Ghost-free Gauss-Bonnet theories of gravity

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In this work we develop a theoretical framework for Gauss-Bonnet modified gravity theories, in which ghost modes can be eliminated at the level of the equations of motion. In particular, after we present how the ghosts can occur at the level of the equations of motion, we employ the Lagrange multiplier technique, and by means of constraints we are able to eliminate the ghost modes from Gauss-Bonnet theories of the forms $f(\mathcal{G})$ and $F(R,\mathcal{G})$. Some cosmological realizations in the context of ghost-free $f(\mathcal{G})$ gravity are presented, by using the reconstruction technique we developed. Finally, we explore the modifications to Newton's law of gravity generated by the ghost-free $f(\mathcal{G})$ theory.

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I. INTRODUCTION

Undoubtedly one of the ultimate goals of theoretical physics is to find a consistent way to describe all the observed interactions under the same theoretical framework. This would require quantizing gravity in some way and to date only string theory seems to provide a complete UV completion of all known particle physics theories. In cosmology, the quantum gravity era controls the preinflationary era, during which gravity is expected to be unified with the other three interactions. It is evident that during this preinflationary era, string theory would be the most appropriate theory to describe the physical laws of our Universe; however it is not easy to prove that this is indeed the case. However some string theory effects could have an impact on the inflationary era, and this impact may be in fact measurable. There exist many theories in modern theoretical cosmology which take into account string-theory-motivated terms in the interaction Lagrangian of the model, such as the scalar-Einstein-Gauss-Bonnet gravity theory [1,2], in which case the Lagrangian is of the form,

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + h(\chi)\mathcal{G} - V(\chi) + \mathcal{L}_{\text{matter}}\right), \tag{1}$$

where $\mathcal{G}$ is the Gauss-Bonnet invariant defined as follows:

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2}$$

The scalar-Einstein-Gauss-Bonnet models are motivated by α' corrections in superstring theories [3], and they serve as a consistent example of how string theory may leave its impact on the primordial acceleration era of the Universe. Another very well-studied class of theories in the same context, is that of $f(\mathcal{G})$ gravity [4–9], in which case the Lagrangian is of the form,

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(\mathcal{G}) + \mathcal{L}_{\text{matter}}\right). \tag{3}$$

These theories contain a function of the Gauss-Bonnet invariant, and therefore the presence of this function generates nontrivial effects in the theory, due to the fact that the effect of the Gauss-Bonnet term does not appear as a total derivative anymore, as in the linear theory of the Gauss-Bonnet scalar. Both these theories belong to a wider class of cosmological models which are known as modified gravity models [10–16], and which generalize the standard Einstein-Hilbert theory. The motivation for studying such theories comes from the fact that in the context of these, several cosmological eras may be described by the same theory in a unified way; see e.g., Ref. [17] in which the unified description of the inflationary and dark energy eras was given in terms of $f(R)$ gravity. In addition, similar studies were presented in terms of scalar-Einstein-Gauss-Bonnet models [18] and $f(\mathcal{G})$ models.

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Due to the importance of the models containing or involving the Gauss-Bonnet scalar, which are string theory motivated in most cases, in this paper we shall address an important shortcoming of these theories, namely the existence of ghosts. Usually, higher-derivative theories contain ghost degrees of freedom (d.o.f.) due to the Ostrogradsky instability; see e.g., Ref. [19]. As was pointed out in Ref. [20], ghost d.o.f. may occur at various levels of the theory, even at the cosmological perturbation level of $F(R,\mathcal{G})$ theories, where superluminal modes $\sim k^4$ occur, where k is the associated wave number. Having these issues in mind, in this paper we shall investigate how the ghosts may be eliminated from $f(\mathcal{G})$ and $F(R,\mathcal{G})$ theories. In particular, by using an appropriate constraint used first in the context of mimetic gravity [21–23], we shall demonstrate that the resulting theories are ghost free. Similar constrained Gauss-Bonnet theories in the context of mimetic gravity were studied in Ref. [24]. Also ghost-free theories were also developed in Refs. [25,26], but in a different context. In this work we shall also consider the cosmological evolution of the resulting theories, and we shall investigate how several cosmological evolutions may be realized by the ghost-free models we will develop, focusing on the dark energy era and inflationary era. Finally, we shall investigate how Newton's law is modified in the context of ghost-free $f(\mathcal{G})$ gravity.

This paper is organized as follows. In Sec. II we address the ghost issue in the context of $f(\mathcal{G})$ gravity. We first demonstrate how ghosts may occur in this theory and we provide two remedy theories, which are ghost-free extensions of $f(\mathcal{G})$ gravity. In Sec. III we investigate how several cosmological evolutions may be realized in the context of the proposed ghost-free $f(\mathcal{G})$ theory. In Sec. IV we discuss how Newton's law is modified in the context of ghost-free $f(\mathcal{G})$ gravity, and finally in Sec. V we briefly investigate how a general $F(R,\mathcal{G})$ theory may be rendered ghost free.

II. GHOST-FREE $f(\mathcal{G})$ GRAVITY

In this section we shall investigate how to obtain ghost-free $f(\mathcal{G})$ gravity, and we shall employ the Lagrange multiplier formalism in order to achieve this. Before getting into the details of our formalism, we will start the presentation by showing explicitly how ghost modes may occur in $f(\mathcal{G})$ gravity at the level of the equations of motion, and the ghost-free version construction of the theory follows.

A. Ghosts in $f(\mathcal{G})$ gravity

In order to investigate if any ghost modes could appear in the $f(\mathcal{G})$ gravity model (3), we investigate the equations of motion, by considering a general variation of the metric of the following form:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{4}$$

Effectively, the variations of $\delta\Gamma^\kappa_{\mu\nu}$, $\delta R_{\mu\nu\lambda\sigma}$, $\delta R_{\mu\nu}$, and δR read,

$$\begin{aligned}
\delta\Gamma^\kappa_{\mu\nu} &= \frac{1}{2} g^{\kappa\lambda}(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}), \\
\delta R_{\mu\nu\lambda\sigma} &= \frac{1}{2}[\nabla_\lambda\nabla_\nu \delta g_{\sigma\mu} - \nabla_\lambda\nabla_\mu \delta g_{\sigma\nu} - \nabla_\sigma\nabla_\nu \delta g_{\lambda\mu} \\
&\quad + \nabla_\sigma\nabla_\mu \delta g_{\lambda\nu} + \delta g_{\mu\rho} R^\rho{}_{\nu\lambda\sigma} - \delta g_{\nu\rho} R^\rho{}_{\mu\lambda\sigma}], \\
\delta R_{\mu\nu} &= \frac{1}{2}[\nabla_\mu \nabla^\rho \delta g_{\nu\rho} + \nabla_\nu \nabla^\rho \delta g_{\mu\rho} - \Box \delta g_{\mu\nu} \\
&\quad - \nabla_\mu\nabla_\nu (g^{\rho\lambda}\delta g_{\rho\lambda}) - 2R^\lambda{}_\nu{}^\rho{}_\mu \delta g_{\lambda\rho} \\
&\quad + R^\rho{}_\mu \delta g_{\rho\nu} + R^\rho{}_\nu \delta g_{\rho\mu}], \\
\delta R &= -\delta g_{\mu\nu} R^{\mu\nu} + \nabla^\mu\nabla^\nu \delta g_{\mu\nu} - \Box(g^{\mu\nu}\delta g_{\mu\nu}).
\end{aligned} \tag{5}$$

Accordingly the variation of the Gauss-Bonnet scalar $\delta\mathcal{G}$ reads,

$$\begin{aligned}
\delta\mathcal{G} &= 2R(-\delta g_{\mu\nu} R^{\mu\nu} + \nabla^\mu\nabla^\nu \delta g_{\mu\nu} - \nabla^2(g^{\mu\nu}\delta g_{\mu\nu})) \\
&\quad + 8R^{\rho\sigma} R^\mu{}_\rho{}^\nu{}_\sigma \delta g_{\mu\nu} - 4(R^{\rho\nu}\nabla_\rho\nabla^\mu + R^{\rho\mu}\nabla_\rho\nabla^\nu)\delta g_{\mu\nu} \\
&\quad + 4R^{\mu\nu}\nabla^2 \delta g_{\mu\nu} + 4R^{\rho\sigma}\nabla_\rho\nabla_\sigma(g^{\mu\nu}\delta g_{\mu\nu}) \\
&\quad - 2R^{\mu\rho\sigma\tau} R^\nu{}_{\rho\sigma\tau}\delta g_{\mu\nu} - 4R^{\rho\mu\sigma\nu}\nabla_\rho\nabla_\sigma \delta g_{\mu\nu}.
\end{aligned} \tag{6}$$

Then for the $f(\mathcal{G})$ gravity model (3), by varying the action with respect to the metric tensor $g_{\mu\nu}$, we obtain the following equations of motion:

$$\begin{aligned}
0 &= \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R\right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2} g^{\mu\nu} f(\mathcal{G}) \\
&\quad + (-2RR^{\mu\nu} + 8R^{\rho\sigma} R^\mu{}_\rho{}^\nu{}_\sigma - 2R^{\mu\rho\sigma\tau} R^\nu{}_{\rho\sigma\tau}) f'(\mathcal{G}) \\
&\quad + 2(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)(R f'(\mathcal{G})) - 4\nabla^\mu\nabla_\rho(R^{\rho\nu} f'(\mathcal{G})) \\
&\quad - 4\nabla^\nu\nabla_\rho(R^{\rho\mu} f'(\mathcal{G})) + 4\Box(R^{\mu\nu} f'(\mathcal{G})) \\
&\quad + 4g^{\mu\nu}\nabla_\rho\nabla_\sigma(R^{\rho\sigma} f'(\mathcal{G})) - 4\nabla_\rho\nabla_\sigma(R^{\rho\mu\sigma\nu} f'(\mathcal{G})).
\end{aligned} \tag{7}$$

By using the Bianchi identities,

$$\begin{aligned}
\nabla^\rho R_{\rho\tau\mu\nu} &= \nabla_\mu R_{\nu\tau} - \nabla_\nu R_{\mu\tau}, \\
\nabla^\rho R_{\rho\mu} &= \frac{1}{2}\nabla_\mu R, \\
\nabla_\rho\nabla_\sigma R^{\mu\rho\nu\sigma} &= \Box R^{\mu\nu} - \frac{1}{2}\nabla^\mu\nabla^\nu R + R^{\mu\rho\nu\sigma} R_{\rho\sigma} - R^\mu{}_\rho R^{\nu\rho}, \\
\nabla_\rho\nabla_\sigma R^{\rho\sigma} &= \frac{1}{2}\Box R,
\end{aligned} \tag{8}$$

we can rewrite Eq. (7) as follows:

$$0 = \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2}g^{\mu\nu}f(\mathcal{G}) + (-2RR^{\mu\nu} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} + 4R^{\mu}{}_{\rho}R^{\nu\rho} + 4R^{\rho\sigma}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma})f'(\mathcal{G}) + 2R\nabla^{\mu}\nabla^{\nu}f'(\mathcal{G}) - 2g^{\mu\nu}R\Box f'(\mathcal{G}) - 4R^{\rho\nu}\nabla^{\mu}\nabla_{\rho}f'(\mathcal{G}) - 4R^{\rho\mu}\nabla^{\nu}\nabla_{\rho}f'(\mathcal{G}) + 4R^{\mu\nu}\Box f'(\mathcal{G}) + 4g^{\mu\nu}R^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}f'(\mathcal{G}) - 4R^{\rho\mu\sigma\nu}\nabla_{\rho}\nabla_{\sigma}f'(\mathcal{G}). \tag{9}$$

Also in four dimensions, we have the following identity:

$$0 = \frac{1}{2}g^{\mu\nu}\mathcal{G} - 2RR^{\mu\nu} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} + 4R^{\mu}{}_{\rho}R^{\nu\rho} + 4R^{\rho\sigma}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}. \tag{10}$$

Then Eq. (9) takes the following form:

$$0 = \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu}_{\text{matter}} + \frac{1}{2}g^{\mu\nu}(f(\mathcal{G}) - \mathcal{G}f'(\mathcal{G})) + 2R\nabla^{\mu}\nabla^{\nu}f'(\mathcal{G}) - 2g^{\mu\nu}R\Box f'(\mathcal{G}) - 4R^{\rho\nu}\nabla^{\mu}\nabla_{\rho}f'(\mathcal{G}) - 4R^{\rho\mu}\nabla^{\nu}\nabla_{\rho}f'(\mathcal{G}) + 4R^{\mu\nu}\Box f'(\mathcal{G}) + 4g^{\mu\nu}R^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}f'(\mathcal{G}) - 4R^{\rho\mu\sigma\nu}\nabla_{\rho}\nabla_{\sigma}f'(\mathcal{G}). \tag{11}$$

We now rewrite Eq. (11) in the following form:

$$\begin{aligned} 0 &= \frac{1}{2\kappa^2}\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\text{matter}\,\mu\nu} + \frac{1}{2}g_{\mu\nu}(f(\mathcal{G}) - \mathcal{G}f'(\mathcal{G})) + D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}f'(\mathcal{G}), \\ D_{\mu\nu}{}^{\tau\eta} &\equiv (\delta_{\mu}{}^{\tau}\delta_{\nu}{}^{\eta} + \delta_{\nu}{}^{\tau}\delta_{\mu}{}^{\eta} - 2g_{\mu\nu}g^{\tau\eta})R + (-4g^{\rho\tau}\delta_{\mu}{}^{\eta}\delta_{\nu}{}^{\sigma} - 4g^{\rho\tau}\delta_{\nu}{}^{\eta}\delta_{\mu}{}^{\sigma} + 4g_{\mu\nu}g^{\rho\tau}g^{\sigma\eta})R_{\rho\sigma} + 4R_{\mu\nu}g^{\tau\eta} - 2R_{\rho\mu\sigma\nu}(g^{\rho\tau}g^{\sigma\eta} + g^{\rho\eta}g^{\sigma\tau}). \end{aligned} \tag{12}$$

Having in mind that,

$$g^{\mu\nu}D_{\mu\nu}{}^{\tau\eta} = 4\left(-\frac{1}{2}g^{\tau\eta}R + R^{\tau\eta}\right), \tag{13}$$

we find in component form,

$$\begin{aligned} D_{00}{}^{00} &= 2R - 2g_{00}g^{00}R - 8R^{0}{}_{0} + 4g_{00}R^{00} + 4g^{00}R_{00} - 4R^{0}{}_{0}{}^{0}{}_{0}, \\ D_{ij}{}^{00} &= 4g_{ij}R^{00} - 4R^{0}{}_{i}{}^{0}{}_{j} - 2g_{ij}g^{00}R + 4R_{ij}g^{00}. \end{aligned} \tag{14}$$

If we choose the gauge in which $g_{0i} = 0$, then the quantity $D_{00}{}^{00}$ vanishes but $D_{ij}{}^{00}$ does not vanish in general. This indicates that Eq. (11) includes the fourth derivative of the metric with respect to the cosmic time coordinate and therefore ghost modes might appear. We may see the existence of ghost modes explicitly, by considering perturbations. Let a solution of Eq. (11) be $g_{\mu\nu} = g^{(0)}_{\mu\nu}$ and we denote the curvatures and connections given by $g^{(0)}_{\mu\nu}$ by using the index "(0)." Then in order to investigate if any ghost could exist, we may consider the variation of Eq. (11) around the solution $g^{(0)}_{\mu\nu}$ as follows: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$. For the variation of $\delta g_{\mu\nu}$, we may impose the following gauge condition:

$$0 = \nabla^{\mu}\delta g_{\mu\nu}. \tag{15}$$

Then Eq. (6) reduces to,

$$\delta\mathcal{G} = 2R(-\delta g_{\mu\nu}R^{\mu\nu} - \nabla^2(g^{\mu\nu}\delta g_{\mu\nu})) + 8R^{\rho\sigma}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\delta g_{\mu\nu} + 4R^{\mu\nu}\nabla^2\delta g_{\mu\nu} + 4R^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}(g^{\mu\nu}\delta g_{\mu\nu}) - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau}\delta g_{\mu\nu} - 4R^{\rho\mu\sigma\nu}\nabla_{\rho}\nabla_{\sigma}\delta g_{\mu\nu}. \tag{16}$$

Even if we impose the condition $\delta g^{\mu}{}_{\mu} = 0$, Eq. (16) has the following form:

$$\delta\mathcal{G} = -2RR^{\mu\nu}\delta g_{\mu\nu} + 8R^{\rho\sigma}R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\delta g_{\mu\nu} + 4R^{\mu\nu}\nabla^2\delta g_{\mu\nu} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau}\delta g_{\mu\nu} - 4R^{\rho\mu\sigma\nu}\nabla_{\rho}\nabla_{\sigma}\delta g_{\mu\nu}, \tag{17}$$

which also contains the second derivative of the metric $g_{\mu\nu}$ with respect to the cosmic time coordinate. Under the perturbation $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$, the term $D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}f'(\mathcal{G})$ takes the following form:

$$D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}f'(\mathcal{G}) \to D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}f'(\mathcal{G}^{(0)}) + D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}(f''(\mathcal{G}^{(0)})\delta\mathcal{G}) + \cdots, \tag{18}$$

which contains the fourth derivative of the metric $g_{\mu\nu}$ with respect to the cosmic time coordinate, and therefore the perturbed equation (12) may have a ghost mode. Note that in Eq. (18), the "$\cdots$" represents the terms arising from the variation of $D_{\mu\nu}{}^{\tau\eta}\nabla_{\tau}\nabla_{\eta}$. The propagating mode is a scalar expressed by the Gauss-Bonnet invariant as it is clear from Eq. (12). Having presented explicitly how a ghost mode may occur in $f(\mathcal{G})$ gravity, we now demonstrate how the ghost modes may be eliminated or avoided in this theory. This is the subject of the next subsection.

B. Development of ghost-free $f(\mathcal{G})$ gravity

In this subsection, we consider how we can avoid the ghost in $f(\mathcal{G})$ gravity. To this end, we rewrite the action of Eq. (3) by introducing an auxiliary field χ as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + h(\chi)\mathcal{G} - V(\chi) + \mathcal{L}_{\text{matter}}\right). \tag{19}$$

Then by varying the action (19) with respect to the auxiliary field χ, we obtain the following equation:

$$0 = h'(\chi)\mathcal{G} - V'(\chi), \tag{20}$$

which can be solved with respect to χ as a function of the Gauss-Bonnet invariant $\mathcal{G}$ as $\chi = \chi(\mathcal{G})$. Then by substituting the obtained expression into Eq. (20), we reobtain the action of Eq. (3) with $f(\mathcal{G})$ being equal to,

$$f(\mathcal{G}) = h(\chi(\mathcal{G}))\mathcal{G} - V(\chi(\mathcal{G})). \tag{21}$$

On the other hand, by varying the action (20) with respect to the metric tensor, we obtain,

$$0 = \frac{1}{2\kappa^2}\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\mathrm{matter}\,\mu\nu} - \frac{1}{2}g_{\mu\nu}V(\chi) + D_{\mu\nu}{}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi), \tag{22}$$

with $D_{\mu\nu}{}^{\tau\eta}$ being defined in Eq. (12). Since χ can be given by a function of the Gauss-Bonnet invariant $\mathcal{G}$, Eq. (22) is the fourth-order differential equation for the metric, which may actually generate the ghost modes. Equation (22) indicates that the propagating scalar mode is quantified in terms of χ. Then in order for the scalar mode to not be a ghost, we may add a canonical kinetic term of χ in the action (19) as in the model of Eq. (1) [1], where we have chosen the mass dimension of χ to be unity. Then instead of Eqs. (20) and (22), we obtain,

$$0 = \Box\chi + h'(\chi)\mathcal{G} - V'(\chi), \tag{23}$$

$$0 = \frac{1}{2\kappa^2}\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\mathrm{matter}\,\mu\nu} + \frac{1}{2}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\partial_\rho\chi\partial^\rho\chi + V(\chi)\right) + D_{\mu\nu}{}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi). \tag{24}$$

Since the equations derived above do not contain higher than second-order derivatives, if we impose initial conditions for the quantities $g_{\mu\nu}$, $\dot{g}_{\mu\nu}$, χ, and $\dot{\chi}$ on a spatial hypersurface of constant cosmic time, the evolution of $g_{\mu\nu}$ and χ is uniquely determined, and as it is clear from Eq. (24), these could not be ghosts. In the model of Eq. (1), we have introduced a new dynamical d.o.f., namely χ, but if we want to reduce the dynamical d.o.f., we may impose a constraint as in the mimetic gravity case [21–23], by introducing the Lagrange multiplier field λ, as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \lambda\left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2}\right) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + h(\chi)\mathcal{G} - V(\chi) + \mathcal{L}_{\mathrm{matter}}\right), \tag{25}$$

where μ is a constant with mass-dimension one. Then, by varying the above action (25) with respect to λ, we obtain the constraint,

$$0 = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2}. \tag{26}$$

Then due to the fact that the kinetic term becomes a constant, the kinetic term in the action of Eq. (25) can be absorbed into the redefinition of the scalar potential $V(\chi)$ as follows:

$$\tilde{V}(\chi) \equiv \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + V(\chi) = -\frac{\mu^4}{2} + V(\chi), \tag{27}$$

and we can rewrite the action of Eq. (25) as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \lambda\left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2}\right) + h(\chi)\mathcal{G} - \tilde{V}(\chi) + \mathcal{L}_{\mathrm{matter}}\right). \tag{28}$$

For the model of Eq. (28), in addition to Eq. (26), we have the following two equations of motion:

$$0 = -\frac{1}{\sqrt{-g}}\partial_\mu(\lambda g^{\mu\nu}\sqrt{-g}\partial_\nu\chi) + h'(\chi)\mathcal{G} - \tilde{V}'(\chi), \tag{29}$$

$$0 = \frac{1}{2\kappa^2}\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}T_{\mathrm{matter}\,\mu\nu} - \frac{1}{2}\lambda\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\tilde{V}(\chi) + D_{\mu\nu}{}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi), \tag{30}$$

where we have also used Eq. (26). By multiplying Eq. (30) with $g^{\mu\nu}$, we obtain,

$$0 = \frac{R}{2\kappa^2} + \frac{1}{2}T_{\mathrm{matter}} + \frac{\mu^4}{2}\lambda - 2\tilde{V}(\chi) - 4\left(-R^{\tau\eta} + \frac{1}{2}g^{\tau\eta}R\right)\nabla_\tau\nabla_\eta h(\chi), \tag{31}$$

where we used Eq. (26) and $T_{\mathrm{matter}} \equiv g^{\mu\nu}T_{\mathrm{matter}\,\mu\nu}$. Equation (31) can be solved with respect to the Lagrange multiplier field λ, and the result is,

$$\lambda = -\frac{2}{\mu^4}\left(\frac{R}{2\kappa^2} + \frac{1}{2}T_{\mathrm{matter}} - 2\tilde{V}(\chi) - 4\left(-R^{\tau\eta} + \frac{1}{2}g^{\tau\eta}R\right)\nabla_\tau\nabla_\eta h(\chi)\right). \tag{32}$$

We expect that the model (28) could not contain a ghost mode. And actually by using perturbations of the metric, we now show explicitly that indeed the model (28) is ghost free. Let the general solutions of Eqs. (26), (29), and (30) be $g^{(0)}_{\mu\nu}$, $\chi^{(0)}$, and $\lambda^{(0)}$ and we consider the following perturbation:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \quad \chi = \chi^{(0)} + \delta\chi, \quad \lambda = \lambda^{(0)} + \delta\lambda. \tag{33}$$

Then Eqs. (26), (29), and (30) can be written as

$$0 = \partial^\mu\chi^{(0)}\partial_\mu\delta\chi - \delta g_{\mu\nu}\partial^\mu\chi^{(0)}\partial^\nu\chi^{(0)}, \tag{34}$$

$$\begin{aligned}
0=&\frac{g^{(0)\rho\sigma}\delta g_{\rho\sigma}}{2\sqrt{-g^{(0)}}}\partial_\mu\left(\lambda^{(0)}g^{(0)\mu\nu}\sqrt{-g^{(0)}}\partial_\nu\chi^{(0)}\right)-\frac{1}{\sqrt{-g^{(0)}}}\partial_\mu\left(\delta\lambda g^{(0)\mu\nu}\sqrt{-g^{(0)}}\partial_\nu\chi^{(0)}\right)\\
&+\frac{1}{\sqrt{-g^{(0)}}}\partial_\mu\left(\lambda^{(0)}g^{(0)\mu\rho}\delta g_{\rho\sigma}g^{(0)\sigma\nu}\sqrt{-g^{(0)}}\partial_\nu\chi^{(0)}\right)-\frac{1}{2\sqrt{-g^{(0)}}}\partial_\mu\left(\lambda^{(0)}g^{(0)\mu\nu}g^{(0)\rho\sigma}\delta g_{\rho\sigma}\sqrt{-g^{(0)}}\partial_\nu\chi^{(0)}\right)\\
&-\frac{1}{\sqrt{-g^{(0)}}}\partial_\mu\left(\lambda^{(0)}g^{(0)\mu\nu}\sqrt{-g^{(0)}}\partial_\nu\delta\chi\right)+h''(\chi^{(0)})\delta\chi\mathcal{G}^{(0)}-\tilde{V}''(\chi^{(0)})\delta\chi\\
&+h'(\chi^{(0)})(2R^{(0)}(-\delta g_{\mu\nu}R^{(0)\mu\nu}+\nabla^{(0)\mu}\nabla^{(0)\nu}\delta g_{\mu\nu}-\Box^{(0)}(g^{(0)\mu\nu}\delta g_{\mu\nu}))+8R^{(0)\rho\sigma}R^{(0)\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\delta g_{\mu\nu}\\
&-4(R^{(0)\rho\nu}\nabla^{(0)}_\rho\nabla^{(0)\mu}+R^{(0)\rho\mu}\nabla^{(0)}_\rho\nabla^{(0)\nu})\delta g_{\mu\nu}+4R^{(0)\mu\nu}\Box^{(0)}\delta g_{\mu\nu}+4R^{(0)\rho\sigma}\nabla^{(0)}_\rho\nabla^{(0)}_\sigma(g^{(0)\mu\nu}\delta g_{\mu\nu})\\
&-2R^{(0)\mu\rho\sigma\tau}R^{(0)\nu}{}_{\rho\sigma\tau}\delta g_{\mu\nu}-4R^{(0)\rho\mu\sigma\nu}\nabla^{(0)}_\rho\nabla^{(0)}_\sigma\delta g_{\mu\nu}),
\end{aligned}\tag{35}$$

$$\begin{aligned}
0=&\frac{1}{2\kappa^2}\bigg(-\frac{1}{2}(\nabla^{(0)}_\mu\nabla^{(0)\rho}\delta g_{\nu\rho}+\nabla^{(0)}_\nu\nabla^{(0)\rho}\delta g_{\mu\rho}-\Box^{(0)}\delta g_{\mu\nu}-\nabla^{(0)}_\mu\nabla^{(0)}_\nu(g^{(0)\rho\lambda}\delta g_{\rho\lambda})\\
&-2R^{(0)\lambda}{}_{\nu}{}^{\rho}{}_{\mu}\delta g_{\lambda\rho}+R^{(0)\rho}{}_{\mu}\delta g_{\rho\nu}+R^{(0)\rho}{}_{\nu}\delta g_{\rho\mu})+\frac{1}{2}R^{(0)}\delta g_{\mu\nu}+\frac{1}{2}g^{(0)}_{\mu\nu}(-\delta g_{\rho\sigma}R^{(0)\rho\sigma}+\nabla^{(0)\rho}\nabla^{(0)\sigma}\delta g_{\rho\sigma}-\Box^{(0)}(g^{(0)\rho\sigma}\delta g_{\rho\sigma}))\bigg)\\
&+\frac{1}{2}\delta T_{\mathrm{matter}\,\mu\nu}-\frac{1}{2}\delta\lambda\partial_\mu\chi^{(0)}\partial_\nu\chi^{(0)}-\frac{1}{2}\lambda^{(0)}\partial_\mu\delta\chi\partial_\nu\chi^{(0)}-\frac{1}{2}\lambda^{(0)}\partial_\mu\chi^{(0)}\partial_\nu\delta\chi-\frac{1}{2}\delta g_{\mu\nu}\tilde{V}(\chi^{(0)})-\frac{1}{2}g^{(0)}_{\mu\nu}\tilde{V}'(\chi^{(0)})\delta\chi\\
&+\{-2(\delta g_{\mu\nu}g^{(0)\tau\eta}-g^{(0)}_{\mu\nu}g^{(0)\tau\zeta}\delta g_{\zeta\xi}g^{(0)\xi\eta})R^{(0)}+(\delta_\mu{}^\tau\delta_\nu{}^\eta+\delta_\nu{}^\tau\delta_\mu{}^\eta-2g^{(0)}_{\mu\nu}g^{(0)\tau\eta})(-\delta g_{\zeta\xi}R^{(0)\zeta\xi}+\nabla^{(0)\zeta}\nabla^{(0)\xi}\delta g_{\zeta\xi}\\
&-\Box^{(0)}(g^{(0)\zeta\xi}\delta g_{\zeta\xi}))+4(g^{(0)\rho\zeta}\delta g_{\zeta\xi}g^{(0)\xi\tau}\delta_\mu{}^\eta\delta_\nu{}^\sigma+g^{(0)\rho\zeta}\delta g_{\zeta\xi}g^{(0)\xi\tau}\delta_\nu{}^\eta\delta_\mu{}^\sigma+\delta g_{\mu\nu}g^{(0)\rho\tau}g^{(0)\sigma\eta}+g^{(0)}_{\mu\nu}g^{(0)\rho\zeta}\delta g_{\zeta\xi}g^{(0)\xi\tau}g^{(0)\sigma\eta}\\
&+g^{(0)}_{\mu\nu}g^{(0)\rho\tau}g^{(0)\sigma\zeta}\delta g_{\zeta\xi}g^{(0)\xi\eta})R^{(0)}_{\rho\sigma}+2(-g^{(0)\rho\tau}\delta_\mu{}^\eta\delta_\nu{}^\sigma-g^{(0)\rho\tau}\delta_\nu{}^\eta\delta_\mu{}^\sigma+g^{(0)}_{\mu\nu}g^{(0)\rho\tau}g^{(0)\sigma\eta})(\nabla^{(0)}_\rho\nabla^{(0)\xi}\delta g_{\sigma\xi}+\nabla^{(0)}_\sigma\nabla^{(0)\xi}\delta g_{\rho\xi}\\
&-\Box^{(0)}\delta g_{\rho\sigma}-\nabla^{(0)}_\rho\nabla^{(0)}_\sigma(g^{(0)\zeta\xi}\delta g_{\zeta\xi})-2R^{(0)\zeta}{}_{\sigma}{}^{\xi}{}_{\rho}\delta g_{\zeta\xi}+R^{(0)\xi}{}_{\rho}\delta g_{\xi\sigma}+R^{(0)\xi}{}_{\sigma}\delta g_{\xi\rho})-4R^{(0)}_{\mu\nu}g^{(0)\tau\zeta}\delta g_{\zeta\xi}g^{(0)\xi\eta}\\
&+4g^{(0)\tau\eta}(\nabla^{(0)}_\mu\nabla^{(0)\xi}\delta g_{\nu\xi}+\nabla^{(0)}_\nu\nabla^{(0)\xi}\delta g_{\mu\xi}-\Box^{(0)}\delta g_{\mu\nu}-\nabla^{(0)}_\mu\nabla^{(0)}_\nu(g^{(0)\zeta\xi}\delta g_{\zeta\xi})-2R^{(0)\zeta}{}_{\nu}{}^{\xi}{}_{\mu}\delta g_{\zeta\xi}+R^{(0)\xi}_\mu\delta g_{\xi\nu}+R^{(0)\xi}_\nu\delta g_{\xi\mu})\\
&-2(\nabla^{(0)}_\sigma\nabla^{(0)}_\mu\delta g_{\nu\rho}-\nabla^{(0)}_\sigma\nabla^{(0)}_\rho\delta g_{\nu\mu}-\nabla^{(0)}_\nu\nabla^{(0)}_\mu\delta g_{\sigma\rho}+\nabla^{(0)}_\nu\nabla^{(0)}_\rho\delta g_{\sigma\mu}+\delta g_{\rho\xi}R^{(0)\xi}{}_{\mu\sigma\nu}-\delta g_{\mu\xi}R^{(0)\xi}{}_{\rho\sigma\nu})g^{(0)\rho\tau}g^{(0)\sigma\eta}\\
&-4R^{(0)}_{\rho\mu\sigma\nu}(g^{(0)\rho\zeta}\delta g_{\zeta\xi}g^{(0)\xi\tau}g^{(0)\sigma\eta}+g^{(0)\rho\tau}g^{(0)\sigma\zeta}\delta g_{\zeta\xi}g^{(0)\xi\eta})\}\nabla^{(0)}_\tau\nabla^{(0)}_\eta h(\chi^{(0)})\\
&-\frac{1}{2}D^{(0)\tau\eta}_{\mu\nu}g^{(0)\zeta\xi}(\nabla^{(0)}_\tau\delta g_{\eta\xi}+\nabla^{(0)}_\eta\delta g_{\tau\xi}-\nabla^{(0)}_\xi\delta g_{\tau\eta})\partial_\zeta h(\chi^{(0)})+D^{(0)\tau\eta}_{\mu\nu}\nabla^{(0)}_\tau\nabla^{(0)}_\eta(h'(\chi^{(0)})\delta\chi).
\end{aligned}\tag{36}$$

On the other hand, Eq. (32) yields,

$$\begin{aligned}
\delta\lambda=&-\frac{2}{\mu^4}\bigg(\frac{1}{2\kappa^2}(-\delta g_{\rho\sigma}R^{(0)\rho\sigma}+\nabla^{(0)\rho}\nabla^{(0)\sigma}\delta g_{\rho\sigma}-\Box^{(0)}(g^{(0)\rho\sigma}\delta g_{\rho\sigma}))+\frac{1}{2}\delta T_{\mathrm{matter}}-2\tilde{V}'(\chi^{(0)})\delta\chi-4\bigg(-\frac{1}{2}(g^{(0)\eta\xi}\nabla^{(0)\tau}\nabla^{(0)\rho}\delta g_{\xi\rho}\\
&+g^{(0)\tau\xi}\nabla^{(0)\eta}\nabla^{(0)\rho}\delta g_{\xi\rho}-g^{(0)\tau\xi}g^{(0)\eta\zeta}\Box^{(0)}\delta g_{\xi\zeta}-\nabla^{(0)\tau}\nabla^{(0)\eta}(g^{(0)\rho\lambda}\delta g_{\rho\lambda})-2R^{(0)\lambda\eta\rho\tau}\delta g_{\lambda\rho}+R^{(0)\rho\tau}g^{(0)\eta\xi}\delta g_{\rho\xi}+R^{(0)\rho\eta}g^{(0)\tau\xi}\delta g_{\rho\xi})\\
&+\frac{1}{2}R^{(0)}g^{(0)\tau\zeta}g^{\eta\xi}\delta g_{\zeta\xi}+\frac{1}{2}g^{(0)\tau\eta}(-\delta g_{\rho\sigma}R^{(0)\rho\sigma}+\nabla^{(0)\rho}\nabla^{(0)\sigma}\delta g_{\rho\sigma}-\Box^{(0)}(g^{(0)\rho\sigma}\delta g_{\rho\sigma})\big)\bigg)\nabla^{(0)}_\tau\nabla^{(0)}_\eta h(\chi^{(0)})\\
&-4(g^{(0)\tau\zeta}\delta g_{\zeta\xi}g^{(0)\xi\alpha}g^{(0)\eta\beta}+g^{(0)\eta\zeta}\delta g_{\zeta\xi}g^{(0)\xi\alpha}g^{(0)\tau\beta})\left(-R^{(0)}_{\alpha\beta}+\frac{1}{2}g^{(0)}_{\alpha\beta}R^{(0)}\right)\nabla^{(0)}_\tau\nabla^{(0)}_\eta h(\chi^{(0)})-4\left(-R^{(0)\tau\eta}+\frac{1}{2}g^{(0)\tau\eta}R^{(0)}\right)\\
&\times\left(-\frac{1}{2}D^{(0)\tau\eta}_{\mu\nu}g^{(0)\zeta\xi}(\nabla^{(0)}_\tau\delta g_{\eta\xi}+\nabla^{(0)}_\eta\delta g_{\tau\xi}-\nabla^{(0)}_\xi\delta g_{\tau\eta})\partial_\eta h(\chi^{(0)})+D^{(0)\tau\eta}_{\mu\nu}\nabla^{(0)}_\tau\nabla^{(0)}_\eta(h'(\chi^{(0)})\delta\chi)\right)\bigg).
\end{aligned}\tag{37}$$

By substituting Eq. (37) into Eq. (36), we may eliminate $\delta\lambda$. The obtained equation contains first and second derivatives of $\delta g_{\mu\nu}$ and χ, especially the first and second derivatives with respect to the cosmic time t. We can choose $\chi^{(0)}$ to be,

$$\chi^{(0)}=\mu^2 t.\tag{38}$$

Then Eq. (34) takes the following form:

$$0 = \delta\dot{\chi} - \mu^2 \delta g_{tt}, \tag{39}$$

and we also have $\delta\ddot{\chi} = \mu^2 \delta\dot{g}_{tt}$. Then we can further eliminate the variation terms $\delta\dot{\chi}$ and $\delta\ddot{\chi}$, and the obtained equation contains the first and second derivatives of $\delta g_{\mu\nu}$ with respect to the cosmic time t, but does not include the first and second derivative terms $\delta\chi$ again with respect to time t. Then by providing the initial conditions for $\delta g_{\mu\nu}$, $\delta\dot{g}_{\mu\nu}$, and χ on a spatial hypersurface, we can determine the time evolution of $\delta g_{\mu\nu}$ uniquely up to the gauge invariance corresponding to the general covariance of the model, and the corresponding constraints. This indicates that the number of physical d.o.f. is only two. Equation (39) also indicates that χ is not dynamical and the time evolution of χ is given by Eq. (39). Therefore, no additional d.o.f. occur, compared to the standard Einstein-Hilbert gravity, and in effect, no ghost modes actually occur in the theory. Having demonstrated that the modified $f(\mathcal{G})$ gravity theory can be rendered ghost free, let us consider several examples of cosmological evolutions which can be realized in the context of this theory. This is the subject of the next subsection.

C. Boundary terms of ghost-free $f(\mathcal{G})$ gravity

In the present paper, our main motivation for deriving the ghost-free equations of motion is their cosmological applications, so we are not concerned with the boundary terms. We shall come to this issue soon; however it is worth discussing the effects of certain boundary terms when one is interested in working on spacetimes with boundaries. In this case, for spacetimes M with boundary ∂M, the variation of the action (28) induces the following terms on the boundary:

$$\begin{aligned}\delta S_{\text{boundary}} = \int_{\partial M} d^3x\sqrt{-q}\Big[&\frac{1}{2\kappa^2}\{n^\mu\nabla^\nu\delta g_{\mu\nu} - n^\rho\nabla_\rho(g^{\mu\nu}\delta g_{\mu\nu})\} + h(\chi)\{Rn^\mu\nabla^\nu\delta g_{\mu\nu} + Rn^\nu\nabla^\mu\delta g_{\mu\nu} - (\nabla^\mu R)n^\nu\delta g_{\mu\nu} - (\nabla^\nu R)n^\mu\delta g_{\mu\nu}\\ &-4n_\rho(R^{\rho\nu}\nabla^\mu\delta g_{\mu\nu} + R^{\rho\mu}\nabla^\nu\delta g_{\mu\nu}) + 4(n^\mu(\nabla_\rho R^{\rho\nu}) + n^\nu(\nabla_\rho R^{\rho\mu}))\delta g_{\mu\nu} + 4n_\rho R^{\mu\nu}\nabla^\rho\delta g_{\mu\nu} - 4n^\rho(\nabla_\rho R^{\mu\nu})\delta g_{\mu\nu}\\ &+4n_R^{\rho\sigma}\nabla_\sigma(g^{\mu\nu}\delta g_{\mu\nu}) - 4n_\sigma(\nabla_\rho R^{\rho\sigma})g^{\mu\nu}\delta g_{\mu\nu} - 4n_\rho R^{\rho\mu\sigma\nu}\nabla_\sigma\delta g_{\mu\nu} + 4n_\sigma(\nabla_\rho R^{\rho\mu\sigma\nu})\delta g_{\mu\nu}\} + h'(\chi)\{-R(\nabla^\mu\chi)n^\nu\delta g_{\mu\nu}\\ &-R(\nabla^\nu\chi)n^\mu\delta g_{\mu\nu} + 4(n^\mu R^{\rho\nu}(\nabla_\rho\chi) + n^\nu R^{\rho\mu}(\nabla_\rho\chi))\delta g_{\mu\nu} - 4n^\rho R^{\mu\nu}(\nabla_\rho\chi)\delta g_{\mu\nu} - 4n_\sigma R^{\rho\sigma}(\nabla_\rho\chi)g^{\mu\nu}\delta g_{\mu\nu}\\ &+4n_\sigma R^{\rho\mu\sigma\nu}(\nabla_\rho\chi)\delta g_{\mu\nu}\} - \lambda n^\mu\partial_\mu\chi\Big],\end{aligned} \tag{40}$$

where n_μ is a unit vector ($n_\mu n_\mu = 1$ if n_μ is spacelike and $n_\mu n_\mu = -1$ if n_μ is timelike) which is perpendicular and directed outward at the boundary, $l_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu$ is the induced metric on the boundary and l is the determinant of $l_{\mu\nu}$. In order for the variational principle to be well defined, we need to require $\delta S_{\text{boundary}} = 0$, which cannot be realized because $\delta S_{\text{boundary}}$ includes both $\delta g_{\mu\nu}$ without a derivative and $\nabla_\sigma\delta g_{\mu\nu}$. In order to avoid this problem, we can add Gibbons-Hawking boundary terms [27],

$$wS_{\text{GH}} = \frac{1}{\kappa^2}\int_{\partial M} d^3\sqrt{-l}l^{\mu\nu}\nabla_\mu n_\nu, \tag{41}$$

for the part corresponding to the Einstein-Hilbert term, which is proportional to $\frac{1}{\kappa^2}$ in (40) or Myers-like boundary terms [28],

$$\begin{aligned}&S_{\text{M}} = 2\int_{\partial M} d^3x\sqrt{-l}h(\chi)\left\{\frac{1}{3}(2KK_{\mu\rho}K^{\rho\nu} + K_{\rho\sigma}K^{\rho\sigma}K - 2K_{\mu\rho}K^{\rho\sigma}K_\sigma{}^\mu - K^3) - G_l^{\mu\nu}K_{\mu\nu}\right\},\\ &K_{\mu\nu} \equiv l^\rho{}_\mu l^\sigma{}_\nu\nabla_\rho n_\sigma,\end{aligned} \tag{42}$$

for the terms proportional to $h(\phi)$ but not to $h'(\phi)$. For some recent useful applications of Gibbons-Hawking-like terms in Euclidean gravity, see Refs. [29,30]. In particular, the terms proportional to $h'(\phi)$ and λ give some boundary conditions for the scalar fields χ and λ, which could be, e.g.,

$$h'(\chi) = 0, \quad \lambda = 0, \tag{43}$$

which could correspond to the boundary conditions chosen in Refs. [29,30].

However in most cosmologies with a homogeneous and isotropic metric, the most characteristic type of metric chosen is a Friedmann-Robertson-Walker (FRW) metric, with or without spatial curvature. In the flat FRW case, the topological spaces not excluded by current are the three-torus which is flat in four-dimensional spacetime and the infinitely large three-dimensional Euclidean flat plane, in which case no boundaries occur, unless some strong finite-time singularity occurs in the future. In that case, the singularities which lead to geodesic incompleteness, like the big rip, may eventually lead to certain forms of boundaries on the spacelike hypersurface on which the singularities occur, but this effect is hard to quantify with Gibbons-Hawking terms because the end of a future timelike geodesic is highly nontrivial to define from a mathematical point of view, so no induced metric can be defined on it, and actually closed timelike curves can occur and at the same time be absorbed in the same notion as the future singularity. Some useful treatment of these issues can be found in [31]. So we refrain from discussing the boundary terms issue further, which is however useful for noncosmological applications.

III. FRW COSMOLOGY IN GHOST-FREE $f(\mathcal{G})$ GRAVITY

In this section, we consider the cosmology produced by the ghost-free $f(\mathcal{G})$ gravity model of Eq. (28). In particular, we show that it is possible to realize any cosmological era of the Universe, by using the model under consideration. Specifically, we will try to realize the late- and early-time acceleration eras.

A. A reconstruction technique for model building

Let us first demonstrate how the equations of motion of the model (28) are modified in the case where the metric is a flat FRW metric with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2. \tag{44}$$

For this metric, we have,

$$\begin{aligned}
&\Gamma^t_{ij} = a^2 H \delta_{ij}, \qquad \Gamma^i_{jt} = \Gamma^i_{tj} = H\delta^i_j, \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk},\\
&R_{itjt} = -(\dot{H} + H^2)a^2\delta_{ij}, \qquad R_{ijkl} = a^4H^2(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}),\\
&R_{tt} = -3(\dot{H} + H^2), \qquad R_{ij} = a^2(\dot{H} + 3H^2)\delta_{ij},\\
&R = 6\dot{H} + 12H^2, \qquad \text{other components} = 0,\\
&\mathcal{G} = 24H^2(\dot{H} + H^2),
\end{aligned} \tag{45}$$

where $H \equiv \frac{\dot{a}}{a}$. We also assume that λ and χ depend solely on the cosmic time t, that is, $\lambda = \lambda(t)$ and $\chi = \chi(t)$. We also assume $T_{\text{matter}\,\mu\nu} = 0$ just for simplicity. Then a solution of Eq. (26) is given as

$$\chi = \mu^2 t. \tag{46}$$

In effect, the (t,t) component and (i,j) component of Eq. (30) yield,

$$0 = -\frac{3H^2}{2\kappa^2} - \frac{\mu^4\lambda}{2} + \frac{1}{2}\tilde{V}(\mu^2 t) - 12\mu^2 H^3 h'(\mu^2 t), \tag{47}$$

$$0 = \frac{1}{2\kappa^2}(2\dot{H} + 3H^2) - \frac{1}{2}\tilde{V}(\mu^2 t) + 4\mu^4 H^2 h''(\mu^2 t) + 8\mu^2(\dot{H} + H^2)Hh'(\mu^2 t). \tag{48}$$

On the other hand, Eq. (29) gives,

$$0 = \mu^2\dot{\lambda} + 3\mu^2 H\lambda + 24H^2(\dot{H} + H^2)h'(\mu^2 t) - \tilde{V}'(\mu^2 t). \tag{49}$$

Equation (47) can be solved with respect to λ as follows:

$$\lambda = -\frac{3H^2}{\mu^4\kappa^2} + \frac{1}{\mu^4}\tilde{V}(\mu^2 t) - \frac{24}{\mu^2}H^3 h'(\mu^2 t). \tag{50}$$

Then by substituting Eq. (50) into Eq. (49), we reobtain Eq. (49). On the other hand, Eq. (48) can be solved with respect to $\tilde{V}(\mu^2 t)$ as follows:

$$\tilde{V}(\mu^2 t) = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + 8\mu^4 H^2 h''(\mu^2 t) + 16\mu^2(\dot{H} + H^2)Hh'(\mu^2 t), \tag{51}$$

which tells that for arbitrary $h(\chi)$, if the potential $\tilde{V}(\chi)$ is assumed to be equal to,

$$\tilde{V}(\chi) = \left[\frac{1}{\kappa^2}(2\dot{H} + 3H^2) + 8\mu^4 H^2 h''(\mu^2 t) + 16\mu^2(\dot{H} + H^2)Hh'(\mu^2 t)\right]_{t=\frac{\chi}{\mu^2}}, \tag{52}$$

then an arbitrary cosmological evolution of the Universe with Hubble rate $H = H(t)$ can be realized. By combining Eqs. (50) and (51), we also obtain,

$$\lambda = \frac{2\dot{H}}{\mu^4\kappa^2} + 8H^2 h''(\mu^2 t) + \frac{8}{\mu^2}(2\dot{H} - H^2)Hh'(\mu^2 t). \tag{53}$$

Basically the above procedure is a reconstruction method for the model (28) and by using this method it is possible to realize an arbitrarily given cosmological evolution. In the next subsection we shall use this reconstruction method.

B. Early- and late-time accelerating universe cosmologies with ghost-free $f(\mathcal{G})$ gravity

In this subsection, we consider some examples of models which describe an accelerating universe. As a first example, we consider a de Sitter spacetime realization, in which case the Hubble rate H is a constant $H = H_0$. Then by using Eq. (52), for an arbitrarily chosen function $h(\chi)$, the corresponding scalar potential is given by,

$$\tilde{V}(\chi) = \frac{3H_0^2}{\kappa^2} + 8\mu^4 H_0^2 h''(\chi) + 16\mu^2 H_0^3 h'(\chi). \tag{54}$$

Equation (53) also indicates how the Lagrange multiplier λ in this model behaves, and it is equal to,

$$\lambda(t) = 8H_0^2 h''(\mu^2 t) - \frac{8}{\mu^2}H_0^3 h'(\mu^2 t). \tag{55}$$

Then by appropriately choosing the functional form of $h(\chi)$, we can obtain several different ghost-free $f(\mathcal{G})$ models which can realize a de Sitter evolution. Next we consider the model which mimics the Λ cold dark matter (ΛCDM) model, in which case the Hubble rate H is given by,

$$H = H_0 \coth\left(\frac{3}{2}H_0 t\right). \tag{56}$$

At late times, that is in the limit $t \to +\infty$, H in Eq. (56) behaves as follows:

$$H \to H_0, \tag{57}$$

which corresponds to an asymptotic de Sitter spacetime. On the other hand, at early times, that is in the limit $t \to 0$, the Hubble rate behaves as follows:

$$H \to \frac{3}{2t}, \tag{58}$$

which corresponds to a matter- or dust-dominated universe. Then by using Eq. (52), we find,

$$\tilde{V}(\chi) = \frac{3H_0^2}{\kappa^2} + 8\mu^4 H_0^2 \coth^2\left(\frac{3H_0}{2\mu^2}\chi\right) h''(\chi) + 16\mu^2 H_0^3 \left(1 - \frac{1}{2\sinh^2(\frac{3H_0}{2\mu^2}\chi)}\right) \coth\left(\frac{3H_0}{2\mu^2}\chi\right) h'(\chi), \tag{59}$$

and from Eq. (53) we can determine the functional form of the Lagrange multiplier λ, which is,

$$\lambda = \frac{3H_0^2}{\mu^4\kappa^2 \sinh^2(\frac{3}{2}H_0 t)} + 8H_0^2 \coth^2\left(\frac{3}{2}H_0 t\right) h''(\mu^2 t) - \frac{8H_0^3}{\mu^2}\left(1 + \frac{4}{\sinh^2(\frac{3}{2}H_0 t)}\right) \coth\left(\frac{3}{2}H_0 t\right) h'(\mu^2 t). \tag{60}$$

The model of Eq. (56), which is generated in the context of ghost-free $f(\mathcal{G})$ gravity by the scalar potential of Eq. (59), realizes the ΛCDM model without introducing any dark matter perfect fluid. Therefore, the model incorporates the cosmological constant part, corresponding to an equation-of-state (EoS) parameter being equal to $w = -1$, and also incorporates the CDM part, corresponding to an EoS parameter exactly equal to $w = 0$. Thus we have realized the present accelerating expansion of the Universe by using the ghost-free $f(\mathcal{G})$ gravity model. Notably, the cosmological evolution (56) can be realized in the context of ghost-free $f(\mathcal{G})$ gravity by using a function $h(\chi)$ and an arbitrary parameter μ^2. In the case of the standard Einstein-Hilbert gravity, the FRW equations have the following form:

$$\frac{3}{\kappa^2}H^2 = \rho_{\mathrm{total}}, \qquad -\frac{1}{\kappa^2}(2\dot{H} + 3H^2) = p_{\mathrm{total}}, \tag{61}$$

where ρ_{total} and p_{total} are the total energy density and the total pressure. In effect, the total EoS parameter w_{total} defined by $w_{\mathrm{total}} = \frac{p_{\mathrm{total}}}{\rho_{\mathrm{total}}}$ is equal to,

$$w_{\mathrm{total}} = -1 - \frac{2\dot{H}}{3H^2}. \tag{62}$$

We should note that the effective total EoS parameter w_{total} includes the contributions of all the fluid components of the Universe like dark energy, dark matter, and so on. The Planck 2018 results [32], constrain the Hubble constant, which is the present value of the Hubble rate, as follows: $H_{\mathrm{present}} = (67.4 \pm 0.5)\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Also the matter density parameter is constrained as $\Omega_{\mathrm{m}} = 0.315 \pm 0.007$ and finally, the dark energy EoS parameter is constrained as $w_0 = -1.03 \pm 0.03$ although w_0 is different from $w_{\mathrm{total}}^{\mathrm{eff}}$. Since $p_{\mathrm{total}} = (1 - \Omega_{\mathrm{m}}) w_0 \rho_{\mathrm{total}}$, the Planck 2018 results indicate that,

$$w_{\mathrm{total}} = (1 - \Omega_{\mathrm{m}}) w_0 \sim -0.705. \tag{63}$$

Even for a general modified gravity theory, in the case of ghost-free $f(\mathcal{G})$ gravity we developed in this paper, the effective total EoS parameter $w_{\mathrm{total}}^{\mathrm{eff}}$ is defined in Eq. (62), that is,

$$w_{\mathrm{total}}^{\mathrm{eff}} = -1 - \frac{2\dot{H}}{3H^2}. \tag{64}$$

Then in the case of the de Sitter space as in the model (52) in this paper, since the Hubble rate is a constant, $H = H_0$, we find $w_{\mathrm{total}}^{\mathrm{eff}} = -1$. On the other hand, in the case of the model mimicking the ΛCDM model, namely Eq. (56), we find,

$$w_{\mathrm{total}}^{\mathrm{eff}} = -1 - \frac{1}{\cosh^2(\frac{3}{2}H_0 t_{\mathrm{present}})}, \tag{65}$$

where t_{present} is the value of the cosmic time today. In the model (56), the dark matter contribution to the evolution is effectively included. Then the Planck 2018 results (63) constrain the parameters of the model (56). Due to the fact that the observed Hubble constant is $H_{\mathrm{present}} = (67.4 \pm 0.5)\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, by using Eq. (56) we find,

$$H_0 \coth\left(\frac{3}{2}H_0 t_{\mathrm{present}}\right) = (67.4 \pm 0.5)\ \mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{66}$$

On the other hand, combined with Eq. (65), the Planck 2018 results (63) indicate that,

$$\frac{1}{\cosh^2(\frac{3}{2}H_0 t_{\mathrm{present}})} \sim 0.294. \tag{67}$$

Then Eqs. (66) and (67) actually constrain the parameters H_0 and $t_{\rm present}$ of the model, so these can appropriately be chosen so that the constraints are satisfied.

In addition, since the ΛCDM model is still consistent with any constraint obtained from the observations on the current expansion of the Universe, the model (56) mimicking the ΛCDM model should be consistent with the current observational data. In the future, perhaps some deviations from the standard ΛCDM model may be observed. Then by using the formulation of ghost-free $f(\mathcal{G})$ gravity presented in this paper, we can always construct a more realistic model than the ΛCDM model, according to future observations.

As another model, we shall consider the following cosmological model with parameters, $\delta, H_0, H_i, t_s, \mu$, and Λ:

$$H(t) = \delta e^{H_0 - H_i t} \tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda, \tag{68}$$

where the parameters μ and H_i are measured in seconds in natural units, while the parameter δ has dimensions $\sec^{-2}$ in natural units. In addition, the parameter H_0 is considered to be dimensionless. The above model has quite interesting early- and late-time phenomenology if the free parameters are appropriately chosen, since it can qualitatively describe a quasi–de Sitter cosmological evolution at early times and an accelerating era of de Sitter form at late times. Indeed, if the parameter t_s is chosen to be the age of the present Universe, and also if the parameter Λ is chosen to be the present-time cosmological constant, then at early times when $t \ll t_s$, the first term is approximated as follows:

$$H(t) \sim \delta(e^{H_0} - e^{H_0} H_i t) - \Lambda, \tag{69}$$

due to the fact that at early times,

$$\tanh\left(\frac{t_s - t}{\mu}\right) \sim 1. \tag{70}$$

Hence, if H_0 and H_i are appropriately chosen so that $e^{H_0}, H_i \gg \Lambda$, the early-time evolution is a quasi–de Sitter evolution of the form,

$$H(t) \sim \delta(e^{H_0} - e^{H_0} H_i t), \tag{71}$$

and the effective EoS parameter is nearly $w_{\rm total}^{\rm eff} \sim -1$. Accordingly, at late times when $t \sim t_s$, the exponential in Eq. (68) tends to zero, and also we have,

$$\tanh\left(\frac{t_s - t}{\mu}\right) \sim 0. \tag{72}$$

In effect, the Hubble rate is again approximated by an exact de Sitter evolution,

$$H(t) \sim \Lambda. \tag{73}$$

The realization of the model (69) in the context of ghost-free $f(\mathcal{G})$ gravity is possible, if the scalar potential is equal to,

$$\begin{aligned} V(\chi(t)) = {}& \frac{3(e^{H_0 - H_i t}\tanh(\frac{t_s - t}{\mu}) + \Lambda)^2 - \frac{2e^{H_0 - H_i t}(H_i \mu \tanh(\frac{t_s - t}{\mu}) + \operatorname{sech}^2(\frac{t_s - t}{\mu}))}{\mu}}{\kappa^2} + 16\mu^2\left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right) \\ & \times h'(\chi)\left(\left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right)^2 - H_i e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) - \frac{e^{H_0 - H_i t}\operatorname{sech}^2(\frac{t_s - t}{\mu})}{\mu}\right) \\ & + 8\mu^4 h''(\chi)\left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right)^2, \end{aligned} \tag{74}$$

and the Lagrange multiplier function $\lambda(\chi(t))$ is chosen as,

$$\begin{aligned} \lambda(\chi(t)) = {}& \frac{2(-H_i e^{H_0 - H_i t}\tanh(\frac{t_s - t}{\mu}) - \frac{e^{H_0 - H_i t}\operatorname{sech}^2(\frac{t_s - t}{\mu})}{\mu})}{\kappa^2 \mu^4} + \frac{8h'(\chi)}{\mu^2}\left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right) \\ & \times\left(2\left(-H_i e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) - \frac{e^{H_0 - H_i t}\operatorname{sech}^2(\frac{t_s - t}{\mu})}{\mu}\right) - \left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right)^2\right) \\ & + 8h''(\chi)\left(e^{H_0 - H_i t}\tanh\left(\frac{t_s - t}{\mu}\right) + \Lambda\right)^2. \end{aligned} \tag{75}$$

By appropriately choosing the function $h(\chi)$, one may obtain different models which can realize the same cosmological evolution (69), so a rich phenomenology can be obtained. The scalar potential at early times is much more simple, since it takes the form,

$$V(\chi(t)) \sim \frac{3(e^{H_0} - e^{H_0}H_i t)^2 - 2e^{H_0}H_i}{\kappa^2} + 8\mu^4 h''(\chi)(e^{H_0} - e^{H_0}H_i t)^2 + 16\mu^2 h'(\chi)(e^{H_0} - e^{H_0}H_i t) \times ((e^{H_0} - e^{H_0}H_i t)^2 - e^{H_0}H_i), \quad (76)$$

while at late times it is approximated by,

$$V(\chi(t)) \sim 8\Lambda^2\mu^4 h''(\chi) + 16\Lambda^3\mu^2 h'(\chi) + \frac{3\Lambda^2}{\kappa^2}. \quad (77)$$

The most interesting feature of the ghost-free model can be seen by looking at Eqs. (76) and (77), due to the presence of the function $h(\chi)$ in both equations. This means that by appropriately choosing the function $h(\chi)$ so that a viable early-time phenomenology is obtained, this choice will affect the late-time phenomenology to some extent, not via the late-time Hubble rate, but certainly through the scalar potential and the Lagrange multiplier function λ. Therefore, quite interesting phenomenologies may be obtained, due to the fact that during the two eras the EoS parameter is nearly $w_{\text{total}}^{\text{eff}} \sim -1$, and hence the potential and the Lagrange multiplier function may affect other observable quantities and render the model more compatible with the observational data. Work is in progress in this direction.

Before closing this section we should note that other cosmological evolutions can be realized in the context of the ghost-free $f(\mathcal{G})$ theory developed in this paper. For example, consider the symmetric bounce with Hubble rate,

$$H(t) = e^{\alpha t^2}, \quad (78)$$

which a well-known bounce cosmology [33,34]. The symmetric bounce has interesting phenomenology, since in the limit $t \to -\infty$, the EoS parameter is approximately, $w_{\text{total}}^{\text{eff}} \sim -1$, which is a nearly de Sitter phase. After that and as the bouncing point at $t = 0$ is approached, the Universe experiences quintessential acceleration which gradually turns into a decelerating expansion. Near the bouncing point, the Universe experiences another nearly de Sitter accelerating era, and as the cosmic time grows it is followed by a phantom accelerating era, which eventually tends to a nearly de Sitter expansion at $t \to \infty$. It is conceivable that the most interesting part of this bounce cosmology, from a phenomenological point of view, is the contracting
phase. This cosmological evolution can be realized by the scalar potential,

$$V(\chi) = \frac{e^{\alpha t^2}(8\kappa^2\mu^4 e^{\alpha t^2} h''(\chi) + 16\kappa^2\mu^2 e^{\alpha t^2}(e^{\alpha t^2} + 2\alpha t)h'(\chi) + 3e^{\alpha t^2} + 4\alpha t)}{\kappa^2}, \quad (79)$$

where $\chi = t\mu^2$, and also by the Lagrange multiplier function $\lambda(\chi)$,

$$\lambda(\chi) = 8e^{2\alpha t^2} h''(\chi) + \frac{8e^{\alpha t^2}(4\alpha t e^{\alpha t^2} - e^{2\alpha t^2})h'(\chi)}{\mu^2} + \frac{4\alpha t e^{\alpha t^2}}{\kappa^2\mu^4}, \quad (80)$$

where in both Eqs. (79) and (80), the function $h(\chi)$ is arbitrary. Thus in the context of the formalism we developed, we do not have a single model realizing the symmetric bounce, but rather a class of models which can realize this cosmological evolution. In principle, the choice of the function $h(\chi)$ can be made in such a way that the phenomenological constraints can be satisfied. We do not discuss this topic further for brevity, but it is conceivable that there is much room for realizing interesting phenomenologies.

IV. NEWTON'S LAW IN GHOST-FREE $f(\mathcal{G})$ GRAVITY

In this section we shall consider Newton's law in the context of ghost-free $f(\mathcal{G})$ gravity and we shall investigate how it is modified in the ghost-free theory. Some alternative solutions in the context of general Gauss-Bonnet theories can be found in Refs. [35,36]. In order to consider the correction to Newton's law, we assume that the geometric background is flat, by considering the limit of $H \to 0$ in the last section. This is because we like to consider Newton's law at scales much smaller in comparison to the cosmological scales, which are of the order $\sim \frac{1}{H}$ in an asymptotically de Sitter spacetime during the present era of the Universe. Then Eq. (51) or Eq. (52) indicates that $\tilde{V}(\chi) = 0$ although $h(\chi)$ can be an arbitrary function in general. Therefore Eq. (50) suggests that $\lambda = \lambda^{(0)} = 0$. We also assume that the gauge condition (15) holds true. Then by using Eqs. (38), (34), (35), (36), and (37), we obtain,

$$0 = -\mu^2\partial_t\delta\chi - \mu^4\delta g_{tt}, \quad (81)$$

$$0 = \mu^2\partial_t\delta\lambda - h'(\chi^{(0)})\Box^{(0)}(\eta^{\mu\nu}\delta g_{\mu\nu}), \quad (82)$$

$$\begin{aligned}0 = &-\frac{1}{4\kappa^2}(-\Box^{(0)}\delta g_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + g^{(0)}_{\mu\nu}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma})) + \frac{1}{2}\delta T_{\mathrm{matter}\,\mu\nu} - \frac{1}{2}\mu^4\delta_{t\mu}\delta_{t\nu}\delta\lambda \\ &+ \mu^4\{2\eta_{\mu\nu}\Box^{(0)}(\eta^{\zeta\xi}\delta g_{\zeta\xi}) + 2(-\eta^{\rho t}\delta_\mu{}^t\delta_\nu{}^\sigma - \eta^{\rho t}\delta_\nu{}^t\delta_\mu{}^\sigma + \eta_{\mu\nu}\eta^{\rho t}\eta^{\sigma t})(-\Box^{(0)}\delta g_{\rho\sigma} - \partial_\rho\partial_\sigma(\eta^{\zeta\xi}\delta g_{\zeta\xi})) \\ &- 4(-\Box^{(0)}\delta g_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\zeta\xi}\delta g_{\zeta\xi})) - 2(\partial_t\partial_\mu\delta g_{\nu t} - \partial_t^2\delta g_{\nu\mu} - \partial_\nu\partial_\mu\delta g_{tt} + \partial_\nu\partial_t\delta g_{t\mu})\}h''(\chi^{(0)}), \end{aligned} \tag{83}$$

$$\delta\lambda = -\frac{2}{\mu^4}\left(-\frac{1}{2\kappa^2}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}) + \frac{1}{2}\delta T_{\mathrm{matter}} - 2\mu^4(\Box^{(0)}\delta g_{tt} + \partial_t^2(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + \Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}))h''(\chi^{(0)})\right). \tag{84}$$

By substituting Eq. (84) into Eq. (83), we obtain,

$$\begin{aligned}0 = &-\frac{1}{4\kappa^2}(-\Box^{(0)}\delta g_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + g^{(0)}_{\mu\nu}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma})) + \frac{1}{2}\delta T_{\mathrm{matter}\,\mu\nu} \\ &+ \delta_{t\mu}\delta_{t\nu}\left\{-\frac{1}{2\kappa^2}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}) + \frac{1}{2}\delta T_{\mathrm{matter}} - 2\mu^4(\Box^{(0)}\delta g_{tt} + \partial_t^2(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + \Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}))h''(\chi^{(0)})\right\} \\ &+ \mu^4\{2\eta_{\mu\nu}\Box^{(0)}(\eta^{\zeta\xi}\delta g_{\zeta\xi}) + 2(-\eta^{\rho t}\delta_\mu{}^t\delta_\nu{}^\sigma - \eta^{\rho t}\delta_\nu{}^t\delta_\mu{}^\sigma + \eta_{\mu\nu}\eta^{\rho t}\eta^{\sigma t})(-\Box^{(0)}\delta g_{\rho\sigma} - \partial_\rho\partial_\sigma(\eta^{\zeta\xi}\delta g_{\zeta\xi})) \\ &- 4(-\Box^{(0)}\delta g_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\zeta\xi}\delta g_{\zeta\xi})) - 2(\partial_t\partial_\mu\delta g_{\nu t} - \partial_t^2\delta g_{\nu\mu} - \partial_\nu\partial_\mu\delta g_{tt} + \partial_\nu\partial_t\delta g_{t\mu})\}h''(\chi^{(0)}). \end{aligned} \tag{85}$$

We shall consider a static point gravitational source for the matter at the spatial origin, that is,

$$\delta T_{\mathrm{matter}\,tt} = M\delta^{(3)}(\boldsymbol{x}), \quad \text{other components of } \delta T_{\mathrm{matter}\,\mu\nu} = 0, \tag{86}$$

where $(\boldsymbol{x}) = (x^i)$. In the following two subsections, we shall investigate how Newton's law is modified in the context of Lagrange-multiplier-constrained Einstein-Hilbert gravity and in the context of ghost-free $f(\mathcal{G})$ gravity.

A. Newton's law for Lagrange-multiplier-constrained Einstein-Hilbert gravity

Let us first consider the constrained Einstein-Hilbert gravity case, in which case $h(\chi) = 0$. Then Eq. (85) reduces to,

$$\begin{aligned}0 = &-\frac{1}{4\kappa^2}\{-\Box^{(0)}\delta g_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + g^{(0)}_{\mu\nu}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma})\} + \frac{1}{2}\delta T_{\mathrm{matter}\,\mu\nu} \\ &+ \delta_{t\mu}\delta_{t\nu}\left\{-\frac{1}{2\kappa^2}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}) + \frac{1}{2}\delta T_{\mathrm{matter}}\right\}. \end{aligned} \tag{87}$$

The (t,t), (i,j), and (t,i) components of Eq. (87) yield,

$$0 = -\frac{1}{4\kappa^2}\{-\Box^{(0)}\delta g_{tt} - \partial_t^2(\eta^{\rho\lambda}\delta g_{\rho\lambda})\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma})\}, \tag{88}$$

$$0 = -\frac{1}{4\kappa^2}\{-\Box^{(0)}\delta g_{ij} - \partial_i\partial_j(\eta^{\rho\lambda}\delta g_{\rho\lambda}) + \delta_{ij}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma})\}, \tag{89}$$

$$0 = -\frac{1}{4\kappa^2}\{-\Box^{(0)}\delta g_{ti} - \partial_t\partial_i\nu(\eta^{\rho\lambda}\delta g_{\rho\lambda})\}, \tag{90}$$

and Eq. (84) has the following form:

$$\delta\lambda = -\frac{2}{\mu^4}\left(-\frac{1}{2\kappa^2}\Box^{(0)}(\eta^{\rho\sigma}\delta g_{\rho\sigma}) + \frac{1}{2}\delta T_{\mathrm{matter}}\right). \tag{91}$$

We now assume that,

$$\delta g_{tt} = A(r), \qquad \delta g_{ij} = B(r)\delta_{ij} + C(r)x^i x^j, \qquad \delta g_{ti} = 0, \tag{92}$$

where $r = \sqrt{\sum_{i=1,2,3}(x^i)^2}$. Then Eq. (90) is trivially satisfied and since,

$$\begin{aligned}
\eta^{\rho\lambda}\delta g_{\rho\lambda} &= -A + 3B + r^2 C, \\
\triangle(x^i x^j C(r)) &= 2\delta_{ij}C(r) + \frac{6x^i x^j}{r}C'(r) + x^i x^j C''(r) \\
\partial_i\partial_j(-A + 3B + r^2 C) &= \frac{\delta_{ij}}{r}(-A' + 3B' + 2rC + r^2 C') + \frac{x^i x^j}{r^3}(A' - rA'' - 3B' + 3rB'' + 3r^2 C' + r^3 C'') \\
\triangle(-A + 3B + r^2 C) &= \frac{1}{r}(-2A' - rA'' + 6B' + 3rB'' + 6rC + 6r^2 C' + r^3 C''') \qquad (93)
\end{aligned}$$

Eqs. (88) and (89) have the following forms:

$$0 = \frac{1}{4\kappa^2}\triangle(3B + r^2 C), \qquad (94)$$

$$0 = -\frac{1}{4\kappa^2}\left\{-\frac{\delta_{ij}}{r}(A' + rA'' - B' - 2rB'' - 2rC - 5r^2 C' - r^3 C'') - \frac{x^i x^j}{r^3}(A' - rA'' - 3B' + 3rB'' + 9r^2 C' + 2r^3 C'')\right\}. \qquad (95)$$

In effect, we have,

$$0 = 3B + r^2 C, \qquad (96)$$

$$0 = A' + rA'' - B' - 2rB'' - 2rC - 5r^2 C' - r^3 C'', \qquad (97)$$

$$0 = A' - rA'' - 3B' + 3rB'' + 9r^2 C' + 2r^3 C''. \qquad (98)$$

By using Eq. (96), we can eliminate B from Eqs. (97) and (98), so we get,

$$0 = A' + rA'' - 2r^2 C' - \frac{r^3}{3}C'', \qquad (99)$$

$$0 = A' - rA'' + 6r^2 C' + r^2 C''. \qquad (100)$$

By also eliminating C from Eqs. (99) and (100), we obtain,

$$0 = 4A' + 2rA''. \qquad (101)$$

Under the boundary condition that $A \to 0$ when $r \to \infty$, the solution of Eq. (101) is given by,

$$A = \frac{A_0}{r}, \qquad (102)$$

with a constant A_0. Then Eq. (97) takes the following form:

$$0 = \frac{A_0}{r^2} - \frac{1}{3r^3}(r^6 C')', \qquad (103)$$

and a solution of the above equation is,

$$C = -\frac{A_0}{2r^3}. \qquad (104)$$

In effect, Eq. (96) indicates that,

$$B = \frac{A_0}{6r}, \qquad (105)$$

where we have assumed that the boundary condition $B, C \to 0$ when $r \to \infty$ holds true. Equation (91) also suggests that,

$$\delta\lambda = -\frac{2}{\mu^4}\left(-\frac{4\pi A_0}{2\kappa^2}\delta^{(3)}(\boldsymbol{x}) + \frac{1}{2}M\delta^{(3)}(\boldsymbol{x})\right). \qquad (106)$$

If we put $\delta\lambda = 0$, we find,

$$A_0 = \frac{\kappa^2 M}{4\pi}, \qquad (107)$$

which reproduces the standard Newtonian potential $\phi_{\rm Newton}$, that is,

$$\phi_{\rm Newton} \equiv \frac{A}{2} = \frac{\kappa^2 M}{8\pi} = \frac{GM}{r}, \qquad (108)$$

where $G = \frac{\kappa^2}{8\pi}$ is Newton's gravitational constant. We should note, however, that Eq. (106) indicates that there is an infinite number of solutions, which do not always reproduce the standard Newton's law if $\delta\lambda \neq 0$. In addition, Eq. (82) indicates that $0 = \partial_t\delta\lambda$ if $h = 0$, which corresponds to the Einstein-Hilbert gravity case. Therefore, if we put $\delta\lambda = 0$ as an initial condition, then the term $\delta\lambda$ always vanishes, and the model reproduces the standard Newton's law.

B. Newton's law in ghost-free $f(\mathcal{G})$ gravity

Let us now investigate how Newton's law is modified in the context of the ghost-free $f(\mathcal{G})$ gravity model (28). First we assume that Eq. (92) holds true in this case too. Then the general solutions of Eqs. (81) and (82) are given by,

$$\begin{aligned}
\delta\chi &= -\mu^2 t A(r) + c_1(\boldsymbol{x}), \\
\delta\lambda &= \frac{1}{\mu^2 r^2}h(\mu^2 t)(r^2(-A(r) + 3B(r) + r^2 C(r))')' + c_2(\boldsymbol{x}),
\end{aligned} \qquad (109)$$

where $c_1(\boldsymbol{x})$ and $c_2(\boldsymbol{x})$ appear by integrating with respect to t, and these can be determined by Eq. (83). However if we assume spherical symmetry, then $c_1(\boldsymbol{x})$ and $c_2(\boldsymbol{x})$ should depend on $\boldsymbol{x}$ via the radial coordinate r, that is, $c_1(\boldsymbol{x}) = c_1(r)$ and $c_2(\boldsymbol{x}) = c_2(r)$. On the other hand, Eq. (84) has the following form:

$$\delta\lambda = -\frac{2}{\mu^4}\left\{-\frac{1}{2\kappa^2}(r^2(-A(r)+3B(r)+r^2C(r))')' - \frac{M}{2}\delta^{(3)}(\boldsymbol{x}) - \frac{2\mu^4}{r^2}(r^2(3B(r)+r^2C(r))')'h''(\mu^2 t)\right\}. \tag{110}$$

By comparing $\delta\lambda$ from Eq. (109) with Eq. (110), for arbitrary $h(\chi)$, we find $A(r) = 3B(r) + r^2C(r) = 0$ and $c_2(r) = -\frac{M}{2}\delta^{(3)}(\boldsymbol{x})$. If surely $A(r) = 0$, the result is in conflict with the resulting Newton's law of the constrained Einstein gravity case, given in Eq. (108). This indicates that the assumption (92) is not satisfied and the correction to Newton's law should be time dependent, which could constrain μ^2, $h(\chi)$, and/or $\tilde{V}(\chi)$, so that the correction could be consistent with any experiment or observation. Equation (85) indicates that the correction to Newton's law in the case of Einstein-Hilbert gravity is proportional to the parameter μ^4 and the function $h(\chi)$, and therefore if μ^4 or $h(\chi)$ is small enough, the constraint for Newton's law is always satisfied. For the case of the model (56) which mimics the ΛCDM model, as long as we consider Newton's law at scales much smaller than the cosmological scales $\sim\frac{1}{H}$ and as long as $h(\chi)$ is small, the constraint for Newton's law is independent of the cosmological constraints. So the constraints (66) and (67) can be imposed without restricting μ and $h(\chi)$.

V. GHOST-FREE $F(R,\mathcal{G})$ GRAVITY

As a final task we shall demonstrate how to obtain a ghost-free $F(R,\mathcal{G})$ theory of gravity. The vacuum $F(R,\mathcal{G})$ gravity action is,

$$S = \int d^4x\sqrt{-g}F(R,\mathcal{G}), \tag{111}$$

where $F(R,\mathcal{G})$ is a function of the scalar curvature R and $\mathcal{G}$ stands for the Gauss-Bonnet invariant given in Eq. (2). It was claimed that this model (111) has ghost instabilities [20], so let us see how ghost d.o.f. are manifested at the level of the equations of motion. By introducing two auxiliary fields Φ and Θ, the action of Eq. (111) can be rewritten as follows:

$$S = \int d^4x\sqrt{-g}\left\{\frac{\Phi R}{2\kappa^2} + \Theta\mathcal{G} - V(\Phi,\Theta)\right\}, \tag{112}$$

where we have introduced the gravitational coupling κ in order to make Φ and Θ dimensionless. By varying the action (112) with respect to Φ and Θ, we obtain,

$$\frac{R}{2\kappa^2} = \frac{\partial V(\Phi,\Theta)}{\partial\Phi}, \qquad \mathcal{G} = \frac{\partial V(\Phi,\Theta)}{\partial\Theta}, \tag{113}$$

which can be algebraically solved with respect to Φ and Θ, that is, $\Phi = \Phi(R,\mathcal{G})$ and $\Theta = \Theta(R,\mathcal{G})$. Then by substituting the obtained expressions for $\Phi = \Phi(R,\mathcal{G})$ and $\Theta = \Theta(R,\mathcal{G})$ into Eq. (112), we obtain the action (111) with,

$$F(R,\mathcal{G}) = \Phi(R,\mathcal{G})R + \Theta(R,\mathcal{G})\mathcal{G} - V(\Phi(R,\mathcal{G}),\Theta(R,\mathcal{G})). \tag{114}$$

In order to investigate the properties of the action (112), we work in the Einstein frame, so under a conformal transformation of the form $g_{\mu\nu} \to e^{\phi}g_{\mu\nu}$, the curvatures are transformed as follows [17,37]:

$$\begin{aligned}
R_{\zeta\mu\rho\nu} \to & \left\{R_{\zeta\mu\rho\nu} - \frac{1}{2}(g_{\zeta\rho}\nabla_\nu\nabla_\mu\phi + g_{\mu\nu}\nabla_\rho\nabla_\zeta\phi - g_{\mu\rho}\nabla_\nu\nabla_\zeta\phi - g_{\zeta\nu}\nabla_\rho\nabla_\mu\phi)\right.\\
&\left. + \frac{1}{4}(g_{\zeta\rho}\partial_\nu\phi\partial_\mu\phi + g_{\mu\nu}\partial_\rho\phi\partial_\zeta\phi - g_{\mu\rho}\partial_\nu\phi\partial_\zeta\phi - g_{\zeta\nu}\partial_\rho\phi\partial_\mu\phi) - \frac{1}{4}(g_{\zeta\rho}g_{\mu\nu} - g_{\zeta\nu}g_{\mu\rho})\partial^\sigma\phi\partial_\sigma\phi\right\},\\
R_{\mu\nu} \to & R_{\mu\nu} - \frac{1}{2}(2\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Box\phi) + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\sigma\phi\partial_\sigma\phi,\\
R \to & \left(R - 3\Box\phi - \frac{3}{2}\partial^\sigma\phi\partial_\sigma\phi\right)e^{-\phi}.
\end{aligned} \tag{115}$$

Therefore the Gauss-Bonnet invariant $\mathcal{G}$ in Eq. (2) is transformed in the following way:

$$\mathcal{G} \to e^{-2\phi}\left[\mathcal{G} + \nabla_\mu\left\{4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\nu\phi + 2(\partial^\mu\phi\Box\phi - (\nabla_\nu\nabla^\mu\phi)\partial^\nu\phi) + \partial_\nu\phi\partial^\nu\phi\partial^\mu\phi\right\}\right]. \tag{116}$$

Then by writing $\Phi = e^{-\phi}$, the action of Eq. (112) can be rewritten by taking into account the conformal transformation $g_{\mu\nu} \to e^{\phi} g_{\mu\nu}$ as follows:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left(R - \frac{3}{2}\partial^\sigma\phi\partial_\sigma\phi\right) + \Theta\mathcal{G} - \partial_\mu\Theta\left\{4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\nu\phi + 2(\partial^\mu\phi\Box\phi - (\nabla_\nu\nabla^\mu\phi)\partial^\nu\phi) + \partial_\nu\phi\partial^\nu\phi\partial^\mu\phi\right\} - e^{2\phi}V(e^{-\phi},\Theta)\right\}. \quad (117)$$

This action (117) may have ghost d.o.f. due to the existence of Θ. As in the last section, we might eliminate the ghost d.o.f. by writing Θ as $\Theta = e^{\theta}$ and add a constraint to the action (117) by using the Lagrange multiplier field λ, in the following way:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left(R - \frac{3}{2}\partial^\sigma\phi\partial_\sigma\phi - \lambda(\partial_\mu\theta\partial^\mu\theta + \mu^2)\right) + e^{\theta}\mathcal{G} - e^{\theta}\partial_\mu\theta\left\{4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\nu\phi + 2(\partial^\mu\phi\Box\phi - (\nabla_\nu\nabla^\mu\phi)\partial^\nu\phi) + \partial_\nu\phi\partial^\nu\phi\partial^\mu\phi\right\} - e^{2\phi}V(e^{-\phi}, e^{\theta})\right\}. \quad (118)$$

As in the previous section, the scalar fields θ and λ are not dynamical d.o.f. and the dynamical d.o.f. are actually the metric and the scalar field ϕ, as in the standard $F(R)$ gravity, therefore no ghost d.o.f. occur in the theory.

VI. CONCLUSIONS

The focus of this work was to alleviate the ghost problem of the modified gravity theories containing the Gauss-Bonnet scalar $\mathcal{G}$. In particular, we studied two kinds of theories, namely $f(\mathcal{G})$ gravity and $F(R,\mathcal{G})$ gravity. In both cases we investigated how the ghost d.o.f. may appear even at the level of the equations of motion, by using perturbations of the metric, and as we demonstrated, ghost d.o.f. haunt both of the aforementioned modified gravity theories. In both cases, we provided a theoretical remedy by using the Lagrange multiplier formalism which provides constraints in terms of the Lagrange multipliers. As we demonstrated, our formalism leads to the elimination of the ghost d.o.f. in both the $f(\mathcal{G})$ gravity and $F(R,\mathcal{G})$ gravity theories, and thus the resulting theories can in principle produce ghost-free primordial curvature perturbations. Specifically, in the $F(R,\mathcal{G})$ gravity case, this was a serious issue due to the fact that modes $\sim k^4$ occurred in the master equation which governed the evolution of the primordial curvature perturbations. For the case of the ghost-free $f(\mathcal{G})$ gravity theory, we investigated how accelerating cosmologies can be realized by these theories. The formalism which we presented can be used as a reconstruction technique, and as we demonstrated there is room for rich model building, since in principle any cosmological evolution can be realized by a number of different ghost-free $f(\mathcal{G})$ theories, due to the freedom provided by the Lagrange multiplier formalism. A future direction for the results we presented, is to provide a concrete formalism to study the inflationary period which can be technically difficult, due to the presence of the Lagrange multiplier. Work is in progress in this direction.

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[1] S. Nojiri, S. D. Odintsov, and M. Sasaki, Phys. Rev. D **71**, 123509 (2005).
[2] S. Nojiri, S. D. Odintsov, and M. Sami, Phys. Rev. D **74**, 046004 (2006).
[3] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[4] S. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005).
[5] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **73**, 084007 (2006).
[6] B. M. Leith and I. P. Neupane, J. Cosmol. Astropart. Phys. 05 (2007) 019.
[7] B. Li, J. D. Barrow, and D. F. Mota, Phys. Rev. D **76**, 044027 (2007).
[8] G. Kofinas and E. N. Saridakis, Phys. Rev. D **90**, 084044 (2014).
[9] S. Y. Zhou, E. J. Copeland, and P. M. Saffin, J. Cosmol. Astropart. Phys. 07 (2009) 009.

[10] S. Capozziello and M. De Laurentis, Phys. Rep. **509**, 167 (2011).
[11] V. Faraoni and S. Capozziello, *Beyond Einstein Gravity* (Springer, Dordrecht, 2011), Vol. 170.
[12] G. J. Olmo, Int. J. Mod. Phys. D **20**, 413 (2011).
[13] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rep. **692**, 1 (2017).
[14] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011).
[15] S. Nojiri and S. D. Odintsov, eConf C0602061, 06 (2006), http://inspirehep.net/record/891788; Int. J. Geom. Methods Mod. Phys. **04**, 115 (2007).
[16] A. de la Cruz-Dombriz and D. Saez-Gomez, Entropy **14**, 1717 (2012).
[17] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[18] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **98**, 044039 (2018).
[19] R. P. Woodard, Scholarpedia **10**, 32243 (2015).
[20] A. De Felice and T. Suyama, J. Cosmol. Astropart. Phys. 06 (2009) 034.
[21] A. H. Chamseddine and V. Mukhanov, J. High Energy Phys. 11 (2013) 135.
[22] S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A **29**, 1450211 (2014).
[23] J. Dutta, W. Khyllep, E. N. Saridakis, N. Tamanini, and S. Vagnozzi, J. Cosmol. Astropart. Phys. 02 (2018) 041.
[24] A. V. Astashenok, S. D. Odintsov, and V. K. Oikonomou, Classical Quantum Gravity **32**, 185007 (2015).
[25] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Lett. B **775**, 44 (2017).
[26] S. Park and R. P. Woodard, Phys. Rev. D **99**, 024014 (2019).
[27] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[28] R. C. Myers, Phys. Rev. D **36**, 392 (1987).
[29] R. G. Cai, B. Hu, and S. Koh, Phys. Lett. B **671**, 181 (2009).
[30] B. H. Lee, W. Lee, and D. Ro, Phys. Lett. B **762**, 535 (2016).
[31] F. J. Tipler, Ann. Phys. (N.Y.) **108**, 1 (1977).
[32] N. Aghanim *et al.* (Planck Collaboration), arXiv:1807.06209.
[33] R. Brandenberger and P. Peter, Found. Phys. **47**, 797 (2017).
[34] J. de Haro and Y. F. Cai, Gen. Relativ. Gravit. **47**, 95 (2015).
[35] G. Antoniou, A. Bakopoulos, and P. Kanti, Phys. Rev. Lett. **120**, 131102 (2018).
[36] B. H. Lee, W. Lee, and D. Ro, Phys. Rev. D **99**, 024002 (2019).
[37] K. i. Maeda, Phys. Rev. D **39**, 3159 (1989).