Space of initial data for self-similar Schwarzschild solutions of the Einstein equations

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The Einstein constraint equations describe the space of initial data for the evolution equations, dictating how space should curve within spacetime. Under certain assumptions, the constraints reduce to a scalar quasilinear parabolic equation on the sphere with various singularities and nonlinearity being the prescribed scalar curvature of space. We focus on self-similar Schwarzschild solutions. Those describe, for example, the initial data of black holes. We construct the space of initial data for such solutions and show that the event horizon is related with global attractors of such parabolic equations. Lastly, some properties of those attractors and its solutions are explored.

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I. INTRODUCTION

The Einstein equations model gravity through spacetime as ten coupled partial differential equations. Six of those evolve space in time, whereas the other four constrain the initial data or, intuitively, dictate how space is curved and embedded in the bigger framework of spacetime.

We focus on *time-symmetric* spacetime, namely, solutions such that the embedding of space in spacetime is trivial and hence its extrinsic curvature vanishes. Hence, those four constraint equations are reduced to only one that indicates how space can bend intrinsically, known as the Einstein Hamiltonian constraint. See Ref. [1].

Exact solutions of such an equation with a prescribed function T_{00} describing its energy density are called *pressureless perfect fluids*, that is, fluids without pressure, viscosity, and heat conduction. See Chap. 4 in Ref. [2]. Such fluids are commonly used in stellar models for idealized distributions of matter, such as stars or black holes. See Ref. [3].

A simple case among the perfect fluids are the *spherically symmetric* ones. Mathematically, space is described by a three-dimensional Riemannian manifold $\mathcal{S}$ with metric g. Assume that the space $\mathcal{S}$ can be written in spherical coordinates, that is, $\mathcal{S} \coloneqq \mathbb{R}_+ \times \mathbb{S}^2$ with $r \in \mathbb{R}_+$ being the radial foliation of two-dimensional spheres $(\theta, \phi) \in \mathbb{S}^2$. In the *shear-free* case, the metric splits as

$$g = u^2 dr^2 + r^2 \omega, \tag{1.1}$$

where ω is the standard metric in $\mathbb{S}^2$, and the component $u = u(r, \theta, \phi)$ is the unknown. For a list of known exact spherically symmetric solutions, see Tables I and II in Ref. [4].

Computing the scalar curvature $R(g)$ of $\mathcal{S}$, Bartnik [5] claimed that u satisfies the following parabolic equation, which fails to be parabolic at $r = 0$:

$$2ru_r = u^2 \Delta_{\mathbb{S}^2} u + u + \frac{r^2 R(g) - 2}{2} u^3. \tag{1.2}$$

This parabolic curvature equation was also computed in the Appendix of Smith [6] and is a pure geometric fact of the chosen space $\mathcal{S}$ and metric g with no relation to the Einstein equations. This connection is made by prescribing a matter model given by a smooth function T_{00} and relating it with the scalar curvature $R(g)$ through the Einstein Hamiltonian constraint equation,

$$R(g) = 16\pi T_{00}, \tag{1.3}$$

as in the work by Rendall [1] and Bartnik [5].

A simple solution of (1.2) is given by the Schwarzschild metric, obtained in vacuum, $R(g) \equiv 0$, and when u is independent of the angle variables. This yields a solution $u(r) = (1 - 1/r)^{-1/2}$, which blows up at $r_1 \coloneqq 1$. Note that this solution is only valid for $r > r_1$, since for $r \leq r_1$ the metric is not Riemannian. It models the exterior of black holes, where the surface r_1 is known as the *event horizon*, a singularity due to coordinates choice, whereas a *physical singularity* occurs at $r_0 \coloneqq 0$. For a mathematical theory of black holes, see the work by Chandrasekhar [7].

We seek to understand the interior structure of black holes. In particular, we construct their initial data, as well as relate the interior and the event horizon. The nature of interior of black holes is still debatable. Some possibilities are the constructions of Schwarzschild in Eq. (35) of

Ref. [8], Synge's formulas (4.2) or (4.6) in Ref. [9], or Florides in Eq. (2.13) of Ref. [10].

An alternative approach, in order to mimic the exterior Schwarzschild solution to its interior, is to require the same blowup rate looking from the exterior $(r \to 1^+)$ and interior $(r \to 1^-)$ of the event horizon. This was proposed by Fiedler *et al.* [11], and they showed that a plethora of angle-dependent metrics can occur inside a black hole, with the same horizon. Indeed, plugging the Schwarzschild solution in (1.1), we obtain the metric $g = |1 - 1/r|^{-1} dr^2 + r^2 \omega$, which is Riemannian and solves (1.2) for $r < 1$ with a prescribed curvature $R(g)$. For example, if $R(g) = 4/r^2$ and the solution is independent of the angle variables, then $u(r) = (1/r - 1)^{-1/2}$ is a solution of (1.2) for $r < 1$. This type of solution models the interior of black holes, in which the metric blows up at the event horizon at r_1 with the same rate in the exterior and interior, and has a curvature singularity at r_0.

We are interested in *Schwarzschild self-similar interior solutions* of (1.2), as

$$u(r, \theta, \phi) = \left(\frac{1}{r} - 1\right)^{-\frac{1}{2}} v(r, \theta, \phi) \tag{1.4}$$

for $r < 1$. In particular, we construct the space of initial data in order to rigorously study the dynamics of Einstein evolution equations, such as the stability of black holes. The term $(1/r - 1)^{-1/2}$ is the interior Schwarzschild blow up rate of the solution u.

Through the self-similar glasses (1.4), v satisfies the following equation for some prescribed scalar curvature $R(g)$, given by (1.3),

$$2(1 - r) v_r = v^2 \Delta_{\mathbb{S}^2} v - v + \frac{r^2 R(g) - 2}{2} v^3. \tag{1.5}$$

Note that the parabolicity of the equation breaks down at the even horizon $r_1 := 1$, since there is no radial derivative. Moreover, it is the backward heat equation for $r > r_1$, which is not well posed. To overcome such a problem outside the horizon, Smith [12] uses the coordinates system u satisfying Eq. (1.2).

For $r > r_1$ and certain curvature $R(g)$, u was constructed by Smith [12] using the Schwarzschild self-similar exterior solutions of (1.2) given by $u = (1 - 1/r)^{-1/2} v$. For example, he constructed the metric for certain choices of $R(g)$, and therefore our interior construction can be glued to an exterior solution with such a smooth metric. For $r \in (r_0, r_1)$ with $r_0 > 0$, and curvature $R(g) = (\lambda + 2)/r^2$ with $\lambda \in \mathbb{R}_+$, it was shown that there are several nonspherical symmetric solutions in the radial direction bifurcating from the solution $v \equiv 1$, by Fiedler *et al.* [11].

It is the aim of this paper to study the structure at the event horizon $r_1 := 1$ from a dynamical point of view, depending on the interior of a static black hole.

For that, rescale the equation through $r = 1 - e^{-2t}$ so that the breakdown of parabolicity at $r_1 := 1$ is now represented as $t \to \infty$ in

$$v_t = v^2 \left[\Delta_{\mathbb{S}^2} v + \frac{r^2 R(g) - 2}{2} v - \frac{1}{v} \right]. \tag{1.6}$$

Note that this is a degenerate quasilinear parabolic equation, and hence one can study its initial value problem with initial data at $t = 0$, corresponding to $r = 0$.

We are dealing with time-independent solutions of the Einstein equations. Even though t is usually called time in parabolic equations, its interpretation here is different: it is a rescaled radial distance from the black hole singularity at $t = 0$ such that the event horizon $r_1 := 1$ occurs at $t = \infty$.

We recall that horizons occur at spheres in the spatial foliation for some fixed radius, which is a minimal surface such that no other leaf has positive mean curvature; see Ref. [12]. Since each leaf $\mathbb{S}^2$ has mean curvature $H = 2/(ru) = [2(1 - r)^{1/2}]/[r^{3/2} v]$, horizons occur either at $r_1 := 1$ or whenever v is unbounded.

The main result is now presented: the construction of the space of initial data for such solutions with only one horizon, including the relation of the horizon with global attractors, describing the structure of the metric at r_1.

Equation (1.6) generates a semiflow denoted by $(t, v_0) \mapsto v(t)$ in the phase space $X := C^{2\alpha+\beta}(\mathbb{S}^2) \cap \{v > 0\}$ where $\alpha, \beta \in (0, 1)$ are, respectively, a fractional power exponent and the Hölder exponent and a fractional power exponent. See Ref. [13]. Note that from Lemma 1 in Ref. [12] if $v_0 > 0$ then $v(t) > 0$ for $t > 0$ and the space of positive functions is invariant guaranteeing strict parabolicity of (1.6).

We suppose that the prescribed scalar curvature $R(g)$ is such that the semiflow $v(t)$ is *slowly nondissipative*, that is, solutions are global, but they may grow up and become unbounded as $t \to \infty$. Such solutions would have a different grow-up rate than the Schwarzschild solution. Sufficient conditions for semilinear equations are in Ref. [14], whereas conditions for quasilinear equations are still not known. Because of such an assumption, we disregard blow up solutions, that is, solutions with other apparent horizons. We consider solutions with a single horizon at r_1.

Moreover, slowly nondissipativity guarantees the existence of an unbounded global attractor $\mathcal{A}$ of (1.6), which attracts all bounded sets as $t \to \infty$. This attractor can be decomposed as the set of equilibria (bounded and unbounded) and heteroclinics between them. See Ref. [14].

A particular case is when the dynamical system is dissipative, and solutions stay bounded at r_1. Therefore, no grow-up occurs, and v has the same grow-up rate as the Schwarzschild solution. Sufficient growth conditions on R for dissipativity with $r \in (0, r_1)$ are

$$R(r,\theta,\phi,v,0) < \frac{2}{r^2v^2}$$
$$\left|\frac{r^2v^3}{2}R_p\right|\cdot(1+|p|) + \left|\frac{r^2v^3}{2}R - v\right| < f_1(|v|) + f_2(|v|)|p|^\gamma$$
$$\left|\frac{\partial}{\partial_\theta}\left(\frac{r^2v^3}{2}R - v\right)\right| + \left|\frac{\partial}{\partial_\phi}\left(\frac{r^2v^3}{2}R - v\right)\right| < [f_3(|v|) + f_4(|v|,|p|)](1+|p|)^3, \tag{1.7}$$

where the first line above holds for $|v|$ large enough, uniformly in (θ,ϕ); the second line above holds for all (r,θ,ϕ,v,p), given continuous f_1, f_2 and $\gamma < 2$; the third line above holds for f_3 nonnegative continuous and monotonically increasing, f_4 continuous, monotonically increasing in $|v|$ and tends to 0 as $|p| \to \infty$ uniformly with bounded $|v|$. Also, R_p denotes both the derivative of R with respect to v_θ and v_ϕ. See Chap. 5, Sec. III in Ref. [15].

Hence, for any bounded initial data $v_0 \in X$ at $t = 0$ and a scalar curvature $R(g)$ such that $v(t)$ is slowly nondissipative, there exists a metric $v(t,\theta,\phi)$ in phase space for all $t \in (0,\infty)$. Moreover, if R does not depend on r and ∇v, the solution v will approach an equilibrium in $\mathcal{A}$ as $t \to \infty$, due to the existence of a Lyapunov function,

$$L := \int_{\mathbb{S}^2} \frac{|\nabla v|^2}{2} - F d\omega, \tag{1.8}$$

where F is the primitive of $-v^{-1} + (r^2R - 2)v/2$. This yields

$$\frac{dL}{dt} = -\int_{\mathbb{S}^2} \left(\frac{v_t}{v}\right)^2 d\omega \tag{1.9}$$

along trajectories of (1.6). Note that $v > 0$ in the phase-space X. If R depends on ∇v, there exists a Lyapunov function for axially symmetric solutions, as in Ref. [13]. For more general radial foliations, we could possibly obtain a fully nonlinear parabolic equation, instead of (1.2). In this case, a Lyapunov function for axisymmetric solutions can be available by incorporating the weight from Ref. [13] to Ref. [16].

In other words, for any bounded initial data $v_0 \in X$ at the singularity $r_0 := 0$ of self-similar Schwarzschild solutions, there exists a metric $v(r,\theta,\phi)$ for $r \in (0, r_1)$ such that v converges to an equilibrium of $\mathcal{A}$ as $r \to r_1$. This means that self-similar metrics at the horizon $v(r_1,\theta,\phi)$ are given by equilibria $v_1(\theta,\phi) \in \mathcal{A}$ through $v(r_1,\theta,\phi) = v_1(\theta,\phi)$, and the attractor $\mathcal{A}$ describes the possible metrics at r_1. The bounded equilibria v models horizons with same blow up rate as the Schwarzschild solution, whereas the unbounded equilibria yield horizons with a different blow up rate than the Schwarzschild solution.

Then, we use Smith's construction in Ref. [12] with such equilibria $v(r_1,\theta,\phi) \in \mathcal{A}$ as initial data at the horizon r_1, yielding a metric for $r > r_1$, if one supposes that the scalar curvature $R(g)$ is compactly supported and satisfies

$$R(g) < \frac{1}{r^2} \tag{1.10}$$

in $(r_1,\infty) \times \mathbb{S}^2$, and $R(g) = 0$ for $[r_1, r_1 + \delta)$ for $\delta > 0$ small. There is no other horizon for $r > r_1$, due to the choice of the standard spherical metric for the foliation.

The above construction shows the following theorem.

Theorem 1.1. Horizons and Attractors Suppose that space is given by a spherically symmetric Riemmanian manifold $(\mathcal{S}, g)$, that is, $\mathcal{S} := \mathbb{R}_+ \times \mathbb{S}^2$. If the scalar curvature $R(g)$ yields a slowly nondissipative semiflow $v(t)$ of (1.6) satisfying (1.10), then for any function $v_0(\theta,\phi) \in X$, there exists a metric for all $r \in \mathbb{R}_+$ given by

$$g = \frac{v^2(r,\theta,\phi)}{|\frac{1}{r} - 1|} dr^2 + r^2\omega, \tag{1.11}$$

where ω is the standard metric on $\mathbb{S}^2$. Moreover, such solutions display only one horizon at $r_1 := 1$, which is described by a (possibly unbounded) function $v(r_1,\theta,\phi)$, an equilibrium of the unbounded global attractor $\mathcal{A}$ of (1.6).

Above, we consider the solutions v that are possibly unbounded at r_1 and hence have a different grow-up rate compared to the Schwarzschild solution. If instead of assuming R yields a slowly nondissipative semiflow we suppose that it satisfies (1.10), then $v(t)$ is dissipative. Therefore, the attractor $\mathcal{A}$ is bounded, and hence the equilibria are within. In this case, solutions have the same grow-up rate as the Schwarzschild solutions.

Recall that the interior region $r \in [0, r_1)$ of the event horizon does not influence the Cauchy development of the exterior $r \in [r_1, \infty)$ of the horizon; see Ref. [17]. This is a claim about *time*. The above Theorem is a claim about *space*; the event horizon can not be arbitrary for each fixed time, but it depends on the metric v inside the black hole $r \in [0, r_1)$, in particular at the singularity $r_0 = 0$. Therefore, the initial data in the horizon cannot be freely specified as Smith [18] but has additional constraints; namely, it has to be within the global attractor of (1.6).

Therefore, a given shape of the metric at the event horizon imposes that only certain possibilities are allowed for the inside of black holes: elements in the basin of attraction of $v(r_1, \theta, \phi)$, i.e., the stable manifold of such equilibrium. Moreover, the evolution dynamics of the Einstein equations of event horizons might constrain even more what is inside a black hole. Even though we cannot know for sure what is inside black holes, the above method tells us what cannot be inside them.

Bartnik's conjecture, stating that quasispherical metrics are typical in the space of smooth metrics, is still an open problem in its full generality. See Ref. [19]. I also mention that the quasispherical structure is related to the quasilocal mass. Therefore, constructing the phase space for such metrics might shine a light in the problem of properly defining a suitable quasilocal mass through a dynamical approach. For instance, one can consider the length of the curve $v(r)$ within the stable manifold, between the metrics at v_0 and $v(r_1)$.

Also, the metric (1.1) is not as general as Bartnik's original [5], since we consider shear-free metrics, that is, we do not allow mixed terms $\beta^1 dr d\theta$ and $\beta^2 dr d\phi$. Metrics with shear satisfy a coupled system, as in Ref. [20], instead of satisfying a scalar partial differential equation given by (1.2), and hence only local existence is proved. It is not known which sufficient conditions yields a dissipative system, as (1.10), and hence global solutions. Also, the system with shear does not necessarily have a Lyapunov function, and the possible metrics at the event horizon are not necessarily equilibria. Therefore, shear-free metrics allow concrete computations of simplified equations relating global attractors and event horizons.

The inverse problem is of interest; consider a metric $v(r_1, \theta, \phi)$, which is in $\mathcal{A}$ at the event horizon r_1 with prescribed $R(g)$, and find its basin of attraction. Hence, for one given metric at the r_1, one can find a zoo of possibilities of metrics inside the horizon. Similarly, one can prescribe the metric at the r_1 and ask which energy density described by the curvature R realizes such equilibria. A similar problem was treated by Fiedler and Rocha [21].

To study the existence of other apparent horizons and their interplay, as in Ref. [11], one should drop the slowly nondissipative assumption and allow blow up solutions. In this case, the metric v blows up for $r < r_1$, and other horizons occur inside the event horizon. In such a case, the metric between an apparent horizon and an event horizon could be constructed by a heteroclinic within the attractor $\mathcal{A}$.

In other words, we can pin down the space of initial data for the Einstein equations for time-symmetric spherically symmetric self-similar Schwarzschild solutions with one horizon.

Corollary 1.2. Space of Initial data Suppose that space is given by a time-symmetric Riemmanian manifold $(\mathcal{S}, g)$ with spherical coordinates $\mathcal{S} := \mathbb{R}_+ \times \mathbb{S}^2$ having shear-free metric g with standard spherical metric ω in each leaf $\mathbb{S}^2$ and scalar curvature $R(g)$ satisfying (1.10). Then, the space of initial data for self-similar Schwarzschild solutions with one horizon at $r_1 = 1$ is given by g as in (1.11) such that v lies in the set,

$$\mathcal{X} := \{v \in C^0([0,\infty), X) \cap C^1((0,\infty), X) \text{ such that } v(r_1) \in \mathcal{A} \text{ an equilibrium}\}, \tag{1.12}$$

where $X := C^{2\alpha+\beta}(\mathbb{S}^2) \cap \{v > 0\}$.

The stability of black holes has been widely studied over the past years, as in Ref. [22]. Usually, only linear stability is treated. For the nonlinear stability, the problem is open, and knowing which space the initial conditions belong to, and hence satisfy the constraints, identifies what is rigorously meant by a perturbation of a solution of the Einstein equations, and neighborhoods of solutions in the phase space (1.12).

Other types of self-similar solutions can be pursued, following the idea of (1.4), such as Kerr self-similar solutions to model rotating black holes; Reissner-Nordström self-similar solutions to model charged black holes; de Sitter-Schwarzschild self-similar solutions; or a Schwarzschild metric in the exterior of the horizon, and a regular interior in order to model dense stars. Once space of initial data has been constructed for the latter, one can rigorously study the dynamics of stars and its collapse into black holes, as in Ref. [23].

For the Kerr case of rotating black holes, we consider the metric

$$g = u^2 dr^2 + (r^2 + a\cos^2(\theta))\omega + a\left[1 + \frac{r}{r^2 + a\cos^2(\theta)}\right]\sin^4(\theta) d\phi^2, \tag{1.13}$$

where ω is the standard metric in $\mathbb{S}^2$; a is related to the rotation of the hole, recovering the Schwarzschild case when $a = 0$; and the component $u = u(r, \theta, \phi)$ is the unknown.

We have to find the equation the unknown u satisfies through the scalar curvature of such a metric (1.13), in case of trace-free extrinsic curvature. This will decouple the constraints, so u satisfies a scalar equation. Note that above we only have the spatial metric components, which constrains intrinsic properties of space within the set of initial data. The mixed space and time terms, namely, the shift vectors, will come from the extrinsic curvature of the embedding of space in spacetime, which have to solve the momentum constraint equations, constraining the embedding of space within spacetime. Moreover, we would seek *Kerr self-similar solutions* of the type $u = [(r^2 + a\cos^2(\theta))/(r^2 - r + a)]^{-1/2} v$.

On the other hand, note that spatially Reissner-Nordström solutions occur as angle-independent solutions of (1.2) when $R(g) = (4 - 2qr^{-2})r^{-2}$, where $q \geq 0$ is a constant related to the charge of the black hole, yielding the blow up solutions $u(r) = (r^{-1} - qr^{-2} - 1)^{-1/2}$ which blows up at $r_\pm := 2^{-1}(1 \pm \sqrt{1 - 4q})$, known as the event horizon at r_+, and an interior horizon at r_-. Note that if $q = 0$ we recover the Schwarzschild case. Also, the horizons occur in the same distance, $r_+ = r_- = 1/2$, when $q = 1/4$, and correspond to extremal black holes.

The *Reissner-Nordström self-similar solutions*, namely, solutions of (1.2) of the type

$$u(r,\theta,\phi) = \left(\frac{1}{r} - \frac{q}{r^2} - 1\right)^{-\frac{1}{2}} v(r,\theta,\phi), \qquad (1.14)$$

will imply that v satisfies the following equation for some prescribed scalar curvature $R(g)$, given by (1.3):

$$2[(1-r) - qr^{-1}]v_r = v^2\Delta_{\mathbb{S}^2} v - [1 - qr^{-2}]v + \frac{r^2R(g) - 2}{2}v^3. \qquad (1.15)$$

Note that the parabolicity of the equation breaks down at the horizons $r = r_\pm$, since there is no radial derivative. Moreover, strict parabolicity holds for $r(1-r) > q$.

On a similar account, the *Schwarzschild-de Sitter self-similar solutions*, namely, solutions of (1.2) as

$$u(r,\theta,\phi) = \left(\frac{1}{r} + \frac{\Lambda}{3}r^2 - 1\right)^{-\frac{1}{2}} v(r,\theta,\phi), \qquad (1.16)$$

where Λ is the cosmological constant, will imply that v satisfies the following equation for some prescribed scalar curvature $R(g)$, given by (1.3),

$$2[(1-r) - \Lambda 3^{-1} r^3]v_r = v^2\Delta_{\mathbb{S}^2} v - [1 + \Lambda r^2]v + \frac{r^2R(g) - 2}{2}v^3, \qquad (1.17)$$

where strict parabolicity holds for $3r^{-3}(1-r) > \Lambda$.

Note that in the last two equations, modeling Reissner-Nordström and Schwarzschild-de Sitter spaces, even though we can rescale $t(r)$ so that the left-hand sides in (1.15) and (1.17) have no radial dependence, and become v_t, the right-hand side will still have radial terms in the linear term in vacuum, for example. Therefore, we still need a deeper understanding of nonautonomous parabolic equations to develop those interior black hole initial data. For example, for such nonautonomous equations, we would not have decay to an equilibrium due to the lack of a Lyapunov function. Therefore, it is desirable to understand the dynamics within global attractors of nonautonomous equations.

Nevertheless, since the Laplacian is still $O(3)$ equivariant, we can use the symmetry breaking methods of Fiedler *et al.* [11].

Also, the Hamiltonian constraint above is one out of four constraints. In the non–time symmetric case, one can rewrite them as a system of equations, as in Ref. [20], to be studied in the future.

Because of the no-hair theorem, black holes are fully described by their mass, charge, and angular momentum. The Schwarzschild self-similarity studied here describes the possible metrics at the event horizon knowing its mass, if the black hole is chargeless and has no momentum. The proposals above, namely, studying Reissner-Nordström self-similar solutions, and studying the four constraint equations could describe the full space of initial data for black holes.

Therefore, we see that the connection between event horizons and global attractors opens many doors, yielding new problems to be tackled, with the dynamical perspective of the constraint equations.

In the next section, we explore particular cases of the above theorem, when the global attractor can be explicitly computed, yielding the number of possible equilibria metrics, or when certain symmetry within the attractor is known.

II. FURTHER EXPLORATION

From now on, we expose two corollaries describing properties of the global attractor $\mathcal{A}$ at the event horizon: one describes the possible axisymmetric self-similar Schwarzschild metrics at r_1. The other describes some symmetries of a certain metric at r_1. Both are rigorously proved in Refs. [13,24,25].

For the first corollary, we are interested in a more detailed study of the structure of the attractor $\mathcal{A}$ that describes the possible metrics at the event horizon r_1. For such, we consider axially symmetric solutions and suppose that the metric $v(r,\theta)$ is independent of the angle $\phi \in \mathbb{S}^1$.

Axially symmetric solutions in general relativity have been extensively studied and are also known in the literature as *Ernst solutions* due to Ref. [26]. For a collection of case studies, see Ref. [27]. A numerical simulation for the dynamics of interaction, pulsation, or collapse of axisymmetric stars was done in Ref. [28].

Therefore, restricting the semiflow to the invariant subspace of axisymmetric solutions $X_{\text{axi}} \subseteq X$, one obtains a subattractor $\mathcal{A}_{\text{axi}} \subseteq \mathcal{A}$ of the flow of

$$v_t = v^2\left[v_{\theta\theta} + \frac{v_\theta}{\tan(\theta)}\right] - v + \frac{r^2R(g)}{2}v^3 \qquad (2.1)$$

with Neumann boundary conditions in $\theta \in [0,\pi]$.

In this case, the subattractor $\mathcal{A}_{\text{axi}}$ within the axisymmetric subspace X_{axi} can be computed explicitly for any prescribed R satisfying (1.10). This was done for the quasilinear case in Ref. [24] and generalized for the case of a singular boundary in Ref. [13]. We choose a particular scalar curvature so that the attractor at the event horizon r_1 is known.

Corollary 2.1. A Prescribed Scalar Curvature Given the scalar curvature $R=2r^{-2}[v^{-2}+\lambda v^3(v-1)(2-v)]$, where $\lambda \in (\lambda_k, \lambda_{k+1})$ and λ_k is the k^{th} eigenvalue of the spherical Laplacian with $k \in \mathbb{N}_0$, then, the semiflow $v(t)$ is dissipative, the attractor $\mathcal{A}_{\text{axi}}$ is compact, and the axisymmetric self-similar Schwarzschild metric at the event horizon $r_1 := 1$ is given by one of the $2k+3$ equilibria $v_1, \ldots, v_{2k+3}$ within the Chafee-Infante type attractor in Fig. 1, in which points denote bounded equilibria and arrows are heteroclinic connections.

Indeed, the above choice of $R(g)$ yields the equation

$$v_t = v^2 \left[v_{\theta\theta} + \frac{v_\theta}{\tan(\theta)} + \lambda v(v-1)(2-v) \right] \quad (2.2)$$

for $\lambda \in \mathbb{R}$. The unknown $w := v - 1$ satisfies the Chafee-Infante equation with quasilinear diffusion coefficient $(w+1)^2$. Hence, the equilibria $v \equiv 1$, corresponding to Schwazrschild, have the role of the bifurcating equilibria $w \equiv 0$ in the usual Chafee-Infante equation, and each time λ crosses an eigenvalue of the spherical Laplacian, the Schwarzschild solution bifurcates to an axisymmetric solution.

It would be interesting to compute the attractor for the prescribed scalar curvature from Ref. [11], namely, $R = (\lambda+2)/r^2$. This yields a slowly nondissipative nonlinearity with nonhyperbolic equilibria. That is, solutions v might now stay bounded as $r \to r_1$ and grow-up may occur. This will be done in the near future.

Note that in order to construct the attractor $\mathcal{A}_{\text{axi}}$ one needs to know the *zero number* of the difference of solutions $v - v_{k+2}$, where the trivial solution $v_{k+2} \equiv 1$ represents the Schwarzschild solution in self-similar variables. Roughly speaking, one needs to know how many intersections other equilibria have with v_{k+2}. This encodes the information of how much such equilibria deviate from the Schwarzschild solution v_{k+2}, and whenever a solution intersects with the trivial solution, it means that $v(r,\theta) = 1$ and the metric looks like the Schwarzschild solution at that fixed radius r.

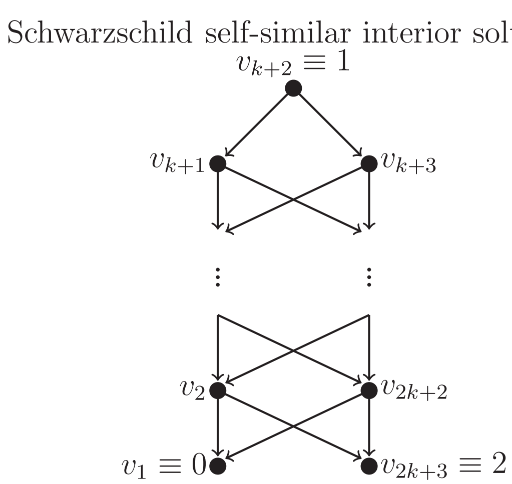

FIG. 1. Global attractor $\mathcal{A}$ of Chafee-Infante type.

The second main result regarding certain elements of the event horizon answers partially the question of how the symmetry of the sphere dictates a symmetry of some solutions in the attractor $\mathcal{A}$. This is done precisely in Ref. [25].

A function $v \in C^1(\mathbb{S}^2)$ has *axial extrema* if its maxima and minima in ϕ occur as an axis from the north to south pole. In other words, if $v_\phi(\theta_0, \phi_0) = 0$ for a fixed $(\theta_0, \phi_0) \in \mathbb{S}^2$, then $v_\phi(\theta, \phi_0) = 0$ for any $\theta \in [0, \pi]$. In that case, the extrema depend only at the position in ϕ. Note that if $R(g)$ is analytic then the solution v of (1.6) is also, as in Ref. [29]. Then, the set of axial extrema is finite, and we denote them by $\{\phi_i\}_{i=0}^N$ where $\phi_0 := \phi_N$.

Axial extrema are *leveled* if all axial maxima ϕ_i have the same value $u(\theta, \phi_i) = M(\theta)$, and all axial minima ϕ_i also have the same value $u(\theta, \phi_i) = m(\theta)$.

Corollary 2.2. Symmetry at the Event Horizon Suppose the scalar curvature R is analytic and $v(r_1, \theta, \phi)$ is an equilibrium of (1.6) within the attractor $\mathcal{A}$ that only has axial extrema $\{\phi_i\}_{i=0}^N$ where $\phi_0 := \phi_N$. Then $\phi_i = (\phi_{i-1} + \phi_{i-1})/2$, and the self-similar Schwarzschild metrics at the event horizon has the reflection symmetry

$$v(r_1, \theta, \phi) = v(r_1, \theta, R_{\phi_i}(\phi)) \quad (2.3)$$

for all $i = 1, \ldots, N$, where $(\theta, \phi) \in [0, 2\pi] \times [\phi_{i-1}, \phi_i]$ and $R_{\phi_i}(\phi) := 2\phi_i - \phi$.

This theorem raises the mathematical question of whether such a result holds for other domains, such as the torus or the hyperbolic disk. Subsequently, it raises the physical question of space foliated by other two-dimensional surfaces than the sphere and if the resulting equation for the scalar curvature is still parabolic and of the same form as (1.2). Even though Hawking's theorem in Ref. [30] states that event horizons for certain black holes are topologically $\mathbb{S}^2$, other stellar objects of interest could carry different topology. For example, it was found numerically that dust collapse might yield a toroidal horizon before reaching its spherical shape in Ref. [31]. Therefore, it is expected that the set $\mathcal{X}$ from Corollary 1.2 and the set of initial data for toroidal foliations are connected to each other in phase space.

Lastly, this theorem was achieved by trying to prove a symmetrization of Gidas, Ni, and Nirenberg type: positive solutions of parabolic equations on the sphere are axial. This has been achieved for elliptic equations with domain being subsets of the sphere; see Ref. [25] and references therein. If this conjecture of symmetrization on the sphere was true, and positive solutions were axial, then we the phase-space X, which consists of positive solutions, would constitute only of axially symmetric functions. Therefore, $X = X_{\text{axi}}$, and $\mathcal{A} = \mathcal{A}_{\text{axi}}$.

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