Erratum: Nonperturbative improvement of the vector current in Wilson lattice QCD [Phys. Rev. D 92, 114503 (2015)]

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It was discovered that the improvement condition proposed in our original manuscript, which is based on equating different discretizations of the vector two-point function, is inconsistent with the power counting in the lattice spacing. Nevertheless, a particular linear combination of the improvement coefficients for the local and point-split currents can still be constrained using this method. We compare the difference of the improvement coefficients obtained by this method with the difference of improvement coefficients obtained using an improvement condition derived from chiral Ward identities.

The proposed $\mathrm{O}(a)$-improvement condition for the local and point-split discretizations of the isovector vector current, $(V)^{\mathrm{l}}_{\mu}(x)$ and $(V)^{\mathrm{c}}_{\mu}(x)$, amounted to solving the linear system

$$\begin{pmatrix} 2f^{\mathrm{ll}}(x_0) - f^{\mathrm{cl}}(x_0) & -f^{\mathrm{lc}}(x_0) \\ -f^{\mathrm{cl}}(x_0) & 2f^{\mathrm{cc}}(x_0) - f^{\mathrm{lc}}(x_0) \end{pmatrix} \begin{pmatrix} c^{\mathrm{l}}_{\mathrm{V}} \\ c^{\mathrm{c}}_{\mathrm{V}} \end{pmatrix} = \frac{1}{a} \begin{pmatrix} g^{\mathrm{cl}}(x_0) - g^{\mathrm{ll}}(x_0) \\ g^{\mathrm{cl}}(x_0) - g^{\mathrm{cc}}(x_0) \end{pmatrix}, \tag{1}$$

where

$$f^{ij}(x_0) = \hat{Z}^i_{\mathrm{V}} \hat{Z}^j_{\mathrm{V}} \int \mathrm{d}^3x \langle (V)^i_1(x_0, \boldsymbol{x}) \partial_\mu T_{1\mu}(0) \rangle, \tag{2}$$

$$g^{ij}(x_0) = \hat{Z}^i_{\mathrm{V}} \hat{Z}^j_{\mathrm{V}} \int \mathrm{d}^3x \langle (V)^i_1(x_0, \boldsymbol{x}) (V)^j_1(0) \rangle \quad \text{and} \quad i, j = \mathrm{l}, \mathrm{c}. \tag{3}$$

The right-hand side of (1) is of order a^0, where a is the lattice spacing. Thus if the determinant of the matrix on the left-hand side (lhs) were $O(a^0)$, the improvement coefficients would turn out to be $O(a^0)$, as they should be. However, the determinant is in fact of $O(a)$. Therefore, to leading order in a, the system is singular and the improvement coefficients $c^{\mathrm{c}}_{\mathrm{V}}$ and $c^{\mathrm{l}}_{\mathrm{V}}$ are not uniquely determined by this equation. In practice, a unique solution to the linear system still exists due to the presence of higher-order lattice artifacts.

In order to investigate the magnitude of the error, we have computed the improvement coefficients independently by imposing an improvement condition derived from enforcing a chiral Ward identity at $\mathrm{O}(a)$ [1,2]. The improvement condition is obtained by imposing

$$\int \mathrm{d}^3y \langle \delta S^{12} A^{23}_{\mathrm{R},i}(y^0, \vec{y}) \mathcal{O}^{31}_{\mathrm{ext}}(0) \rangle = \int \mathrm{d}^3y \langle V^{\mathrm{c},13}_{\mathrm{R},i}(y^0, \vec{y}) \mathcal{O}^{31}_{\mathrm{ext}}(0) \rangle, \tag{4}$$

where

$$\delta S^{12} = -\int_{t_a}^{t_b} \mathrm{d}x^0 \int \mathrm{d}^3x (2m^{12} P^{12}_{\mathrm{R}}(x) - \partial_\mu A^{12}_{\mathrm{R},\mu}(x)) \tag{5}$$

and for any $t_a < y^0 < t_b$ and any small m^3, the partially conserved axial current mass of an auxiliary quenched fermion flavor. The upper indices denote explicitly the flavor of the fermion fields in the bilinears. In order to remove a contact term which arises when the axial current and pseudoscalar density coincide on the lhs of Eq. (4), the limit of the average light quark partially conserved axial current masses $am^{12} \to 0$ must be taken.

We have verified that this condition reproduces the known improvement coefficients in the noninteracting massless theory. Our nonperturbative evaluation for $\beta = 5.3$ results in values $c^{\mathrm{c}}_{\mathrm{V}} = 0.35(3)$ and $c^{\mathrm{l}}_{\mathrm{V}} = -0.22(4)$ for using $y_0/a = 12$, $t_a/a = 6$, $t_b/a = 18$ and $m^3 = m^{12}$. We have used the known nonperturbative renormalization constant for the axial current

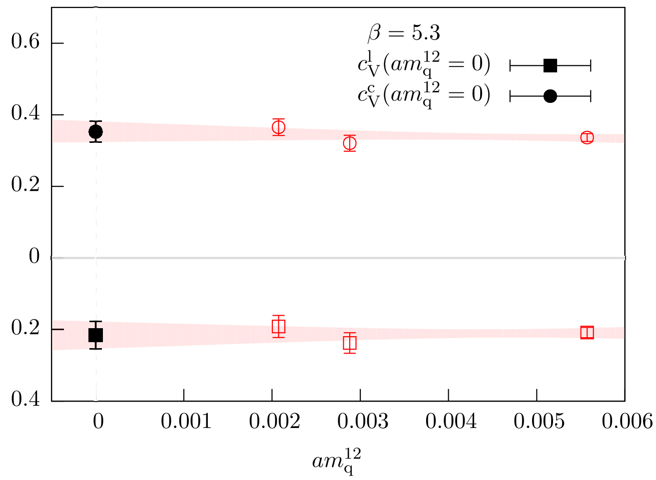

FIG. 1. The chiral limit of Eq. (4) for $\beta = 5.3$, for the conserved (circles) and local vector current (squares).

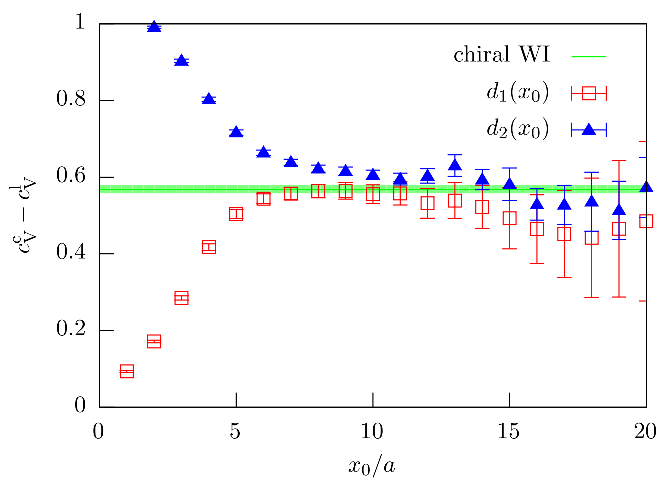

FIG. 2. Estimators d_1 and d_2 for the linear combination $c_{\mathrm{V}}^{\mathrm{c}} - c_{\mathrm{V}}^{l}$ (red points) compared with the determination from the chiral Ward identity (green band) for the E5 ensemble.

[3], while the improvement coefficients multiplying the bare subtracted quark mass, b_{A} and b_{V}, are taken to be their one-loop values [4]. The external current is chosen to be the sum of the vector and tensor bilinears $\mathcal{O}^{31}(x) = V_i^{31}(x) + T_{i0}^{31}(x)$. The chiral extrapolation required to remove the contact term is illustrated in Fig. 1 and appears to be under control. These results are in tension with our original determination of the improvement coefficients at the same value of β, e.g. $c_{\mathrm{V}}^{\mathrm{c}} = 0.232(9)$ and $c_{\mathrm{V}}^{\mathrm{l}} = -0.401(6)$ on ensemble F6.

However, one linear combination, which to leading order in g_0^2 is $c_{\mathrm{V}}^{\mathrm{c}} - c_{\mathrm{V}}^{\mathrm{l}}$, is still well determined by the system Eq. (1). In Fig. 2 the linear combination $c_{\mathrm{V}}^{\mathrm{c}} - c_{\mathrm{V}}^{\mathrm{l}}$ determined using two discretizations of the vector two-point functions (points) is compared with the determination from the improvement condition Eq. (4) (band). Among the many possible estimators for the difference which can be constructed from the two-point functions the two depicted are defined by

$$d_1(x_0) = \frac{g^{\mathrm{ll}}(x_0) - g^{\mathrm{cc}}(x_0)}{f^{\mathrm{l+c,l}}(x_0)}, \tag{6}$$

$$d_2(x_0) = \frac{g^{\mathrm{ll}}(x_0) - g^{\mathrm{cl}}(x_0)}{\frac{1}{2} f^{\mathrm{l+c,l}}(x_0)}, \tag{7}$$

where $f^{\mathrm{l+c,l}} = f^{\mathrm{ll}} + f^{\mathrm{cl}}$, and should agree up to $\mathrm{O}(a)$ ambiguities. As expected, there is good agreement observed in the difference at large Euclidean distances where the contribution of higher-order lattice artifacts is suppressed.

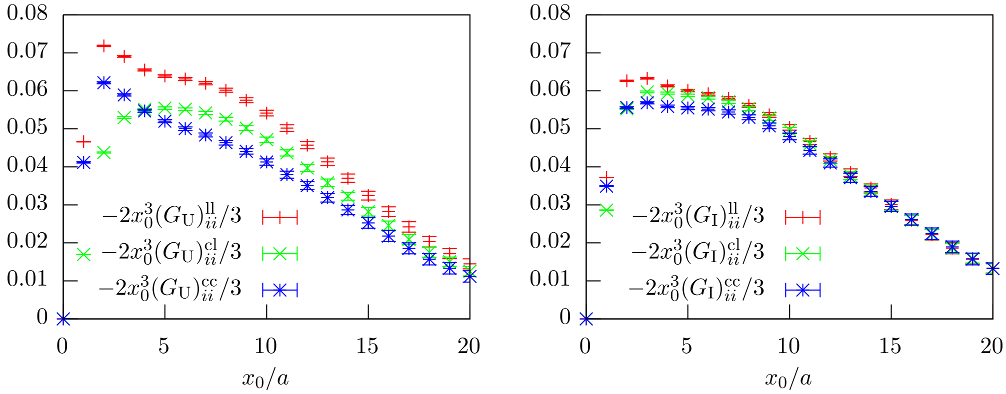

FIG. 3. The unimproved vector two-point function (left) and the improved vector two-point functions using the improvement coefficients determined from the chiral Ward identity Eq. (4) for the E5 ensemble, corresponding to the intermediate lattice spacing.

The effect of the improvement on the vector two-point function with coefficients determined from improvement condition Eq. (4) is shown in Fig. 3. While the improvement tends to enhance the agreement between the different discretizations of the two-point functions, at intermediate distances $5 \leq x_0/a \leq 10$, the agreement is not as striking as when using the originally determined improvement coefficients. We conclude that $O(a^2)$ cutoff effects are still sizable in the vector two-point function at these intermediate distances.

To summarize, the original improvement condition is invalid as a way to determine simultaneously c_V^c and c_V^l, although it remains a valid way to calibrate one of the two if the other is already known. We have used Eq. (4) in subsequent work to determine separately the two improvement coefficients [5].

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