# 2 + 1D loop quantum gravity on the edge

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We develop a new perspective on the discretization of the phase space structure of gravity in $2+1$ dimensions as a piecewise-flat geometry in 2 spatial dimensions. Starting from a subdivision of the continuum geometric and phase space structure into elementary cells, we obtain the loop gravity phase space coupled to a collection of effective particles carrying mass and spin, which measure the curvature and torsion of the geometry. We show that the new degrees of freedom associated to the particlelike elements can be understood as edge modes, which appear in the decomposition of the continuum theory into subsystems and do not cancel out in the gluing of cells along codimension 2 defects. These new particlelike edge modes are gravitationally dressed in an explicit way. This provides a detailed explanation of the relations and differences between the loop gravity phase space and the one deduced from the continuum theory.

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## I. INTRODUCTION

One of the key challenges in trying to define a theory of quantum gravity at the quantum level is to find a regularization that does not drastically break the fundamental symmetries of the theory. This is a challenge in any gauge theory, but gravity is especially challenging, for two reasons. First, one expects that the quantum theory possesses a fundamental length scale, and second, the gauge group contains diffeomorphism symmetry, which affects the nature of the space on which the regularization is applied.

In gauge theories such as QCD, the only known way to satisfy these requirements[1] is to put the theory on a lattice, where an effective finite-dimensional gauge symmetry survives at each scale. One would like to devise such a scheme in the gravitational context. In this paper, we develop a step-by-step procedure achieving this in the context of $2+1$ gravity. We initially expected to find the so-called holonomy-flux discretized phase space, which appears in loop gravity and produces spin networks after quantization. To our surprise, we discovered that there are additional degrees of freedom (d.o.f.) that behave as a collection of particles coupled to the gravitational d.o.f.

In the loop quantum gravity (LQG) framework, gravity is quantized using the canonical approach, with the gravitational d.o.f. expressed in terms of the connection and frame field. The quantum states of geometry are known as "spin networks." In this framework, we can show that geometric operators possess a discrete spectrum. This is, however, only possible after one chooses the quantum states to have support on a graph. Spin network states can be understood as describing a quantum version of *discretized* spatial geometry [1], and the Hilbert space associated to a graph can be related, in the classical limit, to a set of discrete piecewise-flat geometries [2,3]. This means that the LQG quantization scheme consists at the same time of a *quantization* and a *discretization*; moreover, the quantization of the geometric spectrum is entangled with the discretization of the fundamental variables. It has been argued that it is essential to disentangle these two different features [4], especially when one wants to address dynamical issues.

In [4,5], it has been argued that one should understand the discretization as a two step process: a *subdivision* followed by a *truncation*. In the first step one subdivides the systems in fundamental cells, and in the second step one chooses a truncation of d.o.f. in each cell which is consistent with the symmetry of the theory. By focusing first on the classical discretization, before any quantization takes place, several aspects of the theory can be clarified. Let us mention some examples:

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[1]Other than gauge-fixing before regularization.

 

(i) The discretization scheme allows us to study more concretely how to recover the continuum geometry out of the discrete geometry which is the classical picture behind the spin networks [4,5]. In particular, since the discretization is now understood as a truncation of the continuous d.o.f., it is possible to associate a continuum geometry to the discrete data.

(ii) It provides a justification for why in the continuum case the momentum variables are equipped with a zero Poisson bracket, whereas in the discrete case the momentum variables do not commute with each other [4,6,7]. These variables needs to be dressed by the gauge connection and are now understood as charge generators [8].

(iii) It permitted the discovery of duality symmetries that suggests new discrete variables and a new dual formulation of discrete $2+1$ gravity [6]. While the quantization of spin networks leads to spin foam models, the quantization of these dual models can be related to the Dijkgraaf-Witten model [9]. This also allows one to understand, at the classical level, the presence of new dual vacua as advocated by Dittrich and Geiller [7,10].

In this work we revisit these ideas in the context of $2+1$ gravity and deepen the analysis done in [4,6] by focusing on what happens at the location of the curvature defects. Since gravity in $2+1$ dimensions is equivalent to a Chern-Simons theory constructed on a Drinfeld double group, we analyze the phase space truncation of any Chern-Simons theory constructed on a Drinfeld double group and then specialize the analysis to the case where the double group is the inhomogeneous "Poincaré" group DG over a "Lorentz" group $G$. This corresponds to the case of a zero cosmological constant when $G$ is a 3D rotation group. The variables of the theory are a pair of $\mathfrak{g}$-valued connection $\mathbf{A}$ and frame field $\mathbf{E}$, while the Chern-Simons connection $\boldsymbol{\mathcal{A}}$ is simply the sum $\boldsymbol{\mathcal{A}} = \mathbf{A} + \mathbf{E}$.

The aim of the present work is to precisely evaluate such effects in the $2+1$ case. As already emphasized, the procedure of discretization is now understood as a process of *subdivision* of the underlying manifold into cells, followed by a *truncation* of the d.o.f. in each cell. The truncation we chose, and which is adapted to $2+1$ gravity, is to assume that the geometry is locally *flat* $\mathbf{F}(\mathbf{A}) = 0$ and *torsionless* $\mathbf{T} = \mathrm{d}_{\mathbf{A}}\mathbf{E} = 0$ inside each cell. This means that the continuum geometric data $(\mathbf{E}, \mathbf{A})$ is now replaced by a DG structure, that is, a decomposition of the 2D manifold $\Sigma$ into cells $c$ with the transition functions across cells belonging to the group DG preserving the local flatness and torsionlessness conditions of the geometry.

We can think of DG as the local isometry group of our locally flat structure, and hence an analogue of the Poincaré group for flat space. If we allow possible violations of the flatness condition at the vertices $v$ of the cells, this DG structure can be understood as a flat structure on $\Sigma\backslash V$, where $\Sigma$ is the 2D manifold and $V$ is the set of all vertices. In this context, the locally flat geometry on $\Sigma$ is encoded in terms of discrete rotational and translational holonomies which allow the reconstruction of the flat connection on the punctured surface.

At the continuum level, the geometric data forms a phase space with presymplectic structure $\Omega = \int_\Sigma \delta\mathbf{E} \cdot \delta\mathbf{A}$. The main advantage of describing the discretization as a truncation is the fact that one can understand the truncated variables as forming a reduced phase space. This follows from the fact that the truncation is implemented in terms of first-class constraints and is therefore compatible with the Hamiltonian structure. This philosophy has already been applied to gravity in [4,6]. However, these works neglected the contributions from curvature and torsion singularities appearing at the vertices. In the current work we extend the analysis to include these contributions.

### A. A change of paradigm

A key result of this paper is conceptual. In the subdivision process, some of the bulk d.o.f. are replaced by *edge mode* d.o.f., which play a key role in the construction of the full phase space and our understanding of symmetry. The reason this happens is that we propose to implement the procedure of discretization as a rewriting of the theory in terms of specific subsystems. Dealing with subsystems in a gauge theory requires special care with regards to boundaries, where gauge invariance is naively broken and additional d.o.f. must be added in order to restore it.

In other words, what is called a discretization should in fact be seen as a proper way to extend the phase space by adding extra d.o.f.—a generalization of Goldstone modes needed to restore the gauge invariance—that transforms nontrivially under new edge symmetry transformations. The process of subdivision therefore requires a canonical extension of the phase space and converting some momenta into noncommutative charge generators.

The general philosophy is presented in [8] and exemplified in the $3+1$ gravity context in [11–13]. An intuitive reason behind this fundamental mechanism is also presented in [14] and the general idea is, in a sense, already present in [15]. In the $2+1$ gravity context, the edge modes have been studied in great detail in [16,17]. This phenomenon even happens when the boundary is taken to be infinity [18], where these new d.o.f. are the soft modes. One point which is important for us is that these extra d.o.f., which possess their own phase space structure and appear as "dressings" of the gravitationally charged observables, affect the commutation relations of the dressed observables. In a precise sense, this is what happens with the fluxes in loop gravity: the "discretized" fluxes are dressed by the connection d.o.f., implying a different Poisson structure compared to the continuum ones. A nice continuum derivation of this fact is also given in [19].

Once this subdivision and extension of the phase space are done properly, one has to understand the gluing of subregions as the fusion of edge modes across the boundaries. If the boundary is trivial, the fusion merely allows us to extend gauge-invariant observables from one region to another. When several boundaries meet at a corner, there is now the possibility to have residual edge d.o.f. that come from the fusion product. We witness exactly this phenomenon at the vertices of our cellular decomposition, where new d.o.f., in addition to the usual loop gravity ones, are present after regluing.

In our analysis we rewrite the gauge d.o.f., after truncation, as a collection of locally flat "Poincaré" connections $\boldsymbol{\mathcal{A}} = \mathbf{A} + \mathbf{E}$ in each cell. The choice of collection of cells corresponds to the *subdivision*, and the imposition of the flatness condition inside each cell corresponds to the *truncation*. Inside each cell, we introduce a DG-valued 0-form $\mathcal{H}_c(x)$ that parametrizes the flat connection, $\boldsymbol{\mathcal{A}}(x) \equiv \mathcal{H}_c^{-1}(x)\mathrm{d}\mathcal{H}_c(x)$. The quantities $\mathcal{H}_c(x)$ can be used to reconstruct the holonomy of the connection, but they contain more information. This comes from the fact that left and right transformations of the holonomy have different implementations at the level of the connection:

$$\mathcal{H}_c(x) \mapsto \mathcal{G}_c\mathcal{H}_c(x) \Rightarrow \boldsymbol{\mathcal{A}}(x) \mapsto \boldsymbol{\mathcal{A}}(x), \tag{1}$$

$$\mathcal{H}_c(x) \mapsto \mathcal{H}_c(x)\mathcal{G}(x) \\ \Rightarrow \boldsymbol{\mathcal{A}}(x) \mapsto \mathcal{G}^{-1}(x)\boldsymbol{\mathcal{A}}(x)\mathcal{G}(x) + \mathcal{G}^{-1}(x)\mathrm{d}\mathcal{G}(x). \tag{2}$$

In the last case, the standard gauge transformation is expressed as a right action on $\mathcal{H}_c(x)$. However, the first transformation, which acts on the left of $\mathcal{H}_c(x)$, leaves the connection invariant. It therefore corresponds to an additional d.o.f. entering the definition of $\mathcal{H}_c(x)$ which is not contained in $\boldsymbol{\mathcal{A}}$. One can understand the presence of this additional d.o.f. as coming from the presence of a boundary, and the left translation as an edge mode symmetry that has to be implemented when we reconstruct physical observables.

We can now use the variables $\mathcal{H}_c(x)$ to reconstruct our truncated phase space. For each cell, we can express the continuum symplectic potential in terms of the holonomy variables, and the expression can be readily seen as only depending on the fields evaluated at the boundaries of the cell. Summing the symplectic potentials for each cell simplifies the general expression, and the final symplectic potential depends only on the fields evaluated across boundaries.

One recovers, in particular, that holonomies from a cell to its neighbor form a subset of the canonical variables. $\mathbf{X}_c^{c'} \in \mathfrak{g}$ is the flux and $h_{cc'} \in G$ is the holonomy, while the symplectic potential reproduces the loop gravity potential $\Theta^{\mathrm{LQG}} = \sum_{(cc')} \mathrm{Tr}(\delta h_{cc'} h_{cc'}^{-1} \mathbf{X}_c^{c'})$, which gives the holonomy-flux algebra. We also find, however, that there are additional contributions coming from the vertices $v$ where several cells meet. Each vertex carries the phase space structure $\Theta_v$ of a relativistic particle, labeled by the edge modes $\mathcal{H}_v(v) = (h_v(v), \mathbf{y}_v(v)) \in G \ltimes \mathfrak{g}^*$ which are nontrivially coupled to the gravity variables through connectors $h_{vc}$ and $\mathbf{X}_v^c$.

The mass and spin of the effective relativistic particles are determined via a generalization of the Gauss constraint and curvature constraint at the vertices. The appearance of these effective particle d.o.f. from pure gravity is quite interesting and unexpected. The usual loop gravity framework is recovered when the edge mode d.o.f. labeled by $\mathcal{H}_v(v)$ are frozen.

The conceptual shift towards an edge mode interpretation, while not modifying the mathematical structures at all, provides a different paradigm to explore some of the key questions of LQG. For example, the notion of the continuum limit (in a 3 + 1-dimensional theory) attached to subregions could be revisited and clarified in light of this new interpretation, and related to the approach developed in [20,21]. It also strengthens, in a way, the spinor approach to LQG [22–24], which allows one to recover the LQG formalism from d.o.f., the spinors, living on the nodes of the graph. These spinors can be seen as a different parametrization of the edge modes (in a similar spirit to [25,26]).

Edge modes have recently been studied for the purpose of making proper entropy calculations in gauge theory or, more generally, defining local subsystems [15]. Their use could provide some new guidance on understanding the concept of entropy in LQG. They are also relevant to the study of specific types of boundary excitations in condensed matter [27], which could generate some interesting new directions to explore in LQG, just like [20,28].

### B. Comparison to previous work

Discretization (and quantization) of 2 + 1 gravity was already performed some time ago in the Chern-Simons formulation [29–32]. While our results should be equivalent to this formulation, we find them interesting and relevant for several reasons.

Firstly, we work with the gravitational variables and the associated geometric quantities, such as torsion and curvature. Our procedure describes clearly how such objects should be discretized, which is not obvious in the Chern-Simons picture; see e.g., [33] where the link between the combinatorial (quantization) framework and the LQG one was explored.

Secondly, we are using a different discretization procedure than the one used in the combinatorial approach. Instead of considering the reduced graph (flower graph), we use the full graph to generate the spin network and assume that the equations of motion are satisfied in the cells and the disks containing the geometric excitations.

Thirdly, one of the most important differences is in the fact that we *derive* the symplectic structure of the discretized variables from the continuum one, unlike the combinatorial approach which *postulates* it (Fock-Rosly formalism). As we will see, there will be some slight differences in the resulting symplectic potential, which we will comment on in Sec. VI.

Finally, our discretization will apply, with the necessary modifications, to the $3+1$ case as well [34], unlike the combinatorial approach.

### C. Outline

The paper is organized as follows. In Sec. II, we recall the necessary ingredients to define piecewise-homogeneous geometries. We construct the corresponding discrete Chern-Simons connection and establish the first main result: the expression of the Chern-Simons symplectic structure around a curvature defect. In Sec. III, we focus explicitly on a geometric structure suitable for $2+1$ gravity with zero cosmological constant and characterize the connection in terms of the holonomies, which are the relevant variables to characterize the discrete geometry. In Sec. IV, we explicitly determine the phase space structure for the piecewise-flat geometry by discretizing the symplectic potential. We obtain a structure corresponding to classical spin networks coupled to torsion or curvature excitations which behave like relativistic particles. In Sec. V, we discuss constraints and symmetries.

## II. PIECEWISE-DG-FLAT MANIFOLDS IN TWO DIMENSIONS

In this section we define the concept of a piecewise-DG-flat manifold. We describe the cellular decomposition, which is the discrete structure we impose on our manifold. We also define the (continuous) connection relevant to the piecewise-flat geometry and show how it can be expressed in terms of (discrete) holonomies. This allows us to construct the relevant phase space structure in the next section.

### A. The cellular decomposition

Consider a two-dimensional manifold $\Sigma$ without boundary. We introduce a cellular decomposition $\Delta$ of $\Sigma$ made of 0-cells (vertices) denoted $v$, 1-cells (edges) denoted $e$ and 2-cells (cells) denoted $c$. The 1-skeleton $\Gamma \subset \Delta$ is the set of all vertices and edges of $\Delta$. The dual spin network graph $\Gamma^*$ is composed of nodes $c^*$ connected by links $e^*$, such that each node $c^* \in \Gamma^*$ is dual to a 2-cell $c \in \Gamma$ and each link $e^* \in \Gamma^*$ is dual to an edge (1-cell) $e \in \Gamma$.

The edges $e \in \Gamma$ are oriented, and we use the notation $e = (vv')$ to mean that the edge $e$ begins at the vertex $v$ and ends at the vertex $v'$. The inverse edge of $e$, denoted by $e^{-1}$, is the same edge with reverse orientation: $e^{-1} = (v'v)$. The links $e^* \in \Gamma^*$ are also oriented, and we use the notation $e^* = (cc')^*$ to denote that the link $e^*$ connects the nodes $c^*$ and $c'^*$. If the link $e^* = (cc')^*$ is dual to the edge $e = (vv')$,

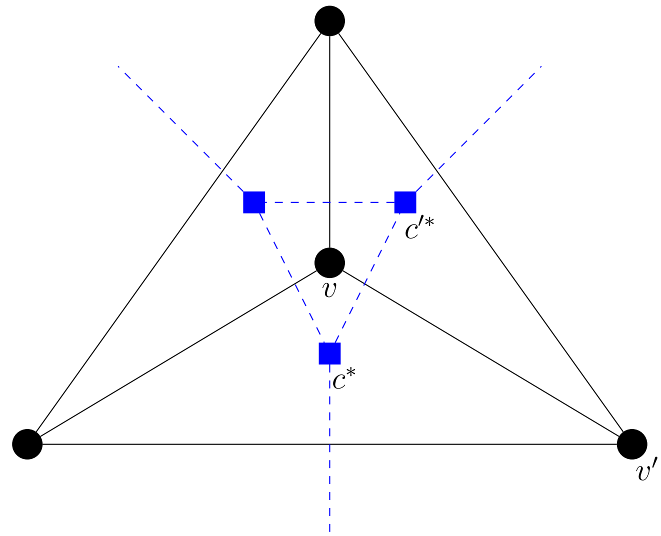


FIG. 1. A simple piece of the cellular decomposition $\Delta$, in black, and its dual spin network $\Gamma^*$, in blue. The vertices $v$ of the 1-skeleton $\Gamma \subset \Delta$ are shown as black circles, while the nodes $c^*$ of $\Gamma^*$ are shown as blue squares. The edges $e \in \Gamma$ are shown as black solid lines, while the links $e^* \in \Gamma^*$ are shown as blue dashed lines. In particular, two nodes $c^*$ and $c'^*$, connected by a link $e^* = (cc')^*$, are labeled, as well as two vertices $v$ and $v'$, connected by an edge $e = (vv') = (cc') = c \cap c'$, which is dual to the link $e^*$.

as in Fig. 1, one can also write $e = (cc') \equiv c \cap c'$, which means that the edge $e$ is the (oriented) intersection of the cells $c$ and $c'$. The orientation is such that $e$ is a counterclockwise rotation of the dual edge $(cc')^*$. In Fig. 1 we show a simple triangulation; however, the cells can be general polygons.

Let DG be a Lie group and $\mathfrak{dg}$ be its Lie algebra. In this section, DG represents the Chern-Simons group; later we will specialize to the case where it is a Drinfeld double. We use a calligraphic font, e.g., $\mathcal{H}$, to denote DG-valued differential forms and a bold calligraphic font, e.g., $\boldsymbol{\mathcal{A}}$, to denote $\mathfrak{dg}$-valued differential forms. We define a $\mathfrak{dg}$-valued connection 1-form $\boldsymbol{\mathcal{A}}$ on $\Sigma$ and its $\mathfrak{dg}$-valued curvature 2-form[2] $\boldsymbol{\mathcal{F}}$:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{A}}) \equiv \mathrm{d}\boldsymbol{\mathcal{A}} + \frac{1}{2}[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}]. \tag{4}$$

In order to define a piecewise-DG-flat geometry on $\Sigma$, we assume that $\boldsymbol{\mathcal{A}}$ is flat ($\boldsymbol{\mathcal{F}} = 0$) inside each cell $c$, and the curvature is restricted to the vertices $v$. We make this notion precise in the following sections.

[2]The graded commutator of two Lie-algebra-valued differential forms $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ is given by

$$[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}] \equiv \boldsymbol{\mathcal{A}} \wedge \boldsymbol{\mathcal{B}} - (-1)^{\deg \boldsymbol{\mathcal{A}} \deg \boldsymbol{\mathcal{B}}} \boldsymbol{\mathcal{B}} \wedge \boldsymbol{\mathcal{A}}, \tag{3}$$

where deg is the degree of the form. In this case, $[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}] = 2\boldsymbol{\mathcal{A}} \wedge \boldsymbol{\mathcal{A}}$ since $\deg \boldsymbol{\mathcal{A}} = 1$.

### B. The DG connection inside the cells

As mentioned above, we make the assumption that the curvature inside each cell $c$ vanishes. Since $c$ is simply connected, this means that the connection 1-form $\boldsymbol{\mathcal{A}}$ can be expressed at any point $x \in c$ as a (left-invariant) Maurer-Cartan form:

$$\boldsymbol{\mathcal{A}}|_c = \mathcal{H}_c^{-1}\mathrm{d}\mathcal{H}_c, \tag{5}$$

where $\mathcal{H}_c$ is a DG-valued 0-form and the notation $|_c$ means the relation is valid in the interior[3] of the cell $c$. It is easy to see that, indeed, $\boldsymbol{\mathcal{F}}|_c = 0$ for this connection.

Let $\mathcal{G}$ be a DG-valued 0-form. Right translations,

$$\mathcal{H}_c(x) \mapsto \mathcal{H}_c(x)\mathcal{G}(x), \tag{6}$$

are gauge transformations that affect the connection in the usual way:

$$\boldsymbol{\mathcal{A}} \mapsto \mathcal{G}^{-1}\boldsymbol{\mathcal{A}}\mathcal{G} + \mathcal{G}^{-1}\mathrm{d}\mathcal{G}. \tag{7}$$

We can also consider left translations,

$$\mathcal{H}_c(x) \mapsto \mathcal{G}_c\mathcal{H}_c(x), \tag{8}$$

acting on $\mathcal{H}_c$ with a constant group element $\mathcal{G}_c \in \mathrm{DG}$. These transformations leave the connection invariant, $\boldsymbol{\mathcal{A}} \mapsto \boldsymbol{\mathcal{A}}$. On the one hand, they label the redundancy of our parametrization of $\boldsymbol{\mathcal{A}}$ in terms of $\mathcal{H}_c$. On the other hand, these transformations can be understood as symmetries of our parametrization in terms of group elements that stem from the existence of new d.o.f. in $\mathcal{H}_c$ beyond the ones in the connection $\boldsymbol{\mathcal{A}}$.

This situation is similar to the situation that arises any time one considers a gauge theory in a region with boundaries [8]. As shown in [8], when we subdivide a region of space we need to add new d.o.f. at the boundaries of the subdivision in order to restore gauge invariance. These d.o.f. are the edge modes, which carry a nontrivial representation of the boundary symmetry group that descends from the bulk gauge transformations.[4]

Now, we can invert (5) and write $\mathcal{H}_c$ using a path-ordered exponential as follows:

$$\mathcal{H}_c(x) = \mathcal{H}_c(c^*)\overrightarrow{\exp}\int_{c^*}^{x}\boldsymbol{\mathcal{A}}, \tag{9}$$

where $\mathcal{H}_c(c^*)$, the value of $\mathcal{H}_c$ at the node $c^*$, is the extra information contained in the edge mode field $\mathcal{H}_c$ that cannot be obtained from the connection $\boldsymbol{\mathcal{A}}$. Left translations can thus be understood as simply translating the value of $\mathcal{H}_c(c^*)$ without affecting the value of $\boldsymbol{\mathcal{A}}$.

[3] That is, inside any open set that does not intersect the boundary of $c$.

[4] As shown in [35], this group also contains the duality group.

### C. The DG connection inside the disks

#### 1. The punctured disk $v^*$

The next step in defining our piecewise-DG-flat geometry is to parametrize the connection around the vertices. In our discrete geometrical setting, the set of vertices is the locus where the curvature is concentrated. It is therefore important to understand the local geometry of the connection in a neighborhood of the vertices $v \in \Gamma$.

An open set containing $v$ forms the interior of a disk, denoted $D_v$. In order to describe the connection around $v$ in a regular manner, it will be necessary to excise an infinitesimal neighborhood of $v$ and consider $A_v \equiv D_v \backslash \{v\}$, which has the topology of an annulus. It will also be necessary to introduce a cut denoted $C_v$ that runs from $v$ to the boundary of $D_v$. This cut and punctured disk will be denoted from now on as $v^* \equiv A_v \backslash C_v$. It will sometimes be referred to as the punctured disk for simplicity.

In order to understand the geometry at play on $v^*$, it is convenient to think of the punctured disk as the interior of a cut annulus where the boundary around $v$ is shrunk to a point. In order to do so, let us introduce Cartesian coordinates that parametrize the cut annulus. It is isomorphic to a rectangle with coordinates $(r_v, \bar{\phi}_v)$ such that $r_v \in (0, R)$ and $\bar{\phi}_v \in [\bar{\alpha}_v - \pi, \bar{\alpha}_v + \pi]$ where $R, \bar{\alpha}_v \in \mathbb{R}$, and where the line at $\bar{\phi} = \bar{\alpha}_v - \pi$ is identified with the line at $\bar{\phi} = \bar{\alpha}_v + \pi$. We also identify the entire line at $r_v = 0$ with the vertex $v$, in the sense that any function $f(r_v, \phi_v)$ evaluated at $r_v = 0$ reduces to a constant value $f(v)$ regardless of the value of $\phi_v$. Then $v^*$ indeed describes a punctured disk of radius $R$,[5] with the puncture located at $v$.

The boundary $\partial v^*$ consists of two curves of length $2\pi$, one at $r_v = 0$ and another at $r_v = R$. We will call the one at $r_v = 0$ the "inner boundary" $\partial_0 v^*$ and the one at $r_v = R$ the "outer boundary" $\partial_R v^*$. In other words

$$\partial v^* = \partial_0 v^* \cup \partial_R v^*, \qquad \partial_0 v^* \equiv \{(r_v, \bar{\phi}_v)|r_v = 0\},$$
$$\partial_R v^* \equiv \{(r_v, \bar{\phi}_v)|r_v = R\}. \tag{10}$$

We will also define a point $v_0$ at $r_v = R$ and $\bar{\phi}_v = \bar{\alpha}_v - \pi$; then the line at $\bar{\phi}_v = \bar{\alpha}_v - \pi$, which extends from $v$ to $v_0$, is the *cut* $C_v$ where we identified the two edges of the rectangle. Note that both $v$ and $v_0$ are on $\partial v^*$ and not inside $v^*$. The punctured disk is shown in Fig. 2.

For brevity of notation, we define a reduced angle function $\phi_v$ such that

$$\phi_v \equiv \frac{\bar{\phi}_v}{2\pi}, \qquad \alpha_v \equiv \frac{\bar{\alpha}_v}{2\pi} \qquad \phi_v \in \left[\alpha_v - \frac{1}{2}, \alpha_v + \frac{1}{2}\right), \tag{11}$$

which will be used from now on.

[5] We assume that $R$ is chosen small enough so that the intersection of any two disks $v^* \cap v'^*$ is empty.

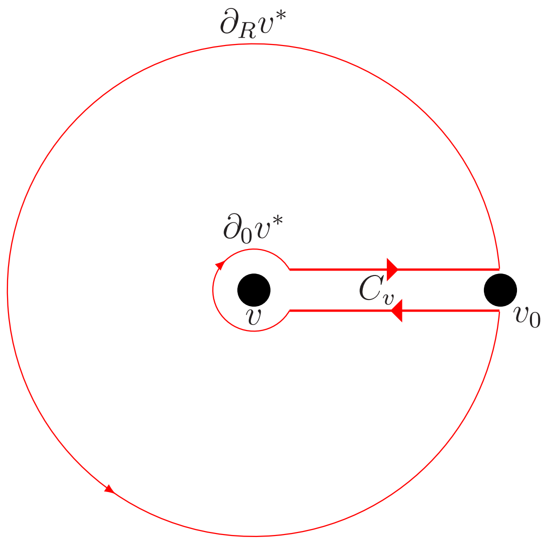


FIG. 2. The punctured disk $v^*$. The figure shows the vertex $v$, cut $C_v$, inner boundary $\partial_0 v^*$, outer boundary $\partial_R v^*$, and reference point $v_0$.

### 2. The distributional curvature

We have assumed that the curvature is concentrated at the vertex. Loosely, this means that the curvature on the full (nonpunctured) disk $D_v$ is distributional:

$$\mathcal{F}|_{D_v} = \mathcal{P}_v \delta(v), \tag{12}$$

where $\mathcal{P}_v$ is some constant element of the Lie algebra $\mathfrak{dg}$ and $\delta(v)$ is the 2-form Dirac distribution concentrated at $v$, defined such that, for any 0-form $f$,

$$\int_{D_v} f\delta(v) \equiv f(v). \tag{13}$$

However, such a formulation is too singular for our purpose. Moreover, it contains some ambiguities. In particular, under the gauge transformation (7) the curvature at $v$ is conjugated: $\mathcal{P}_v \mapsto \mathcal{G}^{-1}\mathcal{P}_v\mathcal{G}$.

It is possible to partially fix this ambiguity by choosing a Cartan subgroup $\mathrm{DH} \subset \mathrm{DG}$, that is, an Abelian subgroup of DG which can serve as a reference for conjugacy classes. Then we demand that $\mathcal{P}_v$ is conjugate to an element $\mathcal{M}_v \in \mathfrak{dh}$ in the corresponding Lie subalgebra. The gauge symmetry is still acting on DH by Weyl transformations $\mathcal{W} \in \mathrm{DW}$, where DW is the subgroup of residual transformations $\mathcal{H} \mapsto \mathcal{W}^{-1}\mathcal{H}\mathcal{W}$ which map DH onto itself. The quotient DH/DW then labels the sets on conjugacy classes. This means that to every vertex $v \in \Gamma$ we attach a conjugacy class labeled by $\mathcal{M}_v \in \mathfrak{dh}$.

### 3. Properly defining the connection and curvature

The proper mathematical formulation of the naive condition (12) is to demand that we have instead a flat connection on the punctured disk $v^*$. That is,

$$\mathcal{F}|_{v^*} = 0. \tag{14}$$

Now, $v^*$ is not simply connected; it possesses a nontrivial homotopy group $\pi_1(v^*) = \mathbb{Z}$ labeling the winding modes. This means that the connection $\mathcal{A}|_{v^*}$ possesses a nontrivial holonomy,

$$\mathrm{Hol}_v = \overrightarrow{\exp} \oint_{S_v} \mathcal{A}, \tag{15}$$

where $S_v$ is any circle in $v^*$ encircling the vertex $v$ once and not encircling any other vertices, and which starts at the cut $C_v$. We demand that this holonomy be in the conjugacy class labeled by $\mathcal{M}_v$. We are therefore looking for a connection on $v^*$ which satisfies

$$\mathcal{F}|_{v^*} = 0, \qquad [\mathrm{Hol}_v] = [\exp \mathcal{M}_v], \tag{16}$$

where the brackets $[\cdot]$ denote the equivalence class under conjugation; that is, $\mathrm{Hol}_v$ is related to $\exp \mathcal{M}_v$ via conjugation with some element $\mathcal{W} \in \mathrm{DW}$.

Such a connection can be conveniently written in terms of a DG-valued 0-form $\mathcal{H}_v$ and an element $\mathcal{M}_v \in \mathfrak{dh}$ in the Cartan subgroup as

$$\begin{aligned}\mathcal{A}|_{v^*} &\equiv (e^{\mathcal{M}_v\phi_v}\mathcal{H}_v)^{-1}\mathrm{d}(e^{\mathcal{M}_v\phi_v}\mathcal{H}_v)\\ &= \mathcal{H}_v^{-1}\mathcal{M}_v\mathcal{H}_v\mathrm{d}\phi_v + \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v.\end{aligned} \tag{17}$$

It is important to note that $\mathcal{H}_v$ is defined on the full disk $D_v$. In particular, it is periodic when going around $v$ and its value $\mathcal{H}_v(v)$ at $v$ is well defined, while $\phi_v$ is defined only on the cut disk and $\mathrm{d}\phi_v$ is defined on the punctured disk.

As for the cells, we again see that gauge transformations are given by right translations

$$\mathcal{H}_v(x) \mapsto \mathcal{H}_v(x)\mathcal{G}(x), \tag{18}$$

while left translations by a constant element $\mathcal{G}_v$ in the Cartan subgroup DH (which thus commutes with $\mathcal{M}_v$),

$$\mathcal{H}_v(x) \mapsto \mathcal{G}_v\mathcal{H}_v(x), \tag{19}$$

leave the connection invariant.

### 4. Calculating the curvature

The curvature can now be obtained in a well-defined way as follows. First, we note that $\mathcal{A}|_{v^*}$ is, in fact, the gauge transformation of a Lagrangian connection $\mathcal{L}_v$ by a DG-valued 0-form $\mathcal{H}_v$:

$$\mathcal{A}|_{v^*} \equiv \mathcal{H}_v^{-1}\mathcal{L}\mathcal{H}_v + \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v, \qquad \mathcal{L}_v \equiv \mathcal{M}_v\mathrm{d}\phi_v,$$
$$[\mathcal{L}_v, \mathcal{L}_v] = 0. \tag{20}$$

Now, the curvature of $\mathcal{L}_v$ is given by

$$\mathcal{F}(\mathcal{L}_v) = \mathrm{d}\mathcal{L}_v = \mathcal{M}_v\mathrm{d}^2\phi_v. \tag{21}$$

Of course, since the exterior derivative satisfies $d^2 = 0$ on $v^*$, the curvature vanishes on $v^*$, as required by (16). However, since $\phi_v$ is not well defined at the origin $v \in D_v$, the term $d^2\phi_v$ might not vanish at $v$ itself. Let us thus perform the following integral over the full disk $D_v$:

$$\int_{D_v} \mathcal{F}(\mathcal{L}_v) = \int_{\partial D_v} \mathcal{L}_v = \mathcal{M}_v \int_{\partial D_v} d\phi_v = \mathcal{M}_v, \quad (22)$$

since the integral over the circle is just 1. Since $\mathcal{F}(\mathcal{L}_v) = 0$ everywhere on $D_v$ except at the origin, and yet its integral over $D_v$ is equal to the finite quantity $\mathcal{M}_v$, we are well within our rights to declare that the curvature takes the form

$$\mathcal{F}(\mathcal{L}_v) = \mathcal{M}_v \delta(v). \quad (23)$$

Next, we gauge-transform $\mathcal{L}_v \mapsto \mathcal{A}|_{v^*}$, obtaining the expression (17). Then the curvature transforms in the usual way:

$$\mathcal{F}(\mathcal{L}_v) \mapsto \mathcal{F}(\mathcal{A})|_{D_v} = \mathcal{H}_v^{-1} \mathcal{F}(\mathcal{L}_v) \mathcal{H}_v \equiv \mathcal{P}_v \delta(v), \quad (24)$$

where we have defined

$$\mathcal{P}_v \equiv \mathcal{H}_v^{-1}(v) \mathcal{M}_v \mathcal{H}_v(v). \quad (25)$$

Thus, Eq. (12) is justified; the curvature may be thought of as taking the form $\mathcal{F}|_{D_v} = \mathcal{P}_v \delta(v)$, with the element $\mathcal{H}_v(v)$ parametrizing the representative of the conjugacy class.

#### *5. Holonomies*

Furthermore, for some subset $K_v \subseteq D_v$ we have

$$\int_{K_v} \mathcal{F}(\mathcal{L}_v) = \int_{\partial K_v} \mathcal{L}_v = \mathcal{M}_v, \quad (26)$$

and by exponentiating and taking the gauge transformation $\mathcal{L}_v \mapsto \mathcal{A}|_{v^*}$ we see that the holonomy of the connection along the loop $\partial K_v$ starting at some point $x \in \partial K_v$ and winding once around $v$ is given by

$$\overrightarrow{\exp} \oint_{\partial K_v} \mathcal{A} = \mathcal{H}_v^{-1}(x) e^{\mathcal{M}_v} \mathcal{H}_v(x), \quad (27)$$

which is indeed conjugate to $e^{\mathcal{M}_v}$. The advantage of the parametrization in terms of $\mathcal{H}_v$ is that even if the notion of a loop starting at $v$ and encircling $v$ once is ill defined, the right-hand side of (27) is still well defined when $x = v$.

The pair $(\mathcal{M}_v, \mathcal{H}_v)$ determines the holonomy, but the reverse is not true. The Cartan subgroup DH acts on the left of $\mathcal{H}_v$ as a symmetry group $\mathcal{H}_v(x) \mapsto \mathcal{G}_v \mathcal{H}_v(x)$, with $\mathcal{G}_v \in$ DH constant, which leaves the connection and the holonomy invariant. However, there is also a left-action of the Weyl group DW:

$$(\mathcal{M}_v, \mathcal{H}_v) \mapsto (\mathcal{W}^{-1} \mathcal{M}_v \mathcal{W}, \mathcal{W}^{-1} \mathcal{H}_v), \quad \mathcal{W} \in \mathrm{DW}, \quad (28)$$

which does not leave $\mathcal{M}_v$ invariant, but fixes its conjugacy class.

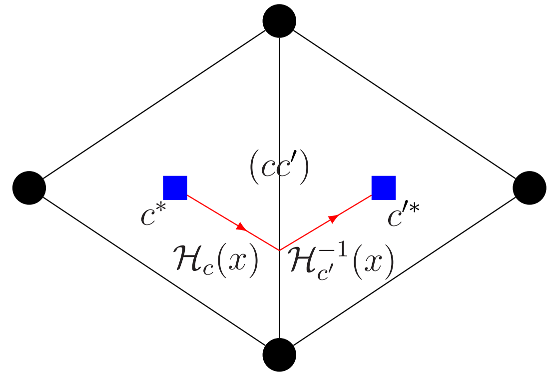


FIG. 3. To get from the node $c^*$ to the adjacent node $c'^*$, we use the group element $\mathcal{H}_{cc'}$. First, we choose a point $x$ somewhere on the edge $(cc') = c \cap c'$. Then, we take $\mathcal{H}_c(x)$ from $c^*$ to $x$, following the first red arrow. Finally, we take $\mathcal{H}_{c'}^{-1}(x)$ from $x$ to $c'^*$, following the second red arrow. Thus $\mathcal{H}_{cc'} = \mathcal{H}_c(x)\mathcal{H}_{c'}^{-1}(x)$. Note that any $x \in c \cap c'$ will do, since the connection is flat and thus all paths are equivalent.

### D. Continuity conditions between cells

Let us consider the link $e^* = (cc')^*$ connecting two adjacent nodes $c^*$ and $c'^*$. This link is dual to the edge $e = (cc') = c \cap c'$, which is the boundary between the two adjacent cells $c$ and $c'$. The connection is defined in the union $c \cup c'$, while in each cell its restriction is encoded in $\mathcal{A}|_c$ and $\mathcal{A}|_{c'}$ as defined above, in terms of $\mathcal{H}_c$ and $\mathcal{H}_{c'}$, respectively.

The continuity equation on the edge $(cc')$ between the two adjacent cells[6] reads

$$\mathcal{A}|_c = \mathcal{H}_c^{-1} d\mathcal{H}_c = \mathcal{H}_{c'}^{-1} d\mathcal{H}_{c'} = \mathcal{A}|_{c'}, \quad \text{on } (cc') = c \cap c'. \quad (29)$$

Since the connections match, this means that the group elements $\mathcal{H}_c$ and $\mathcal{H}_{c'}$ differ only by the action of a left symmetry element. This implies that there exists a group element $\mathcal{H}_{cc'} \in$ DG which is independent of $x$ and provides the change of variables between the two parametrizations $\mathcal{H}_c(x)$ and $\mathcal{H}_{c'}(x)$ on the overlap:

$$\mathcal{H}_{c'}(x) = \mathcal{H}_{c'c} \mathcal{H}_c(x), \qquad x \in (cc') = c \cap c'. \quad (30)$$

Note that $\mathcal{H}_{c'c} = \mathcal{H}_{cc'}^{-1}$. Furthermore, $\mathcal{H}_{cc'}$ can be decomposed as

$$\mathcal{H}_{cc'} = \mathcal{H}_c(x) \mathcal{H}_{c'}^{-1}(x), \quad (31)$$

as illustrated in Fig. 3. The quantity $\mathcal{H}_{cc'}$ is invariant under the right gauge transformation (6), since it is independent of $c$. However, it is not invariant under the left symmetry (8) performed at $c$ and $c'$, under which we obtain

[6]Strictly speaking, one should consider open neighborhoods $U_c, U_{c'}$ of $c$ and $c'$ and consider the overlap condition on the open set $U_c \cap U_{c'}$. We will not dwell too much on this subtlety here, since this is not the main point of our paper, but keep in mind that if necessary one might have to resort to open cell overlaps instead of edges.

$$\mathcal{H}_{c'c} \mapsto \mathcal{G}_{c'}\mathcal{H}_{c'c}\mathcal{G}_c^{-1}. \tag{32}$$

Since this symmetry leaves the connection invariant, this means that $\mathcal{H}_{cc'}$ is *not* the holonomy from $c^*$ to $c'^*$, along the link $(cc')^*$, as it is usually assumed. Instead, from (9) and (31) we have that

$$\mathcal{H}_{cc'} = \mathcal{H}_c(c^*)\left(\overrightarrow{\exp}\int_{c^*}^{c'^*}\boldsymbol{\mathcal{A}}\right)\mathcal{H}_{c'}^{-1}(c'^*) \tag{33}$$

is a dressed gauge-invariant observable. It will be referred to as a *discrete holonomy*, while it is understood that it is a gauge-invariant version of the holonomy.

The map $\boldsymbol{\mathcal{A}} \mapsto \{\mathcal{H}_{cc'}\}$ can be formalized as follows. Let $V_\Gamma$, $E_\Gamma$ be the sets of vertices and edges, respectively, of the 1-skeleton $\Gamma$. Then we can either define the space $\mathcal{P}(\Sigma,\Gamma,\mathrm{DG})/\mathrm{DG}$ of DG-flat connections on $\Sigma\backslash V_\Gamma$ (that is, on the two-dimensional manifold $\Sigma$ with a puncture at each vertex) modulo gauge transformation, or we can define the space

$$\mathcal{D}(\Sigma,\Gamma,\mathrm{DG}) = \mathrm{DG}^{E_\Gamma}/\mathrm{DG}^{V_\Gamma} = \{\mathcal{H}_{cc'}\in\mathrm{DG}\}/\{\mathcal{G}_c\in\mathrm{DG}\} \tag{34}$$

of discrete holonomies at each edge $e = c\cap c'$ modulo global symmetries. The main claim we want to expand upon is that the map $\boldsymbol{\mathcal{A}}/\mathcal{G}\mapsto\{\mathcal{H}_{cc'}\}/\{\mathcal{G}_c\}$ provides an isomorphism between these two structures.

This means that we can think of the space of discrete holonomies alone, before the quotient by the symmetry, as a definition of the space of flat DG-connections modulo gauge transformations that vanish at the vertices of $\Gamma$. The latter space is only formally defined, while the space $\{\mathcal{H}_{cc'}\}$ of discrete connections is well defined.

### E. Continuity conditions between disks and cells

A similar discussion applies when one looks at the overlap $v^*\cap c$ between a punctured disk $v^*$ and a cell $c$. The boundary of this region consists of two truncated edges of length $R$ (the coordinate radius of the disk) touching $v$, plus an arc connecting the two edges, which lies on the boundary of the disk $v^*$. In the following we denote this arc[7] by $(vc)$. It is clear that the union of all such arcs around a vertex $v$ reconstructs the outer boundary $\partial_R v^*$ of the disk, as defined in (10):

$$(vc)\equiv\partial_R v^*\cap c, \qquad \partial_R v^* = \bigcup_{c\ni v}(vc), \tag{35}$$

where $c\ni v$ means "all cells $c$ which have the vertex $v$ on their boundary." It is also useful to introduce the truncated cells:

$$\tilde{c}\equiv c\backslash\bigcup_{v\in c}D_v, \tag{36}$$

where $v\in c$ means "all vertices $v$ on the boundary of the cell $c$." In other words, $\tilde{c}$ is the complement of the union of disks $D_v$ intersecting $c$. The union of all truncated cells reconstructs the manifold $\Sigma$ minus the disks:

$$\bigcup_c\tilde{c} = \Sigma\backslash\bigcup_v D_v. \tag{37}$$

In the intersection $v^*\cap c$ we have two different descriptions of the connection $\boldsymbol{\mathcal{A}}$. On $c$ it is described by the DG-valued 0-form $\mathcal{H}_c$, and on $v^*$ it is described by a DG-valued 0-form $\mathcal{H}_v$. The fact that we have a single-valued connection is expressed in the continuity conditions

$$\boldsymbol{\mathcal{A}}|_{v^*} = \mathcal{H}_v^{-1}\boldsymbol{\mathcal{M}}_v\mathcal{H}_v\mathrm{d}\phi_v + \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v = \mathcal{H}_c^{-1}\mathrm{d}\mathcal{H}_c = \boldsymbol{\mathcal{A}}|_c,$$
$$\text{on } (vc) = \partial v^*\cap c. \tag{38}$$

The relation between the two connections can be integrated. It means that the elements $\mathcal{H}_v(x)$ and $\mathcal{H}_c(x)$ differ by the action of the left symmetry group. In practice, this means that the integrated continuity relation involves a (discrete) holonomy $\mathcal{H}_{cv}$:

$$\mathcal{H}_c(x) = \mathcal{H}_{cv}e^{\boldsymbol{\mathcal{M}}_v\phi_v(x)}\mathcal{H}_v(x), \qquad x\in(vc) \tag{39}$$

where $\phi_v(x)$ is the angle corresponding to $x$ with respect to the cut $C_v$. Isolating $\mathcal{H}_{vc}\equiv\mathcal{H}_{cv}^{-1}$, we find

$$\mathcal{H}_{vc} = e^{\boldsymbol{\mathcal{M}}_v\phi_v(x)}\mathcal{H}_v(x)\mathcal{H}_c^{-1}(x), \tag{40}$$

which is illustrated in Fig. 4.

The quantity $\mathcal{H}_{vc}$ is invariant under right gauge transformations (6) and (18):

$$\mathcal{H}_c(x)\mapsto\mathcal{G}_c\mathcal{H}_c(x), \qquad \mathcal{H}_v(x)\mapsto\mathcal{G}_v\mathcal{H}_v(x). \tag{41}$$

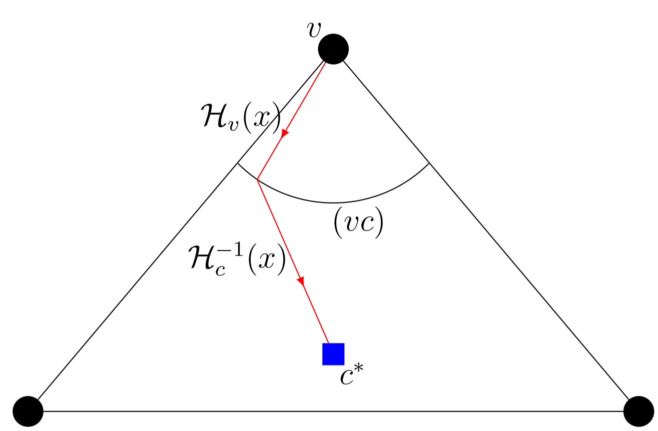


FIG. 4. To get from the vertex $v$ to the node $c^*$, we use the group element $\mathcal{H}_{vc}$. First, we choose a point $x$ somewhere on the arc $(vc) = \partial v^*\cap c$. Then, we use $e^{\boldsymbol{\mathcal{M}}_v\phi_v(x)}$ to rotate from the cut $C_v$ to the angle corresponding to $x$ (rotation not illustrated). Next, we take $\mathcal{H}_v(x)$ from $v$ to $x$, following the first red arrow. Finally, we take $\mathcal{H}_c^{-1}(x)$ from $x$ to $c^*$, following the second red arrow. Thus $\mathcal{H}_{vc} = e^{\boldsymbol{\mathcal{M}}_v\phi_v(x)}\mathcal{H}_v(x)\mathcal{H}_c^{-1}(x)$.

[7]The arc $(vc)$ is dual to the line segment $(vc)^*$ connecting the vertex $v$ with the node $c^*$, just as the edge $e$ is dual to the link $e^*$.

However, under left symmetry transformations (8) and (19), the connection is left invariant, and we get

$$\mathcal{H}_{vc} \mapsto \mathcal{G}_v \mathcal{H}_{vc} \mathcal{G}_c^{-1}. \tag{42}$$

Note also that the translation in $\phi_v(x)$ can be absorbed into the definition of $\mathcal{H}_v$, so that the transformation

$$\phi_v(x) \mapsto \phi_v(x) + \beta_v, \qquad \mathcal{H}_v(x) \mapsto e^{-\mathcal{M}_v \beta_v} \mathcal{H}_v(x), \tag{43}$$

is also a symmetry under which (39) is invariant. The connection $\mathcal{A}|_{v^*}$ is also invariant under this symmetry.

### F. Summary

In conclusion, the connection $\mathcal{A}$ is defined on every point of the manifold $\Sigma$ as follows. Inside each cell $c$, we have a flat connection $\mathcal{A}|_c$. Since the cell $c$ is simply connected, this connection can be written in terms of a DG-valued 0-form $\mathcal{H}_c$:

$$\mathcal{A}|_c = \mathcal{H}_c^{-1} \mathrm{d}\mathcal{H}_c. \tag{44}$$

Inside each punctured disk $v^*$, we have a flat connection $\mathcal{A}|_{v^*}$ parametrized by a DG-valued 0-form $\mathcal{H}_v$ and a constant element $\mathcal{M}_v$ of the Cartan subalgebra $\mathfrak{dh}$:

$$\mathcal{A}|_{v^*} = \mathcal{H}_v^{-1} \mathcal{M}_v \mathcal{H}_v \mathrm{d}\phi_v + \mathcal{H}_v^{-1} \mathrm{d}\mathcal{H}_v. \tag{45}$$

The continuity of the connection $\mathcal{A}$ along the boundaries between cells and other cells or disks is expressed in the relations

$$\mathcal{H}_{cc'} = \mathcal{H}_c(x) \mathcal{H}_{c'}^{-1}(x), \qquad x \in (cc') \equiv c \cap c', \tag{46}$$

$$\mathcal{H}_{vc} = e^{\mathcal{M}_v \phi_v(x)} \mathcal{H}_v(x) \mathcal{H}_c^{-1}(x), \qquad x \in (vc) \equiv \partial v^* \cap c. \tag{47}$$

### G. The Chern-Simons symplectic potential

The first goal of this paper is the construction of the symplectic potential for the Chern-Simons connection in terms of the discrete data $\{\mathcal{H}_{cc'}\}_{c,c' \in \Delta}$ or the discrete data $\{\mathcal{H}_{cv}, \mathcal{M}_v\}_{(c^*,v) \in \Gamma^* \times \Gamma}$. In the continuum, the Chern-Simons symplectic structure is given by[8]

$$\Omega_\Sigma(\mathcal{A}) = \int_\Sigma \omega(\mathcal{A}), \qquad \omega(\mathcal{A}) = \delta\mathcal{A} \cdot \delta\mathcal{A}. \tag{49}$$

We are interested in the computation of the Chern-Simons symplectic potential for a disk $D$, which has the symplectic form $\Omega_D(\mathcal{A}) \equiv \int_D \omega(\mathcal{A})$. We refer the reader to [29,36] for earlier references exploring the same question.

Omitting the subscript $v$ for brevity, we consider the case where the connection $\mathcal{A}$ inside $D$ can be written as the gauge transformation of a Lagrangian connection $\mathcal{L}$ by a DG-valued 0-form $\mathcal{H}$:

$$\mathcal{A} \equiv \mathcal{H}^{-1} \mathcal{L} \mathcal{H} + \mathcal{H}^{-1} \mathrm{d}\mathcal{H}, \tag{50}$$

where we assume that $\mathcal{L}$ belongs to a Lagrangian subspace. This means that $\delta\mathcal{L} \cdot \delta\mathcal{L} = 0$, so that $\omega(\mathcal{L}) = 0$, and where we have omitted the subscripts $v$ for brevity. One first evaluates the variation

$$\delta\mathcal{A} = \mathcal{H}^{-1}(\delta\mathcal{L} + \mathrm{d}_\mathcal{L} \boldsymbol{\Delta}\mathcal{H})\mathcal{H}, \tag{51}$$

where $\mathrm{d}_\mathcal{L}$ denotes the covariant differential $\mathrm{d}_\mathcal{L} \equiv \mathrm{d} + [\mathcal{L}, \cdot]$, and we have used the shorthand notation

$$\boldsymbol{\Delta}\mathcal{H} \equiv \delta\mathcal{H}\mathcal{H}^{-1} \tag{52}$$

for the right-invariant Maurer-Cartan variational form, described in more detail in Appendix A. Under this assumption, we can evaluate the Chern-Simons symplectic form:

$$\omega(\mathcal{A}) = 2\delta(\mathcal{F}(\mathcal{L}) \cdot \boldsymbol{\Delta}\mathcal{H}) + \mathrm{d}(\boldsymbol{\Delta}\mathcal{H} \cdot \mathrm{d}\boldsymbol{\Delta}\mathcal{H}) - 2\mathrm{d}\delta(\mathcal{L} \cdot \boldsymbol{\Delta}\mathcal{H}), \tag{53}$$

where $\mathcal{F}(\mathcal{L}) \equiv \mathrm{d}\mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}]$ is the curvature of $\mathcal{L}$. The derivation of this important formula is spelled out in Appendix B. Furthermore, in the particular case considered above, the Lagrangian connection satisfies

$$\mathcal{L} \equiv \mathcal{M}\mathrm{d}\phi, \qquad \mathcal{F}(\mathcal{L}) = \mathcal{M}\delta(v), \qquad \mathcal{M} \in \mathfrak{dh}. \tag{54}$$

In this case, the symplectic form associated with a disk $D$ centered at $v$ may be further simplified to

$$\Omega_D(\mathcal{A}) = \oint_{\partial D} \Delta\tilde{\mathcal{H}} \cdot \mathrm{d}\boldsymbol{\Delta}\tilde{\mathcal{H}} - 2 \oint_{\partial D} \delta(\mathcal{H}^{-1}(v)\mathcal{M}\mathcal{H}(v) \cdot \boldsymbol{\Delta}\tilde{\mathcal{H}})\mathrm{d}\phi, \tag{55}$$

where we have defined $\tilde{\mathcal{H}}(x) \equiv \mathcal{H}^{-1}(v)\mathcal{H}(x)$, such that $\tilde{\mathcal{H}}(v) = 1$, and used the "Leibniz rule" (A2) for the Maurer-Cartan form,

$$\boldsymbol{\Delta}\tilde{\mathcal{H}}(x) = \mathcal{H}^{-1}(v)(\boldsymbol{\Delta}\mathcal{H}(x) - \boldsymbol{\Delta}\mathcal{H}(v))\mathcal{H}(v). \tag{56}$$

This constitutes the first main technical result of this paper.

The goal of this paper is to study in depth this formula and the consequences it has when one starts to glue together different regions associated to a collection of topological disks. In general this is a formidable task, and we will pursue it under simplifying assumptions. Mainly, we will need to choose a group DG such that the first term $\oint_{\partial D} \Delta\tilde{\mathcal{H}} \cdot \mathrm{d}\boldsymbol{\Delta}\tilde{\mathcal{H}}$ can be written as an exact variational form, and thus the symplectic form can be written as the variation of a symplectic potential, $\Omega_D = \delta\Theta$.

[8]The dot product is defined for any two Lie algebra elements $\mathcal{A}$, $\mathcal{B}$ with components $\mathcal{A}^i \equiv \mathrm{Tr}(\mathcal{A}\boldsymbol{\tau}^i)$ and $\mathcal{B}^i \equiv \mathrm{Tr}(\mathcal{B}\boldsymbol{\tau}^i)$, where $\boldsymbol{\tau}^i$ are the generators of the Lie algebra, as

$$\mathcal{A} \cdot \mathcal{B} \equiv \mathrm{Tr}(\mathcal{A} \wedge \mathcal{B}) = \mathcal{A}^i \wedge \mathcal{B}_i. \tag{48}$$

## III. GRAVITY: SPECIALIZING TO $G \ltimes \mathfrak{g}^*$

### A. Introduction

In order to move forward and explore the discretization of the Chern-Simons symplectic form, we will focus on theories for which the Chern-Simons group DG is specifically chosen to be a Drinfeld double which is locally of the form $\mathrm{DG} \equiv G \times G^*$, with $G^*$ a group dual to $G$. At the Lie algebra level, this means that the Lie algebra of the double, $\mathfrak{dg}$, possesses the structure of a Manin triple: $\mathfrak{dg} = \mathfrak{g} \oplus \mathfrak{g}^*$ is equipped with a nondegenerate pairing $\langle , \rangle$ which is ad-invariant, i.e., $\langle [\mathbf{A}, \mathbf{B}], \mathbf{C} \rangle = \langle \mathbf{A}, [\mathbf{B}, \mathbf{C}] \rangle$, and which is such that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic; i.e., the pairing restricted to $\mathfrak{g}$ or $\mathfrak{g}^*$ is null.

It turns out that all theories of Euclidean gravity in $2+1$ dimensions correspond to the Chern-Simons theory of a double where the factor $G$ is simply the group SU(2), while the factor $G^*$ is another SU(2), the 2D Borel group $\mathrm{A}N_2$, or simply the Abelian group $\mathbb{R}^3$, depending on the sign of the cosmological constant.

In this work we will focus, for simplicity, on the case of a zero cosmological constant. This means that we will restrict ourselves to the study of a double of trivial topology, where $G$ is a simple group and $G^* = \mathfrak{g}^*$ is an Abelian group. In this case, the double is simply the semidirect product

$$\mathrm{DG} \equiv G \ltimes \mathfrak{g}^*. \tag{57}$$

Note that DG is also isomorphic to the cotangent bundle $T^*G$, which shows that the natural pairing on DG descends from the duality pairing of vector and covector fields $TG \times T^*G$, since the Lie algebra of $G$ can be viewed as the set of right-invariant vector fields.

When $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1)$, this group reduces to the 3D Euclidean group ISU(2) or $2+1$D Poincaré group ISU(1,1), respectively, which is the group of isometries of $2+1$D flat gravity.[9] Since we work at the classical level, none of our derivations depends on the fact that the $G = \mathrm{SU}(2)$, so we will keep $G$ general; we just need $G$ to be equipped with a nontrivial trace, denoted Tr and incorporated into the dot product:

$$\mathbf{A} \cdot \mathbf{B} \equiv \mathrm{Tr}(\mathbf{A} \wedge \mathbf{B}) \equiv A^i \wedge B_i, \qquad A^i \equiv \mathrm{Tr}(\mathbf{A}\boldsymbol{\tau}^i), \qquad B^i \equiv \mathrm{Tr}(\mathbf{B}\boldsymbol{\tau}^i), \tag{58}$$

where $\boldsymbol{\tau}^i$ are the generators of the Lie algebra. Keeping in mind the applications to $2+1$D gravity, we will call the group DG the "Euclidean" group for reference.

### B. The Euclidean group DG

The transformations between cells will be given by DG group elements, and the connection 1-form $\boldsymbol{\mathcal{A}}$ will be valued in the Lie algebra $\mathfrak{dg}$. This algebra is generated by the rotation generators $\mathbf{J}_i$ and the translation generators $\mathbf{P}_i$, where $i = 1, \ldots, \dim \mathfrak{g}$. The generators have the Lie brackets and Killing form

$$[\mathbf{P}_i, \mathbf{P}_j] = 0, \qquad [\mathbf{J}_i, \mathbf{J}_j] = C_{ij}{}^k \mathbf{J}_k, \qquad [\mathbf{J}_i, \mathbf{P}_j] = C_{ij}{}^k \mathbf{P}_k, \qquad \langle \mathbf{J}_i, \mathbf{P}_j \rangle = \delta_{ij}. \tag{59}$$

Here $C_{ij}{}^k$ denotes the structure constants[10] of $\mathfrak{g}$. We see that both the rotation algebra $\mathfrak{g}$ generated by $\mathbf{J}_i$ and the translation algebra $\mathfrak{g}^*$ generated by $\mathbf{P}_i$ are subalgebras of $\mathfrak{dg}$. However, $\mathfrak{g}$ is non-Abelian, while $\mathfrak{g}^*$ is Abelian and a normal subalgebra. The pairing $\langle , \rangle$ identifies the translation subalgebra with the dual of the Lie algebra (which is why we denote the translation algebra by $\mathfrak{g}^*$). The metric $\delta_{ij}$ involved in the definition of the pairing is a Killing metric; the tensor $C_{ijk} \equiv C_{ij}{}^l \delta_{lk}$ is fully antisymmetric.

By exponentiating this algebra, we get

$$\mathrm{DG} \cong G \ltimes \mathfrak{g}^*, \tag{60}$$

where $\mathfrak{g}^*$ is an Abelian normal subgroup. This means that every element $\mathcal{H} \in \mathrm{DG}$ may be uniquely decomposed, using the so-called "Cartan decomposition," into a pair

$$\mathcal{H} \equiv e^{\mathbf{y}} h, \qquad (h, \mathbf{y}) \in G \times \mathfrak{g}^*. \tag{61}$$

To avoid confusion, throughout the paper we will be using a calligraphic font (e.g., $\mathcal{H}$) for DG elements, bold calligraphic font (e.g., $\boldsymbol{\mathcal{M}}$) for $\mathfrak{dg}$ elements, Roman font (e.g., $h$) for $G$ elements and bold Roman font (e.g., $\mathbf{y}$) for $\mathfrak{g}^*$ elements.

The product rule is such that

$$e^{\mathbf{y}} e^{\mathbf{y}'} = e^{\mathbf{y}+\mathbf{y}'}, \qquad h e^{\mathbf{y}} = e^{h\mathbf{y}h^{-1}} h. \tag{62}$$

This means that products and inverse elements of DG elements are given in terms of $G$ and $\mathfrak{g}^*$ elements by

$$\mathcal{H}\mathcal{H}' = e^{\mathbf{y}} h e^{\mathbf{y}'} h' = e^{\mathbf{y}+h\mathbf{y}'h^{-1}} hh', \tag{63}$$

$$\mathcal{H}^{-1} = h^{-1} e^{-\mathbf{y}} = e^{-h^{-1}\mathbf{y}h} h^{-1}, \tag{64}$$

and by combining them together we get

$$\mathcal{H}^{-1}\mathcal{H}' = h^{-1} e^{-\mathbf{y}} e^{\mathbf{y}'} h' = e^{h^{-1}(\mathbf{y}'-\mathbf{y})h} h^{-1} h'. \tag{65}$$

[9] See [37–39] for more details on the correspondence between doubles and $2+1$D gravity.

[10] For $\mathfrak{g} = \mathfrak{su}(2)$ we have $C_{ijk} = \epsilon_{ijk}$, the Levi-Civita tensor, and $\delta_{ij}$ the Euclidean metric. In this case we can define $\boldsymbol{\tau}_i \equiv -i\boldsymbol{\sigma}_i/2$ to be the generators of $\mathfrak{su}(2)$, where $\boldsymbol{\sigma}_i$ are the Pauli matrices. The generators satisfy the algebra $[\boldsymbol{\tau}_i, \boldsymbol{\tau}_j] = \epsilon_{ij}^k \boldsymbol{\tau}_k$. The normalized trace is $\mathrm{Tr} \equiv -2\mathrm{tr}$ where tr is the usual matrix trace, and it satisfies $\mathrm{Tr}(\boldsymbol{\tau}_i \boldsymbol{\tau}_j) = \delta_{ij}$.

### C. The $G$ connection and frame field inside the cells

Our $\mathfrak{dg}$-valued connection 1-form $\mathcal{A}$ can be decomposed in terms of the generators of the algebra as follows:

$$\mathcal{A} \equiv A^i \mathbf{J}_i + E^i \mathbf{P}_i. \tag{66}$$

Bearing in mind the Chern-Simons formulation of gravity, it will be convenient to interpret $\mathbf{A} \equiv A^i \mathbf{J}_i$ as a $\mathfrak{g}$-valued connection 1-form and $\mathbf{E} \equiv E^i \mathbf{P}_i$ as a $\mathfrak{g}^*$-valued (co)frame field. Accordingly, the curvature 2-form $\mathcal{F}$ of the DG connection $\mathcal{A}$ can also be decomposed into the sum $\mathcal{F} = \mathbf{F} + \mathbf{T}$ of a "rotational" curvature $\mathbf{F}$ (referred to as the $G$-curvature) and "translational" curvature (referred to as the $G$-torsion). The $G$ curvature $\mathbf{F}$ and torsion $\mathbf{T}$ are 2-forms defined as

$$\mathbf{F}(\mathbf{A}) \equiv \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A},\mathbf{A}], \qquad \mathbf{T}(\mathbf{A},\mathbf{E}) \equiv \mathrm{d}_{\mathbf{A}}\mathbf{E} \equiv \mathrm{d}\mathbf{E} + [\mathbf{A},\mathbf{E}]. \tag{67}$$

If $\mathcal{A}$ is flat, $\mathcal{F}(\mathcal{A}) = 0$, we automatically get a vanishing $G$ curvature and torsion:

$$\mathbf{F}(\mathbf{A}) = \mathbf{T}(\mathbf{A},\mathbf{E}) = 0. \tag{68}$$

In other words, a flat DG geometry corresponds to a flat and torsionless $G$ geometry.

Now, recall from (5) that inside the cell $c$ we have

$$\mathcal{A}|_c = \mathcal{H}_c^{-1}\mathrm{d}\mathcal{H}_c, \tag{69}$$

where $\mathcal{H}_c$ is a DG-valued 0-form. The group element $\mathcal{H}_c^{-1}(c)\mathcal{H}_c(x)$ defines the DG holonomy of the connection $\mathcal{A}$ from $c^*$ to $x$. We can decompose $\mathcal{H}_c$ in terms of a rotation $h_c$ and a translation $\mathbf{y}_c$ using the Cartan decomposition:

$$\mathcal{H}_c \equiv e^{\mathbf{y}_c} h_c, \qquad h_c \in G, \qquad \mathbf{y}_c \in \mathfrak{g}. \tag{70}$$

Plugging it into $\mathcal{A}$, we get

$$\mathcal{A}|_c = h_c^{-1}\mathrm{d}\mathbf{y}_c h_c + h_c^{-1}\mathrm{d}h_c. \tag{71}$$

It is easy to see that the first term is a pure rotation, that is, proportional to $\mathbf{J}_i$, while the second term is a pure translation, that is, proportional to $\mathbf{P}_i$. Recalling that $\mathcal{A} = A^i\mathbf{J}_i + E^i\mathbf{P}_i$, we deduce that the corresponding $\mathfrak{g}$-valued connection and frame field are

$$\mathbf{A}|_c = h_c^{-1}\mathrm{d}h_c, \qquad \mathbf{E}|_c = h_c^{-1}\mathrm{d}\mathbf{y}_c h_c. \tag{72}$$

As before, we have two types of transformations. Gauge transformations (right translations), labeled by DG-valued 0-forms $(g, \mathbf{x})$ and given by

$$\mathbf{A} \mapsto g^{-1}\mathbf{A}g + g^{-1}\mathrm{d}g, \qquad \mathbf{E} \mapsto g^{-1}(\mathbf{E} + \mathrm{d}_{\mathbf{A}}\mathbf{x})g, \tag{73}$$

act on $(h_c, \mathbf{y}_c)$ as follows from (6):

$$h_c(x) \mapsto h_c(x)g(x), \quad \mathbf{y}_c(x) \mapsto \mathbf{y}_c(x) + (h_c\mathbf{x}h_c^{-1})(x). \tag{74}$$

Symmetry transformations (left translations), labeled by constant DG elements $(g_c, \mathbf{z}_c)$ assigned to the cell $c$, leave the connection and frame invariant, $(\mathbf{A},\mathbf{E}) \mapsto (\mathbf{A},\mathbf{E})$, and act on $(h_c, \mathbf{y}_c)$ as follows from (8):

$$h_c(x) \mapsto g_c h_c(x), \qquad \mathbf{y}_c(x) \mapsto \mathbf{z}_c + g_c\mathbf{y}_c(x)g_c^{-1}. \tag{75}$$

### D. The $G$ connection and frame field inside the disks

From (17), the connection $\mathcal{A}$ inside the punctured disk $v^*$ is labeled by a DG-valued 0-form $\mathcal{H}_v$ and an element $\mathcal{M}_v$ of the Cartan subalgebra $\mathfrak{dh}$. We may decompose $\mathcal{M}_v$ as follows:

$$\mathcal{M}_v = \mathbf{M}_v + \mathbf{S}_v, \qquad \mathbf{M}_v \in \mathfrak{h}, \qquad \mathbf{S}_v \in \mathfrak{h}^*, \tag{76}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The connection $\mathcal{A}$ inside the punctured disk $v^*$ is then given by

$$\mathcal{A}|_{v^*} = \mathcal{H}_v^{-1}(\mathbf{M}_v + \mathbf{S}_v)\mathcal{H}_v\mathrm{d}\phi_v + \mathcal{H}_v^{-1}\mathrm{d}\mathcal{H}_v. \tag{77}$$

Using the Cartan decomposition

$$\mathcal{H}_v \equiv e^{\mathbf{y}_v} h_v, \tag{78}$$

we can unpack the $\mathfrak{dg}$-valued connection into the corresponding $\mathfrak{g}$-valued connection and frame field in $v^*$:

$$\begin{aligned} \mathbf{A}|_{v^*} &= h_v^{-1}\mathbf{M}_v h_v \mathrm{d}\phi_v + h_v^{-1}\mathrm{d}h_v, \\ \mathbf{E}|_{v^*} &= h_v^{-1}((\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v])\mathrm{d}\phi_v + \mathrm{d}\mathbf{y}_v)h_v. \end{aligned} \tag{79}$$

We may similarly decompose the momentum $\mathcal{P}_v$ defined in (25):

$$\mathcal{P}_v \equiv \mathcal{H}_v^{-1}(v)\mathcal{M}_v\mathcal{H}_v(v) = \mathbf{p}_v + \mathbf{j}_v, \tag{80}$$

where $\mathbf{p}_v, \mathbf{j}_v \in \mathfrak{g}$ represent the momentum and angular momentum respectively:

$$\mathbf{p}_v \equiv h_v^{-1}\mathbf{M}_v h_v, \qquad \mathbf{j}_v \equiv h_v^{-1}(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v])h_v. \tag{81}$$

Then, by decomposing (12), we may obtain the (naive) distributional $\mathfrak{g}$-valued curvature and torsion 2-forms:

$$\mathbf{F}|_{v^*} = \mathbf{p}_v\delta(v), \qquad \mathbf{T}|_{v^*} = \mathbf{j}_v\delta(v). \tag{82}$$

Finally, we see that gauge transformations (right translations) (18) labeled by DG-valued 0-forms $(g, \mathbf{x})$ act on $(h_v, \mathbf{y}_v)$ the same way they act on $(h_c, \mathbf{y}_c)$:

$$h_v(x) \mapsto h_v(x) g(x), \quad \mathbf{y}_v(x) \mapsto \mathbf{y}_v(x) + (h_v \mathbf{x} h_v^{-1})(x). \quad (83)$$

The symmetry transformations (left translations) (19), labeled by constant elements $(g_v, \mathbf{z}_v)$ of the Cartan subgroup DH, leave $\mathbf{M}_v$ and $\mathbf{S}_v$ invariant, while they transform $(h_v, \mathbf{y}_v)$ by

$$h_v(x) \mapsto g_v h_v(x), \qquad \mathbf{y}_v(x) \mapsto \mathbf{z}_v + g_v \mathbf{y}_v(x) g_v^{-1}. \quad (84)$$

We discuss these two types of transformations in the context of the relativistic particle at $v$ in Appendix C.

As we have seen, the boundary conditions between cells and disks can be expressed as continuity equations either across the edge $(cc') \equiv c \cap c'$ bounding two cells or across the arc $(vc) \equiv \partial v^* \cap c$ bounding the interface of a disk and a cell. Although both continuity conditions are similar, the one across the arcs is more involved.

### E. Continuity conditions between cells

For $x \in (cc') = c \cap c'$ we have from (31)

$$\mathcal{H}_{c'}(x) = \mathcal{H}_{c'c} \mathcal{H}_c(x), \qquad x \in (cc'). \quad (85)$$

The holonomies $\mathcal{H}_{cc'}$ and $\mathcal{H}_c$ can be decomposed into a rotational and translational part using the Cartan decomposition, as usual[11]:

$$\mathcal{H}_{cc'} = e^{\mathbf{y}_c^{c'}} h_{cc'}, \qquad \mathcal{H}_c = e^{\mathbf{y}_c} h_c. \quad (86)$$

Note that under the exchange of indices we have, from (64),

$$\mathcal{H}_{cc'} = \mathcal{H}_{c'c}^{-1}, \quad h_{cc'} = h_{c'c}^{-1}, \quad \mathbf{y}_c^{c'} = -h_{cc'} \mathbf{y}_{c'}^{c} h_{c'c}. \quad (87)$$

Using these quantities and the rules (63) and (65), (85) can be split into a rotational and translational part:

$$h_{c'}(x) = h_{c'c} h_c(x), \quad \mathbf{y}_{c'}(x) = h_{c'c}(\mathbf{y}_c(x) - \mathbf{y}_c^{c'}) h_{cc'}, \quad (88)$$

for $x \in (cc')$. These relations show that the rotational and translational holonomies $(h_{cc'}, \mathbf{y}_c^{c'})$ are *invariant* under the gauge transformation (74) (since it is independent of $c$). On the other hand, the discrete "holonomies" $(h_{cc'}, \mathbf{y}_c^{c'})$ transform nontrivially under the global symmetries (75):

$$\begin{aligned} h_{c'c} &\mapsto \tilde{h}_{c'c} \equiv g_{c'} h_{c'c} g_c^{-1}, \\ \mathbf{y}_c^{c'} &\mapsto \tilde{\mathbf{y}}_c^{c'} \equiv g_c \mathbf{y}_c^{c'} g_c^{-1} + \mathbf{z}_c - \tilde{h}_{cc'} \mathbf{z}_{c'} \tilde{h}_{c'c}. \end{aligned} \quad (89)$$

This may also be obtained from the Cartan decomposition of (32), $\tilde{\mathcal{H}}_{c'c} \equiv \mathcal{G}_{c'} \mathcal{H}_{c'c} \mathcal{G}_c^{-1}$.

[11]The index placement in $\mathcal{H}_{cc'}$ reflects that this is a transformation mapping objects at $c$ to objects at $c'$, while $\mathbf{y}_c^{c'}$ denotes a transformation based at $c$.

### F. Continuity conditions between disks and cells

For $x \in (vc) = \partial v^* \cap c$ one has the continuity condition (39):

$$\mathcal{H}_c(x) = \mathcal{H}_{cv} e^{\boldsymbol{\mathcal{M}}_v \phi_v(x)} \mathcal{H}_v(x), \qquad x \in (vc). \quad (90)$$

The holonomies $\mathcal{H}_{cv}$ and $\mathcal{H}_v$ can be decomposed into a rotational and translational part as we did for $\mathcal{H}_{cc'}$ and $\mathcal{H}_c$ above, and given (76), we can write

$$e^{\boldsymbol{\mathcal{M}}_v \phi_v(x)} = e^{\mathbf{S}_v \phi_v(x)} e^{\mathbf{M}_v \phi_v(x)}, \quad (91)$$

where $\mathbf{S}_v \phi_v(x)$ is the translational part and $e^{\mathbf{M}_v \phi_v(x)}$ the rotational part.[12] Using these quantities, the continuity relations (90) can be split into a rotational and translational part:

$$h_c(x) = h_{cv} e^{\mathbf{M}_v \phi_v(x)} h_v(x), \qquad x \in (vc), \quad (92)$$

$$\mathbf{y}_c(x) = h_{cv}(e^{\mathbf{M}_v \phi_v(x)}(\mathbf{y}_v(x) + \mathbf{S}_v \phi_v(x)) e^{-\mathbf{M}_v \phi_v(x)} - \mathbf{y}_v^c) h_{vc}, \quad x \in (vc). \quad (93)$$

The quantities $h_{vc}$ and $\mathbf{y}_v^c$ are invariant under the gauge transformation (right translation) (74) and (83). However, under the symmetry transformation (left translation) (75) and (84), with $g_c \in G$ and $g_v$ in the Cartan subgroup $H$,

$$h_c \mapsto g_c h_c, \qquad \mathbf{y}_c \mapsto \mathbf{z}_c + g_c \mathbf{y}_c g_c^{-1}, \quad (94)$$

$$h_v \mapsto g_v h_v, \qquad \mathbf{y}_v \mapsto \mathbf{z}_v + g_v \mathbf{y}_v g_v^{-1}, \quad (95)$$

we have

$$\begin{aligned} h_{vc} &\mapsto \tilde{h}_{vc} \equiv g_v h_{vc} g_c^{-1}, \\ \mathbf{y}_v^c &\mapsto \tilde{\mathbf{y}}_v^c \equiv g_v \mathbf{y}_v^c g_v^{-1} + \mathbf{z}_v - \tilde{h}_{vc} \mathbf{z}_c \tilde{h}_{cv}, \end{aligned} \quad (96)$$

where we have used the fact that $e^{\mathbf{M}_v}$, $\mathbf{S}_v$, $g_v$ and $\mathbf{z}_v$ all commute with each other. This follows from (42), $\mathcal{H}_{vc} \mapsto \mathcal{G}_v \mathcal{H}_{vc} \mathcal{G}_c^{-1}$, using the Cartan decomposition.

Note also that the continuity relation on $(vc)$ and the connection $\boldsymbol{\mathcal{A}}$ are invariant under the symmetry transformation (43):

$$\phi_v(x) \mapsto \phi_v(x) + \beta_v, \qquad \mathcal{H}_v(x) \mapsto e^{-\boldsymbol{\mathcal{M}}_v \beta_v} \mathcal{H}_v(x). \quad (97)$$

This transformation of $\mathcal{H}_v(x)$ decomposes via (65) as follows:

$$\begin{aligned} \phi_v(x) &\mapsto \phi_v(x) + \beta_v, \\ \mathbf{y}_v(x) &\mapsto e^{-\mathbf{M}_v \beta_v}(\mathbf{y}_v(x) - \mathbf{S}_v \beta_v) e^{\mathbf{M}_v \beta_v}, \\ h_v(x) &\mapsto e^{-\mathbf{M}_v \beta_v} h_v(x). \end{aligned} \quad (98)$$

[12]In comparison, when we decomposed $\mathcal{H} \equiv e^{\mathbf{y}} h$, the translational part was $\mathbf{y}$ and the rotational part was $h$.

Of course, by construction, the relations (92) and (93) are invariant under this transformation. This transformation turns out to be a special case of a more general class of transformations, as shown in Appendix C.

## IV. DISCRETIZING THE SYMPLECTIC POTENTIAL FOR 2+1 GRAVITY

### A. The symplectic potential

#### 1. The BF action

Now that we have expressed the connection and frame field in terms of discretized variables, we would like to construct the phase space structure. For this we take inspiration from the $2+1$ gravity action, as given by $BF$ theory. The $BF$ action is[13]

$$S = \int_M \mathbf{E}\cdot\mathbf{F}(\mathbf{A}), \tag{100}$$

where $M$ is a $2+1$-dimensional spacetime manifold, and the symplectic potential is

$$\Theta = -\int_\Sigma \mathbf{E}\cdot\delta\mathbf{A}, \tag{101}$$

where $\Sigma$ is a spatial slice. To get the discretized version of the symplectic potential, we are going to express the different components in terms of their associated holonomies, in the truncated cells $\tilde{c} \equiv c\backslash \bigcup_{v\in c} v^*$ and punctured disks $v^*$. In other words, we define

$$\Theta_c \equiv -\int_{\tilde{c}} \mathbf{E}\cdot\delta\mathbf{A}, \qquad \Theta_{v^*} \equiv -\int_{v^*} \mathbf{E}\cdot\delta\mathbf{A}, \tag{102}$$

and since $\Sigma\backslash V_\Gamma = \cup_c\, \tilde{c} \cup_v\, v^*$ the total symplectic potential is simply the sum over all cells $c$ and vertices $v$:

$$\Theta = \sum_c \Theta_c + \sum_v \Theta_{v^*}. \tag{103}$$

We will first evaluate $\Theta_c$ and $\Theta_{v^*}$ independently using the flatness condition, and then take advantage of the simplification that occurs thanks to the fact that the transition maps between cells and adjacent cells or disks are DG transformations.

For a quicker derivation using nonperiodic variables, please see Appendix D.

[13]We view both $\mathbf{E}$ and $\mathbf{A}$ as elements of $\mathfrak{g}$, and we define the normalized trace Tr, which satisfies

$$\mathrm{Tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j) = \delta_{ij}, \tag{99}$$

where $\boldsymbol{\tau}_i$ are the generators of $\mathfrak{g}$. Then the dot product is defined as before, $\mathbf{A}\cdot\mathbf{B} \equiv \mathrm{Tr}(\mathbf{A}\wedge\mathbf{B}) = A^i\wedge B_i$.

#### 2. Evaluation of $\Theta_c$

For $\Theta_c$, we have from (72) that $\mathbf{A}|_c = h_c^{-1}\mathrm{d}h_c$ and $\mathbf{E}|_c = h_c^{-1}\mathrm{d}\mathbf{y}_c h_c$, and we find

$$\delta\mathbf{A}|_c = h_c^{-1}(\mathrm{d}\boldsymbol{\Delta}h_c)h_c. \tag{104}$$

Thus[14]

$$-\mathbf{E}\cdot\delta\mathbf{A} = -\mathrm{d}\mathbf{y}_c\cdot\mathrm{d}\boldsymbol{\Delta}h_c = \mathrm{d}(\mathrm{d}\mathbf{y}_c\cdot\boldsymbol{\Delta}h_c), \tag{105}$$

and we may integrate to get $\Theta_c$ as an integral over the boundary of the truncated cell:

$$\Theta_c = \int_{\partial\tilde{c}} \mathrm{d}\mathbf{y}_c\cdot\boldsymbol{\Delta}h_c. \tag{106}$$

In order to integrate this further, we will need to use the continuity conditions, which we will do in Sec. IV B.

#### 3. Evaluation of $\Theta_{v^*}$

For $\Theta_{v^*}$, we have from (79) that $\mathbf{A}|_{v^*} = h_v^{-1}\mathrm{d}h_v + h_v^{-1}\mathbf{M}_v h_v \mathrm{d}\phi_v$, and similarly for the frame field $\mathbf{E}|_{v^*} = h_v^{-1}(\mathrm{d}\mathbf{y}_v + (\mathbf{S}_v + [\mathbf{M}_v,\mathbf{y}_v])\mathrm{d}\phi_v)h_v$. Therefore one finds

$$\delta\mathbf{A}|_{v^*} = h_v^{-1}(\mathrm{d}\boldsymbol{\Delta}h_v + (\delta\mathbf{M}_v + [\mathbf{M}_v,\boldsymbol{\Delta}h_v])\mathrm{d}\phi_v)h_v. \tag{107}$$

Thus, after some simplification (using $[\mathbf{M}_v,\mathbf{S}_v]=0$),

$$-\mathbf{E}\cdot\delta\mathbf{A} = \mathrm{d}(\mathrm{d}\mathbf{y}_v\cdot\boldsymbol{\Delta}h_v - (\mathbf{y}_v\cdot\delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v,\mathbf{y}_v])\cdot\boldsymbol{\Delta}h_v)\mathrm{d}\phi_v), \tag{108}$$

where we choose as above the polarization $\mathrm{d}\mathbf{y}_v\cdot\mathrm{d}\boldsymbol{\Delta}h_v = -\mathrm{d}(\mathrm{d}\mathbf{y}_v\cdot\boldsymbol{\Delta}h_v)$ for the first term. Now we may integrate. Remembering that $\partial v^* = \partial_0 v^* \cup \partial_R v^*$, we get two contributions, one from the inner boundary $\partial_0 v^*$ and one (with opposite orientation and thus a minus sign) from the outer boundary $\partial_R v^*$:

$$\Theta_{v^*} = \Theta_{\partial_R v^*} - \Theta_{\partial_0 v^*}, \tag{109}$$

where

$$\Theta_{\partial_R v^*} \equiv \int_{\partial_R v^*} (\mathrm{d}\mathbf{y}_v\cdot\boldsymbol{\Delta}h_v - (\mathbf{y}_v\cdot\delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v,\mathbf{y}_v])\cdot\boldsymbol{\Delta}h_v)\mathrm{d}\phi_v). \tag{110}$$

As above, we will need some simplifications in order to integrate $\Theta_{\partial_R v^*}$, which we will do in Sec. IV B. The expression for $\Theta_{\partial_0 v^*}$ is similar except that the boundary

[14]We call this choice the "LQG polarization." One could alternatively write this expression as a total differential using $\mathrm{d}\mathbf{y}_c\cdot\mathrm{d}\boldsymbol{\Delta}h_c = \mathrm{d}(\mathbf{y}_c\cdot\mathrm{d}\boldsymbol{\Delta}h_c)$, that is, with d on $\boldsymbol{\Delta}h_c$ instead of $\mathbf{y}_c$. This leads to the "dual polarization," which we will explore in [40].

condition at $r = 0$ implies that $(h_v, y_v)|_{r=0}$ are constant and equal to $(h_v(v), y_v(v))$. The integrand may then be trivially integrated, and we get

$$-\Theta_{\partial_0 v^*} = \mathbf{y}_v(v) \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v(v)]) \cdot \mathbf{\Delta}(h_v(v)). \tag{111}$$

#### 4. Summary

The total symplectic potential now takes the form of a sum of three contributions:

$$\Theta = \sum_c \Theta_c + \sum_v \Theta_{\partial_R v^*} + \sum_v \Theta_{\partial_0 v^*}, \tag{112}$$

where

$$\Theta_c = \int_{\partial\tilde{c}} \mathrm{d}\mathbf{y}_c \cdot \mathbf{\Delta} h_c, \tag{113}$$

$$\Theta_{\partial_R v^*} = \int_{\partial_R v^*} (\mathrm{d}\mathbf{y}_v \cdot \mathbf{\Delta} h_v - (\mathbf{y}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v]) \cdot \mathbf{\Delta} h_v)\mathrm{d}\phi_v), \tag{114}$$

$$\Theta_{\partial_0 v^*} = \mathbf{y}_v(v) \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v(v)]) \cdot \mathbf{\Delta}(h_v(v)). \tag{115}$$

### B. Rearranging the sums and integrals

From the previous section, we see that the total symplectic potential can be written in terms of the variables $(\mathcal{H}_c, \mathcal{H}_v, \boldsymbol{\mathcal{M}}_v)$ purely as a sum of line integrals plus vertex contributions. In order to simplify this expression, we can break the line integrals into a sum of individual contributions along the boundaries.

Let us recall the construction of Sec. II E. Consider a cell $c$ with $N$ vertices $v_1, \ldots, v_N$ along its boundary. Each vertex $v_i$ is dual to a punctured disk $v_i^*$, and the intersection of the (outer) boundary of that disk with the cell $c$ is the arc

$$(v_i c) \equiv \partial_R v_i^* \cap c. \tag{116}$$

When we remove the intersections of the (full) disks with the cell $c$, we obtain the truncated cells

$$\tilde{c} \equiv c \setminus \bigcup_{i=1}^{N} D_{v_i}. \tag{117}$$

Now, to the cell $c$ there are $N$ adjacent cells $c_1, \ldots, c_N$, truncated into $\tilde{c}_1, \ldots, \tilde{c}_N$, and each such truncated cell intersects $\tilde{c}$ at a truncated edge, denoted with square brackets:

$$[cc_i] \equiv \tilde{c} \cap \tilde{c}_i. \tag{118}$$

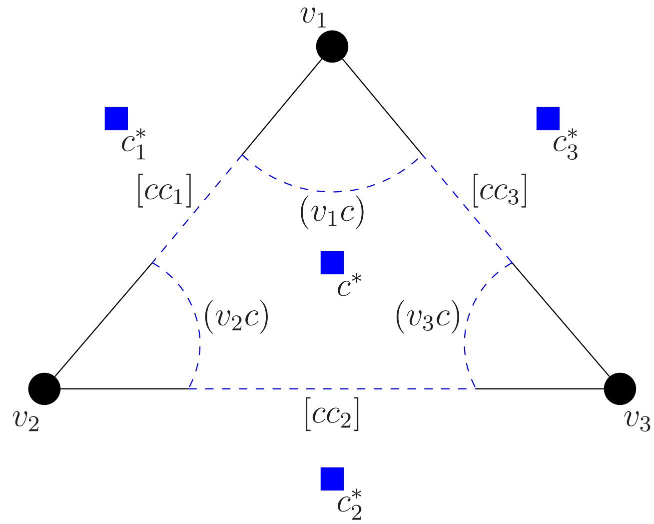


FIG. 5. The blue square in the center is the node $c^*$. It is dual to the cell $c$, outlined in black. In this simple example, we have $N = 3$ vertices $v_1$, $v_2$, $v_3$ along the boundary $\partial c$, dual to three disks $v_1^*$, $v_2^*$, $v_3^*$. Only the wedge $v_i^* \cap c$ is shown for each disk. After removing the wedges from $c$ we obtain the truncated cell $\tilde{c}$, in dashed blue. The cell $c$ is adjacent to three cells $c_i$ (not shown) dual to the three nodes $c_i^*$, in blue. The boundary $\partial\tilde{c}$, in dashed blue, thus consists of three arcs $(v_i c)$ and three truncated edges $[cc_i]$.

We thus see that the boundary $\partial\tilde{c}$ of the truncated cell may be decomposed into a union of truncated edges and arcs:

$$\partial\tilde{c} = \bigcup_{i=1}^{N} ([cc_i] \cup (v_i c)). \tag{119}$$

This is illustrated in Fig. 5. Similarly, given a punctured disk $v^*$ surrounded by $N$ cells $c_1, \ldots, c_N$, its (outer) boundary can be decomposed as a union of arcs:

$$\partial_R v^* = \bigcup_{i=1}^{N} (vc_i). \tag{120}$$

Accordingly, we can now rearrange the first two sums in (112), decomposing the sums over the boundaries $\partial\tilde{c}$ and $\partial_R v^*$ into sums over individual truncated edges and arcs:

$$\Theta = \sum_{[cc']} \Theta_{cc'} + \sum_{(vc)} \Theta_{(vc)} + \sum_v \Theta_{\partial_0 v^*}. \tag{121}$$

The first sum is over all truncated edges $[cc']$ for all pairs of adjacent cells $c$ and $c'$, the second sum is over all the arcs $(vc)$ for all pairs of adjacent vertices $v$ and cells $c$, and the third sum is over all the vertices $v$.

To find the contributions from the edges and arcs, we assume that $\Sigma$ is an oriented surface and choose the counterclockwise orientation of the boundary of each cell. Each edge $[cc']$ is counted twice, once from the direction of $c$ as part of the integral over $\partial\tilde{c}$ from $\Theta_c$ and once from the direction of $c'$ as part of the integral over $\partial\tilde{c}'$ from $\Theta_{c'}$, with opposite orientation resulting in a relative minus sign:

$$\Theta_{cc'} = \int_{[cc']} (\mathrm{d}\mathbf{y}_c \cdot \mathbf{\Delta} h_c - \mathrm{d}\mathbf{y}_{c'} \cdot \mathbf{\Delta} h_{c'}). \tag{122}$$

Similarly, each arc $(vc)$ is counted twice, once from the direction of $v$ as part of the integral over $\partial v^*$ from $\Theta_{\partial_R v^*}$ and once from the direction of $c$ as part of the integral over $\partial\tilde{c}$ from $\Theta_c$, again with opposite orientation:

$$\Theta_{(vc)} = \int_{(vc)} (\mathrm{d}\mathbf{y}_v \cdot \mathbf{\Delta} h_v - (\mathbf{y}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v]) \cdot \mathbf{\Delta} h_v)\mathrm{d}\phi_v - \mathrm{d}\mathbf{y}_c \cdot \mathbf{\Delta} h_c). \tag{123}$$

### C. Simplifying the edge and arc contributions

#### 1. The edge contributions

The calculation of the edge symplectic potential $\Theta_{cc'}$ is similar to [6] and will serve as a warmup to set the stage for the evaluation of the arc symplectic potential below. One first needs to recall the continuity relations (88):

$$h_{c'}(x) = h_{c'c} h_c(x), \qquad \mathbf{y}_{c'}(x) = h_{c'c}(\mathbf{y}_c(x) - \mathbf{y}_c^{c'}) h_{cc'}, \qquad x \in (cc'). \tag{124}$$

Plugging this into (122), we get

$$\Theta_{cc'} = \int_{[cc']} (\mathrm{d}\mathbf{y}_c \cdot \mathbf{\Delta} h_c - h_{c'c}\mathrm{d}\mathbf{y}_c h_{cc'} \cdot \mathbf{\Delta}(h_{c'c} h_c)), \tag{125}$$

where we used the fact that $h_{c'c}$ and $\mathbf{y}_c^{c'}$ are constant (do not depend on $x$) and thus annihilated by d. Next, from the useful identity (A4) we have

$$\Delta(h_{c'c} h_c) = h_{c'c}(\mathbf{\Delta} h_c - \mathbf{\Delta} h_c^{c'}) h_{cc'}, \tag{126}$$

where we have used the notation $\mathbf{\Delta} h_c^{c'} \equiv \delta h_{cc'} h_{c'c}$, introduced in (A6), which emphasizes that it is an algebra element based at $c$. This allows us to cancel $h_{cc'}$ using the cyclicity of the trace and then cancel the first term, simplifying this expression to

$$\Theta_{cc'} = \mathbf{\Delta} h_c^{c'} \cdot \int_{[cc']} \mathrm{d}\mathbf{y}_c, \tag{127}$$

where we took $\mathbf{\Delta} h_c^{c'}$ out of the integral since it is constant. We will perform the final integration in Sec. IV D 2.

#### 2. The arc contributions

We can now evaluate the arc contribution (123). One first recalls the continuity relations (92) and (93):

$$h_c(x) = h_{cv} e^{\mathbf{M}_v \phi_v(x)} h_v(x), \qquad x \in (vc),$$
$$\mathbf{y}_c(x) = h_{cv}(e^{\mathbf{M}_v \phi_v(x)}(\mathbf{y}_v(x) + \mathbf{S}_v \phi_v(x)) e^{-\mathbf{M}_v \phi_v(x)} - \mathbf{y}_v^c) h_{vc}, \qquad x \in (vc).$$

From these relations, and using the fact that $h_{vc}$ and $\mathbf{y}_v^c$ are constant, we find that

$$\Delta h_c = h_{cv}(e^{\mathbf{M}_v \phi_v}(\mathbf{\Delta} h_v + \delta\mathbf{M}_v \phi_v) e^{-\mathbf{M}_v \phi_v} - \mathbf{\Delta} h_v^c) h_{cv}^{-1}, \tag{128}$$

$$\mathrm{d}\mathbf{y}_c = h_{cv} e^{\mathbf{M}_v \phi_v}(\mathrm{d}\mathbf{y}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v])\mathrm{d}\phi_v) e^{-\mathbf{M}_v \phi_v} h_{vc}, \tag{129}$$

where we have denoted again $\mathbf{\Delta} h_v^c \equiv \delta h_{vc} h_{cv}$ since the variational differential is based at $v$. Recall that the arc symplectic potential (123) was

$$\Theta_{(vc)} = \int_{(vc)} (\mathrm{d}\mathbf{y}_v \cdot \mathbf{\Delta} h_v - (\mathbf{y}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v]) \cdot \mathbf{\Delta} h_v)\mathrm{d}\phi_v - \mathrm{d}\mathbf{y}_c \cdot \mathbf{\Delta} h_c). \tag{130}$$

Replacing the expressions for $\mathrm{d}\mathbf{y}_c$ and $\Delta h_c$ in the last term, we get after simplification

$$\Theta_{(vc)} = \mathbf{\Delta} h_v^c \cdot \int_{(vc)} \mathrm{d}(e^{\mathbf{M}_v \phi_v} \mathbf{y}_v e^{-\mathbf{M}_v \phi_v} + \mathbf{S}_v \phi_v) - \delta\mathbf{M}_v \cdot \int_{(vc)} \mathrm{d}\left(\mathbf{y}_v \phi_v + \frac{1}{2}\mathbf{S}_v \phi_v^2\right). \tag{131}$$

We see that the first term depends on $c$, while the second one does not. Thus we can perform the sum over arcs around each disk in the second term [since we had $\sum_{(vc)} \Theta_{(vc)}$ in (121)] and turn it instead into a sum over vertices with an integral along the outer boundary $\partial_R v^*$ for each vertex. In other words, we define

$$\sum_{(vc)} \Theta_{(vc)} \equiv \sum_{(vc)} \Theta_{vc} + \sum_v \Theta_{(v)}, \tag{132}$$

where

$$\Theta_{vc} \equiv \mathbf{\Delta} h_v^c \cdot \int_{(vc)} \mathrm{d}(e^{\mathbf{M}_v \phi_v} \mathbf{y}_v e^{-\mathbf{M}_v \phi_v} + \mathbf{S}_v \phi_v), \tag{133}$$

$$\Theta_{(v)} \equiv -\delta\mathbf{M}_v \cdot \int_{\partial_R v^*} \mathrm{d}\left(\mathbf{y}_v \phi_v + \frac{1}{2}\mathbf{S}_v \phi_v^2\right). \tag{134}$$

#### 3. The vertex contribution

In fact, we may easily integrate $\Theta_{(v)}$, bearing in mind the definition of the punctured disk in Sec. II C 1:

$$\Theta_{(v)} = -\delta\mathbf{M}_v \cdot (\mathbf{y}_v(v_0) + \alpha_v \mathbf{S}_v), \tag{135}$$

where the reader will recall that $v_0$ was the intersection of the cut with the boundary circle at $r_v = R$ and $\phi_v = \alpha_v - \frac{1}{2}$ defines the angle assign to the cut. We may now add this contribution to the term $\Theta_{\partial_0 v^*}$ from (113), obtaining

$$\Theta_v \equiv \Theta_{\partial_0 v^*} + \Theta_{(v)} = (\mathbf{y}_v(v) - \mathbf{y}_v(v_0) - \alpha_v \mathbf{S}_v) \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v(v)]) \cdot \mathbf{\Delta}(h_v(v)). \tag{136}$$

We can finally simplify this expression further. Let us first decompose $\mathbf{y}_v(v_0)$ into a component parallel to $\mathbf{M}_v$ (with $\mathbf{M}_v^2 \equiv \mathbf{M}_v \cdot \mathbf{M}_v$) and another orthogonal to it:

$$\mathbf{y}_v(v_0) \equiv \mathbf{y}_v^{\parallel}(v_0) + \mathbf{y}_v^{\perp}(v_0), \tag{137}$$

where

$$\mathbf{y}_v^{\parallel}(v_0) \equiv \left(\frac{\mathbf{y}_v(v_0) \cdot \mathbf{M}_v}{\mathbf{M}_v^2}\right)\mathbf{M}_v. \tag{138}$$

The term $\mathbf{y}_v(v_0) \cdot \delta\mathbf{M}_v$ only depends on the component $\mathbf{y}_v^{\parallel}(v_0)$. On the other hand, the term $[\mathbf{M}_v, \mathbf{y}_v(v)]$ is left invariant if we translate $\mathbf{y}_v(v)$ by any component parallel to $\mathbf{M}_v$. Hence, we can rewrite the vertex potential as

$$\Theta_v = \mathbf{X}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v]) \cdot \mathbf{\Delta}(h_v(v)), \tag{139}$$

where we have introduced the particle relative position

$$\mathbf{X}_v \equiv \mathbf{y}_v(v) - \mathbf{y}_v^{\parallel}(v_0) - \alpha_v \mathbf{S}_v. \tag{140}$$

The term proportional to $\alpha_v$ can be eliminated by a symmetry transformation of the type (98).

The expression (139) is the usual expression for the symplectic potential of a relativistic particle with mass $\mathbf{M}_v$ and spin $\mathbf{S}_v$ [41,42]. In particular, if we introduce the particle momentum $\mathbf{p}_v$, angular momentum[15] $\mathbf{j}_v$ and position $\mathbf{q}_v$,

$$\mathbf{p}_v \equiv h_v^{-1}(v)\mathbf{M}_v h_v(v), \quad \mathbf{j}_v \equiv h_v^{-1}(v)(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v])h_v(v), \quad \mathbf{q}_v \equiv h_v^{-1}(v)\mathbf{X}_v h_v(v), \tag{141}$$

we can show that the following commutation relations are satisfied[16]:

$$\{p_{vi}, q_v^j\} = \delta_i^j, \quad \{j_v^i, q_v^j\} = C^{ij}{}_k q_v^k, \quad \{j_v^i, p_{vj}\} = -C^{ik}{}_j p_{vk}, \tag{142}$$

where $q_v^i$, $p_{vi}$, $j_v^i$ are the components of $\mathbf{q}_v$, $\mathbf{p}_v$, $\mathbf{j}_v$ with respect to the basis $\boldsymbol{\tau}_i$ of $\mathfrak{g}$ or the dual basis $\boldsymbol{\tau}^i$ of $\mathfrak{g}^*$ (according to the index placement), and $C^{ij}{}_k$ are the structure constants such that $[\boldsymbol{\tau}^i, \boldsymbol{\tau}^j] = C^{ij}{}_k \boldsymbol{\tau}^k$. This shows that, as expected, the momentum $\mathbf{p}_v$ is the generator of translations, the angular momentum $\mathbf{j}_v$ is the generator of rotations, and $\mathbf{q}_v$ is the particle's position.

It can be useful to express the effective particle potential in terms of position and momentum:

$$\Theta_v = \mathbf{q}_v \cdot \delta\mathbf{p}_v - \mathbf{S}_v \cdot \mathbf{\Delta}(h_v(v)). \tag{143}$$

More properties of the vertex potential are presented in Appendix C.

### D. Final integration

In the previous section we have established that the total symplectic potential is given by

$$\Theta = \sum_{[cc']} \Theta_{cc'} - \sum_{(vc)} \Theta_{vc} + \sum_v \Theta_v, \tag{144}$$

where $\Theta_{cc'}$ and $\Theta_{vc}$ are simple line integrals

$$\Theta_{cc'} = \mathbf{\Delta} h_c^{c'} \cdot \int_{[cc']} \mathrm{d}\mathbf{y}_c, \qquad \Theta_{vc} = \mathbf{\Delta} h_v^c \cdot \int_{(vc)} \mathrm{d}(e^{\mathbf{M}_v \phi_v} \mathbf{y}_v e^{-\mathbf{M}_v \phi_v} + \mathbf{S}_v \phi_v), \tag{145}$$

while $\Theta_v$ is the relativistic particle potential (139). We are left with the integration and study of the edge and arc potentials.

#### 1. Source and target points

We have obtained, in (145), two integrals of total differentials over the truncated edges $[cc']$ and the arcs $(vc)$. These integrals are trivial; all we need is to label the source and target points of each edge and arc. One needs to recall that the truncated edges are oriented form the point of view of $c$, while the arcs are oriented from the point of view of $v$ so they have opposite relative orientations as shown in Fig. 6.

These corner points are located along the edges of $\Gamma$ and are determined by the radius $R$ of the disks. We will label the source and target points of the edge $[cc']$ as $\sigma_{cc'}$ and $\tau_{cc'}$ respectively, and the source and target points of the arc $(vc)$ as $\sigma_{vc}$ and $\tau_{vc}$ respectively, where $\sigma$ stands for "source" and $\tau$ for "target." In other words,

$$[cc'] \equiv [\sigma_{cc'} \tau_{cc'}], \qquad (vc) \equiv [\sigma_{vc} \tau_{vc}]. \tag{146}$$

[15]The definition of $\mathbf{j}_v$ given here agrees with the definition (81) given earlier since $\mathbf{X}_v$ differs from $\mathbf{y}_v(v)$ only by a translation in the Cartan, which commutes with $\mathbf{M}_v$.

[16]To clarify the notation, the subscript $v$ denotes the vertex as usual, while $i$, $j$, $k$ are the Lie algebra indices.

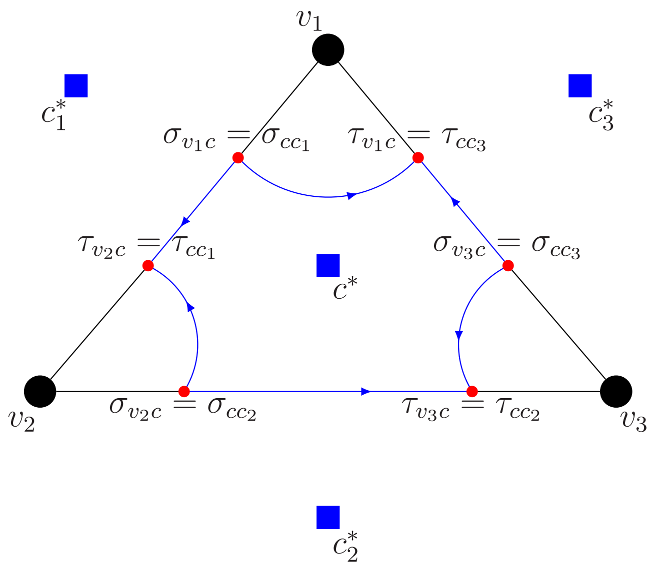


FIG. 6. The intersection points (red circles) of truncated edges and arcs along the oriented boundary $\partial\tilde{c}$ (blue arrows).

Note that these labels are not unique, since the source (target) of an edge is also the source (target) of an arc. More precisely, let us consider a cell $c = [v_1, \ldots, v_N]$, where $v_i$ denote the boundary vertices. This cell is bounded by $N$ other cells $c_1, \ldots, c_N$, which are such that $c \cap c_i = (v_i v_{i+1})$. We then have that

$$\sigma_{v_i c} = \sigma_{cc_i}, \qquad \tau_{v_{i+1}c} = \tau_{cc_i}. \tag{147}$$

This labeling is illustrated in Fig. 6.

### 2. The holonomy-flux algebra

Using these labels, we can perform the integrations in (145). We define fluxes associated to the edges $[cc']$ and arcs $(vc)$ bounding the cell $c$ in terms of the corner variables:

$$\mathbf{X}_c^{c'} \equiv \int_{[cc']} \mathrm{d}\mathbf{y}_c = \mathbf{y}_c(\tau_{cc'}) - \mathbf{y}_c(\sigma_{cc'}), \tag{148}$$

$$\mathbf{X}_v^c \equiv \int_{(vc)} \mathrm{d}(e^{\mathbf{M}_v\phi_v}\mathbf{y}_v e^{-\mathbf{M}_v\phi_v} + \mathbf{S}_v\phi_v) \tag{149}$$

$$= (e^{\mathbf{M}_v\phi_v}\mathbf{y}_v e^{-\mathbf{M}_v\phi_v} + \mathbf{S}_v\phi_v)(\tau_{vc}) - (e^{\mathbf{M}_v\phi_v}\mathbf{y}_v e^{-\mathbf{M}_v\phi_v} + \mathbf{S}_v\phi_v)(\sigma_{vc}). \tag{150}$$

This allows us to write the edges' potential simply as

$$\Theta_{cc'} \equiv \mathbf{\Delta} h_c^{c'} \cdot \mathbf{X}_c^{c'}, \qquad \Theta_{vc} \equiv \mathbf{\Delta} h_v^c \cdot \mathbf{X}_v^c. \tag{151}$$

We see that these two terms correspond to the familiar holonomy-flux phase space for each edge and arc. $\Theta_{cc'}$ is associated to each link $(cc')^*$ in the spin network graph $\Gamma^*$, while $\Theta_{vc}$ is associated to each segment $(vc)^*$ connecting a vertex with a node. The holonomy-flux phase space is the cotangent bundle $T^*G$, and it is the phase space of (classical) spin networks in loop quantum gravity in the case $G = \mathrm{SU}(2)$.

One can prove a relation between a flux and its "inverse." First, note that $\sigma_{cc'} = \tau_{c'c}$. Thus, we have from the continuity condition (88)

$$\mathbf{X}_{c'}^c = \mathbf{y}_{c'}(\tau_{c'c}) - \mathbf{y}_{c'}(\sigma_{c'c}) = -(\mathbf{y}_{c'}(\tau_{cc'}) - \mathbf{y}_{c'}(\sigma_{cc'})) = -h_{c'c}(\mathbf{y}_c(\tau_{cc'}) - \mathbf{y}_c(\sigma_{cc'}))h_{cc'} = -h_{c'c}\mathbf{X}_c^{c'}h_{cc'}.$$

Similarly, from (93) we find that the arc flux as viewed from the point of view of the cell $c$ is

$$\mathbf{X}_c^v \equiv -\int_{(vc)} \mathrm{d}\mathbf{y}_c = -(\mathbf{y}_c(\tau_{vc}) - \mathbf{y}_c(\sigma_{vc})) = -h_{cv}\mathbf{X}_v^c h_{vc}, \tag{152}$$

where the minus sign comes from the opposite orientation of the arc when viewed from $c$ instead of $v^*$.

In conclusion, the fluxes satisfy the relations

$$\mathbf{X}_{c'}^c = -h_{c'c}\mathbf{X}_c^{c'}h_{cc'}, \qquad \mathbf{X}_c^v = -h_{cv}\mathbf{X}_v^c h_{vc}. \tag{153}$$

Note that, if we view $\mathbf{y}_c(\tau_{cc'})$ and $\mathbf{y}_c(\sigma_{cc'})$ as the relative position of the corner $\tau_{cc'}$ from the node $c^*$ to the points $\tau_{cc'}$ and $\sigma_{cc'}$ respectively, then the difference $\mathbf{X}_c^{c'} = \mathbf{y}_c(\tau_{cc'}) - \mathbf{y}_c(\sigma_{cc'})$ is a translation vector from $\sigma_{cc'}$ to $\tau_{cc'}$. In other words, $\mathbf{X}_c^{c'}$ is simply a (translational) holonomy along the truncated edge $[cc']$ dual to the link $(cc')^*$. Similarly, $\mathbf{X}_c^v$ is a vector from $\sigma_{vc}$ to $\tau_{vc}$, along the arc $(vc)$.

Let us label the holonomies $h_\ell \equiv h_{cc'}$ and fluxes $\mathbf{X}_\ell \equiv \mathbf{X}_c^{c'}$, where $\ell \equiv (cc')^*$ is a link in the spin network graph $\Gamma^*$, dual to the edge $(cc') \in \Gamma$. Let us also decompose the flux into components, $\mathbf{X}_\ell \equiv X_\ell^i \boldsymbol{\tau}_i$. Then, from the first term in $\Theta$, one can derive the well-known holonomy-flux Poisson algebra:

$$\{h_\ell, h_{\ell'}\} = 0, \qquad \{X_\ell^i, X_{\ell'}^j\} = \delta_{\ell\ell'}\epsilon^{ij}_k X_\ell^k, \\ \{X_\ell^i, h_{\ell'}\} = \delta_{\ell\ell'}\boldsymbol{\tau}^i h_\ell. \tag{154}$$

This concludes the construction of the total symplectic potential. It can be written solely in terms of discrete data without any integrals, with contributions from truncated edges, arcs, and vertices, as follows:

$$\boxed{\Theta = \sum_{[cc']}\mathbf{\Delta} h_c^{c'} \cdot \mathbf{X}_c^{c'} - \sum_{(vc)}\mathbf{\Delta} h_c^v \cdot \mathbf{X}_c^v + \sum_v(\mathbf{X}_v \cdot \delta\mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v]) \cdot \mathbf{\Delta}(h_v(v))).} \tag{155}$$

## V. THE GAUSS AND CURVATURE CONSTRAINTS

In the continuum theory of $2+1$ gravity, we have the curvature constraint $\mathbf{F}=0$ and the torsion constraint $\mathbf{T}=0$. These constraints are modified when we add curvature and torsion defects; as we have seen in (82), at least naively, a delta function is added to the right-hand side of these constraints: $\mathbf{F}|_{v^*}=\mathbf{p}_v\delta(v)$ and $\mathbf{T}|_{v^*}=\mathbf{j}_v\delta(v)$. We will now show how these constraints are recovered in our formalism to obtain the discrete Gauss (torsion) constraint and the discrete curvature constraint.

### A. The Gauss constraint

#### 1. On the cells

Recall that on the truncated cell $\tilde{c}$ we have

$$\mathbf{A}|_{\tilde{c}}=h_c^{-1}\mathrm{d}h_c,\quad \mathbf{E}|_{\tilde{c}}=h_c^{-1}\mathrm{d}\mathbf{y}_c h_c,\quad \mathbf{T}|_{\tilde{c}}=\mathrm{d}_{\mathbf{A}}\mathbf{E}|_{\tilde{c}}=0. \tag{156}$$

Using the identity

$$\mathrm{d}(h_c\mathbf{E}h_c^{-1})=h_c(\mathrm{d}_{\mathbf{A}}\mathbf{E})h_c^{-1}, \tag{157}$$

we can define the quantity $\mathbf{G}_c$, which represents the Gauss law integrated over the truncated cell $\tilde{c}$:

$$\mathbf{G}_c\equiv\int_{\tilde{c}}h_c(\mathrm{d}_{\mathbf{A}}\mathbf{E})h_c^{-1}=\int_{\tilde{c}}\mathrm{d}(h_c\mathbf{E}h_c^{-1})=\int_{\partial\tilde{c}}h_c\mathbf{E}h_c^{-1} = \int_{\partial\tilde{c}}\mathrm{d}\mathbf{y}_c. \tag{158}$$

The Gauss law $\mathrm{d}_{\mathbf{A}}\mathbf{E}=0$ translates into the condition $\mathbf{G}_c=0$.

As illustrated in Fig. 5, the boundary $\partial\tilde{c}$ consists of truncated edges $[cc']$ and arcs $(vc)$. Thus

$$\mathbf{G}_c=\sum_{c'\ni c}\int_{[cc']}\mathrm{d}\mathbf{y}_c+\sum_{v\ni c}\int_{(vc)}\mathrm{d}\mathbf{y}_c, \tag{159}$$

where $c'\ni c$ means "all cells $c'$ adjacent to $c$" and $v\ni c$ means "all vertices $v$ adjacent to $c$." Using the fluxes defined in (148) and (152), we get

$$\mathbf{G}_c=\sum_{c'\ni c}\mathbf{X}_c^{c'}-\sum_{v\ni c}\mathbf{X}_c^{v}=0, \tag{160}$$

where, as explained before, the minus sign in the second sum comes from the fact that we are looking at the arcs from the cell $c$, not from the vertex $v$, and thus the orientation of the integral is opposite.

In the limit where we shrink the radius $R$ of the punctured disks to zero the points $\mathbf{y}_c(\tau_{vc})$ and $\mathbf{y}_c(\tau_{vc})$ are identified, the arc flux $\mathbf{X}_c^v$ vanishes, and we recover the usual loop gravity Gauss constraint. It is well known, however, that under coarse-graining the naive Gauss constraint is not preserved [43,44]. In our context, this can be interpreted as opening a small disk around the vertices, which carries additional fluxes.

#### 2. On the disks

Similarly, recall that on the *punctured* disks we have

$$\begin{aligned}\mathbf{A}|_{v^*}&=h_v^{-1}\mathrm{d}h_v+(h_v^{-1}\mathbf{M}_v h_v)\mathrm{d}\phi_v,\\ \mathbf{E}|_{v^*}&=h_v^{-1}(\mathrm{d}\mathbf{y}_v+(\mathbf{S}_v+[\mathbf{M}_v,\mathbf{y}_v])\mathrm{d}\phi_v)h_v,\end{aligned} \tag{161}$$

and $\mathrm{d}_{\mathbf{A}}\mathbf{E}|_{v^*}=0$. It will be convenient in this section to introduce nonperiodic variables

$$u_v\equiv e^{\mathbf{M}_v\phi_v}h_v,\quad \mathbf{w}_v\equiv e^{\mathbf{M}_v\phi_v}(\mathbf{y}_v+\mathbf{S}_v\phi_v)e^{-\mathbf{M}_v\phi_v}, \tag{162}$$

in terms of which the connection and frame field can be simply expressed as $\mathbf{A}|_{v^*}=u_v^{-1}\mathrm{d}u_v$ and $\mathbf{E}|_{v^*}=u_v^{-1}\mathrm{d}\mathbf{w}_v u_v$. Using the identity

$$\mathrm{d}(u_v\mathbf{E}u_v^{-1})=u_v(\mathrm{d}_{\mathbf{A}}\mathbf{E})u_v^{-1}, \tag{163}$$

we can evaluate the Gauss constraint $\mathbf{G}_v$ inside the punctured disk $v^*$:

$$\mathbf{G}_v\equiv\int_{v^*}u_v(\mathrm{d}_{\mathbf{A}}\mathbf{E})u_v^{-1}=\int_{v^*}\mathrm{d}(u_v\mathbf{E}u_v^{-1})=\int_{\partial v^*}u_v\mathbf{E}u_v^{-1} = \int_{\partial v^*}\mathrm{d}\mathbf{w}_v. \tag{164}$$

This splits into contributions from the inner and outer boundaries, with opposite signs, and (since we are now using nonperiodic variables) from the cut:

$$\mathbf{G}_v=\int_{\partial_0 v^*}\mathrm{d}\mathbf{w}_v+\int_{\partial_R v^*}\mathrm{d}\mathbf{w}_v+\int_{C_v}\mathrm{d}\mathbf{w}_v. \tag{165}$$

On the inner boundary $\partial_0 v^*$, we use the fact that $\mathbf{y}_v$ takes the constant value $\mathbf{y}_v(v)$ to obtain

$$\begin{aligned}\int_{\partial_0 v^*}\mathrm{d}\mathbf{w}_v&=e^{\mathbf{M}_v\phi_v}(\mathbf{y}_v(v)+\mathbf{S}_v\phi_v)e^{-\mathbf{M}_v\phi_v}\Big|_{\phi_v=\alpha_v-\frac{1}{2}}^{\alpha_v+\frac{1}{2}}\\ &=\mathbf{S}_v+e^{\mathbf{M}_v(\alpha_v-\frac{1}{2})}(e^{\mathbf{M}_v}\mathbf{y}_v(v)e^{-\mathbf{M}_v}-\mathbf{y}_v(v))\\ &\quad\times e^{-\mathbf{M}_v(\alpha_v-\frac{1}{2})}.\end{aligned}$$

On the outer boundary $\partial_R v^*$, we split the integral into separate integrals over each arc $(vc)=(\sigma_{vc}\tau_{vc})$ around $v^*$ and use the definition of the flux (149):

$$\int_{\partial_R v^*}\mathrm{d}\mathbf{w}_v=\sum_{c\in v}\int_{(vc)}\mathrm{d}\mathbf{w}_v=\sum_{c\in v}\mathbf{X}_v^c. \tag{166}$$

On the cut $C_v$, we have contributions from both sides, one at $\phi_v = \alpha_v - \frac{1}{2}$ and another at $\phi_v = \alpha_v + \frac{1}{2}$ with opposite orientation. Since $\mathrm{d}\phi_v = 0$ on the cut, we have

$$\mathrm{d}\mathbf{w}_v|_{C_v} = e^{\mathbf{M}_v\phi_v}\mathrm{d}\mathbf{y}_v e^{-\mathbf{M}_v\phi_v}, \tag{167}$$

and thus

$$\begin{aligned}\int_{C_v}\mathrm{d}\mathbf{w}_v &= \int_{r=0}^{R}(e^{\mathbf{M}_v\phi_v}\mathrm{d}\mathbf{y}_v e^{-\mathbf{M}_v\phi_v}|^{\phi_v=\alpha_v+\frac{1}{2}}_{\phi_v=\alpha_v-\frac{1}{2}}) \\ &= e^{\mathbf{M}_v(\alpha_v-\frac{1}{2})}(e^{\mathbf{M}_v}(\mathbf{y}_v(v_0)-\mathbf{y}_v(v))e^{-\mathbf{M}_v} - (\mathbf{y}_v(v_0) \\ &\quad -\mathbf{y}_v(v)))e^{-\mathbf{M}_v(\alpha_v-\frac{1}{2})},\end{aligned}$$

since $\mathbf{y}_v$ has the value $\mathbf{y}_v(v_0)$ at $r = R$ and $\mathbf{y}_v(v)$ at $r = 0$ on the cut.

Adding up the integrals, we find that the Gauss constraint on the disk is

$$\begin{aligned}\mathbf{G}_v = \mathbf{S}_v &+ e^{\mathbf{M}_v(\alpha_v-\frac{1}{2})}(e^{\mathbf{M}_v}\mathbf{y}_v(v_0)e^{-\mathbf{M}_v} - \mathbf{y}_v(v_0))e^{-\mathbf{M}_v(\alpha_v-\frac{1}{2})} \\ &- \sum_{c\in v}\mathbf{X}^c_v = 0.\end{aligned} \tag{168}$$

The validity of this equation can now be checked from the definition of the fluxes. By performing the sum explicitly and using the fact that $\tau_{vc_i} = \sigma_{vc_{i+1}}$ where $c_i$, $i \in \{1, \ldots, N\}$ are the cells around the disk $v^*$ and

$$\phi_v(\sigma_{vc_{N+1}}) \equiv \phi_v(\sigma_{vc_1}) + 1, \tag{169}$$

we see that this constraint is satisfied identically. Indeed, using the fact that $\mathbf{y}_v$ is periodic, choosing without loss of generality $\phi(\sigma_{vc_1}) \equiv \alpha_v - 1/2$ (that is, the first arc start at the cut), and recalling that $\mathbf{y}_v = \mathbf{y}_v(v_0)$ at the cut, we obtain

$$\begin{aligned}\sum_{c\in v}\mathbf{X}^c_v = \mathbf{S}_v &+ e^{\mathbf{M}_v(\alpha_v-\frac{1}{2})}(e^{\mathbf{M}_v}\mathbf{y}_v(v_0)e^{-\mathbf{M}_v} - \mathbf{y}_v(v_0)) \\ &\times e^{-\mathbf{M}_v(\alpha_v-\frac{1}{2})}.\end{aligned} \tag{170}$$

Remember that we have the decomposition $\mathbf{y}_v(v_0) = \mathbf{y}^{\parallel}_v(v_0) + \mathbf{y}^{\perp}_v(v_0)$. When $\mathbf{M}_v \neq 0$, the previous equation defines the value of $\mathbf{y}^{\perp}_v(v_0)$ in terms of the sum of fluxes $\sum_{c\in v}\mathbf{X}^c_v$.

### B. The curvature constraint

In the previous section we have expressed the Gauss constraints satisfied by the fluxes, which follow from the definition of the fluxes in terms of the translational holonomy variables. Here we do the same for the curvature constraint. We want to find the relations between the discrete holonomies $h_{vc}, h_{cc'}$ and the mass parameters $\mathbf{M}_v$ which express that the curvature is concentrated on the vertices.

The connection on the truncated cell $\tilde{c}$ is

$$\mathbf{A}|_{\tilde{c}^*} = h_c^{-1}\mathrm{d}h_c \Rightarrow \mathbf{F}|_{\tilde{c}^*} = 0. \tag{171}$$

Taking the rotational part of (9), we see that the associated holonomies $h_c$ from a point $x$ to another point $y$ inside $c$ are given by $\overrightarrow{\exp}\int_x^y\mathbf{A} = h_c^{-1}(x)h_c(y)$. One can use the relations (88) to evaluate the holonomy along a path from $x \in c$ to $y \in c'$, where $c$ and $c'$ are adjacent cells:

$$\overrightarrow{\exp}\int_x^y\mathbf{A} = h_c^{-1}(x)h_{cc'}h_{c'}(y). \tag{172}$$

Similarly, the connection on the *punctured* disk is

$$\mathbf{A}|_{v^*} = (e^{\mathbf{M}_v\phi_v}h_v)^{-1}\mathrm{d}(e^{\mathbf{M}_v\phi_v}h_v), \tag{173}$$

and the associated holonomy from a point $x \in v^*$ to a point $y \in v^*$ is given by

$$\overrightarrow{\exp}\int_x^y\mathbf{A} = h_v^{-1}(x)e^{\mathbf{M}_v(\phi_v(y)-\phi_v(x))}h_v(y), \tag{174}$$

while the holonomy between a point $x \in v^*$ and a point $y \in c$ in an adjacent cell $c$ is

$$\overrightarrow{\exp}\int_x^y\mathbf{A} = h_v^{-1}(x)e^{-\mathbf{M}_v\phi_v(x)}h_{vc}h_c(y). \tag{175}$$

Using this, we can evaluate in the holonomy of the curve shown in Fig. 7. This curve goes from a point $x \in c$ to a point $y \in c'$, where both cells $c$, $c'$ are around $v$. By the flatness condition we can evaluate the holonomy from $x$ to $y$ in two ways, and we get

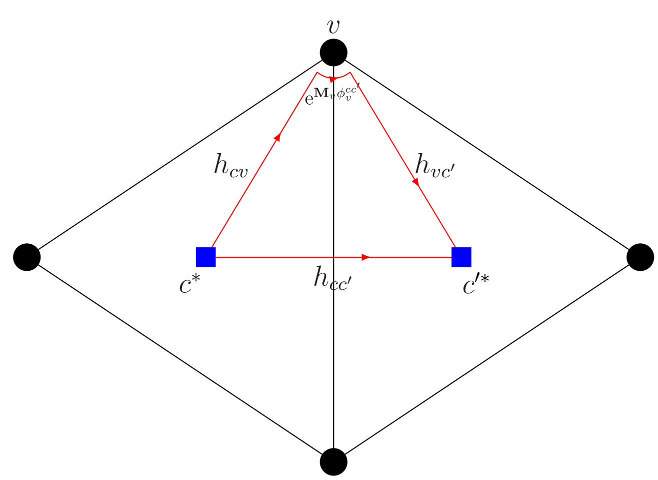


FIG. 7. The holonomy from $c^*$ to $c'^*$ going either directly or through the vertex $v$.

$$\overrightarrow{\exp}\int_x^y \mathbf{A} = h_v^{-1}(x)e^{\mathbf{M}_v\phi_v^{cc'}}h_v(y) = h_v^{-1}(x)h_{vc}h_{cc'}h_{c'v}h_v(y), \tag{176}$$

where we denoted $\phi_v^{cc'} \equiv \phi_v(y) - \phi_v(x)$. We conclude that the flatness of the connection outside the cells implies the following relationship among the discrete holonomies:

$$h_{cc'} = h_{cv}e^{\mathbf{M}_v\phi_v^{cc'}}h_{vc'}. \tag{177}$$

In particular, let the vertex $v$ be surrounded by $N$ cells $c_1, \ldots, c_N$, and take $c_1 \equiv c_{N+1} \equiv c$. If we form a loop of discrete holonomies going from each cell to the next, we find

$$h_{cc_2}\cdots h_{c_N c} = h_{cv}e^{\mathbf{M}_v}h_{vc}, \tag{178}$$

since $\phi_v^{cc_2} + \cdots + \phi_v^{c_n c} = 1$. By moving all of the terms to the left-hand side, we obtain the curvature constraint:

$$F_v \equiv h_{vc}h_{cc_2}\cdots h_{c_N c}h_{cv}e^{-\mathbf{M}_v} = 1. \tag{179}$$

### C. Generators of symmetries

In conclusion, we have obtained two Gauss constraints on the cells $c$ and disks $v^*$,

$$\mathbf{G}_c \equiv \sum_{c'\ni c}\mathbf{X}_c^{c'} - \sum_{v\ni c}\mathbf{X}_c^v = 0, \tag{180}$$

$$\mathbf{G}_v \equiv \mathbf{S}_v + e^{\mathbf{M}_v(\alpha_v - \frac{1}{2})}(e^{\mathbf{M}_v}\mathbf{y}_v(v_0)e^{-\mathbf{M}_v} - \mathbf{y}_v(v_0))e^{-\mathbf{M}_v(\alpha_v-\frac{1}{2})} - \sum_{c\in v}\mathbf{X}_v^c = 0, \tag{181}$$

and a curvature constraint at the vertex $v$,

$$F_v \equiv h_{vc}h_{cc_2}\cdots h_{c_N c}h_{cv}e^{-\mathbf{M}_v} = 1. \tag{182}$$

Note that all of these constraints are satisfied identically in our construction, as shown above. We will now see that the Gauss constraint generates the rotational part of the left translation symmetry transformation given by (75) and (84). In order to do so, we look for transformations $(\delta_{\boldsymbol{\beta}_c}, \delta_{\boldsymbol{\beta}_v}, \delta_{\mathbf{x}_v})$ such that

$$I_{\delta_{\boldsymbol{\beta}_c}}\Omega = -\boldsymbol{\beta}_c\cdot\delta\mathbf{G}_c, \qquad I_{\delta_{\boldsymbol{\beta}_v}}\Omega = -\boldsymbol{\beta}_v\cdot\delta\mathbf{G}_v,$$
$$I_{\delta_{\mathbf{x}_v}}\Omega = -\mathbf{x}_v\cdot\boldsymbol{\Delta}F_c. \tag{183}$$

We will need the explicit expression for the total symplectic form $\Omega \equiv \delta\Theta$:

$$\Omega = \sum_{(cc')}\Omega_{cc'} - \sum_{(vc)}\Omega_{vc} + \sum_v \Omega_v, \tag{184}$$

where the arc and edge contributions are

$$\Omega_{cc'} \equiv \frac{1}{2}[\boldsymbol{\Delta}h_c^{c'}, \boldsymbol{\Delta}h_c^{c'}]\cdot\mathbf{X}_c^{c'} - \boldsymbol{\Delta}h_c^{c'}\cdot\delta\mathbf{X}_c^{c'},$$
$$\Omega_{vc} \equiv \frac{1}{2}[\boldsymbol{\Delta}h_v^c, \boldsymbol{\Delta}h_v^c]\cdot\mathbf{X}_v^c - \boldsymbol{\Delta}h_v^c\cdot\delta\mathbf{X}_v^c,$$

while the vertex contribution is

$$\Omega_v \equiv \delta\mathbf{X}_v\cdot\delta\mathbf{M}_v - (\delta\mathbf{S}_v + [\delta\mathbf{M}_v, \mathbf{X}_v] + [\mathbf{M}_v, \delta\mathbf{X}_v])\cdot\boldsymbol{\Delta}(h_v(v)) + -\frac{1}{2}(\mathbf{S}_v + [\mathbf{M}_v, \mathbf{X}_v])\cdot[\boldsymbol{\Delta}(h_v(v)), \boldsymbol{\Delta}(h_v(v))].$$

#### *1. The Gauss constraint at the nodes*

Consider the infinitesimal version of the symmetry transformation (75) acting on $h_{cc'}$, $h_{cv}$, $\mathbf{y}_c$ and $\mathbf{y}_v$, with $g_c \equiv e^{\boldsymbol{\beta}_c}$:

$$\delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}h_{cc'} = \boldsymbol{\beta}_c h_{cc'}, \qquad \delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}h_{cv} = \boldsymbol{\beta}_c h_{cv},$$
$$\delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}\mathbf{y}_c = \mathbf{z}_c + [\boldsymbol{\beta}_c, \mathbf{y}_c], \qquad \delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}\mathbf{y}_v = 0. \tag{185}$$

From (148) and (152), we see that the fluxes $\mathbf{X}_c^{c'}$ and $\mathbf{X}_c^v$ transform as follows:

$$\delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}\mathbf{X}_c^{c'} = [\boldsymbol{\beta}_c, \mathbf{X}_c^{c'}], \qquad \delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}\mathbf{X}_c^v = [\boldsymbol{\beta}_c, \mathbf{X}_c^v]. \tag{186}$$

Note that the translation parameter $\mathbf{z}_c$ cancels out, so this transformation is in fact a pure rotation. Also note that this transformation only affects holonomies and fluxes which involve the specific cell $c$ with respect to which we are performing the transformation.

Applying the transformation to $\Omega$, we find

$$I_{\delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)}}\Omega = -\boldsymbol{\beta}_c\cdot\delta\left(\sum_{c'\ni c}\mathbf{X}_c^{c'} - \sum_{v\ni c}\mathbf{X}_c^v\right) = -\boldsymbol{\beta}_c\cdot\delta\mathbf{G}_c, \tag{187}$$

and thus we have proven that the cell Gauss constraint $\mathbf{G}_c$ generates rotations at the nodes. Since the translation parameter $\mathbf{z}_c$ cancels out in this calculation, the translational edge mode symmetry is pure gauge.

#### *2. The Gauss constraint at the vertices*

In order to analyze the Gauss constraint at the vertex one first has to recognize that only the part of $\mathbf{G}_v$ along the Cartan subalgebra is a constraint. The part orthogonal to it can simply be viewed as a definition of $\mathbf{y}_v^{\perp}(v_0)$, as emphasized earlier. Therefore, at the vertex, the only constraint is $\boldsymbol{\beta}_v\cdot\mathbf{G}_v = 0$ for $\beta_v \in \mathfrak{h}$.

We consider the infinitesimal version of the symmetry transformation (84) acting on $h_c$, $h_v$, $\mathbf{y}_c$ and $\mathbf{y}_v$, with $g_v \equiv e^{\boldsymbol{\beta}_v}$:

$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} h_c = 0, \qquad \delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} h_v = \boldsymbol{\beta}_v h_v, \qquad \delta_{(\mathbf{z}_c,\boldsymbol{\beta}_c)} \mathbf{y}_c = 0,$$
$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} \mathbf{y}_v = \mathbf{z}_v + [\boldsymbol{\beta}_v, \mathbf{y}_v]. \tag{188}$$

Note that $\mathbf{z}_v$ and $\boldsymbol{\beta}_v$ are both in the Cartan subalgebra. From (96), (148) and (149), we see that the holonomies $h_{cc'}$, $h_{vc}$, and $h_v(v)$ and the fluxes $\mathbf{X}_c^{c'}$, $\mathbf{X}_v^c$ and $\mathbf{X}_v$ transform as follows:

$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} h_{cc'} = 0, \qquad \delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} h_{vc} = \boldsymbol{\beta}_v h_{vc},$$
$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} h_v(v) = \boldsymbol{\beta}_v h_v(v), \tag{189}$$

$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} \mathbf{X}_c^{c'} = 0, \qquad \delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} \mathbf{X}_v^c = [\boldsymbol{\beta}_v, \mathbf{X}_v^c],$$
$$\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)} \mathbf{X}_v = [\boldsymbol{\beta}_v, \mathbf{X}_v], \tag{190}$$

where we used the fact that $\mathbf{z}_v$ and $\boldsymbol{\beta}_v$ commute with $\mathbf{M}_v$ and $\mathbf{S}_v$. Again, the translation parameter $\mathbf{z}_v$ cancels out, so the transformation is in fact a pure rotation on the holonomies and fluxes. Also note that this transformation only affects holonomies and fluxes which involve the specific vertex $v$ with respect to which we are performing the transformation.

Applying the transformation to $\Omega$, we find

$$I_{\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)}} \Omega = -\boldsymbol{\beta}_v \cdot \delta\left(\mathbf{S}_v - \sum_{c\in v} \mathbf{X}_v^c\right), \tag{191}$$

where we have again used the fact that $\mathbf{z}_v$ and $\boldsymbol{\beta}_v$ commute with $\mathbf{M}_v$ and $\mathbf{S}_v$, as well as the Jacobi identity. Importantly, since $[\boldsymbol{\beta}_v, e^{\mathbf{M}_v}] = 0$ and $\delta\boldsymbol{\beta}_v = 0$, we have that

$$\boldsymbol{\beta}_v \cdot \delta(e^{\mathbf{M}_v} \mathbf{y}_{v_0}(v) e^{-\mathbf{M}_v} - \mathbf{y}_{v_0}(v)) = 0, \tag{192}$$

where we used the fact that the trace in the definition of the dot product is cyclic. Therefore we see that, in fact,

$$I_{\delta_{(\mathbf{z}_v,\boldsymbol{\beta}_v)}} \Omega = -\boldsymbol{\beta}_v \cdot \delta \mathbf{G}_v. \tag{193}$$

Thus, we have proven that the disk Gauss constraint $\mathbf{G}_v$ generates rotations at the vertices. Again, since the translation parameter $\mathbf{z}_v$ cancels out in this calculation, the edge mode translations at the vertices are pure gauge.

### 3. The curvature constraint

We are left with the translation symmetry generated by the curvature constraint (182). As before, let the vertex $v$ be surrounded by $N$ cells $c_1, \ldots, c_N$, and take $c_1 \equiv c_{N+1} \equiv c$. Then we define

$$H_{vc} \equiv h_{vc} h_{cc_2} \cdots h_{c_N c}, \qquad H_{cv} \equiv H_{vc}^{-1}, \tag{194}$$

such that (182) becomes $F_v = H_{vc} h_{cv} e^{-\mathbf{M}_v} = 1$. Now, let $\mathbf{x}_v$ be a transformation parameter, and let us consider the transformation

$$\delta_{\mathbf{x}_v} \mathbf{X}_c^{c'} = -H_{cv} \mathbf{x}_v H_{vc}, \qquad \delta_{\mathbf{x}_v} \mathbf{X}_v^c = \mathbf{x}_v - e^{-\mathbf{M}_v} \mathbf{x}_v e^{\mathbf{M}_v},$$
$$\delta_{\mathbf{x}_v} \mathbf{X}_v = -\mathbf{x}_v^{\parallel}, \tag{195}$$

where we denoted

$$\mathbf{x}_v^{\parallel} \equiv \left(\frac{\mathbf{x}_v \cdot \mathbf{M}_v}{\mathbf{M}_v^2}\right) \mathbf{M}_v, \tag{196}$$

and where the only fluxes affected are $\mathbf{X}_v^c$, $\mathbf{X}_v$ and $\mathbf{X}_c^{c'}$ corresponding to links surrounding the vertex $v$. Then we find that

$$I_{\delta_{\mathbf{x}_v}} \Omega_{cc'} = -\mathbf{x}_v \cdot H_{vc} \Delta h_c^{c'} H_{cv}, \qquad I_{\delta_{\mathbf{x}_v}} \Omega_v = -\mathbf{x}_v \cdot \delta \mathbf{M}_v, \tag{197}$$

$$I_{\delta_{\mathbf{x}_v}} \Omega_{vc} = \mathbf{x}_v \cdot (\boldsymbol{\Delta} h_v^c - e^{\mathbf{M}_v} \Delta h_v^c e^{-\mathbf{M}_v}), \tag{198}$$

and the total symplectic form transforms as

$$\begin{aligned} I_{\delta_{\mathbf{x}_v}} \Omega &= -\mathbf{x}_v \cdot \left( \sum_{i=1}^{N} H_{vc_i} \Delta h_{c_i}^{c_{i+1}} H_{c_i v} \right. \\ &\qquad \left. + \boldsymbol{\Delta} h_v^c - e^{\mathbf{M}_v} \Delta h_v^c e^{-\mathbf{M}_v} - \delta \mathbf{M}_v \right) \\ &= -\mathbf{x}_v \cdot \boldsymbol{\Delta} F_v, \end{aligned}$$

which shows that $\delta_{\mathbf{x}_v}$ is the translation symmetry associated with $F_v$. Quite remarkably, this symmetry can be used to simplify the expression for $\mathbf{X}_v$ and the vertex Gauss constraint. Let us consider the transformation parameter

$$\mathbf{x}_v \equiv -\frac{1}{N_v} (e^{\mathbf{M}_v(\alpha_v + \frac{1}{2})} \mathbf{y}_v^{\perp}(v_0) e^{-\mathbf{M}_v(\alpha_v + \frac{1}{2})}) + \mathbf{y}_v^{\parallel}(v_0), \tag{199}$$

where $N_v$ is the number of cells around $v$. Note that

$$\mathbf{x}_v - e^{-\mathbf{M}_v} \mathbf{x}_v e^{\mathbf{M}_v} = \mathbf{x}_v^{\perp} - e^{-\mathbf{M}_v} \mathbf{x}_v^{\perp} e^{\mathbf{M}_v}, \tag{200}$$

since the parallel part $\mathbf{x}_v^{\parallel}$ commutes with $\mathbf{M}_v$. We define the new set of fluxes $\tilde{\mathbf{X}} \equiv \mathbf{X} + \delta_{\mathbf{x}_v} \mathbf{X}$. Then from the expressions (140) and (170), we can see that the transformed fluxes at the vertices and arcs satisfy the simpler relations

$$\tilde{\mathbf{X}}_v = \mathbf{y}_v(v) - \alpha_v \mathbf{S}_v, \qquad \sum_{c \ni v} \tilde{\mathbf{X}}_v^c = \mathbf{S}_v. \tag{201}$$

Taking $\alpha_v = 0$, which we can always do due to the symmetry transformation (98), we may simplify even more and obtain simply $\tilde{\mathbf{X}}_v = \mathbf{y}_v(v)$.

One has to beware that the transformed fluxes $\tilde{\mathbf{X}}_c^{c'}$ on the links no longer satisfy the inverse relations (153): $\tilde{\mathbf{X}}_c^{c'} \neq -h_{cc'}\tilde{\mathbf{X}}_{c'}^{c}h_{c'c}$, since the expressions on both sides of this equation now involve the differences between $\mathbf{x}_v$ and $\mathbf{x}_{v'}$.

## VI. SUMMARY AND OUTLOOK

Two types of transformations acting on the group elements arise in our discretization. Right translations $\mathcal{H} \mapsto \mathcal{H}\mathcal{G}$, with $\mathcal{G}$ a group-valued 0-form, correspond to the familiar gauge transformation of the connection. However, left translations $\mathcal{H} \mapsto \mathcal{G}\mathcal{H}$, with $\mathcal{G}$ constant and possibly restricted to the Cartan subgroup, are instead a symmetry which leaves the connection invariant.

The appearance of gauge-invariant observables that transform nontrivially under a new global symmetry is understood in the continuum as the appearance of new *edge mode* d.o.f. related to the presence of boundaries appearing in the subdivision of a large gauge system into subsystems [8,14]. This point is usually overlooked and it is often assumed that we can work in "a gauge" where we fixed the elements $\mathcal{H}_c(c^*)$ and $\mathcal{H}_v(v)$ to the identity.

What this procedure often ignores is the fact that gauge transformations are transformations in the kernel of the symplectic structure, while symmetries are not. So before we can postulate that we can "gauge fix" a symmetry, we have to ensure that the variable we are gauge fixing does not possess a conjugate variable. Otherwise, "gauge" fixing would imply reducing the number of d.o.f. This means that we cannot decide beforehand if an edge mode variable can be dismissed as a gauge variable—not until we understand its role in the canonical structure.

We have established, from first principles, a connection between the continuous $2+1$ gravity phase space, constrained to be flat and torsionless outside defects, and the loop gravity phase space. We have shown that the vertices carry additional d.o.f., behaving as a collection of relativistic particles coupled to gravity.

This provides a new picture, where pure $2+1$ gravity can be equivalently described using defects carrying "Poincaré" charges. This opens a new perspective on how one can address the continuum limit, similar to the one proposed in [21], and opens up the possibility to directly connect the holonomy-flux variables with the torsion and curvature holonomies measuring the presence of these defects.

This work opens many new directions to explore. One of the main challenges is to develop a similar picture in the $3+1$-dimensional case. We expect that it is possible to rewrite the $3+1$-dimensional loop gravity phase space in terms of pointlike defects. Note that, in this case, the defects are expected to be of codimension 3 instead of codimension 2. The proof of this conjecture is left for future work [34].

Finally, it is interesting to revisit the choice of polarization that we made when constructing the symplectic potential. Another natural choice is the dual loop polarization introduced in [6]. In that case, one expects the piecewise-flat geometry phase space to reduce to that of (the classical version of) group networks [9]. In this dual picture, holonomies and fluxes switch places: the fluxes are on the links and the holonomies are on their dual edges. This dual formulation will be discussed in a companion paper, in preparation [40].

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## APPENDIX A: THE MAURER-CARTAN FORM ON FIELD SPACE

We define the Maurer-Cartan form on field space as follows:

$$\mathbf{\Delta} g \equiv \delta g g^{-1}, \tag{A1}$$

where $g$ is a Lie-group-valued 0-form and $\delta$ is the variation, or differential on field space. Note the bold font on $\Delta$, denoting that $\Delta g$ is valued in the corresponding Lie algebra.

Seen as an operator, $\Delta$ satisfies the "Leibniz rule"

$$\Delta(hg) = \mathbf{\Delta} h + h(\mathbf{\Delta} g)h^{-1}, \tag{A2}$$

and the inversion rule

$$\mathbf{\Delta}(g^{-1}) = -g^{-1}(\mathbf{\Delta} g)g. \tag{A3}$$

Combining them together, we get the useful identity

$$\mathbf{\Delta}(h^{-1}g) = h^{-1}(\mathbf{\Delta} g - \mathbf{\Delta} h)h. \tag{A4}$$

Also, it is easy to see that

$$\delta(\mathbf{\Delta} g) = \frac{1}{2}[\mathbf{\Delta} g, \mathbf{\Delta} g]. \tag{A5}$$

Sometimes we will denote a holonomy as, e.g., $h_{cc'}$ where $c$ is the source cell and $c'$ is the target cell for parallel transport. In this case, our notational convention ensures that the subscripts are always compatible with adjacent subscripts, as in $h_{c'c}\mathrm{d}\mathbf{y}_c^{c'}h_{cc'}$ for example. However, since we have $\mathbf{\Delta} h_{cc'} = \delta h_{cc'}h_{c'c}$, the proper subscript for this expression is $c$, since it is located at $c$, while $c'$ is just an internal point which the two holonomies happen to pass through. We will thus employ, in such cases, the more appropriate notation

$$\mathbf{\Delta} h_c^{c'} \equiv \mathbf{\Delta}(h_{cc'}), \tag{A6}$$

where the internal point $c'$ is now a superscript, much like in the notation for $\mathbf{y}_c^{c'}$; indeed, the latter was employed for a similar reason.

## APPENDIX B: EVALUATION OF THE CHERN-SIMONS SYMPLECTIC POTENTIAL

The connection $\boldsymbol{\mathcal{A}}$ inside the disk $D$ is

$$\boldsymbol{\mathcal{A}} = \mathcal{H}^{-1}\boldsymbol{\mathcal{L}}\mathcal{H} + \mathcal{H}^{-1}\mathrm{d}\mathcal{H}, \tag{B1}$$

where $\boldsymbol{\mathcal{L}}$ is the Lagrangian connection such that $\boldsymbol{\mathcal{L}} \cdot \boldsymbol{\mathcal{L}} = 0$ and $\mathcal{H}$ is a $G$-valued 0-form. Its variation is

$$\delta\boldsymbol{\mathcal{A}} = \mathcal{H}^{-1}(\delta\boldsymbol{\mathcal{L}} + \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H})\mathcal{H}, \tag{B2}$$

where $\mathrm{d}_{\boldsymbol{\mathcal{L}}}$ denotes the covariant differential $\mathrm{d}_{\boldsymbol{\mathcal{L}}} \equiv \mathrm{d} + [\boldsymbol{\mathcal{L}}, \cdot]$, and we have used the shorthand notation $\mathbf{\Delta}\mathcal{H} \equiv \delta\mathcal{H}\mathcal{H}^{-1}$ for the Maurer-Cartan form, introduced in Appendix A.

Let us evaluate the Chern-Simons symplectic form. We have

$$\omega(\boldsymbol{\mathcal{A}}) = 2\delta\boldsymbol{\mathcal{L}} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} + \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H}, \tag{B3}$$

where we used the fact that $\delta\boldsymbol{\mathcal{L}} \cdot \delta\boldsymbol{\mathcal{L}} = 0$. Now, the curvature associated to $\boldsymbol{\mathcal{L}}$ is $\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \equiv \mathrm{d}\boldsymbol{\mathcal{L}} + \frac{1}{2}[\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{L}}]$, its variation is $\delta\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) = \mathrm{d}_{\boldsymbol{\mathcal{L}}}\delta\boldsymbol{\mathcal{L}}$, and it satisfies the Bianchi identity $\mathrm{d}_{\boldsymbol{\mathcal{L}}}^2 = [\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}), ]$. Using the graded Leibniz rule, we find

$$\delta\boldsymbol{\mathcal{L}} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} = \delta\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot \mathbf{\Delta}\mathcal{H} - \mathrm{d}(\delta\boldsymbol{\mathcal{L}} \cdot \mathbf{\Delta}\mathcal{H}). \tag{B4}$$

In addition, we have on the one hand

$$\mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H}) = \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H}) + \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{L}}, \mathbf{\Delta}\mathcal{H}]), \tag{B5}$$

and on the other hand

$$\mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H}) = \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} + \mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}), \mathbf{\Delta}\mathcal{H}], \tag{B6}$$

and thus

$$\mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}_{\boldsymbol{\mathcal{L}}}\mathbf{\Delta}\mathcal{H} = \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H}) + \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{L}}, \mathbf{\Delta}\mathcal{H}]) - \mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}), \mathbf{\Delta}\mathcal{H}]. \tag{B7}$$

Plugging in, we get

$$\omega(\boldsymbol{\mathcal{A}}) = 2\delta\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot \mathbf{\Delta}\mathcal{H} - 2\mathrm{d}(\delta\boldsymbol{\mathcal{L}} \cdot \mathbf{\Delta}\mathcal{H}) + \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H}) + \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{L}}, \mathbf{\Delta}\mathcal{H}]) - \mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}), \mathbf{\Delta}\mathcal{H}]. \tag{B8}$$

Now, from (A5) we know that

$$\delta\mathbf{\Delta}\mathcal{H} = \frac{1}{2}[\mathbf{\Delta}\mathcal{H}, \mathbf{\Delta}\mathcal{H}], \tag{B9}$$

and thus, this time using the graded Leibniz rule on field space, we find

$$\delta\boldsymbol{\mathcal{L}} \cdot \mathbf{\Delta}\mathcal{H} = \delta(\boldsymbol{\mathcal{L}} \cdot \mathbf{\Delta}\mathcal{H}) - \frac{1}{2}\boldsymbol{\mathcal{L}} \cdot [\mathbf{\Delta}\mathcal{H}, \mathbf{\Delta}\mathcal{H}], \tag{B10}$$

$$\delta\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot \mathbf{\Delta}\mathcal{H} = \delta(\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot \mathbf{\Delta}\mathcal{H}) - \frac{1}{2}\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot [\mathbf{\Delta}\mathcal{H}, \mathbf{\Delta}\mathcal{H}]. \tag{B11}$$

Plugging into $\omega(\boldsymbol{\mathcal{A}})$, the triple products all cancel[17] and we get the general expression

$$\omega(\boldsymbol{\mathcal{A}}) = 2\delta(\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot \mathbf{\Delta}\mathcal{H}) + \mathrm{d}(\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H}) - 2\mathrm{d}\delta(\boldsymbol{\mathcal{L}} \cdot \mathbf{\Delta}\mathcal{H}). \tag{B12}$$

Next, we specialize to the case where the Lagrangian connection $\boldsymbol{\mathcal{L}}$ is an Abelian connection of the form considered above:

$$\boldsymbol{\mathcal{L}} \equiv \boldsymbol{\mathcal{M}}\mathrm{d}\phi, \qquad \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) = \boldsymbol{\mathcal{M}}\delta(v), \qquad \boldsymbol{\mathcal{M}} \in \mathfrak{d}\mathfrak{h}, \tag{B13}$$

where $v$ is the center of the disk $D$. Integrating (B12) over the disk, we get

$$\Omega_D(\boldsymbol{\mathcal{A}}) \equiv \int_D \omega(\boldsymbol{\mathcal{A}}) = 2\delta(\boldsymbol{\mathcal{M}} \cdot \mathbf{\Delta}\mathcal{H}(v)) + \oint_{\partial D} (\mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H} - 2\delta(\boldsymbol{\mathcal{M}} \cdot \mathbf{\Delta}\mathcal{H})\mathrm{d}\phi). \tag{B14}$$

Using the useful identity (A4) we find

$$\mathbf{\Delta}\mathcal{H} = \mathcal{H}(v)\mathbf{\Delta}(\mathcal{H}(v)^{-1}\mathcal{H})\mathcal{H}(v)^{-1} + \mathbf{\Delta}\mathcal{H}(v), \tag{B15}$$

which allows us to write

$$\Omega_D(\boldsymbol{\mathcal{A}}) = 2\delta(\boldsymbol{\mathcal{M}} \cdot \mathbf{\Delta}\mathcal{H}(v)) + \oint_{\partial D} \mathbf{\Delta}\mathcal{H} \cdot \mathrm{d}\mathbf{\Delta}\mathcal{H} + \\ - 2\oint_{\partial D} \delta(\mathcal{H}(v)^{-1}\boldsymbol{\mathcal{M}}\mathcal{H}(v) \cdot \mathbf{\Delta}(\mathcal{H}(v)^{-1}\mathcal{H}) + \boldsymbol{\mathcal{M}} \cdot \mathbf{\Delta}\mathcal{H}(v))\mathrm{d}\phi.$$

[17] Since $\mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{L}}, \mathbf{\Delta}\mathcal{H}] = -\boldsymbol{\mathcal{L}} \cdot [\mathbf{\Delta}\mathcal{H}, \mathbf{\Delta}\mathcal{H}]$ and $\mathbf{\Delta}\mathcal{H} \cdot [\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}), \mathbf{\Delta}\mathcal{H}] = -\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{L}}) \cdot [\mathbf{\Delta}\mathcal{H}, \mathbf{\Delta}\mathcal{H}]$ due to graded commutativity on field space.

Since $\delta(\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\Delta}\mathcal{H}(v))$ is a constant with respect to $\phi$, the second term in the second integral becomes trivial and we see that it exactly cancels the first term (recall that $\int \mathrm{d}\phi = 1$). We are thus left with

$$\Omega_D(\boldsymbol{\mathcal{A}}) = \oint_{\partial D} \Delta\mathcal{H}\cdot \mathrm{d}\boldsymbol{\Delta}\mathcal{H} - 2\oint_{\partial D} \delta(\mathcal{H}(v)^{-1}\boldsymbol{\mathcal{M}}\mathcal{H}(v)\cdot\boldsymbol{\Delta}(\mathcal{H}(v)^{-1}\mathcal{H}))\mathrm{d}\phi. \tag{B16}$$

Next, we introduce the DG-valued 0-form $\tilde{\mathcal{H}}(x)\equiv \mathcal{H}(v)^{-1}\mathcal{H}(x)$, which satisfies $\tilde{\mathcal{H}}(v)=1$. Then we can write the symplectic form as

$$\Omega_D(\boldsymbol{\mathcal{A}}) = \oint_{\partial D} \Delta(\mathcal{H}(v)\tilde{\mathcal{H}})\cdot \mathrm{d}\boldsymbol{\Delta}(\mathcal{H}(v)\tilde{\mathcal{H}}) - 2\oint_{\partial D} \delta(\mathcal{H}(v)^{-1}\boldsymbol{\mathcal{M}}\mathcal{H}(v)\cdot\boldsymbol{\Delta}\tilde{\mathcal{H}})\mathrm{d}\phi. \tag{B17}$$

Finally, from (A2) and (A4) we see that

$$\begin{aligned}\boldsymbol{\Delta}(\mathcal{H}(v)\tilde{\mathcal{H}}) &= \mathcal{H}(v)(\boldsymbol{\Delta}\tilde{\mathcal{H}} - \boldsymbol{\Delta}(\mathcal{H}(v)^{-1}))\mathcal{H}(v)^{-1},\\ \mathrm{d}\boldsymbol{\Delta}(\mathcal{H}(v)\tilde{\mathcal{H}}) &= \mathcal{H}(v)\mathrm{d}\boldsymbol{\Delta}\tilde{\mathcal{H}}\mathcal{H}(v)^{-1},\end{aligned} \tag{B18}$$

since $\mathcal{H}(v)$ is constant. Thus we obtain

$$\Omega_D(\boldsymbol{\mathcal{A}}) = \oint_{\partial D} \Delta\tilde{\mathcal{H}}\cdot \mathrm{d}\boldsymbol{\Delta}\tilde{\mathcal{H}} - \oint_{\partial D}\mathrm{d}(\boldsymbol{\Delta}(\mathcal{H}(v)^{-1})\cdot\boldsymbol{\Delta}\tilde{\mathcal{H}}) - 2\oint_{\partial D} \delta(\mathcal{H}(v)^{-1}\boldsymbol{\mathcal{M}}\mathcal{H}(v)\cdot\boldsymbol{\Delta}\tilde{\mathcal{H}})\mathrm{d}\phi. \tag{B19}$$

However, the second term vanishes since $\tilde{\mathcal{H}}$ is a periodic function on the circle $\partial D$, and thus we obtain the final expression:

$$\Omega_D(\boldsymbol{\mathcal{A}}) = \oint_{\partial D} \Delta\tilde{\mathcal{H}}\cdot \mathrm{d}\boldsymbol{\Delta}\tilde{\mathcal{H}} - 2\oint_{\partial D} \delta(\mathcal{H}(v)^{-1}\boldsymbol{\mathcal{M}}\mathcal{H}(v)\cdot\boldsymbol{\Delta}\tilde{\mathcal{H}})\mathrm{d}\phi. \tag{B20}$$

## APPENDIX C: THE RELATIVISTIC PARTICLE

In this Appendix we study the relativistic particle symplectic potential

$$\Theta \equiv \mathbf{X}\cdot\delta\mathbf{M} - (\mathbf{S}+[\mathbf{M},\mathbf{X}])\cdot\boldsymbol{\Delta}h, \tag{C1}$$

with the symplectic form

$$\Omega\equiv\delta\Theta = \delta\mathbf{X}\cdot\delta\mathbf{M} - (\delta\mathbf{S}+[\delta\mathbf{M},\mathbf{X}]+[\mathbf{M},\delta\mathbf{X}])\cdot\boldsymbol{\Delta}h - \frac{1}{2}(\mathbf{S}+[\mathbf{M},\mathbf{X}])\cdot[\boldsymbol{\Delta}h,\boldsymbol{\Delta}h]. \tag{C2}$$

We define the momentum $\mathbf{p}$ and angular momentum $\mathbf{j}$ of the particle:

$$\mathbf{p}\equiv h^{-1}\mathbf{M}h\in\mathfrak{g}^*,\qquad \mathbf{j}\equiv h^{-1}(\mathbf{S}+[\mathbf{M},\mathbf{X}])h\in\mathfrak{g}, \tag{C3}$$

which have the variational differentials

$$\delta\mathbf{p} = h^{-1}(\delta\mathbf{M}+[\mathbf{M},\boldsymbol{\Delta}h])h, \tag{C4}$$

$$\delta\mathbf{j} = h^{-1}(\delta\mathbf{S}+[\delta\mathbf{M},\mathbf{X}]+[\mathbf{M},\delta\mathbf{X}]+[\mathbf{S}+[\mathbf{M},\mathbf{X}],\boldsymbol{\Delta}h])h. \tag{C5}$$

We also define the “position”

$$\mathbf{q}\equiv h^{-1}\mathbf{X}h\in\mathfrak{g}, \tag{C6}$$

in terms of which the symplectic potential may be written as

$$\Theta = \mathbf{q}\cdot\delta\mathbf{p} - \mathbf{S}\cdot\boldsymbol{\Delta}h. \tag{C7}$$

### 1. Right translations (gauge transformations)

Let

$$\begin{aligned}&\mathcal{H}\equiv(\mathbf{X},h) = e^{\mathbf{X}}h\in\mathrm{DG},\\ &\mathcal{G}\equiv(g,\mathbf{x})\in\mathrm{DG}\Rightarrow \mathcal{H}\mathcal{G} = e^{\mathbf{X}+h\mathbf{x}h^{-1}}hg.\end{aligned} \tag{C8}$$

This is a right translation, with parameter $\mathcal{G}$, of the group element $\mathcal{H}$, which corresponds to a gauge transformation:

$$h\mapsto hg,\quad \mathbf{X}\mapsto\mathbf{X}+h\mathbf{x}h^{-1},\quad \mathbf{M}\mapsto\mathbf{M},\quad \mathbf{S}\mapsto\mathbf{S}. \tag{C9}$$

It is interesting to translate this action onto the physical variables ($\mathbf{p}$, $\mathbf{q}$, $\mathbf{j}$) which transform as

$$\mathbf{p}\to h^{-1}\mathbf{p}h,\qquad \mathbf{q}\to\mathbf{x}+h^{-1}\mathbf{q}h,\quad \mathbf{j}\to h^{-1}\mathbf{j}h. \tag{C10}$$

This shows that the parameter $h$ labels a rotation of the physical variables, while $\mathbf{x}$ labels a translation of the physical position $\mathbf{q}$. Taking $g\equiv e^{\boldsymbol{\alpha}}$, we may consider transformations labeled by $\mathbf{x}+\boldsymbol{\alpha}\in\mathfrak{dg}$ with $\mathbf{x}\in\mathfrak{g}^*$ a translation parameter and $\boldsymbol{\alpha}\in\mathfrak{g}$ a rotation parameter, given by the infinitesimal version of the gauge transformation:

$$\begin{aligned}&\delta_{(\mathbf{x},\boldsymbol{\alpha})}h = h\boldsymbol{\alpha},\qquad \delta_{(\mathbf{x},\boldsymbol{\alpha})}\mathbf{X} = h\mathbf{x}h^{-1},\qquad \delta_{(\mathbf{x},\boldsymbol{\alpha})}\mathbf{M}=0,\\ &\delta_{(\mathbf{x},\boldsymbol{\alpha})}\mathbf{S}=0.\end{aligned} \tag{C11}$$

Let $I$ denote the interior product on field space, associated with the variational exterior derivative $\delta$. Then one finds that this transformation is Hamiltonian:

$$I_{\delta_{(\mathbf{x},\boldsymbol{\alpha})}}\Omega = -\delta H_{(\mathbf{x},\boldsymbol{\alpha})},\qquad H_{(\mathbf{x},\boldsymbol{\alpha})}\equiv -(\mathbf{p}\cdot\mathbf{x}+\mathbf{j}\cdot\boldsymbol{\alpha}). \tag{C12}$$

This shows that the variable conjugated to $\mathbf{p}$ is the "position" $\mathbf{q} \equiv h^{-1}\mathbf{y}h$, while the angular momentum $\mathbf{j}$ generates right translations on $G$. The Poisson bracket between two such Hamiltonians is given by

$$\{H_{(\mathbf{x},\boldsymbol{\alpha})}, H_{(\mathbf{x}',\boldsymbol{\alpha}')}\} = \delta_{(\mathbf{x},\boldsymbol{\alpha})} H_{(\mathbf{x}',\boldsymbol{\alpha}')} = H_{([\boldsymbol{\alpha},\mathbf{x}']+[\mathbf{x},\boldsymbol{\alpha}'],[\boldsymbol{\alpha},\boldsymbol{\alpha}'])}, \tag{C13}$$

which reproduces, as expected, the symmetry algebra $\mathfrak{dg}$.

### 2. Left translations (symmetry transformations)

Similarly, let

$$\mathcal{H} \equiv (h, \mathbf{X}) \in \mathrm{DG}, \qquad \mathcal{G} \equiv (g, \mathbf{z}) \in \mathrm{DG} \Rightarrow \mathcal{G}\mathcal{H} = e^{\mathbf{z}+g\mathbf{X}g^{-1}} gh. \tag{C14}$$

This is a left translation, with parameter $\mathcal{G}$, of the group element $\mathcal{H}$, which corresponds to a symmetry that leaves the connection invariant:

$$h \mapsto gh, \qquad \mathbf{X} \mapsto \mathbf{z} + g\mathbf{X}g^{-1}, \qquad \mathbf{M} \mapsto g\mathbf{M}g^{-1}, \qquad \mathbf{S} \mapsto g(\mathbf{S} + [\mathbf{z}, \mathbf{M}])g^{-1}. \tag{C15}$$

Note that it commutes with the right translation. The infinitesimal transformation, with $g \equiv e^{\boldsymbol{\beta}}$, is

$$\delta_{(\mathbf{z},\boldsymbol{\beta})} h = \boldsymbol{\beta} h, \quad \delta_{(\mathbf{z},\boldsymbol{\beta})} \mathbf{X} = \mathbf{z} + [\boldsymbol{\beta}, \mathbf{X}], \quad \delta_{(\mathbf{z},\boldsymbol{\beta})} \mathbf{M} = [\boldsymbol{\beta}, \mathbf{M}], \quad \delta_{(\mathbf{z},\boldsymbol{\beta})} \mathbf{S} = [\boldsymbol{\beta}, \mathbf{S}] + [\mathbf{z}, \mathbf{M}]. \tag{C16}$$

Once again, we can prove that this transformation is Hamiltonian:

$$I_{\delta_{(\mathbf{z},\boldsymbol{\beta})}} \Omega = -\delta H_{(\mathbf{z},\boldsymbol{\beta})}, \quad H_{(\mathbf{z},\boldsymbol{\beta})} \equiv -(\mathbf{M}\cdot\mathbf{z} + \mathbf{S}\cdot\boldsymbol{\beta}). \tag{C17}$$

This follows from the fact that

$$\delta_{(\mathbf{z},\boldsymbol{\beta})}(\mathbf{S} + [\mathbf{M}, \mathbf{X}]) = [\boldsymbol{\beta}, \mathbf{S} + [\mathbf{M}, \mathbf{X}]], \tag{C18}$$

which implies that these transformations leave the momentum and angular momentum invariant: $\delta_{(\mathbf{z},\boldsymbol{\beta})}\mathbf{p} = 0 = \delta_{(\mathbf{z},\boldsymbol{\beta})}\mathbf{j}$.

### 3. Restriction to the Cartan subalgebra

In the case discussed in this paper, where $\mathbf{M} \in \mathfrak{h}^*$ and $\mathbf{S} \in \mathfrak{h}$ are in the Cartan subalgebra, we need to restrict the parameter of the left translation transformation to be in $\mathfrak{dh}$. A particular class of transformations of this type is when the parameter is itself a function of $\mathbf{M}$ and $\mathbf{S}$, which we shall denote $F(\mathbf{M}, \mathbf{S})$. One finds that the infinitesimal transformation

$$\delta_F h = \frac{\partial F}{\partial \mathbf{S}} h, \qquad \delta_F \mathbf{y} = \frac{\partial F}{\partial \mathbf{M}} + \left[\frac{\partial F}{\partial \mathbf{S}}, \mathbf{X}\right], \qquad \delta_F \mathbf{M} = 0, \qquad \delta_F \mathbf{S} = 0, \tag{C19}$$

is Hamiltonian:

$$I_{\delta_F} \Omega = -\delta H_F, \qquad H_F \equiv -F(\mathbf{M}, \mathbf{S}). \tag{C20}$$

In particular, taking

$$F(\mathbf{M}, \mathbf{S}) \equiv \frac{\xi}{2}\mathbf{M}^2 + \chi \mathbf{M}\cdot\mathbf{S}, \qquad \xi, \chi \in \mathbb{R}, \tag{C21}$$

we obtain the Hamiltonian transformation

$$\delta_F h = \mathbf{M}\chi h, \qquad \delta_F \mathbf{X} = \mathbf{M}\xi + (\mathbf{S} + [\mathbf{M}, \mathbf{X}])\chi, \qquad \delta_F \mathbf{M} = 0, \qquad \delta_F \mathbf{S} = 0, \tag{C22}$$

corresponding to (C16) with

$$\mathbf{z} = \frac{\partial F}{\partial \mathbf{M}} = \mathbf{M}\xi + \mathbf{S}\chi, \qquad \boldsymbol{\beta} = \frac{\partial F}{\partial \mathbf{S}} = \mathbf{M}\chi. \tag{C23}$$

This may be integrated to

$$h \mapsto e^{\mathbf{M}\chi} h, \qquad \mathbf{X} \mapsto e^{\mathbf{M}\chi}(\mathbf{M}\xi + \mathbf{S}\chi + \mathbf{X})e^{-\mathbf{M}\chi}, \qquad \mathbf{M} \mapsto \mathbf{M}, \qquad \mathbf{S} \mapsto \mathbf{S}. \tag{C24}$$

The Hamiltonians $\mathbf{M}^2$ and $\mathbf{M}\cdot\mathbf{S}$ represent the Casimir invariants of the algebra $\mathfrak{dg}$.

## APPENDIX D: A QUICKER DERIVATION OF THE SYMPLECTIC POTENTIAL

Using the nonperiodic variables

$$u_v \equiv e^{\mathbf{M}_v \phi_v} h_v, \quad \mathbf{w}_v \equiv e^{\mathbf{M}_v \phi_v}(\mathbf{y}_v + \mathbf{S}_v \phi_v)e^{-\mathbf{M}_v \phi_v}, \tag{D1}$$

which were defined in Sec. V A, we may perform the calculation of Sec. IV in a quicker and clearer way. The symplectic potential is given as before by

$$\Theta = \sum_c \Theta_c + \sum_v \Theta_{v^*}, \qquad \Theta_c \equiv -\int_{\tilde{c}} \mathbf{E}\cdot\delta\mathbf{A}, \qquad \Theta_{v^*} \equiv -\int_{v^*} \mathbf{E}\cdot\delta\mathbf{A}. \tag{D2}$$

On the cells, the calculation is the same, and we obtain as before

$$\Theta_c = \int_{\partial\tilde{c}} \mathrm{d}\mathbf{y}_c \cdot \boldsymbol{\Delta} h_c. \tag{D3}$$

On the disks, we have

$$\mathbf{A}|_{v^*} = u_v^{-1}\mathrm{d}u_v, \qquad \mathbf{E}|_{v^*} = u_v^{-1}\mathrm{d}\mathbf{w}_v u_v, \qquad \delta\mathbf{A}|_{v^*} = u_v^{-1}(\mathrm{d}\boldsymbol{\Delta} u_v)u_v, \tag{D4}$$

and thus

$$\Theta_{v^*} = -\int_{v^*} \mathbf{E}\cdot\delta\mathbf{A} = -\int_{v^*} \mathrm{d}\mathbf{w}_v \cdot \mathrm{d}\boldsymbol{\Delta} u_v = \int_{v^*} \mathrm{d}(\mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v) = \int_{\partial v^*} \mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v. \tag{D5}$$

The boundary of the punctured disk decomposes into $\partial v^* = \partial_0 v^* \cup \partial_R v^* \cup C_v$. Since our variables are now nonperiodic, we must also integrate on the cut $C_v$, which we did not need to do before. Thus

$$\Theta_{v^*} \equiv \Theta_{\partial_R v^*} - \Theta_{\partial_0 v^*} - \Theta_{C_v}, \tag{D6}$$

where

$$\Theta_{\partial_R v^*} \equiv \int_{\partial_R v^*} \mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v, \qquad \Theta_{\partial_0 v^*} \equiv \int_{\partial_0 v^*} \mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v,$$
$$\Theta_{C_v} \equiv \int_{\partial_0 v^*} \mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v. \tag{D7}$$

Now, we have

$$\mathrm{d}\mathbf{w}_v = e^{\mathbf{M}_v \phi_v}(\mathrm{d}\mathbf{y}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v])\mathrm{d}\phi_v) e^{-\mathbf{M}_v \phi_v}, \tag{D8}$$

and from (A2) we find

$$\boldsymbol{\Delta} u_v = \boldsymbol{\Delta}(e^{\mathbf{M}_v \phi_v} h_v) = e^{\mathbf{M}_v \phi_v}(\delta \mathbf{M}_v \phi_v + \boldsymbol{\Delta} h_v) e^{-\mathbf{M}_v \phi_v}. \tag{D9}$$

Thus we may write the integrand as

$$\mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v = \mathrm{d}\mathbf{y}_v \cdot (\delta \mathbf{M}_v \phi_v + \boldsymbol{\Delta} h_v) + (\mathbf{S}_v \cdot \delta \mathbf{M}_v \phi_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v]) \cdot \boldsymbol{\Delta} h_v)\mathrm{d}\phi_v. \tag{D10}$$

Integrating this over the inner boundary $\partial_0 v^*$ is easy, since the integrand is evaluated at the vertex $v$, and $\mathbf{y}_v(v)$ and $h_v(v)$ are constant with respect to $\phi_v$, by assumption. The integral is from $\phi_v = \alpha_v - 1/2$ to $\phi_v = \alpha_v + 1/2$, and we immediately get

$$\Theta_{\partial_0 v^*} = \alpha_v \mathbf{S}_v \cdot \delta \mathbf{M}_v + (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v(v)]) \cdot \boldsymbol{\Delta}(h_v(v)). \tag{D11}$$

On the cut $C_v$, we have contributions from both sides, one at $\phi_v = \alpha_v - 1/2$ and another at $\phi_v = \alpha_v + 1/2$, with opposite orientation. Since $\mathrm{d}\phi_v = 0$ on the cut, only the term $\mathrm{d}\mathbf{y}_v \cdot (\delta \mathbf{M}_v \phi_v + \boldsymbol{\Delta} h_v)$ contributes, and we get

$$\Theta_{C_v} = \int_{r=0}^{R} (\mathrm{d}\mathbf{y}_v \cdot (\delta \mathbf{M}_v \phi_v + \boldsymbol{\Delta} h_v)\Big|_{\phi_v = \alpha_v + 1/2} - \mathrm{d}\mathbf{y}_v \cdot (\delta \mathbf{M}_v \phi_v + \boldsymbol{\Delta} h_v)\Big|_{\phi_v = \alpha_v - 1/2}) = \int_{r=0}^{R} \mathrm{d}\mathbf{y}_v \cdot \delta \mathbf{M}_v = (\mathbf{y}_v(v_0) - \mathbf{y}_v(v)) \cdot \delta \mathbf{M}_v,$$

since $\mathbf{y}_v$ has the value $\mathbf{y}_v(v_0)$ at $r = R$ on the cut and $\mathbf{y}_v(v)$ at $r = 0$. The vertex symplectic potential is then obtained by defining $\Theta_v \equiv -(\Theta_{\partial_0 v^*} + \Theta_{C_v})$:

$$\Theta_v = (\mathbf{y}_v(v) - \mathbf{y}_v(v_0) - \alpha_v \mathbf{S}_v) \cdot \delta \mathbf{M}_v - (\mathbf{S}_v + [\mathbf{M}_v, \mathbf{y}_v(v)]) \cdot \boldsymbol{\Delta}(h_v(v)). \tag{D12}$$

In this way, we have immediately obtained $\Theta_v$ right from the beginning via the integration on the inner boundary and the cut, without ever having to invoke the continuity conditions or go through the trouble of collecting terms from different arcs later on, as we did in the main text (see Secs. IV C 2 and IV C 3).

To find the rest of the symplectic potential, we split it into contributions from the edges and arcs:

$$\Theta = \sum_{[cc']} \Theta_{cc'} + \sum_{(vc)} \Theta_{vc} + \sum_{v} \Theta_v, \tag{D13}$$

where

$$\Theta_{cc'} \equiv \int_{[cc']} (\mathrm{d}\mathbf{y}_c \cdot \boldsymbol{\Delta} h_c - \mathrm{d}\mathbf{y}_{c'} \cdot \boldsymbol{\Delta} h_{c'}),$$
$$\Theta_{vc} = \int_{(vc)} (\mathrm{d}\mathbf{w}_v \cdot \boldsymbol{\Delta} u_v - \mathrm{d}\mathbf{y}_c \cdot \boldsymbol{\Delta} h_c). \tag{D14}$$

For the edges, we use the continuity conditions as we did in Sec. IV C and get

$$\Theta_{cc'} = \boldsymbol{\Delta} h_c^{c'} \cdot \int_{[cc']} \mathrm{d}\mathbf{y}_c. \tag{D15}$$

For the arcs we may now use much simpler continuity conditions formulated in terms of $u_v$ and $\mathbf{w}_v$,

$$h_c(x) = h_{cv} u_v(x), \qquad \mathbf{y}_c(x) = h_{cv}(\mathbf{w}_v(x) - \mathbf{y}_v^c) h_{vc}, \quad x \in (vc), \tag{D16}$$

and thus we may simplify $\Theta_{vc}$ in exactly the same way as we did for $\Theta_{cc'}$, and we immediately obtain

$$\Theta_{(vc)} = \boldsymbol{\Delta} h_v^c \cdot \int_{(vc)} \mathrm{d}\mathbf{w}_v. \tag{D17}$$

We have thus reproduced the result (145) without having to go through the many steps we took in the main text. However, the calculation in the main text is more explicit and thus leaves less room for error; the fact that we have obtained the same result in both calculations is a good consistency check.

[1] C. Rovelli, *Quantum Gravity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004).
[2] B. Dittrich and J. P. Ryan, Phase space descriptions for simplicial 4D geometries, Classical Quantum Gravity **28**, 065006 (2011).
[3] L. Freidel and S. Speziale, Twisted geometries: A geometric parametrisation of SU(2) phase space, Phys. Rev. D **82**, 084040 (2010).
[4] L. Freidel, M. Geiller, and J. Ziprick, Continuous formulation of the loop quantum gravity phase space, Classical Quantum Gravity **30**, 085013 (2013).
[5] L. Freidel and J. Ziprick, Spinning geometry = twisted geometry, Classical Quantum Gravity **31**, 045007 (2014).
[6] M. Dupuis, L. Freidel, and F. Girelli, Discretization of 3D gravity in different polarizations, Phys. Rev. D **96**, 086017 (2017).
[7] B. Dittrich and M. Geiller, A new vacuum for loop quantum gravity, Classical Quantum Gravity **32**, 112001 (2015).
[8] W. Donnelly and L. Freidel, Local subsystems in gauge theory and gravity, J. High Energy Phys. 09 (2016) 102.
[9] C. Delcamp, L. Freidel, and F. Girelli, Dual canonical quantization of 3D quantum gravity, arXiv:1803.03246.
[10] B. Dittrich and M. Geiller, Flux formulation of loop quantum gravity: Classical framework, Classical Quantum Gravity **32**, 135016 (2015).
[11] L. Freidel and A. Perez, Quantum gravity at the corner, arXiv:1507.02573.
[12] L. Freidel, A. Perez, and D. Pranzetti, Loop gravity string, Phys. Rev. D **95**, 106002 (2017).
[13] L. Freidel and E. R. Livine, Bubble networks: Framed discrete geometry for quantum gravity, Gen. Relativ. Gravit. **51**, 9 (2019).
[14] C. Rovelli, Why gauge?, Found. Phys. **44**, 91 (2014).
[15] W. Donnelly, Decomposition of entanglement entropy in lattice gauge theory, Phys. Rev. D **85**, 085004 (2012).
[16] M. Geiller, Edge modes and corner ambiguities in 3D Chern-Simons theory and gravity, Nucl. Phys. **B924**, 312 (2017).
[17] M. Geiller, Lorentz-diffeomorphism edge modes in 3D gravity, J. High Energy Phys. 02 (2018) 029.
[18] A. Strominger, Lectures on the infrared structure of gravity and gauge theory, arXiv:1703.05448.
[19] A. S. Cattaneo and A. Perez, A note on the Poisson bracket of 2D smeared fluxes in loop quantum gravity, Classical Quantum Gravity **34**, 107001 (2017).
[20] C. Delcamp and B. Dittrich, Towards a dual spin network basis for (3 + 1)d lattice gauge theories and topological phases, J. High Energy Phys. 10 (2018) 023.
[21] C. Delcamp, B. Dittrich, and A. Riello, Fusion basis for lattice gauge theory and loop quantum gravity, J. High Energy Phys. 02 (2017) 061.
[22] F. Girelli and E. R. Livine, Reconstructing quantum geometry from quantum information: Spin networks as harmonic oscillators, Classical Quantum Gravity **22**, 3295 (2005).
[23] L. Freidel and E. R. Livine, The fine structure of SU(2) intertwiners from U(N) representations, J. Math. Phys. (N.Y.) **51**, 082502 (2010).
[24] E. R. Livine and J. Tambornino, Spinor representation for loop quantum gravity, J. Math. Phys. (N.Y.) **53**, 012503 (2012).
[25] W. Wieland, Conformal boundary conditions, loop gravity and the continuum, J. High Energy Phys. 10 (2018) 089.
[26] W. Wieland, Quantum gravity in three dimensions, Witten spinors and the quantisation of length, Nucl. Phys. **B930**, 219 (2018).
[27] A. P. Balachandran, L. Chandar, and E. Ercolessi, Edge states in gauge theories: Theory, interpretations and predictions, Int. J. Mod. Phys. A **10**, 1969 (1995).
[28] B. Dittrich and M. Geiller, Quantum gravity kinematics from extended TQFTs, New J. Phys. **19**, 013003 (2017).
[29] A. Yu. Alekseev and A. Z. Malkin, Symplectic structure of the moduli space of flat connection on a Riemann surface, Commun. Math. Phys. **169**, 99 (1995).
[30] A. Yu. Alekseev, H. Grosse, and V. Schomerus, Combinatorial quantization of the Hamiltonian Chern-Simons theory, Commun. Math. Phys. **172**, 317 (1995).
[31] A. Yu. Alekseev, H. Grosse, and V. Schomerus, Combinatorial quantization of the Hamiltonian Chern-Simons theory II, Commun. Math. Phys. **174**, 561 (1996).
[32] C. Meusburger and B. J. Schroers, The quantisation of Poisson structures arising in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$, Adv. Theor. Math. Phys. **7**, 1003 (2003).
[33] C. Meusburger and K. Noui, The Hilbert space of 3D gravity: Quantum group symmetries and observables, Adv. Theor. Math. Phys. **14**, 1651 (2010).
[34] L. Freidel, F. Girelli, and B. Shoshany, 3 + 1D loop quantum gravity on the edge (to be published).
[35] L. Freidel and D. Pranzetti, Electromagnetic duality and central charge, Phys. Rev. D **98**, 116008 (2018).
[36] C. Meusburger and B. J. Schroers, Phase space structure of Chern-Simons theory with a non-standard puncture, Nucl. Phys. **B738**, 425 (2006).
[37] C. Meusburger and B. J. Schroers, Poisson structure and symmetry in the Chern-Simons formulation of (2 + 1)-dimensional gravity, Classical Quantum Gravity **20**, 2193 (2003).
[38] C. Meusburger and B. J. Schroers, Generalised Chern-Simons actions for 3D gravity and kappa-Poincaré symmetry, Nucl. Phys. **B806**, 462 (2009).
[39] Á. Ballesteros, F. J. Herranz, and C. Meusburger, Drinfel'd doubles for (2 + 1)-gravity, Classical Quantum Gravity **30**, 155012 (2013).
[40] B. Shoshany, Dual 2 + 1D loop quantum gravity on the edge (to be published).
[41] A. Kirillov, *Lectures on the Orbit Method* (American Mathematical Society, Providence, 2004).
[42] T. Rempel and L. Freidel, Interaction vertex for classical spinning particles, Phys. Rev. D **94**, 044011 (2016).
[43] E. R. Livine, Deformation operators of spin networks and coarse-graining, Classical Quantum Gravity **31**, 075004 (2014).
[44] E. R. Livine, From coarse-graining to holography in loop quantum gravity, Europhys. Lett. **123**, 10001 (2018).