Tracking the pre-inflation era from density perturbation spectra

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One of the great triumphs of the inflationary model is the prediction of the flat power spectrum of the CMB fluctuation. The prediction is based on the assumption of the de Sitter vacuum in the past infinity. However, the true past infinity of the inflation is expected to be dominated by radiation and curvature of the space. We consider the pre-inflation era as dominated by radiation and curvatures as well as inflation potential. We derive the exact solutions for the scalar fields in this era and find an exact power spectra caused by the inflaton vacuum fluctuation. We show that the power spectrum is almost flat for the subhorizon scale and deviates from flat for very high superhorizon fluctuation, which is quite sensitive to the radiation and the curvature in the pre-inflation era.

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I. INTRODUCTION

The inflationary scenario solves the problems of the flatness, horizon, and origin of the fluctuation at the same time. One of the remarkable predictions of inflation is the power spectrum of the cosmic microwave background (CMB). This prediction has been confirmed in great accuracy and provides severe constraints on the inflationary models [1].

However, it has been pointed out that there is a discrepancy with the Hubble parameters if we assume the flat-ΛCDM model [2–4]. Another problem that may be related with the Hubble constant is that the CMB data favor the value $H_0 = (67.6 \pm 0.6)\,\mathrm{km s^{-1} Mpc^{-1}}$ [1], whereas the local measurement favors $H_0 = (73.24 \pm 1.74)\,\mathrm{km s^{-1} Mpc^{-1}}$ [5]. Since we do not know the reason for this discrepancy, it may be wise to find other models.

In inflationary models, the seed fluctuations are explained by the quantum theory of the inflaton. The quantum fluctuation of the inflaton or some other scalar fields related to the inflation (e.g., the fields in the hybrid inflation) freezes to some classical value through the exponential expansion of the Universe. On the theoretical side, it is known that there is an ambiguity in the choice of quantum states for such scalar fields. In de Sitter spacetime, it is known that there is no de Sitter-invariant Fock vacuum state for minimally coupled massless scalar fields, while there exists a family of such states, called α-vacua, for massive scalar fields [6]. Later, it was shown that a de Sitter-invariant state, which is not a Fock vacuum, can be constructed by including the zero mode properly for massless fields [7]. One natural choice is the so-called Bunch-Davies vacuum (also called Euclidean or thermal vacuum in the literature), which is associated with the mode function v_ω satisfying the following condition:

$$\lim_{\eta\to-\infty} v_\omega(\eta) = \frac{1}{\sqrt{2\omega}} e^{-i\omega\eta}, \tag{1.1}$$

namely, the ordinary positive frequency mode in Minkowski spacetime. Although this choice seems plausible, there have been many arguments about the possibility of non-Bunch-Davies vacuum states or even excited states (see, e.g., [8–10]).

However if we have a pre-inflation era, the past infinity of the inflation era may be affected by the history of the pre-inflation evolution. For example, several works (see [11] and references therein) considered a scenario in which the slow-roll inflation is preceded by a fast-roll phase. Assuming that the energy scale of inflation is well below the Planck scale, we expect that the pre-inflation evolution of the Universe is dominated by radiation as well as the effects of spatial curvature. Note that if our Universe started with the quantum to classical transition, we naturally expect that the kinetic energy of the space is the same order as the potential energy caused by the spatial curvature, which is the origin of the flatness problem. A pre-inflation era dominated by radiation component and its effect on primordial spectrum has been considered in [12–14].[1] Also, the effect of the spatial curvature on density perturbation in inflation has been analyzed in several papers [16–20]. They showed that the power spectrum for low l region deviates from flat spectrum. The analysis comparing to Plank results has been done in Refs. [21,22]. However,

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[1] Such an era was introduced earlier in the context of cosmological phase transitions in [15].

as far as we know, there is no simultaneous account of these two effects. Usually we expect that they affect the overall normalization as a transfer function.

In this paper, we consider the effect of both the radiation and the curvature, and show the exact solution of the inflaton (massless scalar field) equation. By quantizing the inflaton, we derive the exact power spectrum and show that it is almost flat except for the very large scale. In the visible scale, the modification seems to be very small but may be observable.

In the next section, we consider the pre-inflation era and find that the scale factor can be written as a Weierstrass elliptic function $\wp(z)$ in conformal time. In Sec. III, we show that the field equation of the inflaton can be written as a Lamé equation so that the solutions can be written using the Weierstrass sigma and zeta functions, $\sigma(z)$ and $\zeta(z)$. By using various formulas concerning these functions, we derive the exact power spectrum. The final section is devoted to discussions.

II. PRE-INFLATION ERA

We consider the Friedmann-Robertson-Walker metric in conformal time,

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 d^2\Omega\right]. \tag{2.1}$$

We do not know much about pre-inflation era. The inflation may start from the quantum to classical transition in gravity. However, the inflaton may be related to the grand unified theories where the energy scale may be lower than the Planck scale. Therefore, it is reasonable to think that our Universe began with many relativistic matters. If our Universe started with the quantum to classical transition, it is reasonable to expect that the kinetic energy of our Universe, which is related to the expansion rate, and the potential energy, which is related to the spatial curvature, are of the same order. The spatial curvature is suppressed during inflation to resolve the flatness problem. In the inflationary scenario, the vacuum energy caused by the potential is the origin of the inflation era. There are many interpretations of the origin of the potential. In the chaotic inflation, the stochastic process is the origin of the initial value of the potential. But here we assume that even before the inflation, the energy caused by the effective potential is present whose value is denoted by $V_0(>0)$. Then the Friedmann equation is given by

$$\left(\frac{1}{a^2}\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.2}$$

where K is the curvature of the space. The energy density is dominated by the radiation. Therefore, we include the radiation density as well as the inflaton potential,

$$\rho = \frac{\rho_r}{a^4} + V_0. \tag{2.3}$$

We assume that quantum to classical transition occurred at Plank scale and the curvature energy is almost the same order of energy of the radiation. Therefore, after the quantum to classical transition of the spacetime, our Universe is radiation and curvature dominant followed by the era of vacuum energy domination. Exponential growth starts when

$$\frac{|K|}{a^2} \sim \frac{8\pi G}{3}V_0. \tag{2.4}$$

We use the following notation:

$$\frac{8\pi G V_0}{3} = H^2, \qquad A = -\frac{K}{H^2}, \qquad B = \frac{\rho_r}{V_0}. \tag{2.5}$$

Then we have

$$H\eta = \int^a \frac{da}{(a^4 + Aa^2 + B)^{1/2}}. \tag{2.6}$$

$a^4 + Aa^2 + B = 0$ has two solutions, $a^2 = -A/2 + \sqrt{(A/2)^2 - B}$ and $a^2 = -A/2 - \sqrt{(A/2)^2 - B}$, which are denoted by $\tilde{e}_2, \tilde{e}_3$, respectively. There are two cases: (i) $A > -2\sqrt{B}$, and (ii) $A < -2\sqrt{B}$. For case (i), the Universe can start from $a = 0$ since the singularities in the integrand of (2.6) are not on the real axis. Therefore we can fix the integration constant as

$$H\eta = \int_0^a \frac{da}{(a^4 + Aa^2 + B)^{1/2}} = \frac{1}{2}\int_0^{a^2} \frac{dx}{(x^3 + Ax^2 + Bx)^{1/2}}, \tag{2.7}$$

where $x = a^2$. By shifting integration variable as $x = y - A/3$, we can remove the quadratic term,

$$H\eta = \int_{e_1}^{a^2 + e_1} \frac{dy}{[4(y-e_1)(y-e_2)(y-e_3)]^{1/2}}, \tag{2.8}$$

where

$$e_1 = \frac{A}{3}, \qquad e_2 = \tilde{e}_2 + \frac{A}{3}, \qquad e_3 = \tilde{e}_3 + \frac{A}{3}, \tag{2.9}$$

which satisfy

$$e_1 + e_2 + e_3 = 0. \tag{2.10}$$

Since (2.8) can be decomposed as

$$H\eta = \int_{e_1}^{\infty} \frac{dy}{[4(y-e_1)(y-e_2)(y-e_3)]^{1/2}} - \int_{a^2+e_1}^{\infty} \frac{dy}{[4(y-e_1)(y-e_2)(y-e_3)]^{1/2}}, \tag{2.11}$$

we can write the inverse relation by using the Weierstrass elliptic function as follows:

$$a(\eta) = [\wp(\omega_1 - \tilde{\eta}) - e_1]^{1/2}, \tag{2.12}$$

where $\wp$ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2}\right], \tag{2.13}$$

and

$$\tilde{\eta} = H\eta. \tag{2.14}$$

ω_1 is one of the half periods and given by

$$\omega_1 = \int_{e_1}^{\infty} \frac{dy}{[4(y-e_1)(y-e_2)(y-e_3)]^{1/2}}. \tag{2.15}$$

It is easy to see that for small η we have

$$a(\eta) \sim \sqrt{\tilde{e}_2\tilde{e}_3}\tilde{\eta} = \sqrt{B}\tilde{\eta}, \tag{2.16}$$

whereas a approaches

$$a(\eta) \sim \frac{1}{\omega_1 - \tilde{\eta}}, \tag{2.17}$$

when $\eta \to \omega_1/H$, which represents the de Sitter phase in conformal time.

For case (ii), the integrand of (2.6) has two singularities on the positive real axis at $a = \sqrt{\tilde{e}_2}$ and $a = \sqrt{\tilde{e}_3}(<\sqrt{\tilde{e}_2})$. Thus, the Universe starts with a finite value $a = \sqrt{\tilde{e}_2}$ and our Universe does not have the "initial singularity." In this case we have

$$a(\eta) = [\wp(\omega_2 - \tilde{\eta}) - e_1]^{1/2}, \tag{2.18}$$

where

$$\omega_2 = \int_{e_2}^{\infty} \frac{dy}{[4(y-e_1)(y-e_2)(y-e_3)]^{1/2}} \tag{2.19}$$

is another half period. Behavior around $\eta \sim 0$ is different from that of case (i),

$$a(\eta) \sim \sqrt{\tilde{e}_2}\left(1 + \frac{\tilde{e}_2 - \tilde{e}_3}{2}\tilde{\eta}^2\right), \tag{2.20}$$

while the de Sitter phase appears around $\tilde{\eta} \sim \omega_2$,

$$a(\eta) \sim \frac{1}{\omega_2 - \tilde{\eta}}. \tag{2.21}$$

III. EXACT SOLUTION OF MASSLESS SCALAR FIELDS AND THE SPECTRUM OF THE DENSITY PERTURBATION

In this section, we solve the equation of massless scalar field exactly on the background spacetime derived in the previous section, and quantize it to calculate the power spectrum. The equation of massless scalar field ψ is given by

$$\frac{\partial^2}{\partial\eta^2}\psi(x,\eta) + \frac{2}{a}\frac{da}{d\eta}\frac{\partial}{\partial\eta}\psi(x,\eta) - \Delta\psi(x,\eta) = 0. \tag{3.1}$$

If we decompose the solution as $\psi(x,\eta) = \chi_k(\eta)\phi_k(x)$, where $\Delta\phi_k(x) = -k^2\phi_k(x)$, the equation for $\chi_k(\eta)$ is

$$\frac{d^2}{d\eta^2}\chi_k + \frac{2}{a}\frac{da}{d\eta}\frac{d}{d\eta}\chi_k + k^2\chi_k = 0. \tag{3.2}$$

We use the variable $\tilde{\eta} = H\eta$ and write the above equation as

$$\chi_k'' + 2\frac{a'}{a}\chi_k' + \tilde{k}^2\chi_k = 0, \tag{3.3}$$

where the prime denotes the derivative with respect to $\tilde{\eta}$ and $\tilde{k} = k/H$. By rescaling χ_k as

$$v_k = a\chi_k, \tag{3.4}$$

we have the following equation for v_k:

$$v_k'' + \left(-\frac{a''}{a} + \tilde{k}^2\right)v_k = 0. \tag{3.5}$$

Inserting (2.12) for the case (i), we find

$$\frac{d^2}{d\tilde{\eta}^2}v_k = [2\wp(\omega_1 - \tilde{\eta}) + e_1 - \tilde{k}^2]v_k. \tag{3.6}$$

The equation for case (ii) is quite similar. We obtain

$$\frac{d^2}{d\tilde{\eta}^2}v_k = [2\wp(\omega_2 - \tilde{\eta}) + e_1 - \tilde{k}^2]v_k. \tag{3.7}$$

We observe that these equations are the Lamé equation

$$\frac{d^2}{dx^2}y(x) = [l(l+1)\wp(x) + h]y(x), \tag{3.8}$$

with $l = 1, h = e_1 - \tilde{k}^2$. The solution of the Lamé equation for $l = 1$ is a classical result [23].[2] For case (i), two independent solutions are

$$v_k = a_0 \frac{\sigma(\omega_1 - \tilde{\eta} + c)}{\sigma(\omega_1 - \tilde{\eta})\sigma(+c)} e^{-(\omega_1 - \tilde{\eta})\zeta(+c)},$$
$$v_{-k} = a_0 \frac{\sigma(\omega_1 - \tilde{\eta} - c)}{\sigma(\omega_1 - \tilde{\eta})\sigma(-c)} e^{-(\omega_1 - \tilde{\eta})\zeta(-c)}, \qquad (3.9)$$

where a_0 is the normalization constant, and $\zeta(z)$ and $\sigma(z)$ are defined as [25]

$$\zeta(z) = \frac{1}{z} + \sum{}' \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right], \qquad (3.10)$$

$$\sigma(z) = z \prod{}' \left[\left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}\right)\right], \qquad (3.11)$$

where $\sum' = \sum_{(m,n)\neq(0,0)}, \prod' = \prod_{(m,n)\neq(0,0)}$, and $\omega = 2m\omega_1 + 2n\omega_2$. They are related to the Weierstrass elliptic function as follows:

$$\wp(z) = -\zeta'(z), \qquad \zeta(z) = \frac{\sigma'(z)}{\sigma(z)}. \qquad (3.12)$$

The value of c is defined as

$$\wp(c) = e_1 - \tilde{k}^2. \qquad (3.13)$$

Expansion around $z = 0$ can be derived as

$$\wp(z) = \frac{1}{z^2} + \cdots,$$
$$\zeta(z) = \frac{1}{z} + \cdots,$$
$$\sigma(z) = z + \cdots. \qquad (3.14)$$

We also point out that $\wp(z)$ is an even function whereas $\zeta(z)$, $\sigma(z)$ are odd functions. Although Eq. (3.13) determines c only up to the periods of $\wp(z)$,[3] the solutions (3.9) are not ambiguous because these are periodic functions with respect to c, namely,

$$v_k(c + 2\omega_i) = v_k(c). \qquad (3.15)$$

This result can be derived by using quasiperiodic properties

$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i,$$
$$\sigma(z + 2\omega_i) = -\sigma(z) \exp[2(z + \omega_i)\eta_i], \qquad (3.16)$$

where $\eta_i = \zeta(\omega_i)$.

For case (ii), the solutions are obtained as

$$v_k = a_0 \frac{\sigma(\omega_2 - \tilde{\eta} + c)}{\sigma(\omega_2 - \tilde{\eta})\sigma(c)} e^{-(\omega_2 - \tilde{\eta})\zeta(c)},$$
$$v_{-k} = a_0 \frac{\sigma(\omega_2 - \tilde{\eta} - c)}{\sigma(\omega_2 - \tilde{\eta})\sigma(-c)} e^{(\omega_2 - \tilde{\eta})\zeta(c)}, \qquad (3.17)$$

which are also periodic with respect to c.

Let us next prove that v_k and v_{-k} are complex conjugates when $B > A^2/4$ [included in case (i)]. By (3.13), we find

$$c = \frac{1}{2}\int_0^\infty \frac{dx}{x^{1/2}(x^2 + Ax + B)^{1/2}} + \frac{1}{2}\int_{-\tilde{k}^2}^0 \frac{dx}{x^{1/2}(x^2 + Ax + B)^{1/2}}$$
$$= \omega_1 + \frac{1}{2}\int_{-\tilde{k}^2}^0 \frac{dx}{x^{1/2}(x^2 + Ax + B)^{1/2}}. \qquad (3.18)$$

The second term is pure imaginary so that

$$(c - \omega_1)^* = -(c - \omega_1), \qquad (3.19)$$

which leads to

$$c^* = 2\omega_1 - c \sim -c, \qquad (3.20)$$

where "∼" denotes the equivalence up to the periods. Note that we also used the fact that ω_1 is real for $B > A^2/4$. Then, it is straightforward to prove

$$v_k^* = v_{-k}, \qquad {}^\forall k > 0. \qquad (3.21)$$

This relation also holds for $A < -2\sqrt{B}$ [case (ii)].

In the case $A > 2\sqrt{B}$, on the other hand, the relation between v_k and v_{-k} depends on k. This is because the complex conjugate of c behaves differently from (3.20) as follows:

$$c^* \sim \begin{cases} c & (\tilde{k}^4 - A\tilde{k}^2 + B < 0) \\ -c & (\tilde{k}^4 - A\tilde{k}^2 + B > 0). \end{cases} \qquad (3.22)$$

This difference can be seen from the second term in Eq. (3.18), whose integrand becomes real near $x = -\tilde{k}^2$, leading to $c^* \sim c$. It follows then that v_k is real for the wave number k such that $c^*(k) \sim c(k)$,

$$v_k^* = v_k, \qquad \tilde{k}^4 - A\tilde{k}^2 + B < 0. \qquad (3.23)$$

[2]The method used in Ref. [23] has been applied to the evolution equation for gravitational waves in Ref. [24].

[3]There is another ambiguity because $\wp(-c) = \wp(c)$. Replacing c with $-c$ corresponds to the change $v_k \leftrightarrow v_{-k}$. We fix this ambiguity later [see (3.27)].

This result shows that the wave function v_k is deformed in this region so that v_k can no longer be regarded as a mode function to quantize. For this reason, we concentrate on the case $A < 2\sqrt{B}$ in the rest of this paper.

We are going to find the normalization of the solutions. We use the following normalization:

$$v_k \frac{d}{d\eta} v_k^* - v_k^* \frac{d}{d\eta} v_k = i. \tag{3.24}$$

This normalization is equivalent to considering $v_k \sim e^{-ik\eta}/\sqrt{2k}$ for the massless scalar field in flat space. By explicit evaluation of (3.9) and (3.17) using the following formulas [25],

$$\sigma(u-v)\sigma(u+v) = -\sigma^2(u)\sigma^2(v)[\wp(u) - \wp(v)],$$
$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}, \tag{3.25}$$

we find

$$v_k \frac{dv_k^*}{d\eta} - v_k^* \frac{dv_k}{d\eta} = -a_0^2 H \wp'(c). \tag{3.26}$$

$\wp'(c)$ is determined up to sign by the differential equation $(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ with the definition of c (3.13). Here we take

$$\wp'(c) = -2i\tilde{k}\sqrt{\tilde{k}^4 - A\tilde{k}^2 + B} \tag{3.27}$$

to ensure that v_k represents the positive frequency mode around $\eta \sim 0$ as is shown in the next paragraph. Then, the normalization condition (3.24) gives

$$a_0 = \frac{1}{\sqrt{2\tilde{k}H}(\tilde{k}^4 - A\tilde{k}^2 + B)^{1/4}}. \tag{3.28}$$

Before considering the power spectrum, we must choose the vacuum state of the quantum field. To do so, we first derive the behavior of the mode function $v_k(\eta)$ in the past infinity. The past infinity corresponds to $\tilde{\eta} = 0$. We rewrite $v_k(\eta)$ as

$$\begin{aligned} v_k(\eta)/v_k(0) &= \exp[\ln\sigma(\omega_1 - \tilde{\eta} + c) - \ln\sigma(\omega_1 + c) \\ &\quad - (\ln\sigma(\omega_1 - \tilde{\eta}) - \ln\sigma(\omega_1)) + \tilde{\eta}\zeta(c)], \\ &= \exp\left[\int_{\omega_1}^{\omega_1-\tilde{\eta}} (\zeta(x+c) - \zeta(x) - \zeta(c))dx\right]. \end{aligned} \tag{3.29}$$

By using (3.25), we have

$$\begin{aligned} v_k(\eta)/v_k(0) &= \exp\left[\frac{1}{2}\int_{\omega_1}^{\omega_1-\tilde{\eta}} \frac{\wp'(x) - \wp'(c)}{\wp(x) - \wp(c)}dx\right] \\ &= \left(\frac{\wp(\omega_1 - \tilde{\eta}) - \wp(c)}{\wp(\omega_1) - \wp(c)}\right)^{1/2} \\ &\quad \times \exp\left[\frac{-\wp'(c)}{2}\int_{\omega_1}^{\omega_1-\tilde{\eta}} \frac{dx}{\wp(x) - \wp(c)}\right]. \end{aligned} \tag{3.30}$$

We evaluate (3.30) for $\tilde{\eta} < \omega_1$. Since $\wp(\omega_1 - \tilde{\eta}) = e_1 + O(\tilde{\eta}^2)$, we obtain

$$v_k(\eta)/v_k(0) \sim \exp[-i\tilde{k}(1 - A/\tilde{k}^2 + B/\tilde{k}^4)^{1/2}\tilde{\eta}] \tag{3.31}$$

for small $\tilde{\eta}$. This result shows that, in the past, v_k behaves as the mode function in the flat spacetime, i.e., $e^{-ik\eta}$, only for large k while the wave number is deformed for small k. So we fix the mode function by considering large k behavior.

We expand the quantum field as

$$\chi(\eta, x) = \frac{1}{a(\eta)}\sum_k (a_k v_k(\eta)\phi_k(x) + a_k^\dagger v_k^*(\eta)\phi_k^*(x)). \tag{3.32}$$

This estimate of the asymptotic behavior is consistent with the normalization by (3.24). We are considering the large $\tilde{k}$ region, where flat space approximation is valid; therefore

$$\begin{aligned} \chi(\eta, x) &= \frac{1}{a}\int \frac{d^3k}{(2\pi)^{3/2}}(a_{\mathbf{k}} v_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger v_k^*(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}) \\ &= \int \frac{d^3k}{(2\pi)^{3/2}}(a_{\mathbf{k}}\chi_k + a_{\mathbf{k}}^\dagger \chi_k^*). \end{aligned} \tag{3.33}$$

By using the explicit solutions (3.9) and (3.17), we get

$$\chi_k^*\chi_k = \frac{v_k^* v_k}{a^2} = \frac{H^2}{2k^3(1 - AH^2/k^2 + BH^4/k^4)^{1/2}}\left(1 + \frac{k^2}{H^2a^2}\right). \tag{3.34}$$

After inflation ($a \gg 1$), this value is frozen to

$$\chi_k^*\chi_k \to \frac{H^2}{2k^3(1 - AH^2/k^2 + BH^4/k^4)^{1/2}}. \tag{3.35}$$

By the usual definition of the power spectrum

$$\mathcal{P}_\chi(k) = \frac{k^3}{2\pi^2}\chi_k^*\chi_k, \tag{3.36}$$

we finally obtain the following power spectrum:

$$\mathcal{P}_\chi(k) = \left(\frac{H}{2\pi}\right)^2 \frac{1}{(1 - AH^2/k^2 + BH^4/k^4)^{1/2}}. \tag{3.37}$$

One of the predictions of this spectrum is that at large scale, i.e., sufficiently small k, the perturbation spectrum goes to 0 whereas it goes to constant value at large k. Small k behavior is understood from (3.31). If we introduce the effective wave number $q(k) = k(1 - A/\tilde{k}^2 + B/\tilde{k}^4)^{1/2}$ to write $v_k(\eta)/v_k(0) \sim e^{-iq\eta}$, we can see that $q(k)$ has the minimum $q_{\rm min} = H\sqrt{2\sqrt{B} - A}$ at $\tilde{k} = B^{1/4}$. Thus, the radiation energy is a kind of infrared cutoff. As a result, the vacuum expectation value of χ^2, which is evaluated as the integral of $\mathcal{P}_\chi(k)/k$, is IR convergent in contrast to the usual de Sitter vacuum case.

When the curvature is negative ($K < 0 \Leftrightarrow A > 0$), there appears an enhancement of the perturbation at small k. As an example, we list a figure (Fig. 1) for open space ($K < 0$) for the values $A = 5 \times 10^{-3}, B = 2 \times 5^4 \times 10^{-8}$. For this parameter, we find that there is a very small deviation from flat space and there is a peak at $k = H\sqrt{2B/A}$, which may be invisible since the length scale is too large. However, if we consider closed universe ($K > 0$), there is no enhancement but a monotonic decrease as k becomes smaller (Fig. 2).

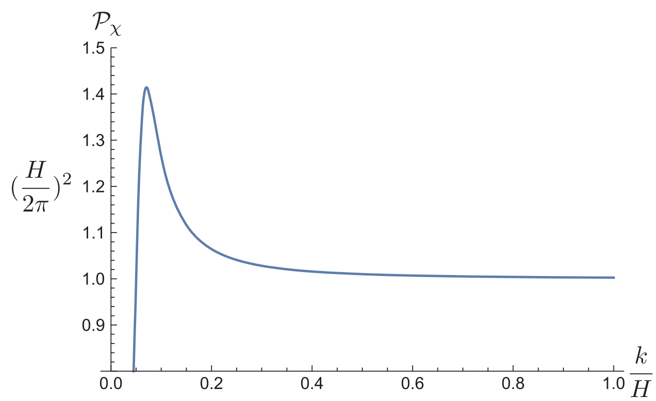

FIG. 1. A plot of the spectrum of $\mathcal{P}_\chi(k)$ normalized by $(H/2\pi)^2$ for open universe $A = 5 \times 10^{-3}, B = 2 \times 5^4 \times 10^{-8}$ at very high superhorizon wavelength. We can see an enhancement of the power spectrum for small k.

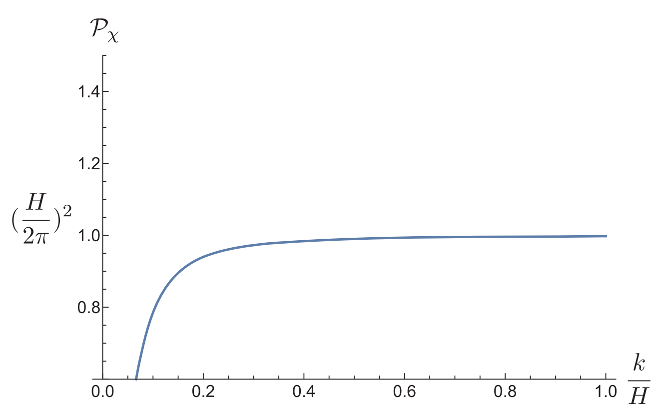

FIG. 2. A plot of the spectrum $\mathcal{P}_\chi(k)$ for closed universe $A = -5 \times 10^{-3}, B = 2 \times 5^4 \times 10^{-8}$ at very high superhorizon wavelength. We see no enhancement of power spectrum for small k.

IV. SUMMARY AND DISCUSSIONS

The usual inflationary scenarios assume that inflation starts from the de Sitter vacuum in the past infinity. We here considered that we have radiation and curvature dominant eras before inflation. These stages affect the in-state vacuum compared with the case of usual inflation. We have shown that the massless free scalar field equation (3.1) in this scenario can be written as Lamé equation (3.6) and can be solved exactly. The solution can be written in terms of Weierstrass elliptic functions and we showed the exact power spectrum of the inflation. It modifies the usual scaling behavior, especially for small k. Power spectrum (3.37) is suppressed as $\mathcal{P}_\chi(k) \propto k^2$ around $k = 0$. This kind of large scale suppression was observed in previous studies on various pre-inflationary scenarios [11–14,17]. Moreover, our result (3.37) exhibits a peak before the suppression when the Universe is open ($K < 0$). The peak is located at $\tilde{k}_{\rm peak} = \sqrt{2B/A}$, so smaller energy density of radiation compared with that of spatial curvature makes the length scale of the peak larger. On the other hand, the height of the peak behaves as $\mathcal{P}_\chi(\tilde{k}_{\rm peak})/\mathcal{P}_\chi(\infty) = (1 - A^2/4B)^{-1/2}$, which means that larger radiation energy makes the peak invisible. Therefore we are in a dilemma: smaller radiation energy (or larger curvature) makes the peak noticeable but leads to longer length scale, where the effect of cosmic variance becomes dominant. Note that this behavior can be understood by comparing the comoving Hubble scale divided by the inflation energy scale $\mathcal{H}/H = a'/a$ with $\tilde{k}_{\rm peak}$. Since $\mathcal{H}$ monotonically decreases in the radiation era while increases in the inflationary epoch, it has a minimum $\mathcal{H}_{\rm min}/H = \sqrt{A + 2\sqrt{B}}$ at $a = B^{1/4}$. The ratio of these two scales can be written as

$$\frac{\mathcal{H}_{\rm min}}{\tilde{k}_{\rm peak} H} = \sqrt{\frac{A}{\sqrt{B}} + \frac{1}{2}\left(\frac{A}{\sqrt{B}}\right)^2}, \qquad (4.1)$$

which shows that the comoving scale $\tilde{k}_{\rm peak}$ can, at least for short time, be inside the Hubble scale when $0 < A/\sqrt{B} < \sqrt{3} - 1$. In this parameter region, the peak height is constrained as $\mathcal{P}_\chi(\tilde{k}_{\rm peak})/\mathcal{P}_\chi(\infty) < \sqrt{2/\sqrt{3}} \approx 1.07$, so we can conclude that the enhancement in the power spectrum is due to large scale modes that are never inside the horizon before and during inflation. Nevertheless, it is interesting that the scalar field equation can be written as a Lamé equation and we could find the solution exactly; and deviation from the scale-invariant spectrum for low multipole l observed in CMB anisotropy may be attributed to this kind of pre-inflationary effect.

There are some problems, however. One is our assumption that the inflaton potential is present as constant even before inflation. There are many scenarios for inflation, in some of which the vacuum energy happens as phase transition. For such a case, we have to consider effective potential before inflation, which may change in accordance with the energy scale. Furthermore, our consideration here lacks how to realize a pre-inflationary era that is dominated by radiation and curvature. We note that in Ref. [18], the authors proposed a mechanism for introducing negative curvature by assuming a preceding old inflationary epoch. Another problem is that we do not know whether it is valid to use free inflaton before inflation. The interaction may change the behavior of the spectrum. However, it is still interesting that the free scalar field can be solved exactly.

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