Prospects of the cosmic scenery in a quintom dark energy model with generalized nonminimal Gauss-Bonnet couplings

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In this work we propose a novel dark energy model in the formalism of quintom scenarios in scalar-tensor theories based on general relativity, taking into account generalized couplings between the scalar fields and Gauss-Bonnet terms. By employing linear stability theory, we reveal the structure of the phase space and analyze the dynamical effects of the Gauss-Bonnet couplings. We show that the present model exhibits various classes of critical points, corresponding to distinct cosmological scenarios that affect the dynamical evolution of the Universe. At the critical points, the cosmological scenarios correspond to either an accelerated expansion or a stiff-fluid case in which the expansion is decelerated, while for some solutions the expansion is neither accelerated nor decelerated. Finally, the present model also exhibits scaling behavior at some of the critical points, in which the dark energy mimics the behavior of matter.

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I. INTRODUCTION

In today's cosmological theory one of the most appealing fundamental questions is related to the existence and dynamical properties of the cosmic accelerated expansion of the Universe, a discovery presented at the end of the last millennium [1–3]. The exposition of such an intriguing phenomenon lead to the development of various models in theoretical physics, astrophysics, and cosmology. One of the most applicable theoretical frameworks for the study of the dark energy phenomenon is represented by scalar-tensor theories of gravity, in which the accelerated expansion can be explained by various modifications of the basic theory of gravity [4,5]. Within these theories, the accelerated expansion is caused by the existence of scalar fields minimally or nonminimally coupled with gravity. Within these constructions, the quintom formalism [6] first appeared as a possible explanation for the crossing of the phantom divide line by the dark energy equation of state, which has been indicated by astrophysical observations [7]. Despite the evolution of the theoretical models, the phenomenology of the crossing of the phantom divide line by the dark energy equation of state is still a curious problem [8]. Various papers have analyzed the dynamical features and physical properties of the quintom paradigm in different cosmological scenarios [9–35]. For a more comprehensive review of the quintom paradigm in different cosmological models, the reader should consult Ref. [6]. In recent times, the quintom cosmological models have been analyzed in the nonminimal coupling scenarios by various authors [36–45].

In scalar-tensor theories of gravity, the Gauss-Bonnet term represents a topological invariant [46] in a four-dimensional space, originating from string theory [47], and has been included in several theoretical models. The inclusion of a Gauss-Bonnet term in cosmological theories was first investigated by Nojiri *et al.* [48], where the viability of such a construction was analyzed and it was shown that the dark energy phenomenon can rise and the cosmic big rip doomsday can be avoided. Within these theories, the extension of modified gravity [49,50] in the direction of a Gauss-Bonnet gravity has occurred naturally [51] as a theory capable of explaining the late-time accelerated expansion of the known Universe. A modified gravity theory based on a Gauss-Bonnet blending term was recently investigated, showing that such a theory is in agreement with the current astrophysical observations [52]. Furthermore, different papers in various theoretical directions have analyzed the effects and implications of Gauss-Bonnet coupling terms [53–75].

In a recent paper [76], Granda and Jimenez studied the dark energy problem in the context of scalar-tensor theories, taking into account nonminimal couplings between the scalar field, Gauss-Bonnet invariant, and gravitation. Within this model, they showed the implications and dynamical effects for the nonminimal couplings in the phase space structure and tested the viability of such a curious scalar construction. Furthermore, several similar approaches that included the addition of Gauss-Bonnet coupling terms were investigated in recent years in the same formalism [56,77–85]. In scalar-tensor theories, the

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coupling of the scalars with Gauss-Bonnet terms represents a viable modification to the action, which has interesting and viable consequences for the late-time dynamical features of the model that are in agreement with the current evolution of the Universe at the level of background dynamics. However, from another perspective [86–88] the addition of a Gauss-Bonnet term to a scalar-tensor theory may be problematic due to the ultraviolet instabilities that affect the scalar and tensor perturbations [89].

Taking into account this direction for the quintom paradigm, we shall further extend the quintom formalism by allowing a nonminimal coupling between the quintessence and phantom scalar fields with Gauss-Bonnet terms. Hence, from a theoretical point of view such a scalar construction represents a natural extension of the present quintom scenarios in the theoretical framework of scalar-tensor theories based on general relativity, and might explain the dark energy phenomenon in the context of current theories related to the observable Universe.

In this paper we propose a novel quintom scenario based on the superposition of nonminimal couplings between the canonical and negative kinetic scalar fields from the quintom action with Gauss-Bonnet terms. After that, we propose the corresponding action for our model in Sec. II, and deduce the Klein-Gordon equations and modified Friedmann relations which describe the dynamics of the present quintom cosmological model. Then, in Sec. III we analyze the dynamical effects of the Gauss-Bonnet couplings by employing a linear stability method, revealing the structure and properties of the six-dimensional phase space. Finally, in Sec. IV we summarize the most important results and present our conclusions related to the present model.

II. DYNAMICAL EQUATIONS FOR THE QUINTOM SCENARIO WITH GAUSS-BONNET COUPLINGS

In what follows, we consider a quintom dark energy model with nonminimal Gauss-Bonnet couplings, where the corresponding action has the following form [48,85]:

$$S = S_m + \int d^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - V_1(\phi) - V_2(\sigma) - F(\phi)\tilde{g}_{\mathrm{GB}} - G(\sigma)\tilde{g}_{\mathrm{GB}}\right], \tag{1}$$

where S_m is the decoupled part of the action describing the matter sector in the cosmological model, R is the scalar curvature, ϕ is the quintessence (canonical) scalar field, σ is the noncanonical phantom field, embedding the potential part in the $V_1(\phi)$ and $V_2(\sigma)$. In this representation, the interaction of the two scalar fields and the Gauss-Bonnet invariant term $\tilde{g}_{\mathrm{GB}}$ is mediated through the analytical form of the two functions $F(\phi)$ and $G(\sigma)$, respectively. Next, in order to perform the variations of the action (1) with respect to the inverse metric and the two quintom fields, we first need to specify the metric corresponding to this cosmological model. In the following calculations, we consider a Friedmann-Robertson-Walker cosmological model with the Roberston-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{2}$$

where $i, j = 1, 2, 3$, $a(t)$ is the cosmic scale factor, and δ_{ij} is the discrete Kronecker symbol, describing a Universe expanding in a homogeneous and isotropic manner and neglecting the cosmic curvature parameter. Concerning the matter sector of the Universe, we consider the case of a barotropic fluid with the equation of state $p_m = w_m\rho_m$, where p_m represents the corresponding pressure, ρ_m is the energy density, and w_m is the constant equation-of-state barotropic parameter. The stress-energy tensor for the matter sector is

$$T^m_{\mu\nu} = (p_m + \rho_m)u_\mu u_\nu + p_m g_{\mu\nu}, \tag{3}$$

where u_μ represents the four-velocity of the comoving frame, $u_0 = \delta^0_0 = 1$ [90]. For the matter sector, the evolution is described by the following continuity equation:

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0, \tag{4}$$

where the dot represents derivation with respect to the cosmic time, while H is the Hubble parameter defined in the usual manner, $H = \dot{a}/a$. For this specific metric, the Gauss-Bonnet invariant, which is defined as $\tilde{g}_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is $\tilde{g}_{\mathrm{GB}} = 24(\dot{H}H^2 + H^4)$.

If we perform the variation of the action (1) with respect to the fields ϕ and σ we can obtain the Klein-Gordon equations, which express the time evolution of the two quintom fields [48],

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV_1(\phi)}{d\phi} + \frac{dF(\phi)}{d\phi}24(\dot{H}H^2 + H^4) = 0, \tag{5}$$

$$-\ddot{\sigma} - 3H\dot{\sigma} + \frac{dV_2(\sigma)}{d\sigma} + \frac{dG(\sigma)}{d\sigma}24(\dot{H}H^2 + H^4) = 0. \tag{6}$$

The modified Friedmann relations for the present quintom cosmological model are obtained by the variation of the action (1) with respect to the inverse metric $g^{\mu\nu}$. For further details of the calculations, the reader can consult Refs. [46,48,85]. Hence, for our cosmological model, we have the following:

$$H^2 = \frac{1}{3}(\rho_m + \rho_{de}), \tag{7}$$

$$-2\dot{H} - 3H^2 = p_m + p_{de}, \tag{8}$$

where

$$\rho_{de} = \frac{1}{2}\dot{\phi}^2 + V_1(\phi) + 24H^3\frac{dF(\phi)}{d\phi}\dot{\phi} - \frac{1}{2}\dot{\sigma}^2 + V_2(\sigma) + 24H^3\frac{dG(\sigma)}{d\sigma}\dot{\sigma}, \tag{9}$$

$$\begin{aligned} p_{de} = {} & \frac{1}{2}\dot{\phi}^2 - V_1(\phi) - 8H^2\frac{dF(\phi)}{d\phi}\ddot{\phi} - 8H^2\frac{d^2F(\phi)}{d\phi^2}\dot{\phi}^2 - 16H\dot{H}\frac{dF(\phi)}{d\phi}\dot{\phi} - 16H^3\frac{dF(\phi)}{d\phi}\dot{\phi} \\ & - \frac{1}{2}\dot{\sigma}^2 - V_2(\sigma) - 8H^2\frac{dG(\sigma)}{d\sigma}\ddot{\sigma} - 8H^2\frac{d^2G(\sigma)}{d\sigma^2}\dot{\sigma}^2 - 16H\dot{H}\frac{dG(\sigma)}{d\sigma}\dot{\sigma} - 16H^3\frac{dG(\sigma)}{d\sigma}\dot{\sigma}. \end{aligned} \tag{10}$$

The dark energy equation of state is then defined as $w_{de} = p_{de}/\rho_{de}$,

$$\begin{aligned} w_{de} = {} & \left(\frac{1}{2}\dot{\phi}^2 - V_1(\phi) - 8H^2\frac{dF(\phi)}{d\phi}\ddot{\phi} - 8H^2\frac{d^2F(\phi)}{d\phi^2}\dot{\phi}^2 - 16H\dot{H}\frac{dF(\phi)}{d\phi}\dot{\phi} - 16H^3\frac{dF(\phi)}{d\phi}\dot{\phi} - \frac{1}{2}\dot{\sigma}^2 - V_2(\sigma) \right. \\ & \left. - 8H^2\frac{dG(\sigma)}{d\sigma}\ddot{\sigma} - 8H^2\frac{d^2G(\sigma)}{d\sigma^2}\dot{\sigma}^2 - 16H\dot{H}\frac{dG(\sigma)}{d\sigma}\dot{\sigma} - 16H^3\frac{dG(\sigma)}{d\sigma}\dot{\sigma}\right) \\ & \times\left(\frac{1}{2}\dot{\phi}^2 + V_1(\phi) + 24H^3\frac{dF(\phi)}{d\phi}\dot{\phi} - \frac{1}{2}\dot{\sigma}^2 + V_2(\sigma) + 24H^3\frac{dG(\sigma)}{d\sigma}\dot{\sigma}\right)^{-1}. \end{aligned} \tag{11}$$

In the following, we introduce the density parameters associated to the quintom dark energy model and the matter component,

$$\Omega_{de} = \frac{\rho_{de}}{3H^2}, \tag{12}$$

$$\Omega_m = \frac{\rho_m}{3H^2}, \tag{13}$$

resulting in the following constraint equation:

$$\Omega_{de} + \Omega_m = 1. \tag{14}$$

The total effective equation of state for the present quintom cosmological model is then written as

$$w_{\rm tot} = \frac{p_m + p_{de}}{\rho_m + \rho_{de}}. \tag{15}$$

Using the above Klein-Gordon equations and the definitions for the dark energy field energy density and pressure, we can show that the quintom component satisfies a continuity equation of the type

$$\dot{\rho_{de}} + 3H(\rho_{de} + p_{de}) = 0, \tag{16}$$

neglecting any interaction between the dark energy component and the matter sector in the energy conservation relations.

In order to study the properties of the present cosmological quintom dark energy model in the phase space, we decompose the potential part into two exponential functions,

$$V(\phi,\sigma) = V_1(\phi) + V_2(\sigma) = V_{1[0]}e^{-\lambda_1\phi} + V_{2[0]}e^{-\lambda_2\sigma}, \tag{17}$$

where $V_{1[0]}, V_{2[0]}, \lambda_1, \lambda_2$ are constant parameters. In the quintom models with nonminimal couplings, this specific decomposition was recently considered in a cosmological construction in teleparallel gravity [41].

Furthermore, for the generalized coupling functions $F(\phi)$ and $G(\sigma)$ we also consider an exponential representation,

$$F(\phi) = F_{[0]}e^{\alpha\phi}, \tag{18}$$

$$G(\sigma) = G_{[0]}e^{\beta\sigma}, \tag{19}$$

where $\alpha, \beta, F_{[0]}, G_{[0]}$ are constants, as in the recent paper by Granda and Jimenez [76]. In the following section, we study the dynamical effects in the phase space and analyze the implications of the Gauss-Bonnet nonminimal couplings encoded in the α and β coefficients.

III. PHASE-SPACE FEATURES AND THE EFFECTS OF GAUSS-BONNET COUPLINGS

In this section, we investigate the cosmic scenery within the present quintom dark energy model, taking into account the exponential behavior of the coupling functions and the potential terms. By employing a linear stability analysis, we reveal the fundamental dynamical properties of the phase space and discuss the possible future evolution within this model. In order to apply the linear stability method, we introduce the following auxiliary variables [76,85]:

$$\Omega_m \equiv s = \frac{\rho_m}{3H^2}, \tag{20}$$

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \tag{21}$$

$$y = \frac{\dot{\sigma}}{\sqrt{6}H}, \tag{22}$$

$$u = \frac{V_1(\phi)}{3H^2}, \tag{23}$$

$$v = \frac{V_2(\sigma)}{3H^2}, \tag{24}$$

$$f = 8H\frac{dF(\phi)}{d\phi}\dot{\phi} = 8H\alpha F(\phi)\dot{\phi}, \tag{25}$$

$$g = 8H\frac{dG(\sigma)}{d\sigma}\dot{\sigma} = 8H\beta G(\sigma)\dot{\sigma}. \tag{26}$$

Using the Friedmann constraint equation (7), we can reduce the dimensionality of our system from seven to six independent auxiliary variables, where

$$s = 1 - x^2 - u - f + y^2 - v - g. \tag{27}$$

Moreover, we note that this particular choice of auxiliary variables imposes a lesser restriction for the potential part encoded in u and v, which also allows negative values to be easily verified.

Next, we introduce the e-folding variable $N \equiv p = \log(a)(' \equiv \frac{d}{dp})$ and express the evolution of the present cosmological scenario as a system of first-order differential equations:

$$x' = \frac{1}{\sqrt{6}}\frac{\ddot{\phi}}{H^2} - x\frac{\dot{H}}{H^2}, \tag{28}$$

$$y' = \frac{1}{\sqrt{6}}\frac{\ddot{\sigma}}{H^2} - y\frac{\dot{H}}{H^2}, \tag{29}$$

$$u' = -\lambda_1 ux\sqrt{6} - 2u\frac{\dot{H}}{H^2}, \tag{30}$$

$$v' = -\lambda_2 yv\sqrt{6} - 2v\frac{\dot{H}}{H^2}, \tag{31}$$

$$f' = f\frac{\dot{H}}{H^2} + \frac{\ddot{\phi}}{H^2}\frac{f}{x\sqrt{6}} + fx\alpha\sqrt{6}, \tag{32}$$

$$g' = g\frac{\dot{H}}{H^2} + \frac{\ddot{\sigma}}{H^2}\frac{g}{y\sqrt{6}} + gy\beta\sqrt{6}. \tag{33}$$

In terms of the auxiliary variables (21)–(26), we can write the following relations:

$$\ddot{\phi} = -\frac{\sqrt{\frac{3}{2}}fH^2}{x} - \frac{\sqrt{\frac{3}{2}}f\dot{H}}{x} + 3H^2\lambda_1 u - 3\sqrt{6}H^2 x, \tag{34}$$

$$\ddot{\sigma} = \frac{\sqrt{\frac{3}{2}}gH^2}{y} + \frac{\sqrt{\frac{3}{2}}g\dot{H}}{y} - 3H^2\lambda_2 v - 3\sqrt{6}H^2 y, \tag{35}$$

$$\begin{aligned}\dot{H} = {} & \frac{1}{-f^2y^2 + 4fx^2y^2 + g^2x^2 + 4gx^2y^2 - 4x^2y^2} \\ & \times (f^2H^2y^2 - 6fH^2x^2y^2w_m - \sqrt{6}fH^2\lambda_1 uxy^2 - 2\sqrt{6}\alpha fH^2x^3y^2 \\ & + 2fH^2x^2y^2 - g^2H^2x^2 - 6gH^2x^2y^2w_m + \sqrt{6}gH^2\lambda_2 vx^2y - 2\sqrt{6}\beta gH^2x^2y^3 + 2gH^2x^2y^2 - 6H^2ux^2y^2w_m - 6H^2vx^2y^2w_m \\ & - 6H^2x^4y^2w_m + 6H^2x^2y^4w_m + 6H^2x^2y^2w_m - 6H^2ux^2y^2 - 6H^2vx^2y^2 + 6H^2x^4y^2 - 6H^2x^2y^4 + 6H^2x^2y^2).\end{aligned} \tag{36}$$

If we consider the definitions for the density parameters, we can arrive at the following relations:

$$w_{de} = \frac{p_{de}}{3H^2(1-\Omega_m)}, \tag{37}$$

$$w_{\mathrm{tot}} = \frac{p_{de} + w_m 3H^2\Omega_m}{3H^2}. \tag{38}$$

Furthermore, considering the auxiliary variables, we can write the dark energy barotropic parameter as

$$
\begin{aligned}
w_{de} = {} & \frac{1}{3(y^2(f^2-4fx^2+4x^2)-g^2x^2-4gx^2y^2)(f+g+u+v+x^2-y^2)} \times [3f^3y^2w_m + 3f^2gy^2w_m + 3f^2uy^2w_m \\
& + 3f^2vy^2w_m - 9f^2x^2y^2w_m - 3f^2y^4w_m - 3f^2y^2w_m - f^2y^2 - 3fg^2x^2w_m - 24fgx^2y^2w_m - 12fux^2y^2w_m \\
& - 12fvx^2y^2w_m - 12fx^4y^2w_m + 12fx^2y^4w_m + 12fx^2y^2w_m - 2\sqrt{6}f\lambda_1uxy^2 - 4\sqrt{6}\alpha fx^3y^2 + 16fx^2y^2 - 3g^3x^2w_m \\
& - 3g^2ux^2w_m - 3g^2vx^2w_m - 3g^2x^4w_m - 9g^2x^2y^2w_m + 3g^2x^2w_m + g^2x^2 - 12gux^2y^2w_m - 12gvx^2y^2w_m - 12gx^4y^2w_m \\
& + 12gx^2y^4w_m + 12gx^2y^2w_m + 2\sqrt{6}g\lambda_2vx^2y - 4\sqrt{6}\beta gx^2y^3 + 16gx^2y^2 - 12ux^2y^2 - 12vx^2y^2 + 12x^4y^2 - 12x^2y^4],
\end{aligned} \tag{39}
$$

while for the effective total equation of state we have

$$
\begin{aligned}
w_{\rm tot} = {} & \frac{1}{3(-y^2(f^2-4fx^2+4x^2)+g^2x^2+4gx^2y^2)} \times [f^2y^2 + 12fx^2y^2w_m + 2\sqrt{6}f\lambda_1uxy^2 + 4\sqrt{6}\alpha fx^3y^2 \\
& - 16fx^2y^2 - g^2x^2 + 12gx^2y^2w_m - 2\sqrt{6}g\lambda_2vx^2y + 4\sqrt{6}\beta gx^2y^3 - 16gx^2y^2 + 12ux^2y^2w_m + 12vx^2y^2w_m \\
& + 12x^4y^2w_m - 12x^2y^4w_m - 12x^2y^2w_m + 12ux^2y^2 + 12vx^2y^2 - 12x^4y^2 + 12x^2y^4].
\end{aligned} \tag{40}
$$

Using the auxiliary variables, the dark energy pressure can be written in the following form:

$$
\begin{aligned}
p_{de} = {} & \frac{1}{-y^2(f^2-4fx^2+4x^2)+g^2x^2+4gx^2y^2} \times [-3f^3H^2y^2w_m - 3f^2gH^2y^2w_m - 3f^2H^2uy^2w_m - 3f^2H^2vy^2w_m \\
& + 9f^2H^2x^2y^2w_m + 3f^2H^2y^4w_m + 3f^2H^2y^2w_m + f^2H^2y^2 + 3fg^2H^2x^2w_m + 24fgH^2x^2y^2w_m + 12fH^2ux^2y^2w_m \\
& + 12fH^2vx^2y^2w_m + 12fH^2x^4y^2w_m - 12fH^2x^2y^4w_m - 12fH^2x^2y^2w_m + 2\sqrt{6}fH^2\lambda_1uxy^2 + 4\sqrt{6}\alpha fH^2x^3y^2 \\
& - 16fH^2x^2y^2 + 3g^3H^2x^2w_m + 3g^2H^2ux^2w_m + 3g^2H^2vx^2w_m + 3g^2H^2x^4w_m + 9g^2H^2x^2y^2w_m - 3g^2H^2x^2w_m \\
& - g^2H^2x^2 + 12gH^2ux^2y^2w_m + 12gH^2vx^2y^2w_m + 12gH^2x^4y^2w_m - 12gH^2x^2y^4w_m - 12gH^2x^2y^2w_m \\
& - 2\sqrt{6}gH^2\lambda_2vx^2y + 4\sqrt{6}\beta gH^2x^2y^3 - 16gH^2x^2y^2 + 12H^2ux^2y^2 + 12H^2vx^2y^2 - 12H^2x^4y^2 + 12H^2x^2y^4].
\end{aligned} \tag{41}
$$

Next, the critical points for the present quintom scenario with nonminimal Gauss-Bonnet couplings are determined by requiring that the right-hand side of Eqs. (28)–(33) are equal to zero. The critical points of our model are listed in the Table I. In the following, we discuss each critical point in detail and search for possible constraints of the model's parameters that give rise to different cosmological behavior, while taking into account the current evolution of the Universe. The existence conditions associated with the critical point imply that the density parameters are physically

TABLE I. The structure of the phase space: the location of the critical points in the current quintom dark energy model with nonminimal Gauss-Bonnet terms.

	x	y	u	v	f	g
$\mathcal{Z}_1$	×	$-\sqrt{x^2-1}$	0	0	0	0
$\mathcal{Z}_2$	×	$+\sqrt{x^2-1}$	0	0	0	0
$\mathcal{Z}_3$	$\frac{\sqrt{\frac{2}{3}}}{\lambda_1}$	0	$\frac{4}{3\lambda_1^2}$	0	0	$\frac{\lambda_1^2-2}{\lambda_1^2}$
$\mathcal{Z}_4$	0	$\frac{\sqrt{\frac{2}{3}}}{\lambda_2}$	0	$-\frac{4}{3\lambda_2^2}$	$\frac{\lambda_2^2+2}{\lambda_2^2}$	0
$\mathcal{Z}_5$	$\frac{\sqrt{6}w_m+\sqrt{6}}{2\alpha}$	$\frac{\sqrt{\frac{3}{2}}(w_m+1)}{\beta}$	0	0	$-\frac{9(w_m-1)(w_m+1)^2}{\alpha^2(3w_m+1)}$	$\frac{9(w_m-1)(w_m+1)^2}{\beta^2(3w_m+1)}$
$\mathcal{Z}_6$	$\frac{\sqrt{6}w_m+\sqrt{6}}{2\alpha}$	$\frac{\sqrt{\frac{3}{2}}(w_m+1)}{\lambda_2}$	0	$\frac{3(w_m^2-1)}{2\lambda_2^2}$	$-\frac{9(w_m-1)(w_m+1)^2}{\alpha^2(3w_m+1)}$	0
$\mathcal{Z}_7$	$\frac{\sqrt{\frac{3}{2}}(w_m+1)}{\lambda_1}$	$\frac{\sqrt{6}w_m+\sqrt{6}}{2\beta}$	$-\frac{3(w_m^2-1)}{2\lambda_1^2}$	0	0	$\frac{9(w_m-1)(w_m+1)^2}{\beta^2(3w_m+1)}$
$\mathcal{Z}_8$	$\frac{\sqrt{6}w_m+\sqrt{6}}{2\lambda_1}$	$\frac{\sqrt{\frac{3}{2}}(w_m+1)}{\lambda_2}$	$-\frac{3(w_m^2-1)}{2\lambda_1^2}$	$\frac{3(w_m^2-1)}{2\lambda_2^2}$	0	0
$\mathcal{Z}_9$	$\frac{(\sqrt{6}-2)\lambda_1\lambda_2^2}{2(\sqrt{6}-3)(\lambda_1^2-\lambda_2^2)}$	$\frac{(\sqrt{6}-2)\lambda_1^2\lambda_2}{2(\sqrt{6}-3)(\lambda_1^2-\lambda_2^2)}$	$\frac{(3\sqrt{2}-2\sqrt{3})\lambda_2^2(\lambda_1^2(\lambda_2^2+6)-6\lambda_2^2)}{6\sqrt{2}(\sqrt{6}-3)(\lambda_1^2-\lambda_2^2)^2}$	$-\frac{(3\sqrt{2}-2\sqrt{3})\lambda_1^2(\lambda_1^2(\lambda_2^2+6)-6\lambda_2^2)}{6\sqrt{2}(\sqrt{6}-3)(\lambda_1^2-\lambda_2^2)^2}$	0	0

viable, which means that the relation $0 \leq \Omega_m = 1 - \Omega_{de} \leq 1$ is satisfied. Moreover, another existence condition is related to the fact that the critical points are in real space with a corresponding denominator different than zero.

The first class of critical points expressed in Table I as $\mathcal{Z}_1$ represents a critical line that is also present in the minimal coupling case [9], corresponding to a decelerated evolution, where we observe an interrelation between the kinetic terms of the two quintom fields. For this critical line, the effective total equation of state is $w_{\rm tot} = +1$ and the dark matter density parameter $\Omega_m = 0$, showing full domination of the dark energy fields. From a physical point of view, we note that we only feel the effects of the kinetic terms of the two quintom fields, without any influence from the potential terms or the nonminimal coupling functions. The existence condition is related to the fact that the location of the critical points is in real space, implying $x^2 - 1 \geq 0$. The eigenvalues for this class of critical points are

$$E_{\mathcal{Z}_1} = (0, \sqrt{6}\alpha x - 6, 6 - \sqrt{6}\lambda_1 x, -\sqrt{6}\beta\sqrt{x^2 - 1} - 6, \sqrt{6}\lambda_2\sqrt{x^2 - 1} + 6, 3 - 3w_m). \quad (42)$$

Since one of the eigenvalues corresponding to the critical line $\mathcal{Z}_1$ is always zero, the linear stability method fails to provide a suitable framework for the study of the stability cases. Hence, one can only apply this method to cases where this critical line exhibits saddle behavior, where one eigenvalue has a positive real part and at least one eigenvalue has a negative real part. Beyond this discussion, one can study the stability only by using a different technique, such as the Lyapunov method or a center manifold. In the present context, due to the high complexity of the phase space, we rely only on linear stability methods for the stability analysis. Analyzing the sign of the eigenvalues corresponding to the $\mathcal{Z}_1$ critical line, we can see that in the case of a pressureless matter component, one eigenvalue is always greater than zero, and $\mathcal{Z}_1$ cannot be stable (it can only be a saddle or unstable). Taking into account the existence conditions and the previous discussion, the critical line $\mathcal{Z}_1$ will exhibit saddle behavior (corresponding to a stiff-fluid scenario) if the following conditions are met for the case of a pressureless matter component:

$$C_{\mathcal{Z}_1} = \left(\left(x \geq 1 \wedge \left(\left(\frac{\sqrt{6}}{\alpha} > x \wedge \alpha < \sqrt{6}\right) \vee \alpha \leq 0\right)\right) \vee \left(x + 1 \leq 0 \wedge \left(\left(\alpha + \sqrt{6} > 0 \wedge \frac{\sqrt{6}}{\alpha} < x\right) \vee \alpha \geq 0\right)\right)\right). \quad (43)$$

We note that this condition is not exclusive, as there are many situations that correspond to saddle physical behavior.

The next critical point, denoted in the table as $\mathcal{Z}_2$, represents an analogous case to $\mathcal{Z}_1$ (a stiff-fluid solution) that is also present in the minimal coupling case [9]. In this case, we note that the kinetic term of the phantom field is positive, with the effective total equation of state $w_{\rm tot} = +1$, and the dark matter density parameter $\Omega_m = 0$. At this critical line we have the same existence condition, $x^2 - 1 \geq 0$, with the eigenvalues

$$E_{\mathcal{Z}_2} = (0, \sqrt{6}\alpha x - 6, 6 - \sqrt{6}\lambda_1 x, \sqrt{6}\beta\sqrt{x^2 - 1} - 6, 6 - \sqrt{6}\lambda_2\sqrt{x^2 - 1}, 3 - 3w_m). \quad (44)$$

Considering the existence conditions and the nonexclusive case where one eigenvalue is real and negative and the other eigenvalue is real and positive, in Fig. 1 we show one possibility in which we obtain saddle behavior for the $\mathcal{Z}_2$ critical line.

The next critical point, denoted in the table as $\mathcal{Z}_3$, represents a solution in which the kinetic and potential terms of the canonical scalar field dominate, along with effects from the Gauss-Bonnet couplings for the phantom field. For this particular solution, the kinetic and potential terms for the quintessence field are related to the parameter λ_1 which describes the strength of the potential part corresponding to the canonical scalar field. Moreover, the constant representing the strength of the potential part of the canonical scalar field is related to the auxiliary variable g, corresponding to the nonminimal coupling for

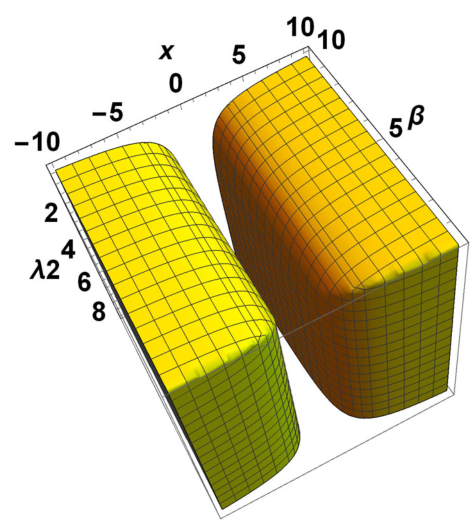

FIG. 1. We have represented here one of the multitude of cases where the critical line $\mathcal{Z}_2$ exhibits saddle behavior.

the phantom field. For this critical point, the dark energy equation of state is equal to the total effective equation of state, $w_{de} = w_{\rm tot} = -\frac{1}{3}$, describing a Universe that is neither accelerating nor decelerating. The density parameter associated with the matter component is equal to $\Omega_m = 0$, a solution dominated completely by the dark energy fields. The single existence condition for this point is related to the relation $\lambda_1 \neq 0$. At this point, we have the eigenvalues

$$E_{\mathcal{Z}_3} = \left(\frac{2\alpha}{\lambda_1} - 2, 2, 2, -1 - i\sqrt{3}, -1 + i\sqrt{3}, -3w_m - 1\right), \quad (45)$$

which represents saddle behavior in the present context.

The $\mathcal{Z}_4$ critical point represents similar behavior to the $\mathcal{Z}_3$ critical point, which is a solution in which the kinetic and potential terms for the phantom field σ dominate, together with the Gauss-Bonnet coupling for the canonical field ϕ. In this case, we notice that the position in phase space is influenced mainly by the strength of the potential term for the noncanonical field represented by the coefficient λ_2. The dark energy field completely dominates the cosmic picture in terms of the density parameters, $\Omega_m = 0$, and the effective equation of state corresponds to neither an accelerated nor a decelerated expansion, $w_{de} = w_{\rm tot} = -\frac{1}{3}$. Analyzing the signs of the eigenvalues,

$$E_{\mathcal{Z}_4} = \left(\frac{2\beta}{\lambda_2} - 2, 2, 2, -1 - i\sqrt{3}, -1 + i\sqrt{3}, -3w_m - 1\right), \quad (46)$$

we notice a saddle behavior, and the dynamical stability is not influenced by any of the coupling parameters or potential constants in this model. At this point, the existence condition is related to the requirement that $\lambda_2 \neq 0$.

In the case of the $\mathcal{Z}_5$ critical point, we notice that the solution represents the domination of the kinetic contribution for the two quintom fields, as well as the manifestation of the nonminimal couplings with the Gauss-Bonnet invariant terms for the scalar fields. Moreover, we can see that the potential terms do not affect the location of this particular solution. For this solution, the matter density parameter is equal to

$$\Omega_m = 1 - \Omega_{de} = \frac{\alpha^2(2\beta^2 + 21) - 21\beta^2 - 9(\alpha^2 - \beta^2)w_m^3 + 3(\alpha^2 - \beta^2)w_m^2 + (\alpha^2(6\beta^2 + 33) - 33\beta^2)w_m}{2\alpha^2\beta^2(3w_m + 1)}, \quad (47)$$

while the effective total equation of state corresponds to a scaling behavior: $w_{\rm tot} = w_{de} = w_m$. At this point, the value of the kinetic term for the quintessence field is affected by the matter equation of state through the coefficient w_m and the parameter α related to the coupling function for the canonical scalar field. For the kinetic term of the noncanonical scalar field, we notice that we have an inverted situation: the coefficient w_m and the parameter β influence this aspect of the solution. Furthermore, the nonminimal Gauss-Bonnet couplings affect this type of solution and the associated dynamical properties. The existence conditions associated with this critical point imply that the location is in real space and the density parameters are physically viable, which satisfies the relations $0 \leq \Omega_m = 1 - \Omega_{de} \leq 1$ and $\alpha, \beta \neq 0$. Taking this physical condition into account, in Fig. 2 we show the existence conditions associated with the previously discussed relation. In the case where the parameter w_m is not set, the eigenvalues at this critical point are complex and the formulas are too cumbersome to be written here. However, if we consider the case of a pressureless matter component ($w_m = 0$) we obtain the following values:

$$E_{\mathcal{Z}_5}^{w_m=0}[1] = 3 - \frac{3\lambda_1}{\alpha}, \quad (48)$$

$$E_{\mathcal{Z}_5}^{w_m=0}[2] = 3 - \frac{3\lambda_2}{\beta}, \quad (49)$$

$$E_{\mathcal{Z}_5}^{w_m=0}[3] = \frac{3}{2}, \quad (50)$$

$$E_{\mathcal{Z}_5}^{w_m=0}[4] = -3, \quad (51)$$

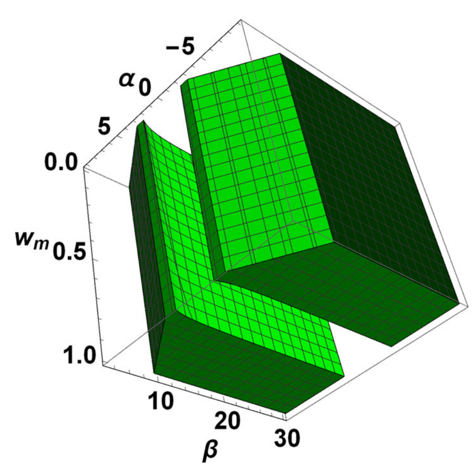

FIG. 2. Various existence conditions for the $\mathcal{Z}_5$ critical point.

$$E_{\mathcal{Z}_5}^{w_m=0}[5] = -\frac{3(\alpha^3\beta(2\beta^2-9)+\sqrt{3}\sqrt{\alpha^6\beta^2(12\beta^4+52\beta^2-477)+\alpha^4(954\beta^4-52\beta^6)-477\alpha^2\beta^6}+9\alpha\beta^3)}{4(\alpha^3\beta(2\beta^2-9)+9\alpha\beta^3)}, \quad (52)$$

$$E_{\mathcal{Z}_5}^{w_m=0}[6] = \frac{3(\alpha^3(9\beta-2\beta^3)+\sqrt{3}\sqrt{\alpha^6\beta^2(12\beta^4+52\beta^2-477)+\alpha^4(954\beta^4-52\beta^6)-477\alpha^2\beta^6}-9\alpha\beta^3)}{4(\alpha^3\beta(2\beta^2-9)+9\alpha\beta^3)}. \quad (53)$$

In the case of radiation $w_m = \frac{1}{3}$, we have the following:

$$E_{\mathcal{Z}_5}^{w_m=1/3}[1] = 4 - \frac{4\lambda_1}{\alpha}, \quad (54)$$

$$E_{\mathcal{Z}_5}^{w_m=1/3}[2] = 4 - \frac{4\lambda_2}{\beta}, \quad (55)$$

$$E_{\mathcal{Z}_5}^{w_m=1/3}[3] = \frac{1}{2}(\sqrt{17}-1), \quad (56)$$

$$E_{\mathcal{Z}_5}^{w_m=1/3}[4] = \frac{1}{2}(-\sqrt{17}-1), \quad (57)$$

$$E_{\mathcal{Z}_5}^{w_m=1/3}[5] = -\frac{\alpha^3\beta(3\beta^2+8)+\sqrt{\alpha^6\beta^2(153\beta^4+1584\beta^2+3136)-16\alpha^4\beta^4(99\beta^2+392)+3136\alpha^2\beta^6}-8\alpha\beta^3}{2(\alpha^3\beta(3\beta^2+8)-8\alpha\beta^3)}, \quad (58)$$

$$E_{\mathcal{Z}_5}^{w_m=1/3}[6] = \frac{\alpha^3(-\beta)(3\beta^2+8)+\sqrt{\alpha^6\beta^2(153\beta^4+1584\beta^2+3136)-16\alpha^4\beta^4(99\beta^2+392)+3136\alpha^2\beta^6}+8\alpha\beta^3}{2(\alpha^3\beta(3\beta^2+8)-8\alpha\beta^3)}. \quad (59)$$

From these expressions, it can be seen that the $\mathcal{Z}_5$ critical point always exhibits saddle behavior, due to the presence of at least one positive eigenvalue and one negative eigenvalue.

The next solution $\mathcal{Z}_6$ represents a situation where the kinetic terms of the quintom fields dominate, together with the potential term associated with the phantom field and the Gauss-Bonnet coupling of the canonical scalar field (quintessence). It can be seen that the barotropic parameter w_m for the matter component dictates the location in phase space, together with an influence from the parameters α and λ_2. These parameters are associated with the coupling function of the quintessence field and the potential strength of the noncanonical scalar field, respectively. This solution also exhibits a scaling behavior, with the barotropic parameter associated with the effective equation of state $w_{\text{tot}} = w_{de} = w_m$. The density parameter for the matter component is

$$\Omega_m = \frac{1}{4}\left(-\frac{6(w_m+1)^2}{\alpha^2} + \frac{36(w_m-1)(w_m+1)^2}{\alpha^2(3w_m+1)} + \frac{6(w_m+1)^2}{\lambda_2^2} + \frac{6-6w_m^2}{\lambda_2^2} + 4\right). \quad (60)$$

As in the previous case, if we take into account the physical existence conditions $0 \le \Omega_m = 1 - \Omega_{de} \le 1$ and $\alpha, \lambda_2 \neq 0$, we obtain Fig. 3. If we set $w_m = 0$ and $\lambda_2 = \alpha$, then we have the following eigenvalues:

$$E_{\mathcal{Z}_6}[1] = \frac{3\beta - 3\alpha}{\alpha}, \quad (61)$$

$$E_{\mathcal{Z}_6}[2] = \frac{3\alpha - 3\lambda_1}{\alpha}, \quad (62)$$

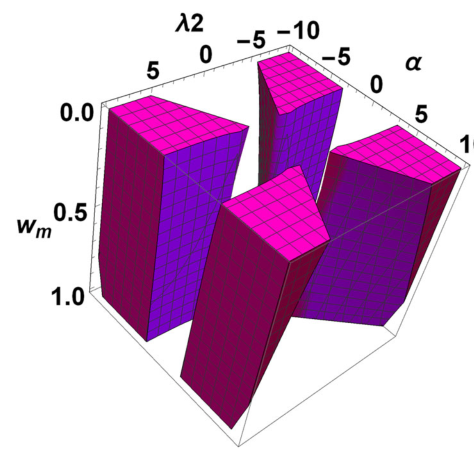

FIG. 3. Different explicit existence conditions for the $\mathcal{Z}_6$ solution.

$$E_{\mathcal{Z}_6}[3,4] = -\frac{3\left(2\alpha^6+9\alpha^4+\sqrt{4\alpha^{12}-252\alpha^{10}-1215\alpha^8\pm 8\sqrt{\alpha^{16}(2\alpha^2+9)^2(4\alpha^4-12\alpha^2+189)}}\right)}{4\alpha^4(2\alpha^2+9)}, \tag{63}$$

$$E_{\mathcal{Z}_6}[5,6] = \frac{3\left(-2\alpha^6-9\alpha^4+\sqrt{4\alpha^{12}-252\alpha^{10}-1215\alpha^8\pm 8\sqrt{\alpha^{16}(2\alpha^2+9)^2(4\alpha^4-12\alpha^2+189)}}\right)}{4\alpha^4(2\alpha^2+9)}, \tag{64}$$

while the matter density parameter $\Omega_m = 1-\frac{15}{2\alpha^2}$. If we consider the existence condition $0 \le \Omega_m = 1-\Omega_{de} \le 1$ and the requirement that this solution exhibits saddle behavior (by considering only the signs of the $E_{\mathcal{Z}_6}[1,2]$ eigenvalues, $E_{\mathcal{Z}_6}[1] > 0, E_{\mathcal{Z}_6}[2] < 0$), we obtain the nonexclusive intervals for the model's parameters depicted in the Fig. 4.

For the $\mathcal{Z}_7$ solution we have a similar behavior as in the previous case: the kinetic terms for the two quintom fields dominate, together with the potential term of the quintessence field and the nonminimal coupling of the noncanonical scalar field. At this point, we note that the kinetic term of the canonical scalar field is related to the matter equation of state and the strength of the potential term associated with the quintessence field. For the kinetic term of the phantom field, we notice a relation with the matter equation of state and the strength of the nonminimal coupling of the noncanonical scalar field. Concerning the potential of the quintessence field, we notice that it is influenced by the matter equation of state and the parameter λ_1 that represents the strength of the potential term of the canonical scalar field. For this solution, the matter density parameter is

$$\Omega_m = \frac{1}{4}\left(\frac{6(w_m+1)^2}{\beta^2} - \frac{36(w_m-1)(w_m+1)^2}{\beta^2(3w_m+1)} - \frac{6(w_m+1)^2}{\lambda_1^2} + \frac{6(w_m^2-1)}{\lambda_1^2} + 4\right), \tag{65}$$

while the total effective equation of state corresponds to a scaling behavior, $w_{\mathrm{tot}} = w_{de} = w_m$. If we take into account the existence conditions for this type of solution, $0 \le \Omega_m = 1-\Omega_{de} \le 1$ and λ_1, $\beta \neq 0$, we arrive at Fig. 5. In the following, we analyze the dynamical stability at this critical point by looking at the signs of the corresponding eigenvalues. As in the previous case, the expressions for the eigenvalues in the general case where $w_m \neq 0$ are too complex to be written here. Hence, we focus on some specific models that lead to stable behavior. If we set $w_m = 0$, $\lambda_2 = 2$, and $\lambda_1 = 1$ we obtain the following simpler expressions for the corresponding eigenvalues:

$$E_{\mathcal{Z}_7}[1] = 3(\alpha-1), \tag{66}$$

$$E_{\mathcal{Z}_7}[2] = 3-\frac{6}{\beta}, \tag{67}$$

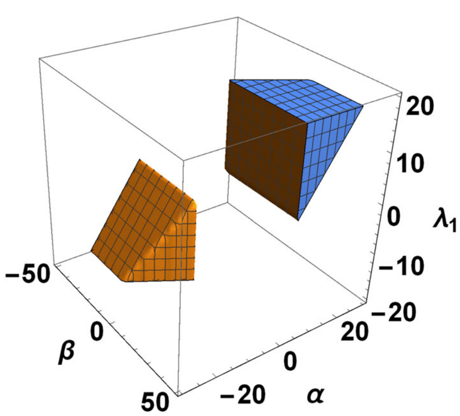

FIG. 4. Nonexclusive conditions where the $\mathcal{Z}_6$ critical point is a saddle point (see the discussion).

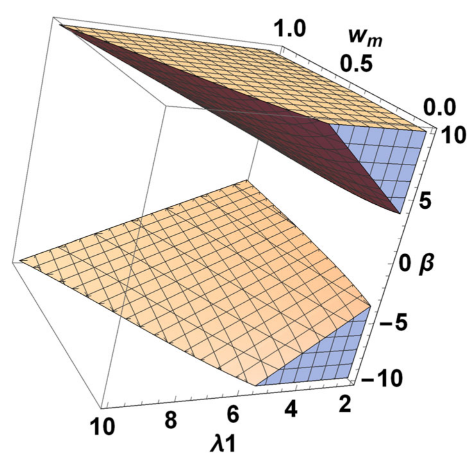

FIG. 5. Various existence conditions associated with the $\mathcal{Z}_7$ critical point.

$$E_{\mathcal{Z}_7}[3] = -\frac{3\left(2\beta^3 + \sqrt{52\beta^6 - 12\beta^4 - 999\beta^2 + 8\sqrt{\beta^4(2\beta^2-9)^2(\beta^4+168\beta^2+36)}} - 9\beta\right)}{4\beta(2\beta^2-9)}, \tag{68}$$

$$E_{\mathcal{Z}_7}[4] = \frac{3\left(-2\beta^3 + \sqrt{52\beta^6 - 12\beta^4 - 999\beta^2 + 8\sqrt{\beta^4(2\beta^2-9)^2(\beta^4+168\beta^2+36)}} + 9\beta\right)}{4\beta(2\beta^2-9)}, \tag{69}$$

$$E_{\mathcal{Z}_7}[5] = -\frac{3\left(2\beta^3 + \sqrt{52\beta^6 - 12\beta^4 - 999\beta^2 - 8\sqrt{\beta^4(2\beta^2-9)^2(\beta^4+168\beta^2+36)}} - 9\beta\right)}{4\beta(2\beta^2-9)}, \tag{70}$$

$$E_{\mathcal{Z}_7}[6] = \frac{3\left(-2\beta^3 + \sqrt{52\beta^6 - 12\beta^4 - 999\beta^2 - 8\sqrt{\beta^4(2\beta^2-9)^2(\beta^4+168\beta^2+36)}} + 9\beta\right)}{4\beta(2\beta^2-9)}, \tag{71}$$

while in this case the matter density parameter is $\Omega_m = \frac{21}{2\beta^2} - 2$. Taking into account the signs of the eigenvalues, we have determined the possible constraints on the parameters α and β that lead to stable scaling behavior, displayed in Fig. 6. By analyzing the evolution of the dynamical equations (28)–(33), we show in Fig. 7 a trajectory towards the $\mathcal{Z}_7$ critical point for specific values of the coupling constants. These conditions are not exclusive, in the sense that we have considered a fine-tuning of the model's parameters so that a stable scenario is present for this scaling solution, in the case where the matter component is a pressureless cold gas and the parameters $\lambda_{1,2}$ are set. Note that in this specific stable case where the parameters $\lambda_{1,2}$, α, and β are set the dark energy density parameter Ω_{de} associated with the $\mathcal{Z}_7$ critical point is not zero.

In the case of the $\mathcal{Z}_8$ critical point we notice the domination of the kinetic and potential terms of the two quintom fields. As it can be seen from Table I, the location in phase space is affected by the matter equation of state and the strength of the potential terms, represented by the values of the parameters λ_1 and λ_2. The first initial existence condition is related to the location in phase space and the requirement that the parameters λ_1 and λ_2 are not null. At this specific point, we have the following value for the matter density parameter:

$$\Omega_m = -\frac{3}{\lambda_1^2} + \frac{3}{\lambda_2^2} + \left(\frac{3}{\lambda_2^2} - \frac{3}{\lambda_1^2}\right) w_m + 1. \tag{72}$$

The second existence condition implies that the density parameter at this point satisfies the relation $0 \le \Omega_m = 1 - \Omega_{de} \le 1$. In the case where the potential parameters are positive $\lambda_{1,2} > 0$ and $w_m = 0$, this condition reduces to evaluating the following expression:

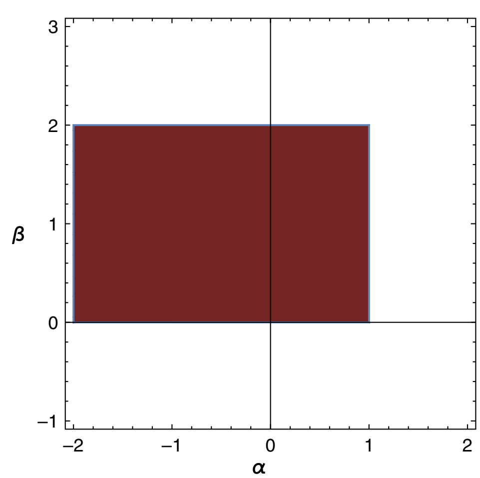

FIG. 6. Nonexclusive conditions associated with the critical point $\mathcal{Z}_7$ where the stable scaling behavior is present: $w_m = 0$, $\lambda_2 = 2, \lambda_1 = 1$.

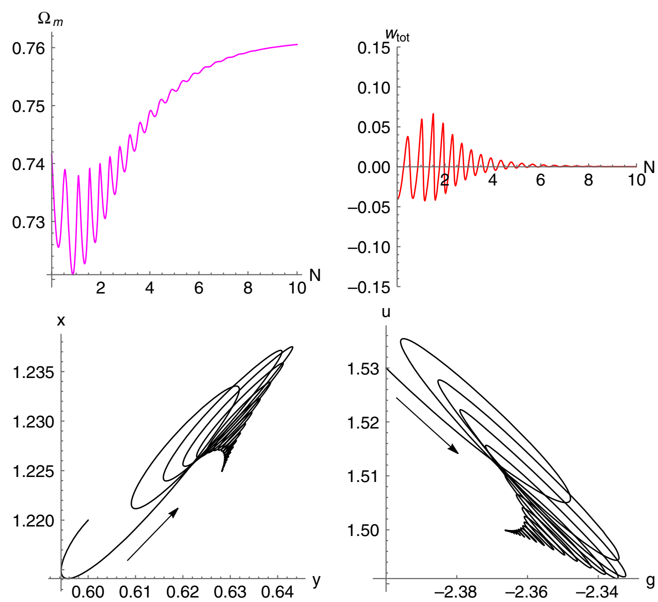

FIG. 7. Numerical evolution of the dynamical system of differential equations toward the $\mathcal{Z}_7$ critical point ($w_m = 0, \lambda_1 = 1, \lambda_2 = 2, \alpha = 0, \beta = 1.95$).

$$C_{\mathcal{Z}_8} = \left(0 < \lambda 1 < \sqrt{3w_m + 3} \wedge \lambda 1 \leq \lambda 2 \leq \sqrt{3}\sqrt{-\frac{\lambda 1^2 + \lambda 1^2 w_m}{\lambda 1^2 - 3w_m - 3}}\right) \vee (\lambda 1 \geq \sqrt{3w_m + 3} \wedge \lambda 2 \geq \lambda 1). \quad (73)$$

For this point, we have the following eigenvalues in the general case, assuming $\lambda_{1,2} > 0$:

$$E_{\mathcal{Z}_8}[1] = \frac{3(\beta - \lambda_2)(w_m + 1)}{\lambda_2}, \quad (74)$$

$$E_{\mathcal{Z}_8}[2] = \frac{3(\alpha - \lambda_1)(w_m + 1)}{\lambda_1}, \quad (75)$$

$$E_{\mathcal{Z}_8}[3] = \frac{1}{4}(-3)\left(\sqrt{9w_m^2 - 2w_m - 7} - w_m + 1\right), \quad (76)$$

$$E_{\mathcal{Z}_8}[4] = \frac{3}{4}\left(\sqrt{9w_m^2 - 2w_m - 7} + w_m - 1\right), \quad (77)$$

$$E_{\mathcal{Z}_8}[5] = -\frac{3(\lambda_1\lambda_2 + \sqrt{w_m - 1}\sqrt{-24\lambda_2^2 + \lambda_1^2(7\lambda_2^2 + 24) + 24(\lambda_1^2 - \lambda_2^2)w_m^2 + (\lambda_1^2(9\lambda_2^2 + 48) - 48\lambda_2^2)w_m} - \lambda_1\lambda_2 w_m)}{4\lambda_1\lambda_2}, \quad (78)$$

$$E_{\mathcal{Z}_8}[6] = \frac{3(-\lambda_1\lambda_2 + \sqrt{w_m - 1}\sqrt{-24\lambda_2^2 + \lambda_1^2(7\lambda_2^2 + 24) + 24(\lambda_1^2 - \lambda_2^2)w_m^2 + (\lambda_1^2(9\lambda_2^2 + 48) - 48\lambda_2^2)w_m} + \lambda_1\lambda_2 w_m)}{4\lambda_1\lambda_2}. \quad (79)$$

In order to determine stability, we investigate the sign of the eigenvalues corresponding to a specific case. We note that if we set $\lambda_1 = \lambda_2$ then the matter density associated with this specific point is equal to 1, describing a full matter-dominated point. Furthermore, in the case of a cold matter component without pressure $w_m = 0$, we find a specific region which corresponds to a stable scaling solution for this critical point, depicted in Fig. 8. Hence, the $\mathcal{Z}_8$ critical point represents a scaling solution ($w_{\rm tot} = w_m$) that can be stable under certain conditions, depending on the values of the parameters.

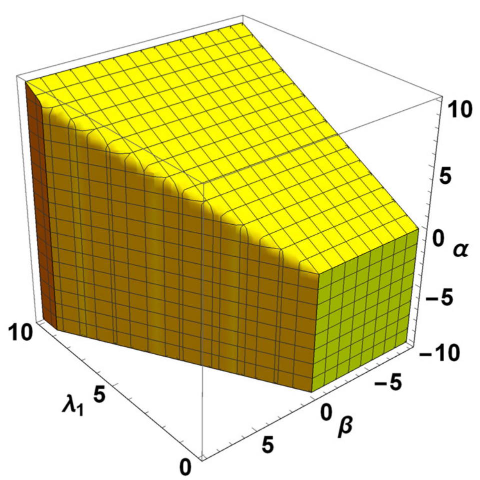

FIG. 8. Nonexclusive regions that correspond to a stable scaling solution at the $\mathcal{Z}_8$ critical point ($w_m = 0, \lambda_2 = \lambda_1$).

Finally, for the last critical point $\mathcal{Z}_9$ we notice that the cosmic picture is dominated by the kinetic and potential terms of the two quintom fields, with the location in phase space being determined by the strength of the potentials given by the values of the two coefficients λ_1 and λ_2. At this point the dark energy completely dominates in terms of density parameters, implying $\Omega_m = 1 - \Omega_{de} = 0$. Hence, from the physical point of view, this solution always satisfies the existence conditions provided by $\lambda_1 \neq \pm\lambda_2$. The total effective equation of state is:

$$w_{\rm tot} = w_{de} = \frac{3\lambda_2^2 - \lambda_1^2(\lambda_2^2 + 3)}{3(\lambda_1^2 - \lambda_2^2)}. \quad (80)$$

In order to have acceleration ($w_{\rm tot} < -\frac{1}{3}$), the parameters λ_1 and λ_2 have to satisfy the following conditions:

$$C_{\mathcal{Z}_9} = \lambda 1 < -\sqrt{\lambda 2^2} \vee -\sqrt{2}\sqrt{\frac{\lambda 2^2}{\lambda 2^2 + 2}} < \lambda 1 < \sqrt{2}\sqrt{\frac{\lambda 2^2}{\lambda 2^2 + 2}} \vee \lambda 1 > \sqrt{\lambda 2^2}. \quad (81)$$

At this point, we obtain the following eigenvalues:

$$E_{\mathcal{Z}_9}[1] = -\frac{3(\sqrt{6}+2)(9\sqrt{6}-22)\lambda_1^2\lambda_2(\beta-\lambda_2)}{2(\sqrt{6}-3)^2(\lambda_1^2-\lambda_2^2)}, \tag{82}$$

$$E_{\mathcal{Z}_9}[2] = -\frac{3(\sqrt{6}+2)(9\sqrt{6}-22)\lambda_1\lambda_2^2(\alpha-\lambda_1)}{2(\sqrt{6}-3)^2(\lambda_1^2-\lambda_2^2)}. \tag{83}$$

The rest of the eigenvalues $E_{\mathcal{Z}_9}[3,4,5,6]$ have very complicated expressions and are difficult to analyze analytically. Hence, in order to extract information related to the dynamical properties for the $\mathcal{Z}_9$ critical point, one has to rely on fine-tuning methods for the parameters of the present model. For example, if we consider the case where $\lambda_1 = 0.1$, $\lambda_2 = 3$, $w_m = 0$, $\alpha = -1$, and $\beta = -3$, we obtain an accelerated solution and a stable critical point where all of the eigenvalues have negative real parts. In Fig. 9 we show the particular crossing of the phantom divide line by the effective equation of state near the $\mathcal{Z}_9$ critical point, which is a specific effect of a quintom scenario. Furthermore, by analyzing the real parts of the eigenvalues of the Jacobian we obtain possible constraints for the constants α and β constants that correspond to a stable scenario (where the real parts of all of the eigenvalues are strictly negative), which we show in Fig. 10. However, this case corresponds to the situation when the potential is negative. It can be shown that for various regions of the model's parameters we can obtain stability for this critical point, leading to an accelerated expansion. If we impose the condition that the potential part is non-negative, then this would require that u and v at the $\mathcal{Z}_9$ critical point are equal to zero, and $\mathcal{Z}_9$ reduces to the $\mathcal{Z}_{1,2}$ critical line for a specific relation between the potential constants λ_1 and λ_2. This situation corresponds to a decelerated expansion, a critical point that represents a particular solution of the $\mathcal{Z}_{1,2}$ type.

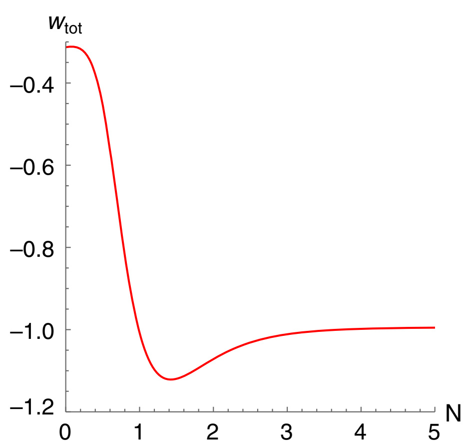

FIG. 9. Crossing of the phantom divide line for the effective equation of state in the evolution of the dynamical system of equations toward the $\mathcal{Z}_9$ critical point ($w_m = 0, \lambda_1 = 0.1, \lambda_2 = 3, \alpha = -1, \beta = -3$).

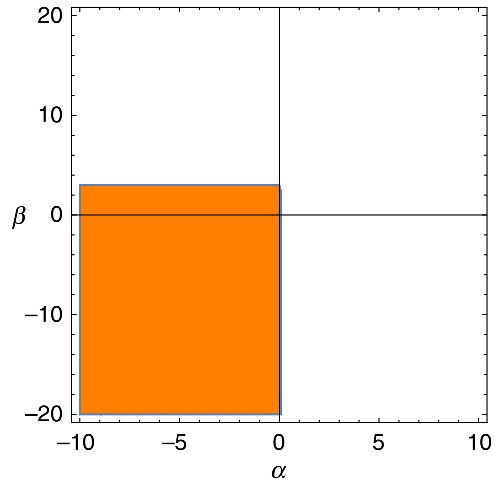

FIG. 10. A possible stable region for the $\mathcal{Z}_9$ solution ($w_m = 0, \lambda_1 = 0.1, \lambda_2 = 3$).

IV. CONCLUSIONS

In this paper we have proposed a new quintom dark energy model, taking into account possible nonminimal interactions between the quintom fields and the Gauss-Bonnet coupling terms. After deducing the Klein-Gordon equations and the modified Friedmann relations for the present quintom dark energy model with independent Gauss-Bonnet couplings for the two scalar fields, we studied the dynamical effects for the evolution of the Universe within this model. The implications of the nonminimal Gauss-Bonnet couplings for the dynamical features of the present quintom scenario have been analyzed by employing the linear stability method, revealing the structure and the basic fundamental properties of the corresponding phase space. Our study revealed that the phase-space structure is complex and exhibits various distinct cosmological scenarios that are capable of explaining the current accelerated expansion of the Universe. For all of the critical points of the dynamical autonomous system of differential equations, we have investigated the stability criteria and determined possible constraints for various parameters of our model that correspond to different types of stability. In our analysis, we have identified several classes of solutions corresponding to the critical points. The first class of dynamical features is represented by stiff-fluid solutions, in which the expansion of the Universe is decelerated due to a positive effective barotropic parameter. We analyzed a second class of dynamical features in which the cosmic expansion is neither accelerated nor decelerated—an interesting type of solution for the present dark energy model—due to a zero effective term on the right-hand side of the acceleration equation. Furthermore, some of the critical points exhibit scaling behavior, in which the dark energy field mimics matter behavior. In these scaling solutions, although the dark energy field dominates the cosmic picture, the total effective equation of state corresponds to a matter epoch, imitating matter behavior at the level of background dynamics. Notice that in our analysis we have studied the critical points in the general case and not taken into account that some of them might have negative potentials, since in many theories the effects of negative potentials have been considered (see Refs. [91–93] and references therein). The reader can observe that the critical points that are related to a negative potential are $\mathcal{Z}_{4,6,8}$. For the $\mathcal{Z}_9$ critical point, if we impose that the potential is non-negative, then this point represents a particular solution of the $\mathcal{Z}_{1,2}$ type, representing a decelerated expansion. Scaling behavior is also valid if we impose that the potential is non-negative, appearing in the case of the saddle dynamics of $\mathcal{Z}_5$ or for the stable solution $\mathcal{Z}_7$. As it can be noted from the results presented in the previous sections, a quintom scenario with Gauss-Bonnet couplings represents an interesting dark energy model capable of explaining the current evolution of the cosmic expansion. We can conclude that in terms of physical properties, the current dynamical dark energy model represents an interesting one, showing a higher complexity of physical features at the level of large-scale background dynamics, which makes it a viable cosmological model in scalar-tensor theories of gravity.

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[1] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner *et al.*, Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter, G. Aldering, G. Goldhaber, R. A. Knop, P. Nugent, P. G. Castro, S. Deustua, S. Fabbro, A. Goobar, D. E. Groom *et al.*, Astrophys. J. **517**, 565 (1999).
[3] P. M. Garnavich, S. Jha, P. Challis, A. Clocchiatti, A. Diercks, A. V. Filippenko, R. L. Gilliland, C. J. Hogan, R. P. Kirshner, B. Leibundgut *et al.*, Astrophys. J. **509**, 74 (1998).
[4] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006).
[5] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003),
[6] Y.-F. Cai, E. N. Saridakis, M. R. Setare, and J.-Q. Xia, Phys. Rep. **493**, 1 (2010).
[7] B. Feng, X.-L. Wang, and X.-M. Zhang, Phys. Lett. B **607**, 35 (2005).
[8] G.-B. Zhao *et al.*, Nat. Astron. **1**, 627 (2017).
[9] Z.-K. Guo, Y.-S. Piao, X.-M. Zhang, and Y.-Z. Zhang, Phys. Lett. B **608**, 177 (2005).
[10] G.-B. Zhao, J.-Q. Xia, M. Li, B. Feng, and X. Zhang, Phys. Rev. D **72**, 123515 (2005).
[11] H. Wei, R.-G. Cai, and D.-F. Zeng, Classical Quantum Gravity **22**, 3189 (2005).
[12] X.-F. Zhang and T. Qiu, Phys. Lett. B **642**, 187 (2006).
[13] X.-F. Zhang, H. Li, Y.-S. Piao, and X.-M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006).
[14] P. Wu and H. Yu, Int. J. Mod. Phys. D **14**, 1873 (2005).
[15] Z.-K. Guo, Y.-S. Piao, X. Zhang, and Y.-Z. Zhang, Phys. Rev. D **74**, 127304 (2006).
[16] M. Alimohammadi and H. M. Sadjadi, Phys. Lett. B **648**, 113 (2007).
[17] J.-Q. Xia, B. Feng, and X.-M. Zhang, Mod. Phys. Lett. A **20**, 2409 (2005).
[18] H. Mohseni Sadjadi and M. Alimohammadi, Phys. Rev. D **74**, 043506 (2006).
[19] M. Kunz and D. Sapone, Phys. Rev. D **74**, 123503 (2006).
[20] B. Feng, M. Li, Y.-S. Piao, and X. Zhang, Phys. Lett. B **634**, 101 (2006).
[21] M. R. Setare and J. Sadeghi, Int. J. Theor. Phys. **47**, 3219 (2008).
[22] W. Wang, Y.-X. Gui, and Y. Shao, Chin. Phys. Lett. **23**, 762 (2006).
[23] R. Lazkoz and G. Leon, Phys. Lett. B **638**, 303 (2006).
[24] W. Zhao and Y. Zhang, Phys. Rev. D **73**, 123509 (2006).
[25] E. N. Saridakis and J. M. Weller, Phys. Rev. D **81**, 123523 (2010).
[26] E. Dil, Adv. High Energy Phys. **2016**, 3740957 (2016).
[27] J. Sadeghi, B. Pourhassan, Z. Nekouee, and M. Shokri, Int. J. Mod. Phys. D **27**, 1850025 (2018).
[28] S. Dutta, M. Lakshmanan, and S. Chakraborty, Int. J. Mod. Phys. D **25**, 1650110 (2016).
[29] E. N. Saridakis, Nucl. Phys. **B830**, 374 (2010).
[30] J. Sadeghi, M. R. Setare, and A. Banijamali, Phys. Lett. B **678**, 164 (2009).
[31] M. R. Setare and E. N. Saridakis, Phys. Lett. B **668**, 177 (2008).
[32] Y.-F. Cai and J. Wang, Classical Quantum Gravity **25**, 165014 (2008).
[33] G. Leon, Y. Leyva, and J. Socorro, Phys. Lett. B **732**, 285 (2014).
[34] M. Marciu, D. M. Ioan, and F. V. Iancu, Int. J. Mod. Phys. D, **28**, 1950018 (2019).
[35] G. Leon, A. Paliathanasis, and J. L. Morales-Martínez, Eur. Phys. J. C **78**, 753 (2018).
[36] J. Sadeghi, M. R. Setare, A. Banijamali, and F. Milani, Phys. Lett. B **662**, 92 (2008).
[37] M. R. Setare, J. Sadeghi, and A. R. Amani, Int. J. Mod. Phys. D **18**, 1291 (2009).

[38] M. R. Setare and E. N. Saridakis, Phys. Lett. B **671**, 331 (2009).
[39] M. R. Setare and M. Sahraee, Gen. Relativ. Gravit. **48**, 119 (2016).
[40] N. Behrouz, K. Nozari, and N. Rashidi, Phys. Dark Universe **15**, 72 (2017).
[41] S. Bahamonde, M. Marciu, and P. Rudra, J. Cosmol. Astropart. Phys. 04 (2018) 056.
[42] M. R. Setare and A. Rozas-Fernandez, Int. J. Mod. Phys. D **19**, 1987 (2010).
[43] K. Nozari, M. R. Setare, T. Azizi, and S. Akhshabi, Acta Phys. Pol. B **41**, 897 (2010).
[44] M. Marciu, Phys. Rev. D **93**, 123006 (2016).
[45] K. Nozari, K. Asadi, and F. Rajabi, Astrophys. Space Sci. **349**, 549 (2014).
[46] S. Nojiri, S. D. Odintsov, and M. Sami, Phys. Rev. D **74**, 046004 (2006).
[47] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B293**, 385 (1987).
[48] S. Nojiri, S. D. Odintsov, and M. Sasaki, Phys. Rev. D **71**, 123509 (2005).
[49] T. Clifton and J. D. Barrow, Classical Quantum Gravity **23**, 2951 (2006).
[50] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 123003 (2005).
[51] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **73**, 084007 (2006).
[52] M. Benetti, S. Santos da Costa, S. Capozziello, J. S. Alcaniz, and M. De Laurentis, Int. J. Mod. Phys. D **27**, 1850084 (2018).
[53] M. Alimohammadi and A. Ghalee, Phys. Rev. D **80**, 043006 (2009).
[54] F. Bauer, J. Sola, and H. Stefancic, J. Cosmol. Astropart. Phys. 12 (2010) 029.
[55] M. Sharif and H. Ismat Fatima, J. Exp. Theor. Phys. **122**, 104 (2016).
[56] T. Koivisto and D. F. Mota, Phys. Rev. D **75**, 023518 (2007).
[57] E. Elizalde, R. Myrzakulov, V. V. Obukhov, and D. Saez-Gomez, Classical Quantum Gravity **27**, 095007 (2010).
[58] A. Escofet and E. Elizalde, Mod. Phys. Lett. A **31**, 1650108 (2016).
[59] A. K. Sanyal, Phys. Lett. B **645**, 1 (2007).
[60] L. Amendola, C. Charmousis, and S. C. Davis, J. Cosmol. Astropart. Phys. 12 (2006) 020.
[61] M. R. Setare and E. N. Saridakis, Phys. Lett. B **670**, 1 (2008).
[62] R. Myrzakulov, D. Saez-Gomez, and A. Tureanu, Gen. Relativ. Gravit. **43**, 1671 (2011).
[63] M. T. Meehan and I. B. Whittingham, J. Cosmol. Astropart. Phys. 12 (2014) 034.
[64] A. De Felice and S. Tsujikawa, Phys. Lett. B **675**, 1 (2009).
[65] E. N. Saridakis, Phys. Rev. D **97**, 064035 (2018).
[66] M. Alimohammadi and A. Ghalee, Phys. Rev. D **79**, 063006 (2009).
[67] S. Nojiri, S. D. Odintsov, and O. G. Gorbunova, J. Phys. A **39**, 6627 (2006).
[68] V. Fayaz, F. Felegary, and H. Hossienkhani, Int. J. Theor. Phys. **52**, 9 (2013).
[69] K. Nozari and N. Rashidi, Int. J. Mod. Phys. D **19**, 219 (2010).
[70] A. R. El-Nabulsi, Astrophys. Space Sci. **327**, 161 (2010).
[71] A. Iqbal and A. Jawad, Adv. High Energy Phys. **2018**, 6139430 (2018).
[72] M. R. Setare, J. Sadeghi, and A. Banijamali, Eur. Phys. J. C **64**, 433 (2009).
[73] C.-J. Feng and X.-Z. Li, Phys. Lett. B **679**, 151 (2009).
[74] A. K. Sanyal, C. Rubano, and E. Piedipalumbo, Gen. Relativ. Gravit. **43**, 2807 (2011).
[75] S. Tsujikawa and M. Sami, J. Cosmol. Astropart. Phys. 01 (2007) 006.
[76] L. N. Granda and D. F. Jimenez, Eur. Phys. J. C **77**, 679 (2017).
[77] L. N. Granda, Mod. Phys. Lett. A **27**, 1250018 (2012).
[78] L. N. Granda, Mod. Phys. Lett. A **28**, 1350117 (2013).
[79] L. N. Granda and D. F. Jimenez, Phys. Rev. D **90**, 123512 (2014).
[80] L. N. Granda, D. F. Jimenez, and C. Sanchez, Int. J. Mod. Phys. D **22**, 1350055 (2013).
[81] L. N. Granda and E. Loaiza, Int. J. Mod. Phys. D **21**, 1250002 (2012).
[82] L. N. Granda, Int. J. Theor. Phys. **51**, 2813 (2012).
[83] T. Koivisto and D. F. Mota, Phys. Lett. B **644**, 104 (2007).
[84] L. Granda and D. Jimenez, Astropart. Phys. **103**, 115 (2018).
[85] L. N. Granda and E. Loaiza, Phys. Rev. D **94**, 063528 (2016).
[86] Z.-K. Guo, N. Ohta, and S. Tsujikawa, Phys. Rev. D **75**, 023520 (2007).
[87] Z.-K. Guo and D. J. Schwarz, Phys. Rev. D **80**, 063523 (2009).
[88] Z.-K. Guo and D. J. Schwarz, Phys. Rev. D **81**, 123520 (2010).
[89] P.-X. Jiang, J.-W. Hu, and Z.-K. Guo, Phys. Rev. D **88**, 123508 (2013).
[90] I. Quiros, R. Garca-Salcedo, T. Gonzalez, F. A. Horta-Rangel, and J. Saavedra, Eur. J. Phys. **37**, 055605 (2016).
[91] R. Giamb, J. Miritzis, and K. Tzanni, Classical Quantum Gravity **32**, 035009 (2015).
[92] I. P. C. Heard and D. Wands, Classical Quantum Gravity **19**, 5435 (2002).
[93] E. J. Copeland, S. Mizuno, and M. Shaeri, Phys. Rev. D **79**, 103515 (2009).
[94] Wolfram Research Inc., Mathematica, http://www.wolfram.com.