, we have performed a test of our conclusions using the general static, spherical, asymptotically flat solution of the vacuum Brans-Dicke equations [41,42,44] (excluding black holes, for which the Brans-Dicke scalar is trivial outside the event horizon [45]). The result, which produces the FJNWBW metric hosting a central naked singularity, confirms the conclusions of Secs. II and IV.

Shedding light on this problem that is two decades old is important for three reasons. First, contrary to two decades ago, there is now a major experimental effort to test, or constrain, gravity at many scales involving cosmology [4,5], supermassive black holes [6], binary systems of compact objects emitting gravitational waves [7], and the Solar System [1–3,9,10]. Since the PPN formalism based on the weak-field limit of gravitational theories [10] is the basis for many of these analyses, one should worry about the exact meaning of this formalism if vacuum Brans-Dicke solutions (which describe wormhole throats or naked singularities) do not go over to the corresponding GR solutions (usually, the Schwarzschild geometry which describes a black hole). Second, a clear picture of the anomalous limit of (electro)vacuum Brans-Dicke theory to GR may help understanding dynamical attractor mechanisms of scalar-tensor to GR gravity. Finally, in the context of the thermodynamics of spacetime [34,35], modified theories of gravity (including scalar-tensor gravity) could correspond to deviations from an equilibrium state corresponding to Einstein theory [36]. The implications of the new picture of the $\omega \to \infty$ limit of (electro)vacuum Brans-Dicke theory for these two areas of research will be discussed in future publications.

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