Black hole entropy from nonproper gauge degrees of freedom: The charged vacuum capacitor

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The semiclassical contribution to the partition function is obtained by evaluating the Euclidean action improved through suitable boundary terms. We address the question of which degrees of freedom are responsible for this contribution. A physical toy model for the gravitational problem is a charged vacuum capacitor. In Maxwell's theory, the gauge sector including ghosts is a topological field theory. When computing the grand canonical partition function with a chemical potential for electric charge in the indefinite metric Hilbert space of the Becchi-Rouet-Stora-Tyutin quantized theory, the classical contribution to the partition function originates from the part of the gauge sector that is no longer trivial due to the boundary conditions required by the physical setup. More concretely, for a planar charged vacuum capacitor with perfectly conducting plates, we identify the degrees of freedom that, in the quantum theory, give rise to additional contributions to the standard blackbody result proportional to the area of the plates and that allow for a microscopic derivation of the thermodynamics of the charged capacitor.

I. INTRODUCTION

The question of which degrees of freedom are responsible for the Bekenstein-Hawking entropy of black holes naturally leads one to study nonproper gauge degrees of freedom, i.e., gauge degrees of freedom that are no longer pure gauge because of nontrivial boundary conditions. (i) The most direct line of reasoning is probably to consider the Hamiltonian formulation of linearized Einstein gravity. The linearized Schwarzschild solution does not involve physical degrees of freedom since the transverse-traceless parts of the spatial metric and its momenta vanish for that solution. (ii) Another argument, which holds on the nonlinear level, concerns the Bekenstein-Hawking entropy of the black hole in three-dimensional anti–de Sitter spacetime where there are no physical bulk gravitons to begin with. (iii) Yet another approach has to do with the type of observables that are involved: in general relativity, the ADM mass is a codimension-two surface integral, with similar properties to electric charge in Maxwell's theory. In particular, it does not involve transverse-traceless variables. Furthermore, the classification of such observables

is directly related to nonproper diffeomorphisms or large gauge transformations.

One possibility is to introduce the nontrivial boundary conditions as dynamical canonical variables in the theory, with suitable additional constraints. This idea goes back to Dirac [1] and has been used in an investigation of the definition of energy, and more generally of the Poincaré generators, in the Hamiltonian formulation of asymptotically flat general relativity [2]. In the context of Yang-Mills theory, it has been implemented for various related questions [3–8], including the infrared problem [9].

These arguments suggest studying the analogue problem in the context of the quantized electromagnetic field, where the role of the black hole is played by the Coulomb solution, the electromagnetic field created by a static point particle source with macroscopic charge Q. Besides being a physical problem in its own right where all conceptual issues are present, the linearity of the problem and the wealth of results readily available in the literature make it directly tractable.

In the first paper of this series [10], a quantum mechanical understanding has been achieved when all polarizations of the photon are quantized in an indefinite metric Hilbert space: the quantum state $|0\rangle^Q$ corresponding to the classical Coulomb solution is a coherent state of null oscillators, made up of a linear combination of longitudinal and temporal photons. In this computation, infrared divergences occur when showing that the expectation value $^Q\langle 0|\hat{\pi}^i(x)|0\rangle^Q$ of the electric field operator is indeed the classical field

produced by a pointlike source: one uses that the Fourier transform of k^{-2} is proportional to $(4\pi r)^{-1}$ which really requires an infrared regularization, $(k^2 + m^2)^{-1}$ giving the Yukawa potential $(4\pi r)^{-1} e^{-mr}$, with $m \to 0^+$.

Unlike ordinary coherent states, null coherent states have the same norm than the standard vacuum, $^Q\langle 0|0\rangle^Q = 1$. Furthermore, the expectation value of the energy of physical photons vanishes. It is in this sense that these states behave like different vacua of the theory.

Rather than quantizing the theory for a fixed charge, what we would like to address here is the computation of the grand canonical partition function,

$$Z(\beta, \mu) = \mathrm{Tr} e^{-\beta(\hat{H} - \mu\hat{Q})}, \qquad (1.1)$$

with a precise understanding of the underlying Hilbert space and thus of the trace that is involved. Again, when trying to deal directly with the electric charge operator,

$$\hat{Q} = -\int_{\partial\mathcal{V}} d\sigma_i \hat{\pi}^i = -\int_{\mathcal{V}} d^3x \partial_i \hat{\pi}^i, \qquad (1.2)$$

in a large volume $\mathcal{V}$, one has to face infrared questions since $-\hat{Q}$ is the zero mode of the longitudinal part of the electric field.

On the classical level, the role of the chemical potential is played by the constant value of $A_0 = -\mu$ at the surface of the body, while a nonvanishing electric charge requires $\pi^r = O(r^{-2})$. In order to take electric charge into account, nontrivial falloff or boundary conditions are thus required.

That longitudinal and temporal photons have an important role to play in topologically nontrivial situations is in agreement with the standard interpretation of the Aharanov-Bohm effect [11] when extrapolated to the quantized electromagnetic field. The approach we will follow here is to start with (A_μ, π^μ) as canonical variables without introducing additional degrees of freedom. For trivial boundary conditions, standard results equivalent to those derived in the framework of reduced phase space quantization are then recovered in the indefinite metric Becchi-Rouet-Stora-Tyutin (BRST) Fock space through the quartet mechanism [12] in the bulk. We will analyze in detail how these results are modified when imposing the boundary conditions that are used in the context of the Casimir effect [13]. For technical reasons, it is then also easier for us here to start with a vacuum capacitor consisting of two large parallel plates instead of a spherical vacuum capacitor, so that one may use Fourier series instead of Bessel functions [14].

Recent work on infrared physics has been driven by new connections in the field summarized in [15]. There is a considerable overlap of ideas and results underlying this computation here and those developed in terms of edge modes in [16–27]. A detailed comparison, also with the considerations in [28], deserves further investigation.

The paper is organized as follows. In the next section, we start by discussing the thermodynamics of a charged vacuum capacitor following the method developed by Gibbons and Hawking [29]: from the Euclidean path integral, it follows that the semiclassical approximation to $\ln Z(\beta, \mu)$ is given by minus the Euclidean action evaluated at the classical solution. The appropriate boundary terms needed for the charged capacitor have already been introduced in the context of charged black holes for instance in [30]. As compared to the one-loop result for the standard blackbody, there is now a contribution proportional to the area coming from the classical saddle point, together with additional contributions at one-loop. The purpose of this paper is to provide a microscopic derivation of the saddle point and the additional contributions to the partition function.

In Sec. III, we point out in what sense the gauge sector of Maxwell's theory can be understood as a topological field theory. It is not really needed for the rest of the paper, but is included in order to better understand the relation with three-dimensional gravity for instance.

In Sec. IV, boundary conditions adapted to perfectly conducting parallel plates, taken at constant z, are imposed. Through a detailed Hamiltonian analysis, we show that the modes with vanishing momenta in the z direction of (A_z, π^z), even though formally longitudinal, are to be considered as physical in the problem at hand. In that sense, we refer to them as nonproper gauge degrees of freedom.

In the quantum theory, we compute in Sec. IV C the contribution of the nonzero modes of the nonproper gauge degrees of freedom to the standard blackbody result. It is proportional to the area of the plates. After turning on the chemical potential for electric charge, a quantum mechanical understanding of the classical thermodynamics of the vacuum capacitor follows from the contribution of the zero mode of the nonproper gauge degrees of freedom.

Additional remarks are relegated to Sec. V. Conventions for mode expansions adapted to the various boundary conditions are given in Appendix A. In order to be self-contained, a summary of standard material on BRST quantization as applied to Maxwell's theory is provided in Appendix B and Appendix C.

II. THERMODYAMICS OF A CHARGED VACUUM CAPACITOR

When making the Legendre transformation of the standard Lagrangian action $S[A_\mu] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$ for $\dot{A}_i$, and after adding the boundary term, $-\oint_{\partial B} d\sigma_i[\pi^i A_0]$, the first-order action is

$$I = \int d^4x[\dot{A}_i\pi^i - \mathcal{H}_0 + A_0\partial_i\pi^i + j^\mu A_\mu],$$

$$\mathcal{H}_0 = \frac{1}{2}(\pi^i\pi_i + B^iB_i), \qquad (2.1)$$

where $B^i = \epsilon^{ijk}\partial_j A_k$, $E^i = -\pi^i$. Alternatively, this action may be obtained from the extended first-order action after eliminating the Lagrange multiplier for the primary constraint and the momentum π^0.

From the viewpoint of constrained Hamiltonian systems, there are two gauge invariant observables in the problem, the reduced phase space energy

$$H^{\text{ph}} = \int d^3x \mathcal{H}^{\text{ph}}, \qquad \mathcal{H}^{\text{ph}} = \frac{1}{2}(\pi_T^i \pi_i^T - A_i^T \Delta A_T^i), \quad (2.2)$$

and also the electric charge

$$Q = -\int_S d\sigma_i \pi_L^i, \qquad (2.3)$$

where S is a closed 2-surface.

Consider a spherical vacuum capacitor consisting of two conducting spheres S_1, S_2 centered at the origin with radii $R_1 < R_2$ and charges q, $-q$. Let us first focus on time-independent fields and assume that there are no sources inside the body. We will assume here that $A_i = 0$, even though the field equations only require $\partial_j F^{ji} = 0$. In this context, there are then no transverse degrees of freedom and

$$A_0 = -\phi = -\frac{q}{4\pi r}, \qquad \pi^i = -\frac{qx^i}{4\pi r^3} \qquad (2.4)$$

for $R_1 < r < R_2$ and zero otherwise.

The thermodynamics can then be obtained from the Euclidean action evaluated on-shell. Since the problem is at fixed electric charge, no improvement boundary terms are needed [31], and

$$I_E = \frac{\beta}{2}\int d^3x \pi_L^i \pi_i^L = \frac{1}{2}c_S\beta q^2, \qquad c_S = \frac{R_2 - R_1}{4\pi R_1 R_2}. \qquad (2.5)$$

Using $\pi_L^i = \partial^i \phi$ and $\Delta\phi = 0$ on-shell for $R_1 < r < R_2$, $\int d^3x \pi_L^i \pi_i^L = \int d^3x \partial_i(\phi\partial^i\phi)$, I_E can also be written in terms of boundary terms as

$$I_E = -\frac{\beta}{2}(\phi|_{S_2} - \phi|_{S_1})Q, \qquad (2.6)$$

where $\phi_S = \frac{q}{4\pi r}$ and $Q = q$ for the problem at hand. This then gives rise to the semiclassical contribution to the partition function,

$$\ln Z(\beta, Q) = -I_E(\beta, Q) + f(\beta), \qquad (2.7)$$

where one would expect $f(\beta)$ to be given by the standard one-loop contribution of physical photons,

$$f_V(\beta) = \frac{1}{3}b_V\beta^{-3}, \qquad b_V = \frac{\pi^2 V}{15}. \qquad (2.8)$$

The analysis below shows however that there are additional contributions

$$f(\beta) = f_V(\beta) + f_A(\beta) - \frac{1}{2}\ln(2\pi\beta),$$

$$f_A(\beta) = \frac{1}{2}b_A\beta^{-2}, \qquad (2.9)$$

with b_A proportional to the area,

$$b_A = \frac{\zeta(3)}{\pi}A, \qquad (2.10)$$

in the case of the planar capacitor.[1] This implies that

$$U = -\frac{\partial \ln Z(\beta, Q)}{\partial \beta} = -f'(\beta) + \frac{1}{2}c_S Q^2. \qquad (2.11)$$

In case this can be inverted to yield $\beta = \beta(U')$, with $U' = U - \frac{1}{2}c_S Q^2$, the entropy is

$$S(U, Q) = \left(1 - \beta\frac{\partial}{\partial\beta}\right)f(\beta)\big|_{\beta=\beta(U')}. \qquad (2.12)$$

Alternatively, in order to deal directly with

$$Z(\beta, \mu) = \text{Tr}e^{-\beta(\hat{H} - \mu\hat{Q})}, \qquad (2.13)$$

one supposes instead that the electric potentials at the boundary are fixed and constant, $\phi|_{S_1} = \phi_1$, $\phi|_{S_2} = \phi_2$ with $\mu = \phi_1 - \phi_2$. Under the additional assumptions that there are no sources inside the body, $\partial^i A_i = 0$ and $A_i^T = 0 = \pi_T^i$, the classical solution is

$$\phi = \frac{1}{R_2 - R_1}\left(R_2\phi_2 - R_1\phi_1 + \frac{\mu R_1 R_2}{r}\right),$$

$$E^i = \frac{\mu R_1 R_2 x^i}{(R_2 - R_1)r^3}. \qquad (2.14)$$

In this situation, following [2] (see also [30]), the action needs to be improved by boundary terms so that this solution is a true extremum of the variational principle,

$$I' = I + \int dt\phi_2 Q - \int dt\phi_1 Q \qquad (2.15)$$

On-shell, the Euclidean action is now

$$I'_E = \frac{\beta}{2}(\phi_2 - \phi_1)Q, \qquad Q = c_S^{-1}\mu. \qquad (2.16)$$

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[1]The conditions under which some of these terms can be neglected will be discussed elsewhere.

This leads to

$$I'_E = -\frac{1}{2}c_S^{-1}\beta\mu^2. \tag{2.17}$$

Together with the one-loop results, one thus finds

$$\ln Z(\beta, \mu) = -I'_E + f(\beta). \tag{2.18}$$

The electric charge is then

$$Q = \beta^{-1}\frac{\partial \ln Z(\beta, \mu)}{\partial \mu} = c_S^{-1}\mu. \tag{2.19}$$

At fixed β, the Legendre transform of $\ln Z(\beta, \mu)$ with respect to μ,

$$\ln Z(\beta, Q) = \left(1 - \mu\frac{\partial}{\partial \mu}\right)\ln Z(\beta, \mu)|_{\mu=\mu(Q)}, \tag{2.20}$$

then leads back to (2.7).

For the case of the so-called exterior problem, the thermodynamics of a charged spherical shell of radius R_1 can be obtained from the above by letting $R_2 \to \infty$ and taking $\phi_2 = 0$.

For two parallel plates P_1, P_2 at $z = 0$ and at $z = L_3$, with charge densities $\frac{q}{A}$ and $-\frac{q}{A}$, one finds under the same assumptions and in the same manner that $\pi^i = -\delta_3^i\frac{q}{A}$, [when $x^i = (x, y, z)$], $\phi = -\frac{q}{A}z$, with $\mu = \frac{L_3 q}{A}$. The only change in the classical part of the above discussion is then the replacement of the geometric factor c_S by

$$c_P = \frac{L_3}{A}. \tag{2.21}$$

What we will study below is the quantum mechanical origin of the semiclassical contribution to the partition function, together with the additional one-loop contributions.

III. GAUGE SECTOR OF ELECTROMAGNETISM AS A TOPOLOGICAL FIELD THEORY

The gauge sector of Maxwell's theory is treated in the context of the Batalin-Fradkin-Vilkovisky Hamiltonian formalism [32–34]. It contains the scalar potential, the longitudinal vector potential, ghosts and their momenta, and thus captures the information on the electric charge in regions where there are no sources. A Witten-type supersymmetric quantum mechanical model [35] is a model for which the whole action, including the kinetic term is BRST exact. We show that this is the case for the gauge sector of Maxwell's theory when treating the spatial dimensions in a formal way.

We follow the reviews [36], chapter 19, and [37] chapter 3, for the BFV treatment of electromagnetism and for supersymmetric quantum mechanics, respectively.

In the nonminimal BFV-BRST approach in which (A_0, π^0) are among the canonical variables, the action to be used in the Hamiltonian path integral for electromagnetism is

$$S = \int dt \int d^3x[\dot{A}_\mu\pi^\mu + \dot{\eta}\mathcal{P} + \dot{\bar{C}}\rho - \mathcal{H}_0 - \{\Omega, K_\xi\}], \tag{3.1}$$

where the BRST invariant Hamiltonian is $H_0 = \int d^3x\mathcal{H}_0$, $\mathcal{H}_0$ is given in (2.1), and the graded Poisson brackets are determined by

$$\{A_\mu(\vec{x}), \pi^\nu(\vec{y})\} = \delta_\mu^\nu\delta^{(3)}(\vec{x}, \vec{y}),$$
$$\{\eta(\vec{x}), \mathcal{P}(\vec{y})\} = -\delta^{(3)}(\vec{x}, \vec{y}) = \{\bar{C}(\vec{x}), \rho(\vec{y})\}. \tag{3.2}$$

The BRST charge is

$$\Omega = -\int d^3x(i\rho\pi^0 + \eta\partial_i\pi^i), \tag{3.3}$$

and the gauge fixing fermion is chosen as

$$K_\xi = -\int d^3x\left(i\bar{C}\partial_k A^k + \mathcal{P}A_0 - \xi\frac{i}{2}\bar{C}\pi^0\right), \tag{3.4}$$

so that

$$\{\Omega, K_\xi\} = \int d^3x\left(\partial_k A^k\pi^0 - \partial_i\pi^i A_0 + i\mathcal{P}\rho \right.$$
$$\left. + i\partial^i\bar{C}\partial_i\eta - \frac{\xi}{2}\pi^0\pi^0\right). \tag{3.5}$$

Eliminating the auxiliary fields $\pi^i \approx F^{i0}$, $\pi^0 \approx \frac{1}{\xi}(\partial_\mu A^\mu)$, $\rho \approx i\dot{\eta}$, $\mathcal{P} \approx -i\dot{\bar{C}}$, gives the covariant gauge fixed Faddeev-Popov action,

$$S = \int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)(\partial^\nu A_\nu) - i\partial^\mu\bar{C}\partial_\mu\eta\right], \tag{3.6}$$

but we will not do so here in order to keep better track of the various degrees of freedom.

Decomposing into transverse and longitudinal fields, $A_i = A_i^T + \partial_i A$, with $A = \frac{\partial^i A_i}{\Delta}$, $\pi^i = \pi_T^i + \frac{1}{\Delta}\partial^i\pi$ with $\pi = \partial_i\pi^i$, the first-order action decomposes into a transverse piece,

$$S^{\text{ph}} = \int dt \int d^3x(\dot{A}_i^T\pi_T^i - \mathcal{H}^{\text{ph}}), \tag{3.7}$$

with $\mathcal{H}^{\mathrm{ph}}$ given in (2.2), and a piece from the gauge sector (including ghosts),

$$S^{\mathrm{gs}} = \int dt \int d^3x (\dot{A}_0 \pi^0 - \dot{A}\pi + \dot{\eta}\mathcal{P} + \dot{\bar{C}}\rho - \mathcal{H}^{\mathrm{gs}}), \quad (3.8)$$

where

$$H^{\mathrm{gs}} = \int d^3x \mathcal{H}^{\mathrm{gs}} = -\frac{1}{2}i\{\Omega, \bar{\Omega}\},$$

$$\bar{\Omega} = 2iK_\xi - i \int d^3x \mathcal{P}\frac{1}{\Delta}\partial_i\pi^i, \quad (3.9)$$

includes the contribution of the longitudinal electric fields, and is explicitly given by

$$\mathcal{H}^{\mathrm{gs}} = -\pi\left(A_0 + \frac{1}{2\Delta}\pi\right) + \pi^0\left(\Delta A - \frac{\xi}{2}\pi^0\right) + i\mathcal{P}\rho - i\bar{C}\Delta\eta. \quad (3.10)$$

Turning on the chemical potential for electric charge can be done through the shift $A_0(t,\vec{x}) \to A_0(t,\vec{x}) - \mu(\vec{x})$ for a time independent external source $\mu(\vec{x})$, since this changes $H^T \to H^T + \int d^3x \mu(\vec{x})\partial_i\pi^i$ and thus to $H^T \to H^T - \mu Q$ for constant μ.

In the case of a constant metric, supersymmetric quantum mechanics is described by the action

$$S^{\mathrm{ss}} = \int dt\left(iB_i\frac{d\phi^i}{dt} - i\bar{\psi}_i\frac{d\psi^i}{dt} + H^{\mathrm{ss}}\right),$$

$$H^{\mathrm{ss}} = \frac{\alpha}{2}g^{ij}B_iB_j + is\frac{\partial V}{\partial\phi^i}g^{ij}B_j - is\bar{\psi}_i g^{ij}\frac{\partial^2 V}{\partial\phi^j\partial\phi^k}\psi^k. \quad (3.11)$$

The entire action is BRST exact

$$S^{\mathrm{ss}} = \int dt\left\{\Omega, \bar{\psi}_i\left[i\frac{d\phi^i}{dt} + g^{ij}\left(\frac{\alpha}{2}B_j + is\frac{\partial V}{\partial\phi^j}\right)\right]\right\}, \quad (3.12)$$

where the BRST charge is

$$\Omega = -iB_i\psi^i, \quad (3.13)$$

and the fundamental Poisson brackets are $\{\phi^i, B_j\} = -i\delta^i_j = -\{\psi^i, \bar{\psi}_j\}$, with all other brackets vanishing. As consequence, the BRST transformations $\mathrm{s} = \{\Omega, \cdot\}$ are explicitly given by

$$\mathrm{s}\phi^i = \psi^i, \quad \mathrm{s}\psi^i = 0, \quad \mathrm{s}\bar{\psi}_i = B_i, \quad \mathrm{s}B_i = 0. \quad (3.14)$$

The Hamiltonian can be written as

$$H^{\mathrm{ss}} = \frac{1}{2}i\{\Omega, \bar{\Omega}\}, \quad \bar{\Omega} = -i\bar{\psi}_i g^{ij}\left(\alpha B_j + 2is\frac{\partial V}{\partial\phi^j}\right), \quad (3.15)$$

with $\bar{\Omega}$ generating the so-called anti-BRST symmetry, $\bar{\mathrm{s}} = \{\bar{\Omega}, \cdot\}$, explicitly given by

$$\bar{\mathrm{s}}\phi^i = \alpha g^{ij}\bar{\psi}_j, \quad \bar{\mathrm{s}}\psi^i = g^{ij}\left(\alpha B_j + 2is\frac{\partial V}{\partial\phi^j}\right),$$

$$\bar{\mathrm{s}}\bar{\psi}_i = 0, \quad \bar{\mathrm{s}}B_i = -2is\bar{\psi}_j g^{jk}\frac{\partial^2 V}{\partial\phi^k\partial\phi^i}. \quad (3.16)$$

The gauge sector can be written as a supersymmetric quantum mechanical model with $H^{\mathrm{gs}} = -H^{ss}$ if $\alpha = s = 1 = \xi$,

$$\phi^i = \begin{pmatrix} A(\vec{x}) \\ A_0(\vec{x}) \end{pmatrix}, \quad \psi^i = \begin{pmatrix} -\eta(\vec{x}) \\ i\rho(\vec{x}), \end{pmatrix},$$

$$B_i = \begin{pmatrix} i\pi(\vec{x}) \\ -i\pi^0(\vec{x}) \end{pmatrix}, \quad \bar{\psi}^i = \begin{pmatrix} i\mathcal{P}(\vec{x}) \\ -\bar{C}(\vec{x}) \end{pmatrix}, \quad (3.17)$$

and

$$g^{ij} = \begin{pmatrix} -\frac{1}{\Delta}\delta^3(\vec{x}, \vec{x}') & 0 \\ 0 & -\delta^3(\vec{x}, \vec{x}') \end{pmatrix},$$

$$V = \int d^3x A\Delta A_0, \quad (3.18)$$

provided spatial integrations by parts are allowed. Formally, DeWitt's condensed notation is used (in the sense that summation over i includes an integration over $\vec{x}$, while δ^i_j includes a Dirac delta function in three dimensions).

This formulation of the gauge sector can be turned into a local topological field theory with a BRST exact Hamiltonian when using the potential π' defined through $\pi = \Delta\pi'$ in (3.8), (3.9), and (3.10).

Such a reformulation is clearly not essential for an understanding of the problem. Nevertheless, it indicates at this stage already that the explicit computation of the partition function involves the value of the exponential at the classical saddle point, the "instanton" solution $\frac{d\phi^i}{dt} = 0$, $\frac{\partial V}{\partial\phi^i} = 0$.

IV. PLANAR VACUUM CAPACITOR

In this main section, the partition function for the vacuum capacitor is computed, after identifying the complete Hilbert space from a constrained Hamiltonian analysis that takes the nontrivial boundary conditions of the physical setup into account. Notations and conventions are fixed in Appendix A. In order to understand how the boundary conditions influence the result, it is instructive to first review the standard and well-known results in the case of periodic boundary conditions. This is done in Appendix B and C, following [36].

A. Spatial boundary conditions

For conducting plates, spatial boundary conditions on the fields have to be imposed that implement $\vec{n} \cdot \vec{B} = 0 = \vec{n} \times \vec{E} = 0$ on the boundary. If $x^i = (x^a, x^3)$ with $a = 1, 2$, this is guaranteed if the mode expansion of (A_a, π^a) contains sines only,

$$
A_c(x^i) = \sum_{n_a} \sum_{n_3>0} A^S_{c,k_a,k_3} \sin k_3 x^3 e^{ik_a x^a},
$$

$$
\pi^d(x^i) = \sum_{n_a} \sum_{n_3>0} \pi^{Sd}_{k_a,k_3} \sin k_3 x^3 e^{ik_a x^a}, \tag{4.1}
$$

with nonvanishing Poisson brackets

$$
\{A^S_{c,k_a,k_3}, \pi^{*Sd}_{k'_a,k'_3}\} = \frac{2\delta^d_c}{V} \prod_{i=1}^{3} \delta_{n_i,n'_i}, \tag{4.2}
$$

where $V = 4L_1 L_2 L_3$. In order for bulk cancellations to work as in the case of periodic boundary conditions, one is forced to use Neumann conditions for (A_3, π^3), so that

$$
A_3(x^i) = \sum_{n_a} \left[A^C_{3,k_a,0} + \sum_{n_3>0} A^C_{3,k_a,k_3} \cos k_3 x^3 \right] e^{ik_a x^a},
$$

$$
\pi^3(x^i) = \sum_{n_a} \left[\pi^{C3}_{k_a,0} + \sum_{n_3>0} \pi^{C3}_{k_a,k_3} \cos k_3 x^3 \right] e^{ik_a x^a}. \tag{4.3}
$$

This implies that

$$
\{A^C_{3,k_a,0}, \pi^{*C3}_{k'_a,0}\} = \frac{1}{V} \prod_{a=1}^{2} \delta_{n_a,n'_a},
$$

$$
\{A^C_{3,k_a,k_3}, \pi^{*C3}_{k'_a,k'_3}\} = \frac{2}{V} \prod_{i=1}^{3} \delta_{n_i,n'_i}, \qquad k_3 > 0. \tag{4.4}
$$

These conditions are consistent with the boundary conditions used in the context of the Casimir effect when one works in radiation gauge $A_0 = 0, \partial^i A_i = 0$ (see e.g., [38]). The boundary conditions on the remaining variables then follow from the Hamiltonian analysis starting from $H_C = \int_B d^3x (\mathcal{H}_0 - A_0 \partial_i \pi^i)$. Indeed, in order to impose the Gauss law in the bulk, (A_0, π^0) should satisfy Dirichlet conditions. In turn, the same then goes for the ghost pairs $(\eta, \mathcal{P})$, $(\bar{C}, \rho)$, and also for (A, π). Again, this is consistent with the conditions in the context of the Casimir effect (e.g., [39] where it is shown that there is a standard supersymmetric cancellation between the zero point energies of the gauge sector, and also [40–42] for related considerations).

B. Degrees of freedom and dynamics

When substituting the mode expansion, the canonical Hamiltonian splits into three pieces,

$$
H_C = H_B + H_W + H_{\mathrm{NPG}}, \tag{4.5}
$$

with a standard bulk piece

$$
H_B = \frac{V}{4} \sum_{n_a,n_3>0} [\pi^{bS}_{k_a,k_3} \pi^{*S}_{b,k_a,k_3} + \pi^{3C}_{k_a,k_3} \pi^{*3C}_{b,k_a,k_3} + k_1^2 (A^S_{2,k_a,k_3} A^{*S}_{2,k_a,k_3} + A^C_{3,k_a,k_3} A^{*C}_{3,k_a,k_3}) + k_2^2 (A^S_{1,k_a,k_3} A^{*S}_{1,k_a,k_3} + A^C_{3,k_a,k_3} A^{*C}_{3,k_a,k_3})
$$
$$
+ k_3^2 (A^S_{1,k_a,k_3} A^{*S}_{1,k_a,k_3} + A^S_{2,k_a,k_3} A^{*S}_{2,k_a,k_3}) + ik_2 k_3 (A^S_{2,k_a,k_3} A^{*C}_{3,k_a,k_3} - A^C_{3,k_a,k_3} A^{*S}_{2,k_a,k_3}) + ik_1 k_3 (A^S_{1,k_a,k_3} A^{*C}_{3,k_a,k_3} - A^C_{3,k_a,k_3} A^{*S}_{1,k_a,k_3})
$$
$$
- k_1 k_2 (A^S_{1,k_a,k_3} A^{*S}_{2,k_a,k_3} + A^S_{2,k_a,k_3} A^{*S}_{1,k_a,k_3})]. \tag{4.6}
$$

The piece

$$
H_W = \frac{V}{2} \sum_{n_a,n_3>0} [A^S_{0,k_a,k_3}(ik_b \pi^{*S}_{b,k_a,k_3} + k_3 \pi^{*3C}_{b,k_a,k_3})], \tag{4.7}
$$

will give rise to the secondary constraints, $-ik_b \pi^S_{b,k_a,k_3} + k_3 \pi^{3C}_{b,k_a,k_3} \approx 0$. As expected and can be easily checked, there are no tertiary constraints.

The most interesting piece from the current perspective is

$$
H_{\mathrm{NPG}} = \frac{V}{2} \sum_{n_a} [\pi^{3C}_{k_a,0} \pi^{3C*}_{k_a,0} + \omega_{k_a}^2 A^C_{3,k_a,0} A^{C*}_{3,k_a,0}],
$$

$$
\omega_{k_a} = \sqrt{k_1^2 + k_2^2}. \tag{4.8}
$$

In summary, we can split degrees of freedom according to whether they are k_3 zero modes or not. In the latter group,

we have (A_b, π^b), (A_0, π^0), $(\eta, \mathcal{P})$, $(\bar{C}, \rho)$, which all satisfy Dirichlet boundary conditions, as well as the $k_3 \neq 0$ modes of (A_3, π^3) satisfying Neumann conditions.

The former group contains $(A^C_{3,k_a,0}, \pi^{3C}_{k_a,0})$, respectively, the fields

$$
\phi(x,y) = \sum_{n_a} A^C_{3,k_a,0} e^{ik_b x^b}, \qquad \pi(x,y) = \sum_{n_a} \pi^C_{3,k_a,0} e^{ik_b x^b}. \tag{4.9}
$$

None of these variables is involved in any of the constraints. They are thus physical. Note that while the associated vector potential and electric fields are formally longitudinal,

$$
A^{\mathrm{NPG}}_i(x,y,0) = \delta_i^3 \phi = \partial_i[z\phi],
$$

$$
\pi^i_{\mathrm{NPG}}(x,y,0) = \delta_3^i \pi = \partial_i[z\pi], \tag{4.10}
$$

this is not really the case since z is restricted to the closed interval $[0, L_3]$. Note also that the Poisson brackets for these variables given in (4.4) and the Hamiltonian (4.8), which are encoded in the bulk first-order action restricted to these degrees of freedom, completely determine the Lagrangian action of a massless scalar in $(2+1)$ dimensions after integrating out the momenta,

$$S^{\text{NPG}} = \frac{L_3}{2} \int dt \int_{-L_1}^{L_1} dx \int_{-L_2}^{L_2} dy [(\dot{\phi})^2 - \partial_a \phi \partial^a \phi]. \quad (4.11)$$

In this context, the electric charge operator, by analogy with the discussion in Sec. II, is taken to be the quantum version of the classical observable

$$Q = -\pi_{0,0,0}^{3C} A = -\int_{-L_1}^{L_1} dx \int_{-L_2}^{L_2} dy \pi, \qquad A = 4L_1 L_2.$$
$$(4.12)$$

which Poisson commutes both with the complete Hamiltonian and all constraints.

C. Partition function

For the nonzero-mode sector of the theory, one can then follow the analysis of the periodic case (fix the gauge, choose suitable variables). The difference is only that the modes involved are restricted to $k_3 > 0$. Up to details related to the standard Casimir effect (which will be addressed elsewhere), one finds that the contribution to the partition function from this sector is the standard blackbody result, Eq. (B29).

For the new sector, we first consider the nonzero modes of the nonproper gauge degrees of freedom, $(A_{3,k_a,0}^C, \pi_{k_a,0}^{3C})$, with $k_a \neq 0$. For them, one defines standard oscillator variables

$$a_{k_a} = \sqrt{\frac{\omega_{k_a} V}{2}} \left(A_{3,k_a,0}^C + \frac{i}{\omega_{k_a}} \pi_{k_a,0}^{3C} \right), \quad (4.13)$$

so that

$$\{a_{k_a}, a_{k_a'}^*\} = -i\delta_{n_a, n_a'}, \qquad H_{\text{NPG}}' = \sum_{n_a}^{'} \omega_{k_a} a_{k_a}^* a_{k_a}. \quad (4.14)$$

The contribution to the partition function,

$$Z_{\text{NPG}}'(\beta, \rho) = \text{Tr} e^{-\beta \rho \hat{H}_{\text{NPG}}'}, \quad (4.15)$$

is given by

$$\ln Z_{\text{NPG}}'(\beta, \rho) = -\sum_{n_a}^{'} \ln(1 - e^{-\beta \rho \omega_{k_a}}). \quad (4.16)$$

The standard approximation then leads to

$$\ln Z_{\text{NPG}}'(\beta, \rho) = -\frac{A}{4\pi^2} \int dk_1 dk_2 \ln \left(1 - e^{-\beta \rho \sqrt{k_1^2 + k_2^2}}\right)$$
$$= \frac{A}{2\pi} \zeta(3)(\beta \rho)^{-2}. \quad (4.17)$$

For the zero mode of the nonproper gauge degrees of freedom, the variables $q = A_{3,0,0}^C \sqrt{V}$, $p = \pi_{3,0,0}^{3C} \sqrt{V}$, have canonical commutation relations, while the Hamiltonian and electric charge observable are given by

$$H_{\text{NPG}}^0 = \frac{1}{2} p^2, \qquad Q = -\sqrt{\frac{A}{L_3}} p. \quad (4.18)$$

It follows that the contribution to the partition function,

$$Z_{\text{NPG}}^0(\beta, \nu, \mu) = \text{Tr} e^{-\beta \nu \hat{H}_{\text{NPG}}^0 + \beta \mu \hat{Q}}, \quad (4.19)$$

of this free particle is

$$\ln Z_{\text{NPG}}^0(\beta, \nu, \mu) = \ln \Delta q - \frac{1}{2} \ln (2\pi \beta \nu) + \frac{\beta \mu^2}{\nu} \frac{A}{2L_3}, \quad (4.20)$$

where Δq denotes the divergent interval of integration over q, which should be dropped.

The starting point Hamiltonian corresponds to $\rho = 1 = \nu$, so that the semiclassical contribution to the partition function discussed in Sec. II is recovered through the last term of Eq. (4.20).

V. DISCUSSION AND PERSPECTIVES

We have used a Hamiltonian approach here in order to keep track of the various degrees of freedom and of their nature. It should be possible to streamline these derivations by using finite temperature Lagrangian path integral methods combined with techniques from topological field theory and extend the considerations here to more complicated nontrivial boundary conditions than the ones we have treated explicitly.

The nontrivial effect is a zero-mode effect, like in the case of Bose-Einstein condensation [43]. The difference is however that in the latter both observables $\hat{H}$ and $\hat{N}$ involve the same degrees of freedom, whereas in our case, the physical Hamiltonian $\hat{H}$ and the electric charge $\hat{Q}$ involve different degrees of freedom. The electromagnetic analog of the semiclassical Bekenstein-Hawking contribution to the partition function comes here from the zero mode of the nonproper gauge degrees of freedom, which are themselves zero modes from the bulk perspective.

Magnetic charge can be treated in the same way when using a magnetic instead of an electric formulation. Both types of charges simultaneously can be understood in a manifestly duality invariant first-order formulation [44] (see also e.g., [45]) which includes an additional quartet [46,47].

The next, in principle straightforward, step is then to generalize the result discussed here to the spherical vacuum capacitor. For linearized gravity around flat space, one can easily adapt the result of [10] and understand the Schwarzschild solution as a coherent state of unphysical gravitons. Generalizing the derivation here should also be tractable and is the object of a follow-up project. This is then what an observer at spatial infinity would see. He would, however, not be able to distinguish between a black hole and a star from that computation alone.

The analysis in this paper in terms of a detailed mode expansions is possible because boundary conditions at both $z = 0$ and $z = L_3$ are specified. The additional scalar field that emerges from the canonical analysis was not associated to a single boundary plane, but to both of these planes together. In the case of black holes, one might wonder whether the role of these planes might be played by boundary conditions at infinity on the one hand and at the horizon on the other. This would be consistent with the fact that one needs "surface terms" both at the black hole horizon and at infinity when using the Hamiltonian action [48] in order to derive the background contribution to the partition function.

It would be interesting to study in more detail how the quantization of the electromagnetic field in this topologically nontrivial setup appears from the viewpoint of large gauge symmetries. In Chern-Simons theories for instance, large gauge symmetries become gobal symmetries of the Wess-Zumino or Liouville theories that describe the residual dynamics in the presence of boundaries. This role is played here by the massless scalar theory, which does indeed possess an infinite number of global symmetries.

The consequences of the present computation, both from a theoretical and an experimental viewpoint should be fully explored. One would need to understand from the current perspective what happens in an interacting theory like QED for instance, how to resum contributions from the gauge sector and to get different charged sectors in the electromagnetic case, and similarly, to go from a flat to a black hole background in the gravitational case.

As we have tried to show in [10] and with this computation here, in order to deal consistently with charged sectors or black holes in the operator formalism, computations are transparent when all polarizations of the four potential or of the metric are quantized in a nonunitary Hilbert space. This is also implicitly the case in the Euclidean path integral formulation when choosing real paths for the Euclidean version of A_0, or for the shift vectors. Since most of the questions on black hole entropy have little to do with transverse-traceless variables but rather with variables from the gauge sector, one might want to take this specific nonunitarity into account when discussing paradoxes related to black hole physics.

APPENDIX A: MODE EXPANSIONS

1. Periodic boundary conditions

Consider first periodic boundary conditions in a box B_P with sides of lengths $2L_i$ and volume $V_P = 8L_1L_2L_3$. Note that in this case, no improvement terms are needed for the gauge fixed Hamiltonian $H_0 + \{\Omega, K_\xi\}$. The fields

$$z^A = (A_0, \pi^0, A_i, \pi^i, \eta, \mathcal{P}, \bar{C}, \rho), \tag{A1}$$

are expanded in terms of Fourier series at fixed time t,

$$z^A(x^i) = \sum_{n_i} z^A_{k_i} e^{ik_i x^i}, \qquad z^A_{k_i} = z^{A*}_{-k_i},$$

$$z^A_{k_i} = \frac{1}{V_P} \int_{B_P} d^3x \, z^A(x^i) e^{-ik_i x^i}, \tag{A2}$$

with $n_i \in \mathbb{Z}$ and $k_i = \frac{\pi n_i}{L_{(i)}}$ (no summation over i). Quadratic integrals are related as

$$\int_{B_P} d^3x \, z^A(x^i) z^B(x^i) = V_P \sum_{n_i} z^A_{k_i} z^{*B}_{k_i}. \tag{A3}$$

The canonical Poisson bracket relations that originate from the kinetic term

$$\int_{B_P} d^3x \, \dot{\phi}(x^i, t)\pi(x^i, t), \tag{A4}$$

for each canonically conjugated pair are then

$$\{z^A_{k_i}, z^{*B}_{k'_i}\} = \frac{\sigma^{AB}}{V_P} \prod_{i=1}^{3} \delta_{n_i, n'_i}, \tag{A5}$$

with all other Poisson brackets following from the middle of Eq. (A2). Here σ^{AB} is the canonical symplectic matrix obtained by combining

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A6}$$

for each canonical pair. Translating back to position space gives

$$\{z^A(x^i), z^B(y^i)\} = \sigma^{AB} \delta_P^{(3)}(x^i, y^i),$$

$$\delta_P^{(3)}(x^i, y^i) = \frac{1}{V_P} \sum_{n_i} e^{ik_i(x^i - y^i)}. \tag{A7}$$

Alternatively, if one replaces the exponentials by sines and cosines in the $z = x^3$ direction,

$$z^A(x^i) = \sum_{n_a} \left[c_{k_a,0}^A + \sum_{n_3 > 0} (c_{k_a,k_3}^A \cos k_3 x^3 + s_{k_a,k_3}^A \sin k_3 x^3) \right]$$
$$\times e^{ik_a x^a}, \tag{A8}$$

with $a = 1, 2$

$$c_{k_a,0}^A = \frac{1}{V_P} \int_{B_P} d^3 x z^A(x^i) e^{-ik_a x^a} = z_{k_a,0}^A, \tag{A9}$$

and, for $k_3 > 0$,

$$\begin{pmatrix} c_{k_a,k_3}^A \\ s_{k_a,k_3}^A \end{pmatrix} = \frac{2}{V_P} \int_{B_P} d^3 x z^A(x^i) e^{-ik_a x^a} \begin{pmatrix} \cos k_3 x^3 \\ \sin k_3 x^3 \end{pmatrix}$$
$$= \begin{pmatrix} z_{k_a,k_3}^A + z_{k_a,k_3}^{*A} \\ i(z_{k_a,k_3}^A - z_{k_a,k_3}^{*A}) \end{pmatrix}. \tag{A10}$$

In this case,

$$\int_{B_P} d^3 x z^A(x^i) z^B(x^i)$$
$$= V_P \sum_{n_a} \left\{ c_{k_a,0}^A c_{k_a,0}^{*B} + \frac{1}{2} \sum_{n_3 > 0} [c_{k_a,k_3}^A c_{k_a,k_3}^{*B} + s_{k_a,k_3}^A s_{k_a,k_3}^{*B} \right.$$
$$\left. + i(c_{k_a,k_3}^A s_{k_a,k_3}^{*B} - s_{k_a,k_3}^A c_{k_a,k_3}^{*B})] \right\}. \tag{A11}$$

and the Poisson brackets are

$$\{c_{k_a,0}^A, c_{k_a',0}^{*B}\} = \frac{\sigma^{AB}}{V_P} \prod_{a=1}^{2} \delta_{n_a, n_a'}, \tag{A12}$$

and, for $k_3, k_3' > 0$,

$$\{c_{k_3,k_a}^A, c_{k_3',k_a'}^{*B}\} = \frac{2\sigma^{AB}}{V_P} \prod_{i=1}^{3} \delta_{n_i, n_i'} = \{s_{k_3,k_a}^A, s_{k_3',k_a'}^{*B}\}, \tag{A13}$$

and all other Poisson brackets vanishing. In these terms, the periodic delta function can be written as

$$\delta_P^{(3)}(x^i, y^i) = \frac{1}{V_P} \sum_{n_a} e^{ik_a x^a} \left[1 + 2 \sum_{n_3 > 0} \cos k_3 (x^3 - y^3) \right]$$
$$= \frac{1}{V_P} \sum_{n_a} e^{ik_a x^a} \left[1 + 2 \sum_{n_3 > 0} \cos k_3 x^3 \cos k y^3 \right.$$
$$\left. + \sin k_3 x^3 \sin k_3 y^3 \right]. \tag{A14}$$

2. Neumann/Dirichlet boundary conditions

Imposing Neumann or Dirichlet boundary conditions on an interval of length L_3 in the $z = x^3$ direction can be achieved by extending the function of $z \in [0, L_3]$ to an even, respectively, odd function of $z \in [-L_3, L_3]$. This amounts to setting s_{k_3,k_a}^A, respectively, c_{k_a,k_3}^A in (A8) to zero, while keeping the definitions of the remaining modes in (A9) and (A10) unchanged (see [49] for an interpretation in terms of second class constraints). These formulas can then be expressed in terms of the real volume $V = 4L_1 L_2 L_3$ of the body B by the substitution $V_P = 2V$. In the Neumann case, we now have

$$\int_B d^3 x z^A(x^i) z^B(x^i) = V \sum_{n_a} \left[c_{k_a,0}^A c_{k_a,0}^{*B} + \frac{1}{2} \sum_{n_3 > 0} c_{k_a,k_3}^A c_{k_a,k_3}^{*B} \right], \tag{A15}$$

while for the Dirichlet case,

$$\int_B d^3 x z^A(x^i) z^B(x^i) = \frac{V}{2} \sum_{n_a, n_3 > 0} s_{k_a,k_3}^A s_{k_a,k_3}^{*B}. \tag{A16}$$

The canonical Poisson brackets now originate from kinetic terms of the form

$$\int_B d^3 x \dot{\phi}(x^i, t) \pi(x^i, t) = \int_{-L_1}^{L_1} dx \int_{-L_2}^{L_2} dy$$
$$\times \int_0^{L_3} dz \dot{\phi}(x^i, t) \pi(x^i, t), \tag{A17}$$

which implies that the brackets of the remaining modes in (A12), (A13) are to be multiplied by 2, or equivalently, in these equations, V_P needs to be replaced by V. In position space, one needs to replace $\delta_P^{(3)}(x^i, y^i)$ in the RHS of (A7) by $\delta_P^{(2)}(x^a, y^a) \Delta_\pm(x^3, y^3)$, with the $+$ corresponding to the Neumann and the $-$ to the Dirichlet case, and where (see e.g., [50], chapter 4)

$$\Delta_\pm(x^3, y^3) = \delta_{2L_3}(x^3 - y^3) \pm \delta_{2L_3}(x^3 + y^3)$$
$$= \frac{1}{2L_3} \sum_{n_3} (e^{ik_3(x^3 - y^3)} \pm e^{ik_3(x^3 + y^3)}), \tag{A18}$$

and also

$$\Delta_+(x^3, y^3) = \frac{1}{L_3} + \frac{2}{L_3} \sum_{n_3 > 0} \cos k_3 x^3 \cos k_3 y^3,$$

$$\Delta_-(x^3, y^3) = \frac{2}{L_3} \sum_{n_3 > 0} \sin k_3 x^3 \sin k_3 y^3. \tag{A19}$$

APPENDIX B: PARTITION FUNCTION FOR PERIODIC BOUNDARY CONDITIONS

When there is no electric potential at the surface of the body, no global electric charge and no nontrivial boundary conditions, the theory is quantized in such a way that the contribution to the partition function from the unphysical

bosonic degrees of freedom (A_0, π^0), (A, π) cancels the one from the ghost degrees of freedom $(\eta, \mathcal{P})$, $(\bar{C}, \rho)$ so that only the physical degrees of freedom (A_i^T, π_T^i) contribute. Let us briefly review these computations. As we are ultimately interested in infrared effects, we keep the volume finite and work with Fourier series including zero modes, instead of Fourier integrals.

1. Nonzero modes

For periodic boundary conditions in a box B_P of volume $V_P = 8L_1 L_2 L_3$, we can adapt the change of variables from Sec. 19.1.6 of [36] to the case of Fourier series instead of Fourier integrals. In this case, $k_i = \frac{\pi n_{(i)}}{L_{(i)}}$ and one defines

$$A_0' = \sum_{\vec{n}}' \frac{1}{\sqrt{2\omega_{\vec{k}} V_P}} [a_{0,\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}],$$

$$\pi^{0'} = i\sum_{\vec{n}}' \sqrt{\frac{\omega_{\vec{k}}}{2V_P}} [(a_{3,\vec{k}} + a_{0,\vec{k}}) e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}], \quad \text{(B1)}$$

$$A_i' = \sum_{\vec{n}}' \frac{1}{\sqrt{2\omega_{\vec{k}} V_P}} [a_{m,\vec{k}} e_{i,\vec{k}}^m e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}],$$

$$\pi^{i'} = -i\sum_{\vec{n}}' \sqrt{\frac{\omega_{\vec{k}}}{2V_P}} [(a_{m,\vec{k}} e_{i,\vec{k}}^m + a_{0,\vec{k}}) e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}], \quad \text{(B2)}$$

$$\eta' = -\sum_{\vec{n}}' \frac{1}{\sqrt{2\omega_{\vec{k}}^3 V_P}} [c_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}],$$

$$\mathcal{P}' = i\sum_{\vec{n}}' \sqrt{\frac{\omega_{\vec{k}}^3}{2V_P}} [\bar{c}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}], \quad \text{(B3)}$$

$$\bar{C}' = -i\sum_{\vec{n}}' \sqrt{\frac{\omega_{\vec{k}}}{2V_P}} [\bar{c}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}],$$

$$\rho' = -\sum_{\vec{n}}' \frac{1}{\sqrt{2\omega_{\vec{k}} V_P}} [c_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}], \quad \text{(B4)}$$

so that

$$A' = -i\sum_{\vec{n}}' \frac{1}{\sqrt{2\omega_{\vec{k}}^3 V_P}} [a_{3,\vec{k}} e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}],$$

$$\pi' = \sum_{\vec{n}}' \sqrt{\frac{\omega_{\vec{k}}^3}{2V_P}} [(a_{3,\vec{k}} + a_{0,\vec{k}}) e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}], \quad \text{(B5)}$$

where $\sum_{\vec{n}}' = \sum_{\vec{n}\neq\vec{0}}$, and $\omega_{\vec{k}} = \sqrt{\vec{k}\cdot\vec{k}}$, while $\{e_{i,\vec{k}}^m\}$ is an orthonormal triad, the first two vectors being transversal and the third longitudinal, $k^i e_{i,\vec{k}}^1 = 0 = e_{i,\vec{k}}^2$ and $e_{i,\vec{k}}^3 = \frac{k_i}{\omega_{\vec{k}}}$.

Finally, there is an additional change of variables to null oscillators,

$$a_{\vec{k}} = a_{3,\vec{k}} + a_{0,\vec{k}}, \qquad b_{\vec{k}} = \frac{1}{2}(a_{3,\vec{k}} - a_{0,\vec{k}}). \quad \text{(B6)}$$

For the nonzero modes, if $a_{a,\vec{k}}$, $a = 1, 2$ are the transverse physical oscillators, while $a_{\Gamma,\vec{k}}^\alpha$, $\alpha = 1, 2, \Gamma = 1, 2$ are the null oscillators of the unphysical sector, with $a_{\Gamma,\vec{k}}^1 = (a_{\vec{k}}, b_{\vec{k}})$ bosonic and $a_{\Gamma,\vec{k}}^2 = (c_{\vec{k}}, \bar{c}_{\vec{k}})$ fermionic, the nonvanishing Poisson brackets are

$$\{a_{a,\vec{k}}, a_{b,\vec{k}'}^*\} = -i\delta_{ab}\delta_{\vec{n},\vec{n}'}, \qquad \{a_{\Gamma,\vec{k}}^\alpha, a_{\Delta,\vec{k}'}^\beta\} = -i\eta_{\Gamma\Delta}\delta^{\alpha\beta}\delta_{\vec{n},\vec{n}'}, \quad \text{(B7)}$$

where indices are lowered (and raised) with $\delta_{ab}, \delta_{\alpha\beta}$ and the indefinite metric $\eta_{\Gamma\Delta}$ given by

$$\eta_{\Gamma\Delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{(B8)}$$

The canonical Poisson brackets of the fields z^A are then equivalent to these nonzero-mode Poisson brackets and the zero-mode brackets:

$$\{A_{0,\vec{0}}, \pi_0^0\} = 1 = \{A_{i,\vec{0}}, \pi_0^i\} = -\{\eta_{\vec{0}}, \mathcal{P}_{\vec{0}}\} = -\{\bar{C}_{\vec{0}}, \rho_{\vec{0}}\}. \quad \text{(B9)}$$

Note that longitudinal fields $A = A'$, $\pi = \pi'$ do not have zero modes, so that the commutation relations for the modes in a box imply $\{A(\vec{x}), \Pi(\vec{y})\} = -[\delta^{(3)}(\vec{x}, \vec{y}) - \frac{1}{V_P}]$. How zero modes for these fields may be re-introduced is briefly discussed in the next section.

With a view towards a subsequent large volume limit and a passage from Fourier series to integrals, zero modes are usually neglected. In this case, $\sum_{\vec{n}} \to \frac{V}{(2\pi)^3}\int d^3k$, $\delta_{\vec{n},\vec{n}'} \to \frac{(2\pi)^3}{V}\delta^{(3)}(\vec{k}, \vec{k}')$. If discrete and continuous Fourier coefficients/oscillators are related by $z_{\vec{k}}^A \to \frac{\sqrt{V}}{(2\pi)^{3/2}} z^A(\vec{k})$, $a_{A\vec{k}} \to \frac{\sqrt{V}}{(2\pi)^{3/2}} a_A(\vec{k})$ for all $a_{a\vec{k}}$, $a_{\Gamma,\vec{k}}^\alpha$, sums over $\vec{n}$ may simply be replaced by integrals over $\vec{k}$ and Kronecker by Dirac deltas in the above expressions for the mode expansions of the fields, the Poisson brackets and quadratic expressions like the Hamiltonian or the BRST charge.

2. Zero modes

The piece of the BRST gauge fixed Hamiltonian (in Feynman gauge $\xi = 1$) $H^1 = H_0 + \{\Omega, K_1\}$ involving the zero modes $z_{\vec{0}}^A$ is $\mathcal{H}_{\vec{0}}^1 = \frac{1}{2}\pi_0^i\pi_{i,\vec{0}} - \frac{1}{2}\pi_0^0\pi_0^0 + i\mathcal{P}_{\vec{0}}\rho_{\vec{0}}$. When $(A_{0,\vec{0}}, \bar{\pi}_0^0)$ are quantized as anti-Hermitian operators and the zero-mode ghosts in the Schrödinger representation (cf. [36], Secs. 15.3.2 and 15.4.4), and after limiting the bosonic zero-mode integrations to intervals $\Delta A_{\mu,\vec{0}}$, their contribution to the partition function would be

$$Z(\beta) = \prod_{\mu=0}^{3} \frac{\Delta A_{\mu,\vec{0}}}{\sqrt{2\pi\beta}} \times \beta \times Z'(\beta), \quad \text{(B10)}$$

with $Z'(\beta)$ the partition function for the nonzero modes. Note also that the piece of the BRST charge involving zero modes is $\Omega_{\vec{0}} = -i\pi_0^0\rho_{\vec{0}}$.

We will proceed differently however and start the analysis from the zero-mode contribution to the classical Lagrangian $L = -\frac{1}{4}\int d^3x F_{\mu\nu}F^{\mu\nu}$. Indeed, $\mathcal{L}_{\vec{0}}[A_{\mu,\vec{0}}] = \frac{1}{2}\dot{A}_{i,\vec{0}}\dot{A}^i_{\vec{0}}$. There then is only the primary constraint $\pi_0^0 \approx 0$, but no secondary constraint. Introducing the zero-mode ghost pair $(\bar{C}_{\vec{0}}, \rho_{\vec{0}})$, the associated BRST charge is $\Omega_{\vec{0}}$ given above. If one would like the theory to also include the zero modes of the other ghost pair, $(\eta_{\vec{0}}, \mathcal{P}_{\vec{0}})$, one can do so by adding a suitable nonminimal sector. This is done by considering the zero-mode Lagrangian as a function of the spurious $\mathcal{P}_{\vec{0}}$, $\mathcal{L}_{\vec{0}} = \mathcal{L}_{\vec{0}}[A_{\mu,\vec{0}}, -\mathcal{P}_{\vec{0}}]$. There then is an additional constraint $-\eta_{\vec{0}} \approx 0$, for which one introduces the ghost pair $(\pi_{\vec{0}}, A_{\vec{0}})$, unrelated to components of $(A_{i,\vec{0}}, \pi_0^i)$. The BRST charge including this nonminimal sector is then

$$\Omega_{\vec{0}} = -(\pi_{\vec{0}}\eta_{\vec{0}} + i\pi_0^0\rho_{\vec{0}}). \tag{B11}$$

Choosing as gauge fixing fermion

$$\frac{1}{2}i\Omega_{\vec{0}} = i\bar{C}_{\vec{0}}\left(-A_{\vec{0}} - \frac{1}{2}\pi_0^0\right) + \mathcal{P}_{\vec{0}}\left(A_{0,\vec{0}} - \frac{1}{2}\pi_{\vec{0}}\right), \tag{B12}$$

the BRST gauge fixed Hamiltonian is $\mathcal{H}_{\vec{0}} = \mathcal{H}^{ph}_{\vec{0}} + \mathcal{H}^{gs}_{\vec{0}}$, with

$$\mathcal{H}^{ph}_{\vec{0}} = \frac{1}{2}\pi_0^i\pi_{i,\vec{0}}, \tag{B13}$$

and $\mathcal{H}^{gs}_{\vec{0}} = -\frac{1}{2}i\{\Omega_{\vec{0}}, \Omega_{\vec{0}}\}$, which is explicitly given by

$$\mathcal{H}^{gs}_{\vec{0}} = -\pi_{\vec{0}}\left(A_{0,\vec{0}} - \frac{1}{2}\pi_0^0\right) - \pi_0^0\left(A_{\vec{0}} + \frac{1}{2}\pi_0^0\right) + i\bar{C}_{\vec{0}}\eta_{\vec{0}} + i\mathcal{P}_{\vec{0}}\rho_{\vec{0}}. \tag{B14}$$

When proceeding in this way, the longitudinal fields (A, π) will also include zero modes. Integrating out momenta can be done consistently including the zero modes. The same applies to the mode expansion of (3.3), (3.9) with the understanding that Δ goes to -1 for zero modes. When defining new variables for zero-modes as for the nonzero modes [without a sum and with $\omega_{\vec{0}} = 1$ in (B1), (B5), (B3), (B4)], and in (B6), the Poisson brackets of the unphysical sector in (B7) also include these zero modes.

When quantizing the unphysical zero-mode pairs,

$$(A_{0,\vec{0}}, \pi_0^0), \qquad (\pi_{\vec{0}}, A_{\vec{0}}), \qquad (\eta_{\vec{0}}, \mathcal{P}_{\vec{0}}), \qquad (\bar{C}_{\vec{0}}, \rho_{\vec{0}}), \tag{B15}$$

in the Dirac-Fock representation, their contribution to the partition function cancels through the same mechanism,

reviewed in Appendix B 3 below, as for the nonzero modes of the unphysical sector. One then remains with the (infinite) contribution of three bosonic free particles encoded in (B13), whose contribution to the partition function is

$$Z(\beta) = \prod_{i=1}^3 \frac{\Delta A_{i,\vec{0}}}{\sqrt{2\pi\beta}} \times Z'(\beta). \tag{B16}$$

3. Bulk cancellations

When inserting the mode expansion reviewed above, the BRST charge is given by

$$\Omega = \sum_{\vec{n}}(c_{\vec{k}}^*a_{\vec{k}} + a_{\vec{k}}^*c_{\vec{k}}). \tag{B17}$$

In Feynman gauge $\xi = 1$, the gauge fixed Hamiltonian

$$H^1 := H_0 + \{\Omega, K_1\} = H^{ph} + H^{gs}, \tag{B18}$$

is given by

$$H^{ph} = \frac{1}{2}\pi_0^i\pi_{i,\vec{0}} + \sum_{\vec{n}}'\omega_{\vec{k}}a^*_{a,\vec{k}}a^a_{\vec{k}}, \qquad H^{gs} = \sum_{\vec{n}}\omega_{\vec{k}}a^\alpha_{\Gamma,\vec{k}}a^\Gamma_{\alpha,\vec{k}}. \tag{B19}$$

Here $\omega_{\vec{k}} = \sqrt{k_ik^i}$ for the nonzero modes, $\omega_{\vec{0}} = 1$ for the zero modes of the unphysical sector, $a_{a,\vec{k}}$, $a = 1, 2$ are the transverse oscillators of the physical sector, while $a^\alpha_{\Gamma,\vec{k}}$ are the bosonic and fermionic null oscillators of the unphysical sector, with nonvanishing (graded) commutation relations

$$[\hat{a}_{a,\vec{k}}, \hat{a}^\dagger_{b,\vec{k}'}] = \delta_{ab}\delta_{\vec{n},\vec{n}'}, \qquad [\hat{a}^\alpha_{\Gamma,\vec{k}}, \hat{a}^{\beta\dagger}_{\Delta,\vec{k}'}] = \eta_{\Gamma\Delta}\delta^{\alpha\beta}\delta_{\vec{n},\vec{n}'}, \tag{B20}$$

where indices are lowered and raised with the appropriate metrics δ_{ab}, $\delta_{\alpha\beta}$, $\eta_{\Gamma\Delta}$ and their inverses.

At this stage, the difference with the partition function for a complex scalar field, and with Bose-Einstein condensation, appears clearly: the observable for which we would like to introduce a chemical potential involves different degrees of freedom than the ones of the Hamiltonian. Furthermore, such a BRST Fock space quantization guarantees that only the physical sector contributes. Indeed, since

$$\hat{H}^1 = \hat{H}^{ph} + \frac{1}{2}[\hat{\Omega}, \hat{\bar{\Omega}}], \qquad \frac{1}{2}\hat{\bar{\Omega}} = \sum_{\vec{n}}\omega_{\vec{k}}(\hat{\bar{c}}^\dagger_{\vec{k}}\hat{b}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}}\hat{\bar{c}}_{\vec{k}}), \tag{B21}$$

it follows that $e^{-\beta\hat{H}^1} = e^{-\beta\hat{H}^{ph}} + [\hat{\Omega}, \hat{M}]$ for some operator $\hat{M}$. The trace to be used for the partition function is the Lefschetz trace, for which the sum over diagonal matrix

elements is weighted by minus one to the power the ghost number of the state. In the context of supersymmetric quantum mechanics, this corresponds to computing the Witten index. The Lefschetz trace of BRST exact operators vanishes, while for a BRST closed operator, it agrees with the Lefschetz trace of the operator in cohomology. Hence, in the current setup, the trace reduces to the trace for the physical Hamiltonian in the physical Hilbert space associated to transverse photons,

$$\text{Tr}_W e^{-\beta \hat{H}_1} = \text{Tr}_{\text{ph}} e^{-\beta \hat{H}^{\text{ph}}}. \tag{B22}$$

Alternatively, in the context of path integral quantization, it is convenient to introduce a collective notation a_A for all the oscillators a_a, a_Γ^α. BRST Fock quantization is implemented by using the holomorphic representation with boundary conditions that fix that creation operators at t', $a_A^*(t') = a_A^*$ and destruction operators at t, $a_A(t) = a_A$, (see e.g., [51,52], and also [53], chapter 9, [36], chapters 15, 16). In order to be able to turn on a chemical potential, we consider the coupling to a source by using

$$\mathcal{H}_{\vec{k}}^j = \omega_{\vec{k}} a_{A,\vec{k}}^* a_{\vec{k}}^A - a_{A,\vec{k}}^* j_{\vec{k}}^A(\tau) - a_{A,\vec{k}} j_{\vec{k}}^{*A}(\tau). \tag{B23}$$

The path integral representation of the kernel $U_{\vec{k}}^j(t', t)$ at fixed $\vec{k}$ of the evolution operator $e^{i(t'-t)\mathcal{H}_{\vec{k}}^j}$ is then given by $U_{\vec{k}}^j(t', t) = e^{iS_{\vec{k}}^j}|_{\text{extr}}$, where the classical action to be used is the one that has a true extremum when taking into account the boundary conditions

$$S_{\vec{k}}^j = \int_t^{t'} d\tau \left[\frac{1}{2i}(\dot{a}_{A,\vec{k}}^* a_{\vec{k}}^A - a_{A,\vec{k}}^* \dot{a}_{\vec{k}}^A) - \mathcal{H}_{\vec{k}}^j \right]$$
$$+ \frac{1}{2i}\left[(a_{A,\vec{k}}^* a_{\vec{k}}^A)(t') + (a_{A,\vec{k}}^* a_{\vec{k}}^A)(t)\right]. \tag{B24}$$

When using that the appropriate extremum is

$$a_{\vec{k}}^A(\tau) = e^{-i\omega_{\vec{k}}(\tau-t)} a_{\vec{k}}^A + i \int_t^\tau d\tau' j_{\vec{k}}^A(\tau') e^{-i\omega_{\vec{k}}(\tau-\tau')},$$
$$a_{\vec{k}}^{*A}(\tau) = e^{-i\omega_{\vec{k}}(t'-\tau)} a_{\vec{k}}^{*A} + i \int_\tau^{t'} d\tau' j_{\vec{k}}^{*A}(\tau') e^{-i\omega_{\vec{k}}(\tau'-\tau)}, \tag{B25}$$

one finds

$$\ln U_{\vec{k}}^j(t', t)$$
$$= a_{A,\vec{k}}^* a_{\vec{k}}^A e^{-i\omega_{\vec{k}}(t'-t)}$$
$$+ i \int_t^{t'} d\tau [a_{A,\vec{k}}^* j_{\vec{k}}^A(\tau) e^{-i\omega_{\vec{k}}(t'-\tau)} + j_{\vec{k}}^{*A}(\tau) e^{-i\omega_{\vec{k}}(\tau-t)} a_{A,\vec{k}}]$$
$$- \int_t^{t'} d\tau \left[\int_t^{t'} d\tau' j_{A,\vec{k}}^*(\tau) \theta(\tau-\tau') e^{-i\omega_{\vec{k}}(\tau-\tau')} j_{\vec{k}}^A(\tau') \right]. \tag{B26}$$

When using a time-independent source j^c and $t' - t = -i\beta$, this gives

$$\ln U_{\vec{k}}^{j^c}(\beta) = a_{A,\vec{k}}^* a_{\vec{k}}^A e^{-\beta\omega_{\vec{k}}} + (a_{A,\vec{k}}^* j_{\vec{k}}^A + j_{A,\vec{k}}^* a_{\vec{k}}^A)\omega_{\vec{k}}^{-1}(1 - e^{-\beta\omega_{\vec{k}}})$$
$$+ j_{A,\vec{k}}^* j_{\vec{k}}^A [\omega_{\vec{k}}^{-1}\beta - \omega_{\vec{k}}^{-2}(1 - e^{-\beta\omega_{\vec{k}}})]. \tag{B27}$$

When evaluating the trace in the holomorphic representation, one should split into physical and unphysical oscillators. For each physical oscillators, there is a pre-factor of $(1 - e^{-\beta\omega_{\vec{k}}})^{-1}$ coming from an appropriate change of variables. As explicitly recalled in Appendix C, these pre-factors cancel for the unphysical oscillators. This cancellation corresponds to the one between the bosonic and fermionic determinants in supersymmetric quantum mechanics. As a result,

$$\text{Tr} e^{-\beta\hat{\mathcal{H}}_{\vec{k}}^{j^c}} = \frac{1}{(1 - e^{-\beta\omega_{\vec{k}}})^2} e^{\omega_{\vec{k}}^{-1}\beta j_{A,\vec{k}}^* j_{\vec{k}}^A} \tag{B28}$$

In the absence of sources, when integrating over all the modes and discarding the infinite contribution of the zero modes of the physical sector, one finds the standard blackbody result,

$$\ln Z'(\beta) = -2 \frac{V}{(2\pi)^3} \int d^3k \ln(1 - e^{-\beta\omega_{\vec{k}}}) = \frac{V\pi^2}{45\beta^3}. \tag{B29}$$

Note that, if instead of the kernel of the evolution operator, one directly computes the trace, the alternating sign in the Lefschetz trace is taken into account through periodic boundary conditions in imaginary time for the ghosts (see e.g., [54] for finite temperature QED or [55] for supersymmetric quantum mechanics), so that all fields satisfy periodic boundary conditions in imaginary time.

Note also that, in real time, indefinite metric quantization is implemented in the path integral through imaginary values for the paths associated to (A_0, π^0) (cf. [36] page 355). In the Euclidean approach, when one substitutes A_0 by $i\tilde{A}_0$, these become then again real paths for $\tilde{A}_0$. Conversely, this means that standard real paths for $\tilde{A}_0$ in the Euclidean approach correspond to using an indefinite metric Hilbert space in real time.

Turning on a chemical potential for electric charge, $H^{\text{gs}} \to H^{\text{gs}} + \mu Q$, with $Q = -\sqrt{\frac{V}{2}}(a_{\vec{0}} + a_{\vec{0}}^*)$ can be done in the above computation through the coupling to the source,

$$j_{\vec{k}}^A = (j_{\vec{k}}^a, j_{\Gamma,\vec{k}}^\alpha) = \left(0, -\mu\sqrt{\frac{V}{2}}\delta_1^\alpha \delta_\Gamma^1 \delta_{\vec{n},\vec{0}}\right) \tag{B30}$$

and its complex conjugate. The result does not change: $e^{\omega_{\vec{k}}^{-1}\beta j_{A,\vec{k}}^* j_{\vec{k}}^A} = 1$ due to the metric $\eta_{\Gamma\Delta}$ used to contract indices of the sources.

APPENDIX C: COHERENT STATES OF QUARTETS

To a pair of bosonic null oscillators,

$$[\hat{a}_\Gamma, \hat{a}^\dagger_\Delta] = \eta_{\Gamma\Delta}, \qquad \eta_{\Gamma\Delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

one associates the coherent states,

$$|a\rangle = e^{\hat{a}^\dagger_\Gamma a^\Gamma}|0\rangle, \qquad \langle a^*| = \langle 0|e^{a^{*\Gamma}\hat{a}_\Gamma}.$$

Their overlap is given by

$$\langle a^*|a\rangle = e^{a^{*\Gamma}a_\Gamma},$$

while the completeness relation is

$$\hat{1} = \int \prod_{\Gamma=1,2} \frac{da^*_\Gamma da_\Gamma}{2\pi i} e^{-a^{*\Delta}a_\Delta}|a\rangle\langle a^*|,$$

with fundamental integral

$$I[j,j^*] = \int \prod_{\Gamma=1,2} \frac{da^*_\Gamma da_\Gamma}{2\pi i} e^{-a^{*\Delta}a_\Delta + a^{*\Delta}j_\Delta + j^{*\Delta}a_\Delta} = e^{j^{*\Delta}j_\Delta}.$$

(C1)

Formulas for a pair of fermionic null oscillators, with anticommutation relations given by $[\hat{c}_\Gamma, \hat{c}^\dagger_\Delta] = \eta_{\Gamma\Delta}$, are the same except for the absence of $(2\pi i)^{-1}$ in the integration measure.

Using the notation $a^\alpha_\Gamma = (a_\Gamma, c_\Gamma)$, for $\alpha = 1$, 2, let $O(a^*; a)$ be the kernel of an operator $\hat{O}$ in the Fock space of a quartet. In this representation, the Lefschetz trace is given by

$$\text{Tr}\hat{O} = \int \prod_{\alpha,\Gamma=1,2} \frac{da^*_{\alpha,\Gamma} da_{\alpha,\Gamma}}{(2\pi i)^{2-\alpha}} O(a^*; a) e^{-a^{*\Delta}_\gamma a^\gamma_\Delta}.$$

For the operator $e^{-\beta\omega\hat{N}}$ with $\hat{N} = \hat{a}^{*\Gamma}_\alpha \hat{a}^\alpha_\Gamma$ the counting operator for quartets, the kernel is

$$\langle a^*|e^{-\beta\omega\hat{N}}|a\rangle = e^{a^{*\mu}_\alpha a^\alpha_\mu e^{-\beta\omega}},$$

so that

$$\text{Tr}e^{-\beta\omega\hat{N}} = \int \prod_{\alpha,\Gamma=1,2} \frac{da^*_{\alpha,\Gamma} da_{\alpha,\Gamma}}{(2\pi i)^{2-\alpha}} e^{-a^{*\Delta}_\gamma a^\gamma_\Delta(1-e^{-\beta\omega})} = 1.$$

For the last equality, the change of variables $a^{(*)}_{\alpha,\Gamma} \to a^{(*)}_{\alpha,\Gamma}(1-e^{-\beta\omega})^{-\frac{1}{2}}$ leads to a vanishing Jacobian because bosonic and fermionic contribution cancels, before using (C1) with vanishing sources.

It also follows that

$$\text{Tr}e^{-\beta\omega\hat{N}_b} = \frac{1}{(1-e^{-\beta\omega})^2}, \qquad (C2)$$

where $\hat{N}_b = \hat{a}^\dagger_\Gamma \hat{a}^\Gamma$ is the number operator for the bosonic part of the quartet, i.e., for a pair of bosonic null oscillators.

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