Addendum to "Critical behavior of $(2+1)$-dimensional QED: $1/N_f$-corrections in an arbitrary nonlocal gauge"

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Dynamical chiral symmetry breaking is studied within $(2+1)$-dimensional QED with N four-component fermions. The leading and next-to-leading orders of the $1/N$ expansion were computed exactly by V. P. Gusynin and P. K. Pyatkovskiy [Phys. Rev. D **94**, 125009 (2016)] and A. V. Kotikov and S. Teber [Phys. Rev. D **94**, no. 11, 114011 (2016)] in an arbitrary nonlocal gauge. In this addendum to the work by Kotikov and Teber, we show that the resummation of the wave-function renormalization constant at the level of the gap equation yields a *complete* cancellation of the gauge dependence of the critical fermion flavour number resulting in $N_c = 2.8469$, which is such that dynamical chiral symmetry breaking takes place for $N < N_c$. The result is in full agreement with one of Gusynin and Pyatkovskiy.

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I. INTRODUCTION

We consider quantum electrodynamics in $2+1$ dimensions (QED_3), which is described by the Lagrangian

$$L = \bar{\Psi}(i\hat{\partial} - e\hat{A})\Psi - \frac{1}{4}F_{\mu\nu}^2, \tag{1}$$

where Ψ is taken to be a four-component complex spinor. In the presence of N fermion flavors, the model has a $U(2N)$ symmetry. A fermion mass term, $m\bar{\Psi}\Psi$, breaks this symmetry to $U(N) \times U(N)$. In a $1/N$ expansion [1,2], the theory is super-renormalizable, and the mass scale is then given by the dimensionful coupling constant, $a = Ne^2/8$, which is kept fixed as $N \to \infty$.

A central issue is related to the value of the critical fermion number, N_c, which is such that DχSB takes place only for $N < N_c$. An accurate determination of N_c is of crucial importance to understand the phase structure of QED_3.

In our studies Refs. [3,4], we followed the approach of Appelquist *et al.* [5], who found that $N_c = 32/\pi^2 \approx 3.24$ by solving the Schwinger-Dyson (SD) gap equation in the Landau gauge using a leading order (LO) $1/N$-expansion. Soon after the analysis of Ref. [5], Nash approximately included next-to-leading-order (NLO) corrections and performed a partial resummation of the wave-function renormalization constant at the level of the gap equation; he found [6] $N_c \approx 3.28$. Recently, upon refining the work of Ref. [7], the NLO corrections could be computed exactly in the Landau gauge, yielding (in the absence of resummation) [4] $N_c \approx 3.29$. More recently, the results of Ref. [4] have been extended in Ref. [3] to an arbitrary nonlocal gauge [8]. Reference [3] then found a residual weak gauge dependence of N_c even after Nash's resummation; it was also noticed in Ref. [3] that, if the weak gauge-dependent terms contributing to N_c were neglected, then the final result would be in perfect agreement with the one of Ref. [9].

The purpose of this short paper is to upgrade the exact results of Ref. [3] and to show the *complete* gauge independence of the critical value N_c in the $1/N^2$ approximation. Following Ref. [3] and after long discussions with Valery Gusynin, we shall modify the expansion prescription used in Ref. [3], which was based on (a NLO correction to) the gap equation to (a NLO correction to) the parameter α of its solution [see Eq. (4) and below it]. This subtle change in the interpretation of the NLO corrections does not affect at all the LO results of Appelquist but significantly modifies the NLO results (see below Sec. III) leading to gauge-invariant N_c values after Nash's resummation.

II. LEADING ORDER

Let us briefly recall the structure and solutions of the LO SD equations; see Ref. [3] for more details. In the LO approximation to the $1/N$ expansion, the SD equation to the fermion propagator has the form

$$\Sigma(p) = \frac{8(2+\xi)a}{N}\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{(k^2+\Sigma^2(k))[(p-k)^2+a|p-k|]} + O(N^{-2}), \tag{2}$$

where $\Sigma(p)$ is the dynamically generated parity-conserving mass.

Following Refs. [7] and [5], we consider the limit of large a and linearize Eq. (2), which yields

$$\Sigma(p) = \frac{8(2+\xi)}{N}\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{k^2|p-k|} + O(N^{-2}). \tag{3}$$

The mass function may then be parametrized as [5]

$$\Sigma(k) = B(k^2)^{-\alpha}, \tag{4}$$

where B is arbitrary and the index α has to be self-consistently determined. Using this ansatz, Eq. (3) reads

$$\Sigma^{(\mathrm{LO})}(p) = \frac{4(2+\xi)B}{N}\frac{(p^2)^{-\alpha}}{(4\pi)^{3/2}}\frac{2\beta}{\pi^{1/2}} + O(N^{-2}), \tag{5}$$

from which the LO gap equation is obtained,

$$1 = \frac{(2+\xi)\beta}{L} + O(L^{-2}) \quad \text{or} \quad \beta^{-1} = \frac{(2+\xi)}{L} + O(L^{-2}), \tag{6}$$

where

$$\beta = \frac{1}{\alpha(1/2-\alpha)} \quad \text{and} \quad L \equiv \pi^2 N. \tag{7}$$

Let us note that the two equations in (6) are completely equal to each other. Solving the gap equation yields

$$\alpha_\pm = \frac{1}{4}\left(1 \pm \sqrt{1-\frac{16(2+\xi)}{L}}\right), \tag{8}$$

which reproduces the solution given by Appelquist *et al.* [5]. The gauge-dependent critical number of fermions, $N_c \equiv N_c(\xi) = 16(2+\xi)/\pi^2$, is such that $\Sigma(p) = 0$ for $N > N_c$ and

$$\Sigma(0) \simeq \exp[-2\pi/(N_c/N-1)^{1/2}] \tag{9}$$

for $N < N_c$. Thus, dynamical chiral symmetry breaking (DχSB) occurs when α becomes complex, that is, for $N < N_c$.

III. NEXT-TO-LEADING ORDER

Evaluating the NLO corrections to the SD equation (2) yields (see Ref. [3]) the gap equation

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8S(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(-\frac{5}{3}+\frac{26}{3}\xi - 3\xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1-\xi)-\xi^2\right)\right] + O(L^{-3}), \tag{10}$$

where

$$\hat{\Pi} = \frac{92}{9} - \pi^2 \tag{11}$$

arises from the two-loop polarization operator in dimension $D = 3$ [10–12].

The factor $S(\alpha, \xi)$ contains the contribution of the most complicated diagrams. As was shown in Ref. [3], it is convenient to extract the most important contributions $\sim\beta$ and $\sim\beta^2$ from the complicated part $S(\alpha, \xi)$. After these calculations, the gap equation takes the equivalent form

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right] + O(L^{-3}), \tag{12}$$

where the new complicated part $\tilde{S}(\alpha, \xi)$ does not contain any positive β powers and can be expanded in series of α^n (and, hence, β^{-n}) starting with $n = 0$.

A. Gap equation

In Ref. [3], we analyzed Eq. (10) at the critical point $\beta = 16$ and found the corresponding critical value L_c. The same results can also be obtained from Eq. (12).

Here, we will follow another strategy. As was already discussed in the Introduction, we will proceed in computing the NLO correction to the parameter β^{-1} of the solution of the SD equation. From (12), we have

$$\beta^{-1} = \frac{2+\xi}{L} + \frac{1}{L^2}\left[\frac{8}{\beta}\tilde{S}(\beta,\xi) - 2(2+\xi)\hat{\Pi} + \left(\frac{2}{3} - \xi\right)(2+\xi)\beta + 4\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right] + O(L^{-3}). \tag{13}$$

From this equation, it is clear that the first term in brackets is of the order of $\sim 1/L$ [as can be seen by solving Eq. (13) iteratively], and thus its contribution is of the order of $\sim 1/L^3$ and should therefore be neglected in the present analysis. So, with NLO accuracy, we obtain that

$$\beta^{-1} = \frac{2+\xi}{L} + \frac{1}{L^2}\left[\left(\frac{2}{3} - \xi\right)(2+\xi)\beta - 2(2+\xi)\hat{\Pi} + 4\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right] + O(L^{-3}). \tag{14}$$

We are now in a position to compute β^{-1} from Eq. (14) as a combination of terms $\sim 1/L$ and $\sim 1/L^2$. This is, however, not so important in the present analysis. Since we are interested in the critical regime, we may derive L_c in a straightforward way from (14) [or equally from Eq. (12) with the condition $\tilde{S}(\beta, \xi) = 0$] by setting $\beta = 16$ and keeping the terms $O(1/L^2)$. This yields

$$L_c^2 - 16(2+\xi)L_c + 32\left[(2+\xi)\hat{\Pi} + 2\xi\left(\frac{20}{3} + 3\xi\right)\right] = 0. \tag{15}$$

Solving Eq. (15), we have two standard solutions:

$$L_{c,\pm} = 8\left(2 + \xi \pm \sqrt{d_1(\xi)}\right), \tag{16a}$$

$$d_1(\xi) = 4 - \frac{8}{3}\xi - 2\xi^2 - \frac{2+\xi}{2}\hat{\Pi}. \tag{16b}$$

Combining these values with the one of $\hat{\Pi}$ in Eq. (11) yields

$$N_c(\xi = 0) = 3.17, \qquad N_c(\xi = 2/3) = 2.91, \tag{17}$$

where "−" solutions are unphysical and there is no solution in the Feynman gauge ($\xi = 1$). The range of ξ values for which there is a solution corresponds to $\xi_- \leq \xi \leq \xi_+$, where $\xi_+ = 0.82$ and $\xi_- = -2.24$.

B. Resummation

Performing Nash's resummation, the gap equation takes the form (see Ref. [3])

$$1 = \frac{8\beta}{3L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\beta\left(\frac{40}{9} + \hat{\Pi}\right)\right] + O(L^{-3}), \tag{18}$$

which displays a strong suppression of the gauge dependence as ξ-dependent terms do exist but they enter the gap equation only through the rest, $\tilde{S}$, which is very small numerically.

In Ref. [3], we have analyzed Eq. (18) at the critical point $\beta = 16$ and found the corresponding critical value L_c. By analogy with the previous subsection, we now proceed in finding the NLO correction to the parameter β^{-1} of the solution of the SD equation. From (18), this yields

$$\beta^{-1} = \frac{8}{3L} + \frac{1}{L^2}\left[\frac{8}{\beta}\tilde{S}(\alpha,\xi) - \frac{16}{3}\left(\frac{40}{9} + \hat{\Pi}\right)\right] + O(L^{-3}). \tag{19}$$

From this equation, it is again clear that the first term in brackets is of the order of $\sim 1/L$ [as can be seen by solving Eq. (19) iteratively], and thus its contribution is $\sim 1/L^3$ and should be neglected in the present analysis. So, we have

$$\beta^{-1} = \frac{8}{3L} - \frac{1}{L^2}\frac{16}{3}\left(\frac{40}{9} + \hat{\Pi}\right) + O(L^{-3}), \tag{20}$$

which is now completely gauge independent.

We now consider Eq. (20) [or, equivalently, Eq. (18) with the condition $\tilde{S}(\beta,\xi) = 0$] at the critical point $\alpha = 1/4$ ($\beta = 16$) keeping all terms $O(1/L^2)$. This yields

$$L_c^2 - \frac{128}{3}L_c + \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi}\right) = 0. \tag{21}$$

Solving Eq. (21), we have two standard solutions,

$$L_{c,\pm} = \frac{64}{3}\left(1 \pm \sqrt{d_2(\xi)}\right), \tag{22a}$$

$$d_2(\xi) = 1 - \frac{3}{16}\left(\frac{40}{9} + \hat{\Pi}\right) = \frac{1}{6} - \frac{3}{16}\hat{\Pi}, \tag{22b}$$

and we have for the "+" solution (the − one is nonphysical):

$$\bar{L}_c = 28.0981, \qquad \bar{N}_c = 2.85. \tag{23}$$

The results of Eq. (23) are in full agreement with the recent results of Ref. [9].

IV. CONCLUSION

We have studied DχSB in QED_3 by including $1/N^2$ corrections to the SD equation exactly and taking into account the full ξ dependence of the gap equation. Following Nash, the wave-function renormalization constant has been resummed at the level of the gap equation leading to a very weak gauge variance of the critical fermion number N_c.

Reconsidering the NLO expansion of Ref. [3], we have implemented a NLO expansion for the parameter β^{-1}, which is related to the index parametrizing the mass function rather than the mass function itself. This prescription allowed us to show that the complicated weakly gauge-variant terms are actually of the order of $1/N^3$ and should be neglected in the present NLO analysis. Thus, the obtained value $N_c = 2.85$ is completely gauge independent and in full agreement with the one of Ref. [9]. Both Refs. [9] and [3] are therefore in perfect agreement and yield order by order fully gauge-invariant methods to compute N_c.

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