New Monte Carlo determination of the critical coupling in ϕ_2^4 theory

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We investigate the nonperturbative features of ϕ^4 theory in two dimensions, using Monte Carlo lattice methods. In particular we determine the ratio $f_0 \equiv g/\mu^2$, where g is the unrenormalized coupling, in the infinite volume and continuum limit. Our final result is $f_0 = 11.055(24)$.

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I. INTRODUCTION

ϕ^4 theory plays an important role in quantum field theory as it represents for example an extremely simplified model for the Higgs sector of the Standard Model.

In $D = 2$ dimensions the theory is super-renormalizable: the coupling constant g has positive mass dimensions $[g] = [\mu_0^2]$, where μ_0 is the (bare) mass parameter of the theory; this means that the ratio $f \equiv g/\mu^2$, where μ^2 is a renormalized squared mass in some given renormalization scheme, is the only physically relevant dimensionless parameter we have to consider. Thanks to the super-renormalizability of the theory we can use the unrenormalized coupling constant in the definition of f, since in any case the renormalization of g amounts to a finite constant.

In this paper we determine the value of $f \equiv g/\mu^2$ at the critical point, that is the value of f computed in the limit in which both g and μ^2 go to zero. We follow the renormalization scheme used in [1,2], adopting the simulation technique introduced in [3,4], namely the *worm algorithm*, and we compute the ratio g/μ^2 using the same strategy implemented in [5]; we present an improvement in the determination of the critical value f_0, obtained thanks to the *gradient flow* [6], a technique that allows us to reach smaller values of the coupling g with respect to our previous work [5].

In the following, after briefly describing the model and the renormalization scheme chosen in order to extract μ^2 at fixed g in the infinite volume limit, we will recall the main steps of the simulations, focusing on the application of the gradient flow; we will then proceed to the continuum limit extrapolation. In the end we will compare our results with our previous determination of the same quantity, and we will draw some conclusions.

II. LATTICE FORMULATION

Let us introduce the ϕ^4 Lagrangian in the Euclidean space,

$$\mathcal{L}_E = \frac{1}{2}(\partial_\nu \phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \frac{g}{4}\phi^4. \tag{1}$$

In $D = 2$ the Euclidean action is

$$\mathcal{S}_E = \int d^2x \mathcal{L}_E. \tag{2}$$

In order to obtain a dimensionless discretized action we put the system on a two-dimensional lattice with spacing a and linear size $L \equiv Na$. By introducing the following parametrization:

$$\hat{\mu}_0^2 = a^2\mu_0^2, \qquad \hat{g} = a^2 g, \tag{3}$$

we have

$$\mathcal{S}_E = \sum_x \left\{ -\sum_\nu \phi_x \phi_{x+\hat{\nu}} + \frac{1}{2}(\hat{\mu}_0^2 + 4)\phi_x^2 + \frac{\hat{g}}{4}\phi_x^4 \right\}, \tag{4}$$

where $\phi_{x\pm\hat{\nu}}$ are fields at neighbor sites in the $\pm\nu$ directions.

In the following we will omit the "hat" on top of lattice parameters: all quantities will be expressed in lattice units; i.e., they become dimensionful when multiplied by appropriate powers of the lattice spacing a.

Numerical simulations are necessarily performed at finite values of a, L and of the bare parameters of the theory: to obtain f_0 we have to extrapolate our results to $L \to \infty$ (infinite volume or thermodynamic limit) and to $a \to 0$ (continuum limit).

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In order to approach the thermodynamic limit we fix g to a given value in lattice units, and this amounts to keep fixed the lattice spacing a; we then simulate the system at several values of $N = L/a$. For each value of N we perform several simulations searching for a value of μ_0^2 such that a certain condition is satisfied. This condition, which we will describe in details later on, is conceived in such a way that by extrapolating $\mu_0^2(g, L/a)$ to the infinite volume limit we get a second order phase transition point in the plane (g, μ_0^2).

As we discuss in [5], in order to safely go to the continuum limit, we have to work out a renormalization of the mass parameter, since μ_0^2 in this limit diverges like $\log(a)$. Adhering to the same renormalization procedure adopted in [1,2], we determine the renormalized squared mass μ^2 putting it equal to the solution, in the infinite volume limit, of the equation

$$\mu^2 = \mu_0^2 + 3gA(\mu^2), \tag{5}$$

where $A(\mu^2)$ is the only 1-particle-irreducible divergent diagram in $D = 2$, which on a $N \times N$ lattice is written as,

$$A(\mu_0^2) = \frac{1}{N^2}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}\frac{1}{4(\sin^2\frac{\pi k_1}{N} + \sin^2\frac{\pi k_2}{N}) + \mu_0^2}. \tag{6}$$

As we stated before, this counterterm cancels the logarithmic divergence of μ_0^2 in the continuum limit, and it should be computed on an infinite lattice. As a matter of fact, for the range of μ_0^2 we have considered, the result for $A(\mu_0^2)$ at $N = 1024$ differs from the value at $N \to \infty$ by a quantity which is several order of magnitude smaller than the statistical error we could attain, and to be on a safe side, in the recursive procedure used to find μ^2 [see Eq. (5)] we always evaluate $A(\mu_0^2)$ on a lattice with linear extension $N = 2048$.

The condition (5) is equivalent to the introduction of a proper divergent mass-squared counterterm in the action. We may finally extrapolate the quantity $f \equiv g/\mu^2$ to $g \to 0$ in order to obtain f_0, the critical value in the continuum limit.

Another useful parametrization of the action is the following:

$$\begin{aligned}\mathcal{S}_E &= -\beta\sum_x\sum_\nu \varphi_x\varphi_{x+\hat{\nu}} + \sum_x[\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2] \\ &= \mathcal{S}_I + \mathcal{S}_{\text{Site}},\end{aligned} \tag{7}$$

where the relations between (μ_0^2, g) and (β, λ) are

$$\phi_x = \sqrt{\beta}\varphi, \qquad \mu_0^2 = 2\frac{1-2\lambda}{\beta} - 4, \qquad g = \frac{4\lambda}{\beta^2}. \tag{8}$$

In Eq. (7) there is an interaction term between neighbor sites, $\mathcal{S}_I$, with a coupling constant of strength β and a term related to a single site, $\mathcal{S}_{\text{Site}}$.

A. Simulations

Now we outline the general computational strategy, focusing in particular on the improvements with respect to our previous work [5].

In our simulations we used the *worm algorithm* [3,4] and used the lattice action given by (7). Operatively we fix a value of λ and L/a and search for a value of β such that the physical condition

$$mL = L/\xi = \text{const} = z \tag{9}$$

is matched for a given and fixed value of z. Condition (9) implies that the second moment correlation length ξ of the system grows linearly with L: thus, when $a/L \to 0$, we arrive at the critical point, where the correlation length ξ diverges if measured in units of the lattice spacing. We then perform several simulations using different values of $N \equiv L/a$; for each couple (λ, N) we obtain a particular value of $\beta_c(\lambda, N)$ such that the condition (9) is satisfied. $\beta_c(\lambda)$ is then obtained by extrapolating our results to $a/L \to 0$. As explained and numerically demonstrated in [5], renormalization group arguments ensure us that for small enough values of a/L we can extrapolate $\beta_c(\lambda, a/L)$ linearly in a/L.

Using the relations in (8) we compute $g(\lambda, \beta_c)$ and $\mu_0^2(\lambda, \beta_c)$. Then, using the renormalization condition (5), we get the value of $\mu^2(g)$ and hence the ratio $f \equiv g/\mu^2$. This procedure is repeated for several values of λ (and hence of g) and finally, in order to obtain f_0, we extrapolate our results to $g \to 0$.

We now focus on the condition (9). In this work we introduce a slight modification in the procedure for the computation of the mass parameter m: it is implicitly defined by the condition

$$R_\rho \equiv \frac{G_\rho(\tau, p^*)}{G_\rho(\tau, 0)} = \frac{m^2}{p^{*2} + m^2}, \tag{10}$$

where p^* is the smallest momentum on the lattice. We decide to compute the propagator $G_\rho(\tau, p)$ using the gradient flow technique in the contest of scalar field theory (see e.g., [6]). In particular, considering the action (7), we introduce a new scalar field $\rho(x, \tau)$ depending on the space-time index x and on the so-called flow-time τ. The flow-time evolution equation of $\rho(x, \tau)$ is

$$\frac{\partial}{\partial\tau}\rho(x, \tau) = \partial^2\rho(x, \tau), \tag{11}$$

where ∂^2 is the Laplace operator acting in the configurations space. If we now impose the Dirichlet boundary conditions, that is $\rho(x, 0) = \varphi_x$, it is easy to write the exact solution, in the momentum space, for the propagator of the field ρ at flow-time τ,

$$G_\rho(\tau, p) = e^{-2\tau p^2} G(p). \tag{12}$$

In this way we obtain a smearing effect of the original fields, since the flow-time exponentially suppresses the ultraviolet modes. To the total flow-time τ we can associate a smearing radius $r_{sm} = \sqrt{2d\tau}$, where d is the dimensionality of the space-time. For a certain value of τ, the ultraviolet suppression effect of the flow-time helps us to obtain values which are closer to the continuum limit. In this way, at fixed λ, we expect to safely extrapolate $\beta_c(\lambda, a/L) \to \beta_c(\lambda, 0)$ using not too large values of L/a with a consequently reduction of both the computational time and the statistical errors.

Some preliminary simulations convinced us to assume the condition $z_\rho = mL = 1$ and to fix the value of τ such that at different a/L values the smearing radius is equal to $L/4$. In this way we take advantage of the smearing effect of the gradient flow and take under control finite volume effects. In Fig. 1 we show the extrapolation of β_c at $\lambda = 0.25$: the red line (upper one) is the extrapolation obtained by fixing $z_\rho = 1$, while the blue line (lower one) is obtained by fixing $z = 4$ (see [5]). We used this small series of simulations as a test of our procedure: our old result (see Table II of [5]) is $\beta_c(\lambda = 0.25) = 0.6586276(98)$, while the new one is $\beta_c(\lambda = 0.25) = 0.6586246(19)$. Note that the two results are perfectly compatible, although the new one has a much smaller statistical error and has been obtained by using lattice with much smaller extension (the biggest lattice in this test has $L/a = 256$ to be compared with $L/a = 768$ of [5]). As one can see, the new results (triangular points) are much closer to the infinite volume limit respect to the old ones. For example, for $L/a = 80$, the difference $\beta_c(\infty) - \beta_c(80)$ is almost 3 times smaller for the new simulation.

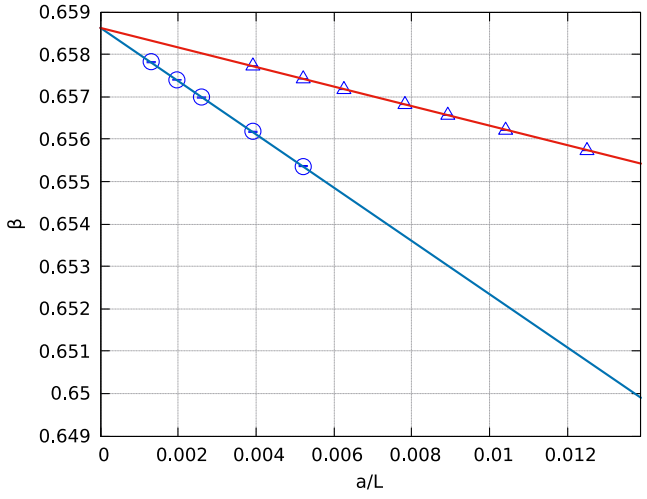

FIG. 1. Comparison between our previous results (lower set of points) and the new ones at $\lambda = 0.25$. See text for a description of the results.

In order to fully appreciate the advantages of using the gradient-flow definition of z, two considerations have to be taken into account:

(1) At a fixed value of λ the behavior of $\beta(\lambda, L)$ for large L is expected to be linear, as it can be deduced from finite size scaling arguments and as it is very well confirmed numerically (see e.g., [5]). For small L, corrections to scaling terms are obviously expected. Numerical evidence show us that, using the gradient-flow definition of z, we obtain a faster approach to linear scaling: this allow us to use smaller lattices to linearly, but safely, extrapolate $\beta_c(\lambda)$ to the infinite volume limit, with a considerable gain in computational time.

(2) In order to obtain $z_\rho = 1$ we tune $\beta(L)$ at fixed λ. The corresponding z computed without using the gradient-flow is a very smooth function of L that can be analytically computed by using Eq. (12). In particular, it is easy to see that for $z_\rho = 1$, corresponding to our choice, the "previous" value of z is a number approximately equal to 1.9. In [5] it was observed that the statistical error on $\beta_c(\lambda, L)$, with a given number of samples, grows as z decreases: in fact, with the choice $z = 1$ we obtained a flatter extrapolation to the infinite volume limit but with too big errors, while the choice $z = 4$ needed greater lattices (and therefore much more computational time), but gives smaller statistical errors. Within this context the choice $z_\rho = 1$ corresponds to an effective value $z \simeq 1.9$ In this way, we have the advantage of an earlier approach to the infinite volume limit with a statistical error which is about a quarter of what we would have obtained by using $z = 1$ without gradient-flow.

In order to obtain the extrapolation at $\lambda \to 0$, we perform our simulations for the coupling values $\lambda = 0.005$, 0.004, 0.003, 0.002, 0.001, 0.00075, 0.0005. At fixed value of λ we simulate the system for several values of L/a, namely: $L/a = 32$, 40, 48, 56, 72, 80, 96, 112, 128, 144, 192, 256, 320, 384. Only for $\lambda = 0.005$, 0.0005 we added simulations at $L/a = 448$, 512. The number of thermalization sweeps for all our simulations is several hundreds times the autocorrelation time of (9). We keep under control the autocorrelation time of our observables using a Python program described in [7], based on [8]. We perform 1000 worm-sweeps between two consecutive measurements and the numbers of measurements varies from 10^4–10^5, according to L and λ.

III. RESULTS

In Table I we report the infinite volume results at the several λ we simulate. These results are obtained performing a linear extrapolation in which the smallest values of L are excluded. All the extrapolations give a final $\chi^2/\text{d.o.f} \sim 1$.

Two values of λ, namely $\lambda = 0.002$, 0.005, are in common with our previous work on the same subject [5]; in particular these were the smallest λ values, and therefore the most critical, reached in that work. We note quite a big discrepancy between the old and the new values of $f(\lambda)$. We exclude that this is due to a failure of our procedure, because the result we obtain at $\lambda = 0.25$ (this result has been obtained just for a check and not used for the final extrapolation) is perfectly compatible with the old one in [5]. We decide anyway to reanalyze our old data, and it came out that just for these two values of λ we underestimated the autocorrelation time of the relevant observable, and therefore the statistical error associated to the measure. The reanalysis of the old data does not change, if not marginally, the conclusions of [5], nor it invalidates the results in the present work, that we carefully checked.

We finally extrapolate $f(\lambda) \to f_0$ as $\lambda \to 0$ by using three functional forms,

$$g_a(\lambda) = f_0^{\text{lin}} + a_1\lambda \tag{13}$$

$$g_b(\lambda) = f_0^{\text{quad}} + b_1\lambda + b_2\lambda^2 \tag{14}$$

$$g_c(\lambda) = f_0^{\text{log}} + c_1\lambda + c_2\lambda^2 + c_3\lambda\log(\lambda). \tag{15}$$

Our final results are

$$f_0^{\text{lin}} = 11.053(13) \quad \chi^2/\text{d.o.f} = 1.79 \tag{16}$$

$$f_0^{\text{quad}} = 11.058(4) \quad \chi^2/\text{d.o.f} = 0.77 \tag{17}$$

$$f_0^{\text{log}} = 11.072(20) \quad \chi^2/\text{d.o.f} = 0.85. \tag{18}$$

For the linear extrapolation we used the four last data points of Table I, while for the quadratic and logarithm extrapolation we used all the data set at our disposal. The results are plotted in Fig. 2.

In [5] the data show no evidence for the presence of a term of the type $\lambda\log(\lambda)$. In the present work we reach much smaller values of λ and we decided to verify if such a term may be present. As it is clear from Fig. 2 no definite conclusion can be drawn: also we observe that the function (15) almost completely overlaps with a simple quadratic fit, excluding the region of very small λ where we have no data. Moreover we note that the linear and the quadratic fit give almost identical results, and we take this agreement as a numerical evidence of the fact that a simple polynomial law is enough to describe the behavior of $f(\lambda)$ near $\lambda \to 0$.

TABLE I. Infinite volume results of β_c and $f(\lambda)$ with different linear lattice sizes.

λ	β_c	$(L/a)_{\text{min}}$	$(L/a)_{\text{max}}$	$f(\lambda)$
0.0005	0.5019535(5)	192	512	10.9920(88)
0.00075	0.5027800(10)	144	384	10.9754(79)
0.001	0.5035613(10)	112	384	10.9258(79)
0.0015	0.5050340(2)	112	384	10.8801(20)
0.002	0.5064156(9)	112	384	10.8304(45)
0.003	0.5089871(7)	128	384	10.7504(15)
0.004	0.5113712(4)	128	384	10.6884(15)
0.005	0.5136155(5)	112	512	10.6435(10)

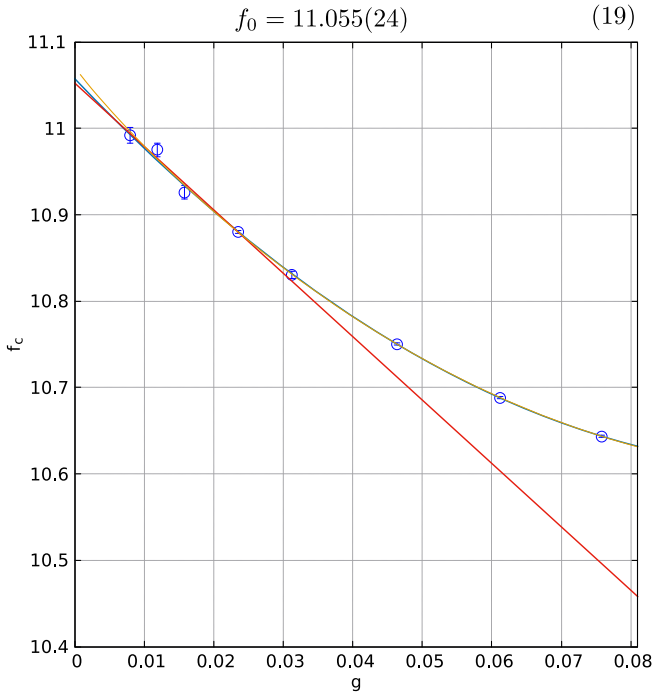

FIG. 2. Final extrapolation of f versus g. Straight red line is (15). The quadratic fit completely overlaps with the "log" fit; only for $g \to 0$ we observe a small increase of the "log" fit (orange curve) with respect to the quadratic one (blue curve).

TABLE II. Sample of the results for the continuum critical parameter f_0 from the literature. DLCQ stands for *discretized light cone quantization*, QSE diagonalization for *quasi–sparse eigenvector* diagonalization, DMRG for *density matrix renormalization group* and for DLCH-FS *diagonalized light-front Hamiltonian in Fock-space representation*.

Method	f_0	Year, Ref.
DLCQ	5.52	1988, [9]
QSE diagonalization	10	2000, [10]
DMRG	9.9816(16)	2004, [11]
Monte Carlo cluster	$10.8^{0.1}_{0.05}$	2009, [2]
Monte Carlo SLAC derivative	10.92(13)	2012, [17]
Uniform matrix product states	11.064(20)	2013, [12]
Monte Carlo worm	11.15(9)	2015, [5]
Borel summability-lattice results	11.00(4)	2015, [16]
DLCH-FS	4.40(12)	2016, [18]
Renormalized Hamiltonian	11.04(12)	2017, [13,14]
Borel summability	11.23(14)	2018, [15]
This work	11.055(24)	2018

We decide to take the mean of the two polynomial results as our final value and the sum in quadrature of all the three errors, including the one coming from the "log fit", to compute the error. We finally quote

$$f_0 = 11.055(24) \tag{19}$$

to be compared to our previous result,

$$f_0 = 11.15(9). \tag{20}$$

The two values are well compatible within 1-sigma level but the new result (16) has a sensibly reduced error.

IV. CONCLUSIONS

In Table II we summarize some of the latest results of f_0 derived with different approaches: the works [9–14] are based on Hamiltonian truncation (variational) methods, Borel summability is applied in [15,16], where in [16] lattice results are used. Finally in [17] lattice theory is simulated by using nonlocal SLAC derivative. Since we are in a good agreement with our previous result, the same considerations are still valid: our result is compatible with the last six determinations (excluding [5,18]) at the 2σ-level, which come from different methods.

The gradient flow technique allows us to reach lower values of λ with respect to our previous work and to obtain a more precise estimation of f_0.

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