Exact conformal field theories from mutually T-dualizable σ-models

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Exact conformal field theories (CFTs) are obtained by using the approach of Poisson-Lie (PL) T-duality in the presence of spectators. We explicitly construct some non-Abelian T-dual σ-models (here as the PL T-duality on a semi-Abelian double) on $2+2$-dimensional target manifolds $M \approx O \times \mathbf{G}$ and $\tilde{M} \approx O \times \tilde{\mathbf{G}}$, where $\mathbf{G}$ and $\tilde{\mathbf{G}}$ as two-dimensional real non-Abelian and Abelian Lie groups act freely on M and $\tilde{M}$, respectively, while O is the orbit of $\mathbf{G}$ in M. The findings of our study show that the original models are equivalent to Wess-Zumino-Witten (WZW) models based on the Heisenberg (H_4) and $GL(2,\mathbb{R})$ Lie groups. In this way, some new T-dual backgrounds for these WZW models are obtained. For one of the duals of the H_4 WZW model, we show that the model is self-dual. In the case of the $GL(2,\mathbb{R})$ WZW model it is observed that the duality transformation changes the asymptotic behavior of solutions from $\mathrm{AdS}_3 \times \mathbb{R}$ to flat space. Then, the structure and asymptotic nature of the dual spacetime of this model including the horizon and singularity are determined. We furthermore get the noncritical Bianchi type III string cosmological model with a nonvanishing field strength from T-dualizable σ-models and show that this model describes an exact CFT (equivalent to the $GL(2,\mathbb{R})$ WZW model). After that, the conformal invariance of T-dual models up to two-loop order (first order in α') is discussed.

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I. INTRODUCTION

The duality symmetries play an important role in string theory. On the one hand, they are specific to string theory and their study has led to important insights in understanding the spacetime geometry from the string point of view. A very important symmetry of string theory or more generally, two-dimensional sigma models, is the T-duality [1]. A study of the T-duality in string theory has led to the discovery of PL T-duality. Klimčik and Ševera in their seminal work [2] proposed a generalization of T-duality, or the so-called PL T-duality, which allows the duality to be performed on a target space without isometries. In Klimčik and Ševera's formalism, PL T-dual sigma models are defined by PL group manifolds which constitute a Drinfeld double [3]. The classification of low-dimensional Drinfeld doubles [4,5] has become a convenient laboratory for investigation of the PL T-duality.

On the other hand, the duality symmetries in Wess-Zumino-Witten (WZW) models have received considerable attention because of the preservation of the conformal symmetry under the Abelian duality [6]. This duality has been investigated in the WZW models [7]. Furthermore, for the case of non-Abelian duality [8], it has been shown that the conformal symmetry is preserved when the trace of the adjoint representation of the isometry group is zero [9].

The WZW model is a well-known construction for obtaining a CFT which describes string propagation on a Lie group. For instance, the natural metric on the Lie group $SL(2,\mathbb{R})$ is precisely the three-dimensional anti-de Sitter metric. Hence, the WZW model based on Lie group $SL(2,\mathbb{R})$ can be considered as an exact CFT describing string propagation on anti-de Sitter space [10]. Up to now, only few examples of PL symmetric σ-models have been treated at the quantum level [11,12]. Furthermore, PL symmetry in the WZW models based on the Lie supergroups have recently been studied in Refs. [13,14]. We also refer the reader to the literatures given in Ref. [15]. In Ref. [11] it has been shown that the duality relates the $SL(2,\mathbb{R})$ WZW model to a constrained σ-model defined by the $SL(2,\mathbb{R})$ group space. We have shown that [16] the PL T-duality relates the H_4 WZW model to a σ-model defined on the dual Lie group $A_2 \oplus 2A_1$. We have also stressed that the dual model is conformally invariant up to two-loop order. Furthermore, we have recently shown that [17] the PL T-duality relates the $SL(2,\mathbb{R})$ WZW model to a σ-model defined on $2+1$-dimensional manifold $M \approx O \times \mathbf{G}$ in which $\mathbf{G}$ is two-dimensional real non-Abelian Lie group A_2, and O as a one-dimensional space is the orbit of $\mathbf{G}$ in M. Accordingly, we have obtained a dual model for the $SL(2,\mathbb{R})$ WZW model yielding a new three-dimensional charged black string which is stationary and asymptotically flat.

The main purpose of this paper is to construct some new non-Abelian T-dual backgrounds for the H_4 and $GL(2,\mathbb{R})$

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WZW models via PL T-duality approach in the presence of spectators. The original models as exact CFTs (the H_4 and $GL(2,\mathbb{R})$ WZW models) are constructed on $2+2$-dimensional target manifold $M \approx O \times \mathbf{G}$ with $\mathbf{G} = A_2$ and dual models on manifold $\tilde{M} \approx O \times \tilde{\mathbf{G}}$ with $\tilde{\mathbf{G}} = 2A_1$, whereas, in [16] T-dual σ-models were only constructed on Lie groups in the absence of spectators. In the present work, two dual models for the H_4 WZW are obtained for one of which we show that the dual model is indeed identical to the same H_4 WZW model. Moreover, we get one dual model for the $GL(2,\mathbb{R})$ WZW for which the structure and asymptotic nature of the spacetime including the horizon and singularity are determined. We also obtain the noncritical Bianchi type III string cosmological model with a nonvanishing field strength from a T-dualizable σ-model on $3+1$-dimensional target manifold $M \approx O \times \mathbf{G}$, in which $\mathbf{G}$ represents three-dimensional decomposable Lie group $A_2 \oplus A_1$, and then we show that this model describes an exact CFT. Finally, we discuss the conformal invariance conditions of the T-dual models up to the first order in α' to introduce new solutions for two-loop B-function equations of the σ-model with a nonvanishing field strength H and the dilaton field in both cases of the absence and presence of a cosmological constant Λ.

This paper is organized as follows. In Sec. II, we present a basic review of the PL T-dual σ-models construction in the presence of spectator fields. In Sec. III, we get the H_4 and $GL(2,\mathbb{R})$ WZW models from T-dualizable σ-models constructed on $2+2$-dimensional target manifolds $M \approx O \times \mathbf{G}$ and $\tilde{M} \approx O \times \tilde{\mathbf{G}}$. In addition, the dual backgrounds for these WZW models together with the structure and asymptotic nature of the dual spacetime of the $GL(2,\mathbb{R})$ WZW including the horizon and singularity are studied. Finally, the non-Abelian T-dualization of the noncritical Bianchi type III string cosmology solution is discussed at the end of Sec. III. In Sec. IV, we investigate the conformal invariance conditions for T-dual models up to two-loop order. Some concluding remarks are given in Sec. V.

II. CONSTRUCTION OF PL T-DUAL σ-MODELS WITH SPECTATORS

We begin this section by reviewing the construction of PL T-dual σ-models in the presence of spectator fields. First of all, for the description of PL T-duality we need to introduce the Drinfeld double group $\mathbf{D}$ [3], which by definition has a pair of maximally isotropic subgroups $\mathbf{G}$ and $\tilde{\mathbf{G}}$ corresponding to the subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively. The generators of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are denoted, respectively, T_a and $\tilde{T}^a$, $a = 1, \ldots, \dim \mathbf{G}$. One says that the Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are compatible if the brackets

$$[T_a, T_b] = f^c{}_{ab} T_c, \qquad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}{}_c \tilde{T}^c,$$
$$[T_a, \tilde{T}^b] = \tilde{f}^{bc}{}_a T_c + f^b{}_{ca} \tilde{T}^c, \tag{1}$$

define a Lie algebra structure on the direct sum vector space $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$. In this case, we say that the Lie algebra $\mathcal{D}$ is the Drinfeld double of $\mathcal{G}$ or, equivalently, of $\tilde{\mathcal{G}}$. Thus, the group $\mathbf{D}$ is called the Drinfeld double of $\mathbf{G}$ (or $\tilde{\mathbf{G}}$). We also note that the Drinfeld double $\mathcal{D}$ is equipped with an invariant inner product $\langle .,. \rangle$ with the following properties

$$\langle T_a, \tilde{T}^b \rangle = \delta_a{}^b,$$
$$\langle T_a, T_b \rangle = \langle \tilde{T}^a, \tilde{T}^b \rangle = 0. \tag{2}$$

In what follows we shall investigate PL T-duality transformations in the presence of spectators [2,18] of a nonlinear σ-model with the following action for a bosonic string, propagating in a d-dimensional spacetime, with the metric $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton field ϕ

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{-h} \left[\frac{1}{2} (h^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4} \alpha' \phi R^{(h)} \right], \tag{3}$$

where $h_{\alpha\beta}$ is the world sheet metric with $R^{(h)}$ the corresponding world sheet curvature scalar and $h = \det h_{\alpha\beta}$. The indices α, β run over (τ, σ), and $\epsilon^{\alpha\beta}$ is an antisymmetric tensor on the world sheet Σ. The dimensionful coupling constant α' turns out to be the inverse string tension. The functions x^μ: $\Sigma \to \mathbb{R}$, $(\mu = 1, \ldots, \dim M)$ are obtained by the composition $x^\mu = X^\mu \circ x$ of a map x: $\Sigma \to M$ and components of a coordinate map X on a chart of M. Here, and in the following, we use the standard light-cone variables on the world sheet, $\sigma^\pm = \tau \pm \sigma$.

Let us now consider a d-dimensional manifold M and some coordinates $x^\mu = (x^i, y^\alpha)$ on it, where $x^i (i = 1, \ldots, \dim \mathbf{G})$ are the coordinates of Lie group $\mathbf{G}$ acting freely from right on M. $y^\alpha (\alpha = 1, \ldots, d - \dim \mathbf{G})$ are the coordinates labeling the orbit O of $\mathbf{G}$ in the target space M. We note that the coordinates y^α do not participate in the PL T-duality transformations and are therefore called spectators [18]. Take a linear (idempotent) map $\mathcal{K}$ from the space $T^*_y M \oplus T_y M \oplus \mathcal{D}$ into itself. It has two eigenspaces $\mathbb{R}_\pm(y^\alpha)$ with eigenvalues ± 1. They are perpendicular to each other according to the bilinear form on $T^*_y M \oplus T_y M \oplus \mathcal{D}$. These eigenspaces may be considered as the graph of a nondegenerate linear map $E^\pm(y)$: $T_y M \oplus \mathcal{G} \to T^*_y M \oplus \tilde{\mathcal{G}}$, such that by translating this graph to the point $g \in \mathbf{G}$ we have

$$g^{-1} \mathbb{R}_\pm(y^\alpha) g = \mathrm{Span}\{X_A \pm E^\pm_{AB}(g, y^\alpha) \tilde{X}^B\}, \tag{4}$$

where $X_A = (T_a, \partial_\alpha)$ and $\tilde{X}^A = (\tilde{T}^a, dy^\alpha)$ are the basis of the spaces $T_y M \oplus \mathcal{G}$ and $T^*_y M \oplus \tilde{\mathcal{G}}$, respectively. In order to determine the $d \times d$ matrix $E^\pm_{AB}(g, y^\alpha)$ we write the spaces $\mathbb{R}_\pm(y^\alpha)$ as follows:

$$g^{-1}\mathbb{R}_{\pm}(y^\alpha)g = \mathrm{Span}\{g^{-1}X_A g \pm E^{\pm}_{AB}(e,y^\alpha)g^{-1}\tilde{X}^B g\}, \quad (5)$$

in which the matrix $E^{\pm}_{AB}(e,y^\alpha)$ is defined as

$$E^{\pm}_{AB}(e,y^\alpha) = \begin{pmatrix} E^{\pm}_{0\,ab}(e,y^\alpha) & F^{\pm^{(1)}}_{a\beta}(e,y^\alpha) \\ F^{\pm^{(2)}}_{\alpha b}(e,y^\alpha) & F_{\alpha\beta}(y^\alpha) \end{pmatrix}. \quad (6)$$

Here, submatrices $E^{\pm}_{0\,ab}(e,y^\alpha)$, $F^{\pm^{(1)}}_{a\beta}(e,y^\alpha)$ and $F^{\pm^{(2)}}_{\alpha b}(e,y^\alpha)$ are functions of the variables y^α and e, where e is the unit element of $\mathbf{G}$. $F_{\alpha\beta}(y^\alpha)$ is also a function of y^α only. Here, and in the following, the minus sign stands for transpose, namely, $E^{+}_{0\,ab} = E^{-}_{0\,ba}$, $F^{+^{(1)}}_{a\beta} = F^{-^{(2)}}_{\beta a}$ and $F^{+^{(2)}}_{\alpha b} = F^{-^{(1)}}_{b\alpha}$.

It is convenient to define matrices $a(g)$, $b(g)$ and the Poisson bracket $\Pi(g)$ in the following way

$$g^{-1}T_a g = a_a{}^b(g)T_b,$$
$$g^{-1}\tilde{T}^a g = b^{ab}(g)T_b + (a^{-1})_b{}^a(g)\tilde{T}^b, \quad (7)$$

$$\Pi^{ab}(g) = b^{ac}(g)(a^{-1})_c{}^b(g). \quad (8)$$

Thus, using (4) and (5) together with (7) one gets

$$E^{\pm}_{AB}(g,y^\alpha) = (A(g) \pm E^{\pm}(e,y^\alpha)B(g))^{-1}_A{}^C \times E^{\pm}_{CD}(e,y^\alpha)(A^{-1})_B{}^D(g), \quad (9)$$

where[1]

$$A(g) = \begin{pmatrix} a(g) & 0 \\ 0 & Id \end{pmatrix}, \qquad B(g) = \begin{pmatrix} b(g) & 0 \\ 0 & 0 \end{pmatrix}. \quad (10)$$

We also define

$$\mathbb{F}^{\pm}_{AB}(g,y^\alpha) = A_A{}^C(g)E^{\pm}_{CD}(g,y^\alpha)A_B{}^D(g). \quad (11)$$

Considering matrix $\mathbb{F}^{\pm}_{AB}(g,y^\alpha)$ in the form

$$\mathbb{F}^{\pm}_{AB}(g,y^\alpha) = \begin{pmatrix} \mathbb{E}^{\pm}_{ab}(g,y^\alpha) & \Phi^{\pm^{(1)}}_{a\beta}(g,y^\alpha) \\ \Phi^{\pm^{(2)}}_{\alpha b}(g,y^\alpha) & \Phi_{\alpha\beta}(y^\alpha) \end{pmatrix}, \quad (12)$$

and then using (6), (9), (10) and (11) one can obtain the backgrounds appearing in the action of original σ-model. They are given in matrix notation by

$$\mathbb{E}^{\pm}(g,y^\alpha) = (E^{\pm^{-1}}_0(e,y^\alpha) \pm \Pi(g))^{-1}, \quad (13)$$

$$\Phi^{\pm^{(1)}}(g,y^\alpha) = \mathbb{E}^{\pm}(g,y^\alpha)(E^{\pm}_0)^{-1}(e,y^\alpha)F^{\pm^{(1)}}(e,y^\alpha), \quad (14)$$

$$\Phi^{\pm^{(2)}}(g,y^\alpha) = F^{\pm^{(2)}}(e,y^\alpha)(E^{\pm}_0)^{-1}(e,y^\alpha)\mathbb{E}^{\pm}(g,y^\alpha), \quad (15)$$

$$\Phi(g,y^\alpha) = F(y^\alpha) - F^{+^{(2)}}(e,y^\alpha)\Pi(g)\mathbb{E}^{+}(g,y^\alpha) \times (E^{+}_0)^{-1}(e,y^\alpha)F^{+^{(1)}}(e,y^\alpha). \quad (16)$$

Let us now introduce the elements $V_\pm$ of subspaces $\mathbb{R}_\pm(y^\alpha)$ as

$$V_\pm \coloneqq \partial_\pm y^\alpha \frac{\partial}{\partial y^\alpha} \mp p^{(\mp)}_\alpha dy^\alpha + \partial_\pm l l^{-1}, \quad (17)$$

where $p^{(\mp)}_\alpha \in T^*_y M$ and $l \in \mathbf{D}$. Inserting the decomposition $l = g\tilde{h}(g \in \mathbf{G}, \tilde{h} \in \tilde{\mathbf{G}})$ [11] into (17) we get

$$V_\pm \coloneqq R^A_\pm X_A + \tilde{R}_{\pm_A}(B^{BA}(g)X_B + A_B{}^A(g)\tilde{X}^B), \quad (18)$$

where $R^A_\pm$ and $\tilde{R}_{\pm_A}$ are the elements of the respective spaces $T_y M \oplus \mathcal{G}$ and $T^*_y M \oplus \tilde{\mathcal{G}}$, and are given by

$$R^A_\pm = (R^a_\pm, \partial_\pm y^\alpha) = ((\partial_\pm g g^{-1})^a, \partial_\pm y^\alpha), \quad (19)$$

$$\tilde{R}_{\pm_A} = ((\partial_\pm \tilde{h}\tilde{h}^{-1})_a, \mp p^{(\mp)}_\alpha). \quad (20)$$

Thus, by using the equations of motion

$$\langle V_\pm, (X_A \mp E^{\mp}_{AB}(g,y^\alpha)\tilde{X}^B)\rangle = 0, \quad (21)$$

we obtain

$$\tilde{R}_{\pm_A} = \pm R^B_\pm \mathbb{F}^{\pm}_{BC}(g,y^\alpha)(A^{-1})_A{}^C(g). \quad (22)$$

Equation (22) can be written in terms of components. They then take the following forms

$$(\partial_+ \tilde{h}\tilde{h}^{-1})_a = (a^{-1})_a{}^c(g)[R^b_+ \mathbb{E}^{+}_{bc}(g,y^\alpha) + \partial_+ y^\alpha \Phi^{+^{(2)}}_{\alpha c}(g,y^\alpha)], \quad (23)$$

$$(\partial_- \tilde{h}\tilde{h}^{-1})_a = -(a^{-1})_a{}^c(g)[\mathbb{E}^{+}_{cb}(g,y^\alpha)R^b_- + \Phi^{+^{(1)}}_{c\beta}(g,y^\alpha)\partial_- y^\beta], \quad (24)$$

and

$$p^{(+)}_\alpha = -[\Phi^{+^{(2)}}_{\alpha b}(g,y^\alpha)R^b_- + \Phi_{\alpha\beta}(g,y^\alpha)\partial_- y^\beta], \quad (25)$$

$$p^{(-)}_\alpha = -[R^a_+ \Phi^{+^{(1)}}_{a\alpha}(g,y^\alpha) + \partial_+ y^\beta \Phi_{\beta\alpha}(g,y^\alpha)]. \quad (26)$$

The above results indicate that the Eq. (21) are nothing but the equations of motion concerning the σ-model described by the following action

[1] Here Id means the identity matrix.

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^- \mathbb{F}^+_{AB}(g, y^\alpha) R^A_+ R^B_-,$$
$$= \frac{1}{2}\int d\sigma^+ d\sigma^- [\mathbb{E}^+_{ab}(g, y^\alpha) R^a_+ R^b_- + \Phi^{+^{(1)}}_{a\beta}(g, y^\alpha) R^a_+ \partial_- y^\beta + \Phi^{+^{(2)}}_{\alpha b}(g, y^\alpha)\partial_+ y^\alpha R^b_- + \Phi_{\alpha\beta}(g, y^\alpha)\partial_+ y^\alpha \partial_- y^\beta]. \quad (27)$$

As we shall see below, one can construct another σ-model (denoted as usual with tilded symbols) which is said to be dual to (27) in the sense of the PL T-duality if the Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ form a pair of maximally isotropic subalgebras of the Lie algebra $\mathcal{D}$. In order to get the dual σ-model one proceeds in an analogous way so that eigenspaces $\mathbb{R}_\pm(y^\alpha)$ are considered as

$$\mathbb{R}_\pm(y^\alpha) = \mathrm{Span}\{\tilde{Y}^A \pm \tilde{E}^{\pm^{AB}}(\tilde{e}, y^\alpha) Y_B\}, \quad (28)$$

where $\tilde{E}^+(y^\alpha)$: $T_y M \oplus \tilde{\mathcal{G}} \to T^*_y M \oplus \mathcal{G}$, $Y_B = (T_a, dy^\alpha)$ and $\tilde{Y}^A = (\tilde{T}^a, \partial_\alpha)$. With a slight abuse of the notation, comparing (5) and (28) we get the matrix form of $\tilde{E}^\pm(\tilde{e}, y^\alpha)$ as [2]

$$\tilde{E}^\pm(\tilde{e}, y^\alpha) = \pm(\mathcal{A} \pm E^\pm(e, y^\alpha)\mathcal{B})^{-1} \times (\mathcal{B} \pm E^\pm(e, y^\alpha)\mathcal{A}), \quad (29)$$

in which

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}. \quad (30)$$

Now, using (28) and inserting the decomposition $l = \tilde{g}h$ into the Eq. (21), one can get the equations of motion for y^α and $\tilde{x}^i$ corresponding to the following action

$$\tilde{S} = \frac{1}{2}\int d\sigma^+ d\sigma^- \tilde{\mathbb{F}}^{+^{AB}}(\tilde{g}, y^\alpha)\tilde{R}_{+_A}\tilde{R}_{-_B}$$
$$= \frac{1}{2}\int d\sigma^+ d\sigma^- [\tilde{\mathbb{E}}^{+^{ab}}(\tilde{g}, y^\alpha)\tilde{R}_{+_a}\tilde{R}_{-_b} + \tilde{\Phi}^{+^{(1)^a}}{}_\beta(\tilde{g}, y^\alpha)\tilde{R}_{+_a}\partial_- y^\beta + \tilde{\Phi}^{+^{(2)^b}}_\alpha(\tilde{g}, y^\alpha)\partial_+ y^\alpha \tilde{R}_{-_b} + \tilde{\Phi}_{\alpha\beta}(\tilde{g}, y^\alpha)\partial_+ y^\alpha \partial_- y^\beta]. \quad (31)$$

The coupling matrices of the dual σ-model are also determined in a similar fashion [2,18]. Using (29) one relates them to those of the original one by

$$\tilde{\mathbb{E}}^\pm(\tilde{g}, y^\alpha) = (E^\pm_0(e, y^\alpha) \pm \tilde{\Pi}(\tilde{g}))^{-1}, \quad (32)$$

$$\tilde{\Phi}^{\pm^{(1)}}(\tilde{g}, y^\alpha) = \pm\tilde{\mathbb{E}}^\pm(\tilde{g}, y^\alpha) F^{\pm^{(1)}}(e, y^\alpha), \quad (33)$$

$$\tilde{\Phi}^{\pm^{(2)}}(\tilde{g}, y^\alpha) = \mp F^{\pm^{(2)}}(e, y^\alpha)\tilde{\mathbb{E}}^\pm(\tilde{g}, y^\alpha), \quad (34)$$

$$\tilde{\Phi}(\tilde{g}, y^\alpha) = F(y^\alpha) - F^{+^{(2)}}(e, y^\alpha) \times \tilde{\mathbb{E}}^+(\tilde{g}, y^\alpha) F^{+^{(1)}}(e, y^\alpha). \quad (35)$$

The actions (27) and (31) correspond to PL T-dual σ-models [2]. Notice that if the group $\mathbf{G}(\tilde{\mathbf{G}})$ besides having free action on $M(\tilde{M})$, acts transitively on it, then the corresponding manifold $M(\tilde{M})$ will be the same as the group $\mathbf{G}(\tilde{\mathbf{G}})$. In this case only the first term appears in the actions (27) and (31).

In the PL T-duality case, dilaton shifts in both models have been obtained by quantum considerations based on a regularization of a functional determinant in a path integral formulation of PL T-duality by incorporating spectator fields [19] (see, also, [20])

$$\phi = \phi_0(y^\alpha) + \log(\det \mathbb{E}^+) - \log(\det E^+_0), \quad (36)$$

$$\tilde{\phi} = \phi_0(y^\alpha) + \log(\det \tilde{\mathbb{E}}^+), \quad (37)$$

where $\phi_0(y^\alpha)$ is just a function of y^α.

III. T-DUALIZABLE σ-MODELS ON 2+2-DIMENSIONAL MANIFOLDS AS EXACT CFTS

In this section, we explicitly construct two pairs of PL T-dual σ-models on $2+2$-dimensional target manifolds $M \approx O \times \mathbf{G}$ and $\tilde{M} \approx O \times \tilde{\mathbf{G}}$, where $\mathbf{G}$ and $\tilde{\mathbf{G}}$ as two-dimensional real non-Abelian and Abelian Lie groups act freely on M and $\tilde{M}$, respectively, while O is the orbit of $\mathbf{G}$ in M with the spectators $y^\alpha = \{y_1, y_2\}$. The Lie algebras of the Lie groups $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are denoted by $\mathcal{A}_2$ and $2\mathcal{A}_1$, respectively. According to Sec. II, having Drinfeld doubles we can construct PL T-dual σ-models on them. The four-dimensional Lie algebra of the Drinfeld double $(\mathcal{A}_2, 2\mathcal{A}_1)$ is given by the following nonzero commutation relations:

$$[T_1, T_2] = T_2, \qquad [T_1, \tilde{T}^2] = -\tilde{T}^2, \qquad [T_2, \tilde{T}^2] = \tilde{T}^1, \quad (38)$$

where $\{T_1, T_2\}$ and $\{\tilde{T}^1, \tilde{T}^2\}$ are the basis of $\mathcal{A}_2$ and $2\mathcal{A}_1$, respectively. Notice that the double $(\mathcal{A}_2, 2\mathcal{A}_1)$ has non-vanishing trace in the adjoint representations. In such a situation, there is usually a conformal anomaly at one-loop associated with non-Abelian T-duality [20]. In what follows, we will also discuss the conformal anomaly appeared in the string effective Lagrangians corresponding to the T-dual models.

In order to calculate the components of right invariant one-forms $R^a_\pm$ on the Lie group A_2 we parametrize an element of A_2 as

$$g = e^{x_1 T_1} e^{x_2 T_2}, \quad (39)$$

where $x^i = \{x_1, x_2\}$ are the coordinates of the Lie group A_2. $R^a_\pm$'s are then derived in the following form

$$R^1_\pm = \partial_\pm x_1, \qquad R^2_\pm = e^{x_1}\partial_\pm x_2. \tag{40}$$

Since the dual Lie group, $2A_1$, is Abelian, by using (7), (8) and (38) it follows that the Poisson bracket $\Pi^{ab}(g)$ on A_2 vanishes. Furthermore, for obtaining the Poisson bracket on the dual group $2A_1$ we first parametrize the Lie group $2A_1$ with coordinates $\tilde{x}^i = \{\tilde{x}_1, \tilde{x}_2\}$ so that its elements are defined as in (39) by replacing untilded quantities with tilded ones. Then, using (7) and (8) for tilded quantities together with (38) the Poisson bracket on $2A_1$ is derived as follows:

$$\tilde{\Pi}_{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 \\ \tilde{x}_2 & 0 \end{pmatrix}. \tag{41}$$

In addition to the right invariant one-forms, to construct the σ-models (27) and (31) on manifolds M and $\tilde{M}$ we need to determine the couplings $\mathbb{E}^+_{ab}(g, y^\alpha)$, $\Phi^{+(1)}_{a\beta}(g, y^\alpha)$, $\Phi^{+(2)}_{\alpha b}(g, y^\alpha)$, and $\Phi_{\alpha\beta}(g, y^\alpha)$. By convenient choices of the background matrices $E^+_{0\,ab}(e, y^\alpha)$, $F^{+(1)}_{a\beta}(e, y^\alpha)$, $F^{+(2)}_{\alpha b}(e, y^\alpha)$ and $F_{\alpha\beta}(y^\alpha)$, we will show that the original models are equivalent to the H_4 and $GL(2,\mathbb{R})$ WZW models. In this way, the new dual backgrounds for these WZW models are obtained.

A. The H_4 WZW model from T-dualizable σ-models and its dual pairs

In this subsection, we obtain two different duals of the H_4 WZW model. In both cases, the original σ-models (which are equivalent to the H_4 WZW model) are constructed on the manifold $M \approx O \times \mathbf{G}$ with $\mathbf{G} = A_2$ acting freely on it, however, the spectator-dependent background matrices are chosen to be different for each model.

Case (1): In this case, we take the background matrices as

$$E^+_{0\,ab} = \begin{pmatrix} 0 & e^{y_1} \\ e^{y_1} & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$
$$F^{+(1)}_{a\beta} = \begin{pmatrix} 0 & 0 \\ e^{y_1} & 0 \end{pmatrix}, \qquad F^{+(2)}_{\alpha b} = \begin{pmatrix} 0 & -e^{y_1} \\ 0 & 0 \end{pmatrix}. \tag{42}$$

As explained above, the Poisson bracket $\Pi^{ab}(g)$ is zero. By using relations (13)–(16) one can get the required couplings which where mentioned above. Finally, using (40) and then (27), the original σ-model is found to be of the form

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^- [-\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1 + e^{x_1+y_1}(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1) + e^{x_1+y_1}(\partial_+ x_2 \partial_- y_1 - \partial_+ y_1 \partial_- x_2)]. \tag{43}$$

By identifying action (43) with the σ-model of the form (3) one can read off the background matrix. Thus, the metric and antisymmetric tensor field corresponding to the action (43) can be written as

$$ds^2 = -2dy_1 dy_2 + 2e^{x_1+y_1} dx_1 dx_2, \tag{44}$$

$$B = e^{x_1+y_1} dx_2 \wedge dy_1. \tag{45}$$

Before proceeding to construct the dual σ-model, let us discuss the conformal invariance of the model (43). The classical canonical equivalence to the σ-models related by PL T-duality was done by Sfetsos in [18] (see, also, [21]). The canonical transformations are essentially classical and the quantum equivalence of the two σ-models has not yet been revealed. Equivalence can hold in some special cases but it fails in most cases. In this respect, checking the equivalence by studying conformal invariance (the vanishing of the Beta-functions) is important. But, since after a classical canonical transformation, the equivalence always holds up to first order in Planck's constant in the semiclassical expansion (corresponding to one-loop order in σ-model language), only the two-loop order is the first real test of quantum equivalence of the two different σ-models related by PL T-duality. For these reasons it is important to check the conformal invariance conditions of our models.

In the σ-model context, the conformal invariance is provided by the vanishing of the B-functions equations [22], which are equivalent to the equations of motion of effective action in the string frame [23]. In four dimensions, the low energy string effective action is

$$S_{\text{eff}} = \int d^4x \sqrt{-G} e^{-\phi} \mathcal{L}_{\text{eff}}, \tag{46}$$

where $G = \det G_{\mu\nu}$, and $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{R} + (\nabla\phi)^2 - \frac{1}{3}H^2 + 2\Lambda. \tag{47}$$

In this expression, $\mathcal{R}$ is the scalar curvature of the metric $G_{\mu\nu}$, and $H_{\mu\nu\rho}$, defined by $H_{\mu\nu\rho} = 1/2(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})$ is the torsion (the field strength) of the field $B_{\mu\nu}$. Λ is a cosmological constant[2] which is vanished for critical

[2]In string theory, the cosmological constant term Λ is related to the dimension of spacetime, d, and the inverse string tension by $\Lambda = (d-26)/3\alpha'$, whereas, here in this paper it is, in some cases, treated as a free parameter.

strings. Our analysis applies also to noncritical strings, i.e., when Λ is different from zero.

Consistency of the string theory requires that the action (3) be defined a conformally invariant quantum field theory. The conditions for conformal invariance can be interpreted as field equations for $G_{\mu\nu}$, $B_{\mu\nu}$, and ϕ of the string effective action [24,25]. The vanishing of the one-loop B-functions equations gives us the conformal invariance conditions of the σ-model (3) up to one-loop order (zeroth order in the inverse string tension α') [26]. These equations are given by

$$B^{G}_{\mu\nu}:\ \mathcal{R}_{\mu\nu} - (H^2)_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \mathcal{O}(\alpha') = 0, \tag{48a}$$

$$B^{B}_{\mu\nu}:\ -\nabla^\lambda H_{\lambda\mu\nu} + H_{\mu\nu}{}^{\lambda}\nabla_\lambda\phi + \mathcal{O}(\alpha') = 0, \tag{48b}$$

$$B^{\phi}:\ \Lambda + \frac{1}{2}\nabla^2\phi - \frac{1}{2}(\nabla\phi)^2 + \frac{1}{3}H^2 + \mathcal{O}(\alpha') = 0. \tag{48c}$$

We have used the conventional notations $(H^2)_{\mu\nu} = H_{\mu\rho\sigma}H^{\rho\sigma}{}_{\nu}$, $H^2 = H_{\mu\nu\sigma}H^{\mu\nu\sigma}$ and $(\nabla\phi)^2 = \partial_\mu\phi\partial^\mu\phi$. In the equation (48a), $\mathcal{R}_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$.

The metric (44) describes a four-dimensional spacetime of signature (2,2).[3] One quickly finds that the only nonzero component of $\mathcal{R}_{\mu\nu}$ is $\mathcal{R}_{y_1y_1} = -1/2$ and then $\mathcal{R} = 0$. Thus, the metric is flat in the sense that its scalar curvature vanishes.[4] For the antisymmetric tensor field (45) one verifies that the only nonzero component of H is $H_{x_1x_2y_1} = (e^{x_1+y_1})/2$. It then follows that $H^2 = 0$ and the only nonzero component of $(H^2)_{\mu\nu}$ is $(H^2)_{y_1y_1} = -1/2$. Inserting the above results in the vanishing of the one-loop B-functions equations (48a)–(48c), the conformal invariance conditions up to one-loop order are satisfied with $\Lambda = 0$ and the dilaton field

$$\phi = \sigma_0 + \sigma_1 y_1, \tag{49}$$

where σ_0 and σ_1 are integration constants. In addition to the conformal invariance of the model (43) up to one-loop order, we are interested in investigating the conformal invariance of the model for higher orders in α'. Instead of this, we show that the model (43) is equivalent to an exact CFT, namely a WZW model based on a Lie group. The WZW models represent exact solutions to the string equations of motion to all orders in α'. One can show that under the coordinate transformation

$$e^{x_1} = a_+, \qquad x_2 = a_-, \qquad y_1 = n, \qquad y_2 = m, \tag{50}$$

action (43) turns into

$$\begin{aligned} S = \frac{1}{2}\int d\sigma^+ d\sigma^- [&-\partial_+ n\partial_- m - \partial_+ m\partial_- n \\ &+ e^n(\partial_+ a_+\partial_- a_- + \partial_+ a_-\partial_- a_+) \\ &+ a_+ e^n(\partial_+ a_-\partial_- n - \partial_+ n\partial_- a_-)], \end{aligned} \tag{51}$$

which is nothing but the action of WZW model based on the Lie group H_4[5] [16] (cf. Appendix). Therefore, the action (43) as an exact CFT describes string propagation on a four-dimensional manifold with (2,2)-signature. We showed that the PL T-duality relates the H_4 WZW model to a σ-model defined on $2+2$-dimensional manifold $M \approx O \times \mathbf{G}$ only when $\mathbf{G}$ is the Lie group A_2.

To continue, we obtain a new dual background for the H_4 WZW model. This background is obtained from a σ-model which is constructed on $2+2$-dimensional manifold $\tilde{M} \approx O \times \tilde{\mathbf{G}}$ with two-dimensional Abelian Lie group $\tilde{\mathbf{G}} = 2A_1$ acting freely on it. In order to construct the dual model in the form (31) we need to determine the dual couplings. Making use of (41) and inserting (42) into Eqs. (32)–(35) they are then read off to be

$$\tilde{\mathbb{E}}^{ab} = \begin{pmatrix} 0 & \frac{1}{e^{y_1}+\tilde{x}_2} \\ \frac{1}{e^{y_1}-\tilde{x}_2} & 0 \end{pmatrix}, \qquad \tilde{\Phi}_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$\tilde{\Phi}^{+(1)^a}{}_{\beta} = \begin{pmatrix} \frac{e^{y_1}}{e^{y_1}+\tilde{x}_2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\Phi}^{+(2)^b}_{\alpha} = \begin{pmatrix} \frac{e^{y_1}}{e^{y_1}-\tilde{x}_2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{52}$$

Putting these pieces together into (31) and using the fact that the components of the right invariant one-forms on $2A_1$ are $\tilde{R}_{\pm_a} = \partial_\pm \tilde{x}_a$, the action of dual σ-model is obtained to be

$$\begin{aligned} \tilde{S} = \frac{1}{2}\int d\sigma^+ d\sigma^- \Big\{ &-\partial_+ y_1\partial_- y_2 - \partial_+ y_2\partial_- y_1 \\ &+ \frac{1}{\Delta}[(e^{y_1} - \tilde{x}_2)\partial_+\tilde{x}_1\partial_-\tilde{x}_2 + (e^{y_1} + \tilde{x}_2)\partial_+\tilde{x}_2\partial_-\tilde{x}_1 \\ &+ e^{y_1}(e^{y_1} - \tilde{x}_2)\partial_+\tilde{x}_1\partial_- y_1 + e^{y_1}(e^{y_1} + \tilde{x}_2)\partial_+ y_1\partial_-\tilde{x}_1] \Big\}, \end{aligned} \tag{53}$$

[3](2,2)-signature often appears in Kleinian geometry as the neutral $(--++)$-signature. Metrics with (2,2)-signature might seem a purely mathematical problem, but there are several physical reasons that motivate this. First of all, two-time physics (cf. [27] for a review) has interesting applications in various areas, like cosmology [28] or M-theory [29]. Moreover, these metrics are intimately related to twistor space [30], which is an important tool in perturbative computations of scattering amplitudes in gauge theories [31].

[4]The metric (44) can be considered as the plane-parallel (pp-) wave in the so-called Rosen coordinates [32]. To this end, one can first use the coordinate transformation $e^{x_1} = \theta + \varphi$, $x_2 = -\theta + \varphi$, $y_1 = u$, $y_2 = -v$ to obtain $ds^2 = 2dudv + 2e^u(-d\theta^2 + d\varphi^2)$, then, after the change of the metric signature by a Wick rotation as $\theta = it$, the resulting metric turns into the pp-wave one in the Rosen coordinates.

[5]The WZW model based on the H_4 Lie group (a different real form of the h_4 Lie algebra of H_4) was, for the first time, introduced by Nappi and Witten [33].

where $\Delta = e^{2y_1} - \tilde{x}_2{}^2$. Comparing the above action with the σ-model action of the form (3), the corresponding line element and antisymmetric field $\tilde{B}$ take the following forms

$$d\tilde{s}^2 = -2dy_1dy_2 + 2\frac{e^{y_1}}{\Delta}(d\tilde{x}_1 d\tilde{x}_2 + e^{y_1} d\tilde{x}_1 dy_1), \tag{54}$$

$$\tilde{B} = -\frac{\tilde{x}_2}{\Delta}(d\tilde{x}_1 \wedge d\tilde{x}_2 + e^{y_1} d\tilde{x}_1 \wedge dy_1). \tag{55}$$

The line element (54) is ill defined at the regions $\tilde{x}_2 = e^{y_1}$ and $\tilde{x}_2 = -e^{y_1}$. We can test whether there are true singularities by calculating the scalar curvature, which is, $\tilde{\mathcal{R}} = 0$. Furthermore, one gets that the only nonzero components of the Ricci tensor and Riemann tensor field are, respectively,

$$\tilde{\mathcal{R}}_{\tilde{x}_2 y_1} = \frac{e^{y_1}}{(e^{y_1} - \tilde{x}_2)^2}, \qquad \tilde{\mathcal{R}}_{y_1 y_1} = -\frac{e^{2y_1} + 6\tilde{x}_2 e^{y_1} + \tilde{x}_2{}^2}{2(e^{y_1} - \tilde{x}_2)^2}, \tag{56}$$

and

$$\begin{aligned}\tilde{\mathcal{R}}_{\tilde{x}_1\tilde{x}_2\tilde{x}_2 y_1} &= \frac{e^{2y_1}}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3},\\ \tilde{\mathcal{R}}_{\tilde{x}_1 y_1 \tilde{x}_2 y_1} &= -\frac{e^{y_1}(e^{2y_1} + 6e^{y_1}\tilde{x}_2 + \tilde{x}_2^2)}{4(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3}.\end{aligned} \tag{57}$$

Then, the other invariant characteristics of spacetime, such as $\tilde{\mathcal{R}}_{\mu\nu}\tilde{\mathcal{R}}^{\mu\nu}$ and the Kretschmann scalar are found to be zero. Therefore, the singular points are not the essential singularities, that is, they can be removed by an appropriate change of coordinates. In order to investigate the conformal invariance conditions of the dual model (53) we look at the vanishing of the one-loop B-functions Eqs. (48a)–(48c). To this end, we find that the only nonzero component of $\tilde{H}$ corresponding to $\tilde{B}$-field (55) is $\tilde{H}_{\tilde{x}_1\tilde{x}_2 y_1} = e^{y_1}/2(e^{y_1} - \tilde{x}_2)^2$; consequently $\tilde{H}^2 = 0$. Hence, Eqs. (48a) and (48b) are satisfied by the new dilaton field

$$\tilde{\phi} = b_0 + b_1 y_1 - \log\left(\frac{\tilde{x}_2 - e^{y_1}}{\tilde{x}_2 + e^{y_1}}\right), \tag{58}$$

where b_0 and b_1 are integration constants. The dilatonic contribution, Eq. (48c), is also satisfied if the cosmological constant of the dual theory is left invariant, that is, $\tilde{\Lambda} = 0$. Thus, it seems that under the non-Abelian T-duality the cosmological constant has been restored from the dual model to the original one.

At the end of this subsection let us discuss the invariance of the string effective Lagrangians corresponding to the σ-models (43) and (53). For these models, the two Lagrangians $\mathcal{L}_{\rm eff}$ and $\tilde{\mathcal{L}}_{\rm eff}$ yield the same expression. They are both equal to zero. The equivalence of Lagrangians holds in spite of the nonvanishing traces of the structure constants corresponding to the double $(\mathcal{A}_2, 2\mathcal{A}_1)$. Moreover, in the case of this example one can show that the integration weights $\sqrt{-G}e^{-\phi}$ and $\sqrt{-\tilde{G}}e^{-\tilde{\phi}}$ are not equal. The reason behind this can be interpreted in two ways: firstly, the dilaton obtained in (58) does not follow the formula (37). Second, due to the particularity of our model, there is a possibility of absorbing the anomalous terms into dilaton shift which is the same as a diffeomorphism transformation. An analysis similar to this has been carried out in Ref. [34] in a strict field theory sense, regardless of the relationship between σ-models and string theory effective actions. Notice that the Eqs. (36) and (37) are the only transformations which lead to a proportionality between the integration weights $\sqrt{-G}e^{-\phi}$ and $\sqrt{-\tilde{G}}e^{-\tilde{\phi}}$ [20].

Case (2): The self-duality of the H_4 WZW model

The self-duality of the WZW model under PL T-duality, as well as the $SU(N)$ WZW model, has already been discussed in [35]. It turns out that the dual to the WZW model is again the same WZW model. Here we shall show that the H_4 WZW model is self-dual. Let us now choose the spectator-dependent background matrices as

$$E^+_{0\,ab} = \begin{pmatrix} 0 & -y_1 \\ y_1 & 0 \end{pmatrix}, \qquad F^{+(1)}_{a\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$F^{+(2)}_{\alpha b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = 0. \tag{59}$$

Then, using the fact that $\Pi(g) = 0$ and utilizing formulas (13)–(16) together with (40) and (27), the original σ-model is, in this case, obtained to be of the form

$$\begin{aligned}S = \frac{1}{2}\int d\sigma^+ d\sigma^- [&-y_1 e^{x_1}(\partial_+ x_1 \partial_- x_2 - \partial_+ x_2 \partial_- x_1)\\ &+ e^{x_1}(\partial_+ y_1 \partial_- x_2 + \partial_+ x_2 \partial_- y_1)\\ &- \partial_+ x_1 \partial_- y_2 - \partial_+ y_2 \partial_- x_1].\end{aligned} \tag{60}$$

As it is seen, the action (60) is indeed identical to the action of H_4 WZW model (cf. Appendix).

Analogously to Case (1), the dual model is constructed on the manifold $\tilde{M} \approx O \times \tilde{\mathbf{G}}$. The Poisson bracket on $\tilde{\mathbf{G}}$ is given by formula (41), and thus by inserting (59) into Eqs. (32)–(35) the dual couplings are computed to be

$$\tilde{\mathbb{E}}^{ab} = \frac{1}{y_1 + \tilde{x}_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{\Phi}_{\alpha\beta} = \frac{1}{y_1 + \tilde{x}_2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\tilde{\Phi}^{+(1)a}{}_{\beta} = \frac{1}{y_1 + \tilde{x}_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\Phi}^{+(2)b}_{\alpha} = \frac{1}{y_1 + \tilde{x}_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{61}$$

Finally, inserting these into formula (31) the dual σ-model is obtained to be in the following form

$$\tilde{S} = \frac{1}{2}\int d\sigma^+ d\sigma^- \frac{1}{(y_1+\tilde{x}_2)}[\partial_+\tilde{x}_1\partial_- y_1 + \partial_+ y_1\partial_-\tilde{x}_1 + \partial_+\tilde{x}_2\partial_- y_2 + \partial_+ y_2\partial_-\tilde{x}_2 + \partial_+\tilde{x}_1\partial_-\tilde{x}_2 - \partial_+\tilde{x}_2\partial_-\tilde{x}_1 - \partial_+ y_1\partial_- y_2 + \partial_+ y_2\partial_- y_1]. \quad (62)$$

The line element and antisymmetric tensor field corresponding to action (62) can be cast in the forms

$$d\tilde{s}^2 = \frac{2}{(y_1+\tilde{x}_2)}(d\tilde{x}_1 dy_1 + d\tilde{x}_2 dy_2), \quad (63)$$

$$\tilde{B} = \frac{1}{(y_1+\tilde{x}_2)}(d\tilde{x}_1\wedge d\tilde{x}_2 - dy_1\wedge dy_2). \quad (64)$$

By using (63) and (64), the conformal invariance conditions of the model (62) are satisfied with zero cosmological constant and dilaton field that supports the dual background is found to be

$$\tilde{\phi} = c_0 + c_1\log(y_1+\tilde{x}_2), \quad (65)$$

where c_0 and c_1 are the constants of integration. On the one hand, it is interesting to note that if we write (65) as $e^{-\tilde{\phi}} = \varrho_0/(y_1+\tilde{x}_2)$, then metric (63) may be expressed as[6]

$$\tilde{G}_{\mu\nu} = e^{-\tilde{\phi}}\tilde{\eta}_{\mu\nu}, \quad (66)$$

in which $\tilde{\eta}_{\mu\nu} = 2d\tilde{x}_1 dy_1 + 2d\tilde{x}_2 dy_2$. The formula (66) indicates a conformal transformation[7] between the dual metric $\tilde{G}_{\mu\nu}$ and flat metric $\tilde{\eta}_{\mu\nu}$, and $e^{-\tilde{\phi}}$ is a smooth, nonvanishing function of the spacetime which is called a conformal factor. We note that the conformal transformations do change geometry and they are entirely different from coordinate transformations. This is crucial since conformal transformations may lead to a different physics [36]. If we use the coordinate transformation

$$\tilde{x}_1 = m, \qquad \tilde{x}_2 = a_+, \qquad y_1 = e^{-n} - a_+, \qquad y_2 = a_- + m, \quad (67)$$

then the dual background can be cast to [16]

$$d\tilde{s}^2 = -2dn\,dm + 2e^n da_+ da_-,$$
$$\tilde{B} = -a_+ e^n dn\wedge da_-.$$
$$\tilde{\phi} = \varsigma_0 + \varsigma_1 n, \quad (68)$$

where ς_0 and ς_1 are arbitrary constants. Here we have ignored the total derivative terms that appeared in the $\tilde{B}$-field part. Indeed, the solution (68) is identical to the background of the original σ-model action (60). *Thus, we showed that the H_4 WZW model does remain invariant under the non-Abelian T-duality transformation, that is, the model is self-dual.*

B. The $GL(2,\mathbb{R})$ WZW model from T-dualizable σ-models and its dual pair

We shall show that the original σ-model (27) on the $2+2$-dimensional manifold $M \approx O\times \mathbf{G}$ can be equalled to the $GL(2,\mathbb{R})$ WZW model. Similar to previous examples, the isometry group $\mathbf{G}$ that is being dualized is A_2. The only difference is in choosing the spectator-dependent background matrices. In this regard, the non-Abelian T-dual geometry of the $GL(2,\mathbb{R})$ WZW model is determined.

1. The original σ-model as the $GL(2,\mathbb{R})$ WZW model

Here, we choose

$$E^+_{0\,ab} = \begin{pmatrix} 0 & \frac{1}{2}e^{-2y_1} \\ \frac{1}{2}e^{-2y_1} & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$
$$F^{+(1)}_{a\beta} = \begin{pmatrix} 0 & 0 \\ -e^{-2y_1} & 0 \end{pmatrix}, \qquad F^{+(2)}_{\alpha b} = \begin{pmatrix} 0 & e^{-2y_1} \\ 0 & 0 \end{pmatrix}, \quad (69)$$

where b is a nonzero real constant. Inserting relations (69) into (13)–(16) and then using (40) together with (27), the original σ-model is worked out to be

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^-\Big[\partial_+ y_1\partial_- y_1 + b\partial_+ y_2\partial_- y_2 + \frac{1}{2}e^{x_1-2y_1}(\partial_+ x_1\partial_- x_2 + \partial_+ x_2\partial_- x_1) + e^{x_1-2y_1}(\partial_+ y_1\partial_- x_2 - \partial_+ x_2\partial_- y_1)\Big]. \quad (70)$$

Now, if one uses the following coordinate transformation

$$e^{x_1} = \theta_-, \qquad x_2 = \theta_+, \qquad y_1 = \theta_3, \qquad y_2 = \theta, \quad (71)$$

then, action (70) becomes

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^-\Big[b\partial_+\theta\partial_-\theta + \partial_+\theta_3\partial_-\theta_3 + \frac{1}{2}e^{-2\theta_3}(\partial_+\theta_-\partial_-\theta_+ + \partial_+\theta_+\partial_-\theta_-) + \theta_- e^{-2\theta_3}(\partial_+\theta_3\partial_-\theta_+ - \partial_+\theta_+\partial_-\theta_3)\Big]. \quad (72)$$

Using the integration by parts over the fourth term of action, it is concluded that the action (72) is nothing but the

[6] Here we have set $\varrho_0 = 1$.

[7] The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system x^μ on the manifold M, but they preserve the angles between vectors which lead to a conservation of the (global) causal structure of the manifold [36].

action of WZW model based on the Lie group $GL(2,\mathbb{R})$ (cf. Appendix). Hence, the original σ-model (70) can be described as an exact CFT.

The line element and the antisymmetric tensor field corresponding to action (70) are, respectively, given by

$$ds^2 = dy_1{}^2 + b dy_2{}^2 + e^{x_1-2y_1} dx_1 dx_2, \tag{73}$$

$$B = -e^{x_1-2y_1} dx_2 \wedge dy_1. \tag{74}$$

To better understand action (70) we diagonalize the corresponding metric. Let

$$e^{x_1} = \frac{1}{l}(t - l\varphi), \qquad x_2 = -l(t + l\varphi),$$
$$e^{y_1} = \frac{l}{r}, \qquad y_2 = z, \tag{75}$$

where l is constant with the dimension of length. Then, the metric (73) and the field strength corresponding to the B-field (74) shall, respectively, become

$$ds^2 = -\frac{r^2}{l^2}dt^2 + r^2 d\varphi^2 + \frac{1}{r^2}dr^2 + b dz^2, \tag{76}$$

$$H = -\frac{r}{l} dt \wedge d\varphi \wedge dr. \tag{77}$$

The metric (76) describes a four-dimensional Lorentz-signature spacetime if b is considered positive and when b is negative, the metric has (2,2)-signature. One immediately finds that the scalar curvature of the metric is $\mathcal{R} = -6$, and the nonzero components of $\mathcal{R}_{\mu\nu}$ are $\mathcal{R}_{nn} = -2g_{nn}$ where $n = (t, \varphi, r)$. Since $\mathcal{R}_{zz} = 0$, this spacetime cannot be described as an AdS_4 space. On the other hand, the metric is a direct product of $\mathbb{R}$ associated with the coordinate z and the three-dimensional metric of (t, φ, r), which is nothing but the AdS_3 space. Hence, the metric (76) corresponds to $\mathrm{AdS}_3 \times \mathbb{R}$. Furthermore, using (76) and (77) it is concluded that the only nonzero components of $(H^2)_{\mu\nu}$ are $(H^2)_{tt} = (2r^2)/l^2, (H^2)_{\varphi\varphi} = -2r^2$ and $(H^2)_{rr} = -2/r^2$. Finally, one verifies the Eqs. (48a) and (48b) with the following dilaton field

$$\phi = \zeta_0 + \zeta_1 z, \tag{78}$$

where ζ_0 and ζ_1 are integration constants. Also, computing $H^2 = -6$ the dilatonic contribution in Eq. (48c) is satisfied provided that $\zeta_1{}^2 = b(2\Lambda - 4)$. Notice that the metric (76) has no horizon and no curvature singularity. Indeed, this solution is everywhere regular including $r = 0$. Consider now two killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$ corresponding to the time translational and the rotational isometries of the metric (76), respectively. The killing field $\partial/\partial t$ becomes null at $r = 0$ and it is time-like for the whole range $r > 0$, while the killing field $\partial/\partial\varphi$ is everywhere spacelike except for $r = 0$. In addition, there is another killing field such as $(1/b)\partial/\partial z$ so that it is timelike for $b < 0$, and remains spacelike for $b > 0$.

2. The dual σ-model

Similar to the construction of dual σ-models for H_4 WZW, the dual manifold is, here, assumed to be $\tilde{M} \approx O \times \tilde{\mathbf{G}}$ in which $\tilde{\mathbf{G}} = 2A_1$. So, the Poisson structure on $2A_1$ does follow the relation (41). In order to obtain the dual σ-model for $GL(2,\mathbb{R})$ WZW, we use the action (31). The dual coupling matrices can be obtained by inserting (41) and (69) into (32)–(35). They are then read

$$\tilde{\mathbb{E}}^{ab} = \begin{pmatrix} 0 & \frac{1}{\tilde{x}_2 + \frac{1}{2}e^{-2y_1}} \\ \frac{1}{-\tilde{x}_2 + \frac{1}{2}e^{-2y_1}} & 0 \end{pmatrix}, \qquad \tilde{\Phi}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$
$$\tilde{\Phi}^{+(1)a}{}_{\beta} = \begin{pmatrix} -\frac{e^{-2y_1}}{\tilde{x}_2 + \frac{1}{2}e^{-2y_1}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\Phi}^{+(2)b}_{\alpha} = \begin{pmatrix} \frac{e^{-2y_1}}{\tilde{x}_2 - \frac{1}{2}e^{-2y_1}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{79}$$

Finally, the dual σ-model to the $GL(2,\mathbb{R})$ WZW model is found to be

$$\tilde{S} = \frac{1}{2}\int d\sigma^+ d\sigma^- \Big\{\partial_+ y_1 \partial_- y_1 + b\partial_+ y_2 \partial_- y_2 + \frac{1}{\bar{\Delta}}\Big[\Big(\frac{1}{2}e^{-2y_1} - \tilde{x}_2\Big)\partial_+\tilde{x}_1\partial_-\tilde{x}_2 + \Big(\frac{1}{2}e^{-2y_1} + \tilde{x}_2\Big)\partial_+\tilde{x}_2\partial_-\tilde{x}_1$$
$$- e^{-2y_1}\Big(\frac{1}{2}e^{-2y_1} - \tilde{x}_2\Big)\partial_+\tilde{x}_1\partial_- y_1 - e^{-2y_1}\Big(\frac{1}{2}e^{-2y_1} + \tilde{x}_2\Big)\partial_+ y_1\partial_-\tilde{x}_1\Big]\Big\}, \tag{80}$$

where $\bar{\Delta} = \frac{1}{4}e^{-4y_1} - \tilde{x}_2^2$. The line element and antisymmetric field corresponding to this action may be expressed as

$$d\tilde{s}^2 = dy_1{}^2 + b dy_2{}^2 + \frac{e^{-2y_1}}{\bar{\Delta}}(d\tilde{x}_1 d\tilde{x}_2 - e^{-2y_1} d\tilde{x}_1 dy_1), \tag{81}$$

$$\tilde{B} = -\frac{\tilde{x}_2}{\bar{\Delta}}(d\tilde{x}_1 \wedge d\tilde{x}_2 - e^{-2y_1} d\tilde{x}_1 \wedge dy_1). \tag{82}$$

The scalar curvature of the metric is

$$\tilde{\mathcal{R}} = -\frac{2(11e^{-4y_1} + 28\tilde{x}_2 e^{-2y_1} + 12\tilde{x}_2^2)}{(e^{-2y_1} - 2\tilde{x}_2)^2}. \tag{83}$$

As it can be seen from formulas (81) and (83), the region $\tilde{x}_2 = \frac{1}{2}e^{-2y_1}$ is a true curvature singularity (in what follows we will discuss the structure and asymptotic nature of the dual spacetime including the horizon and singularity). We deduce that the only nonzero component of the field strength corresponding to the $\tilde{B}$-field (82) is $\tilde{H}_{\tilde{x}_1\tilde{x}_2y_1} = -(2e^{-2y_1})/(e^{-2y_1} - 2\tilde{x}_2)^2$; consequently $\tilde{H}^2 = -6(e^{-2y_1} + 2\tilde{x}_2)^2/(e^{-2y_1} - 2\tilde{x}_2)^2$. Thus, Eqs. (48a) and (48b) are satisfied by the new dilaton field

$$\tilde{\phi} = \lambda_0 + \lambda_1 y_2 + \log\left(\frac{2\tilde{x}_2 + e^{-2y_1}}{2\tilde{x}_2 - e^{-2y_1}}\right), \tag{84}$$

where λ_0 and λ_1 are arbitrary constants. Also, the dilatonic contribution in (48c) is vanished if the cosmological constant of the dual theory does satisfy in $\lambda_1{}^2 = b(2\tilde{\Lambda} - 4)$.

As in the first example of subsection A, we now discuss the presence of anomalous terms breaking the proportionality between the original and dual string effective actions. The string effective Lagrangians corresponding to the σ-models (70) and (80) are found to be

$$\mathcal{L}_{\text{eff}} = 4\Lambda - 8, \tag{85}$$

$$\tilde{\mathcal{L}}_{\text{eff}} = 4\tilde{\Lambda} - 8 - \frac{64\tilde{x}_2 e^{2y_1}}{(1 - 2\tilde{x}_2 e^{2y_1})^2}. \tag{86}$$

The last term of Eq. (86) is not invariant under PL T-duality transformation and therefore the two Lagrangians $\mathcal{L}_{\text{eff}}$ and $\tilde{\mathcal{L}}_{\text{eff}}$ are not equal. This anomaly is due to the nonvanishing traces of the structure constants of the double $(\mathcal{A}_2, 2\mathcal{A}_1)$; furthermore, the dilaton field obtained in (84) does not follow the transformation (37).

The dilaton field (84) is well behaved for the ranges $\tilde{x}_2 < -\frac{1}{2}e^{-2y_1}$ and $\tilde{x}_2 > \frac{1}{2}e^{-2y_1}$. We also note that a dilaton field can easily be found for the range $-\frac{1}{2}e^{-2y_1} < \tilde{x}_2 < \frac{1}{2}e^{-2y_1}$ by shifting λ_0 by an imaginary constant $(\lambda_0 \to \lambda_0 + i\pi)$.

For the range $\tilde{x}_2 < -\frac{1}{2}e^{-2y_1}$ we consider $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} = -e^X$. Then, we introduce the following coordinate transformation

$$\tilde{x}_1 = Y + \frac{1}{2}(W + e^W), \qquad \tilde{x}_2 = -e^X\left(1 + \frac{e^{-W}}{2}\right),$$
$$y_1 = \frac{1}{2}(W - X), \qquad y_2 = V. \tag{87}$$

Under this transformation, the dual background now looks as follows

$$d\tilde{s}^2 = bdV^2 + \frac{1}{4}(dW^2 + dX^2) + \frac{1}{e^W + 1}dXdY, \tag{88}$$

$$\tilde{B} = -\frac{2e^W + 1}{2(e^W + 1)}dX \wedge dY, \tag{89}$$

$$\tilde{\phi} = \lambda_0 + \lambda_1 V + \log\left(\frac{e^W}{e^W + 1}\right). \tag{90}$$

Here we have ignored the terms concerning $\tilde{B}$-field which are contributed to the Lagrangian as the total derivatives. Notice that there is no singularity for the metric (88). In fact, this was expected since the solutions (81), (82), and (84) are, in this case, defined only for the range $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} < 0$. As explained above, the true singularity of the metric (81) occurs at $\tilde{x}_2 = \frac{1}{2}e^{-2y_1}$, a region which is located out of the range $\tilde{x}_2 + \frac{1}{2}e^{-2y} < 0$. The background (88)–(90) can be simplified by performing a coordinate transformation. Let us now consider the transformation $e^W = 1/(r-1)$ so that it requires that $1 < r < \infty$. In addition, we introduce the following linear transformation

$$X = -2\left(t + \frac{x}{\sqrt{3}}\right), \qquad Y = \left(t - \frac{x}{\sqrt{3}}\right), \qquad V = z. \tag{91}$$

By applying the above transformation to the solutions (88), (89), and (90), one obtains the forms of the dual spacetime metric, antisymmetric field strength, and dilaton field in new coordinate base $\{t, x, r, z\}$ as

$$d\tilde{s}^2 = -\left(1 - \frac{2}{r}\right)dt^2 + \left(1 - \frac{2}{3r}\right)dx^2 + \frac{2}{\sqrt{3}}dtdx + \left(1 - \frac{1}{r}\right)^{-2}\frac{dr^2}{4r^2} + bdz^2, \tag{92}$$

$$\tilde{H}_{rtx} = \frac{1}{\sqrt{3}r^2}, \tag{93}$$

$$\tilde{\phi} = \lambda_0 + \lambda_1 z - \log r. \tag{94}$$

We note that this solution is valid only for the range $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} < 0$ or $1 < r < \infty$. In order to have a solution with the range $0 < r < 1$ we have to look at the second case, where $\tilde{x}_2 - \frac{1}{2}e^{-2y_1} > 0$. In this case it is assumed that $\tilde{x}_2 - \frac{1}{2}e^{-2y_1} = e^X - e^{-2y_1}$ for which $X + 2y_1 > 0$. Analogously, we introduce the transformation

$$\tilde{x}_1 = Y - \frac{1}{2}(e^W - W), \qquad \tilde{x}_2 = e^X\left(1 - \frac{e^{-W}}{2}\right),$$
$$y_1 = \frac{1}{2}(W - X), \qquad y_2 = V, \tag{95}$$

in which $W = X + 2y_1 > 0$, i.e., $e^W > 1$. We then define the transformation $e^W = 1/(1-r)$ so that it requires that $0 < r < 1$. Using these results and also utilizing the linear transformation (91) one concludes that the solution given by Eqs. (81), (82), and (84) is nothing but the solution given by (92)–(94). Thus, the obtained solutions to both the valid ranges $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} < 0$ and $\tilde{x}_2 - \frac{1}{2}e^{-2y_1} > 0$ can be expressed as a solution in the form of Eqs. (92)–(94) only with $0 < r < \infty$. Analogously, for the range $-\frac{1}{2}e^{-2y_1} < \tilde{x}_2 < \frac{1}{2}e^{-2y_1}$ one can consider $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} = e^X$ to obtain the same results presented in (92)–(94).

One can simply check that the solution (92)–(94) does satisfy the Eqs. (48a)–(48c). Considering this solution for the whole spacetime, $0 < r < \infty$, one sees that the metric components (92) are ill behaved at $r = 0$ and $r = 1$. Looking at the scalar curvature, which is $\tilde{\mathcal{R}} = 2(4r - 7)/r^2$, we find that $r = 0$ is a curvature singularity. Notice that the singularity at $r = 0$ corresponds to the same true singularity at the region $\tilde{x}_2 = \frac{1}{2}e^{-2y_1}$ which mentioned above. We furthermore see that $r = 1$ is also an event horizon. The cross term appeared in the metric is constant and thus for large r one can show that the metric is asymptotically flat. For large r the metric (92) approaches the following asymptotic solution

$$d\tilde{s}^2 = -dt^2 + dx^2 + \frac{2}{\sqrt{3}}dtdx + \frac{dr^2}{4r^2} + bdz^2. \quad (96)$$

Performing a convenient coordinate transformation, the metric (96) can be simply diagonalized. We also note that the sign of b changes the signature of metric. If we introduce the new coordinates $(\hat{t}, \hat{x}, \hat{r}, \hat{z})$ by the transformation

$$r = e^{2\hat{r}}, \qquad t = \frac{\sqrt{3}}{2}\hat{t}, \qquad x = \hat{x} - \frac{\hat{t}}{2}, \qquad z = \frac{\hat{z}}{\eta}, \quad (97)$$

then, (96) will become

$$d\tilde{s}^2 = \begin{cases} -d\hat{t}^2 + d\hat{x}^2 + d\hat{r}^2 + d\hat{z}^2 & \text{for } b = \eta^2 \\ -d\hat{t}^2 + d\hat{x}^2 + d\hat{r}^2 - d\hat{z}^2 & \text{for } b = -\eta^2 \end{cases}. \quad (98)$$

As it is seen for $b > 0$ the metric has (1,3)-signature, while the signature is (2,2) when b is negative. *Thus, we have shown that the non-Abelian T-duality transformation (here as the PL T-duality on a semi-Abelian double) changes the asymptotic behavior of solutions from* $\mathrm{AdS}_3 \times \mathbb{R}$ *to flat space*.

C. The non-Abelian T-duality of noncritical Bianchi type III string cosmological model (the $GL(2,\mathbb{R})$ WZW model)

In this subsection, we show that the noncritical Bianchi type III string cosmology solution with a nonvanishing field strength and an appropriate dilaton field can be described by the $GL(2,\mathbb{R})$ WZW model. In fact, we shall obtain the $GL(2,\mathbb{R})$ WZW model from a T-dualizable σ-model constructed on a $3+1$-dimensional manifold $M \approx O \times \mathbf{G}$, in which $\mathbf{G}$ is three-dimensional decomposable Lie group $A_2 \oplus A_1$ acting freely on M. In this case, the non-Abelian T-duality of the model is studied here. The dual Lie group $\tilde{\mathbf{G}}$ is considered to be three-dimensional Abelian Lie group $3A_1$. We note that the Lie algebra $\mathcal{A}_2 \oplus \mathcal{A}_1$ is isomorphic to the Lie algebra of Bianchi type III. Hence, six-dimensional Lie algebra of the Drinfeld double $(\mathcal{A}_2 \oplus \mathcal{A}_1, 3\mathcal{A}_1)$ is defined by the following commutation relations:

$$[T_1, T_2] = T_2, \qquad [T_3, .] = 0, \qquad [T_1, \tilde{T}^2] = -\tilde{T}^2,$$
$$[T_2, \tilde{T}^2] = \tilde{T}^1, \qquad [\tilde{T}^3, .] = 0. \quad (99)$$

Taking a convenient element of the Lie group $A_2 \oplus A_1$ such as $g = e^{(\ln x_1)T_1}e^{x_2T_2}e^{x_3T_3}$ we immediately find that $R^1_\pm = \partial_\pm x_1/x_1$, $R^2_\pm = x_1\partial_\pm x_2$, $R^3_\pm = \partial_\pm x_3$. In order to study the non-Abelian T-duality of noncritical Bianchi type III string cosmological model, we consider the orbit O as a one-dimensional space with time coordinate $y^\alpha = \{t\}$. Now, one can choose the spectator-dependent background matrices as

$$E^+_{0\,ab} = \begin{pmatrix} 0 & -\frac{a_0^2}{2}e^{-2t} & 0 \\ -\frac{a_0^2}{2}e^{-2t} & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad F^{+(1)}_{a\beta} = \begin{pmatrix} 0 \\ a_0^2 e^{-2t} \\ 0 \end{pmatrix},$$
$$F^{+(2)}_{\alpha b} = \begin{pmatrix} 0 & -a_0^2 e^{-2t} & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = -a_0^2, \quad (100)$$

for some constants a_0, b, and then use (27) to obtain the following background

$$ds^2 = -a_0^2 dt^2 - a_0^2 e^{-2t}dx_1 dx_2 + b{dx_3}^2, \quad (101)$$

$$B = a_0^2 x_1 e^{-2t} dx_2 \wedge dt. \quad (102)$$

Comparing (101) and the general form of the string cosmology metric

$$ds^2 = -g_{00}^2(t)dt^2 + \sum_{a,b=1}^{3} R_\mu{}^a R_\nu{}^b g_{ab}(t) dx^\mu\, dx^\nu, \quad (103)$$

one concludes that (101) is nothing but the Bianchi type III string cosmology metric. This metric has (2,2)-signature if b is considered positive. One can easily check that the metric (101) and the field strength corresponding to the B-field (102) ($H_{x_1x_2t} = a_0^2 e^{-2t}/2$) along with the dilaton field $\phi = v_0 + v_1 x_3$ (for some constants v_0, v_1) make up a solution for the vanishing of the one-loop B-functions equations (48a)–(48c). It is also interesting to note that the corresponding action to (101) and (102) is equivalent to the $GL(2,\mathbb{R})$ WZW model. This means that the obtained background can be described as an exact CFT.

The dual model is constructed on $3+1$-dimensional manifold $\tilde{M} \approx O \times \tilde{\mathbf{G}}$ with $\tilde{\mathbf{G}} = 3A_1$. Finally, using (99) and (100) together with Eqs. (32)–(35) the dual background is obtained to be

$$d\tilde{s}^2 = -a_0^2 dt^2 + \frac{1}{b}{dx_3}^2 + \frac{a_0^2 e^{-2t}}{\tilde{x}_2^2 - \frac{a_0^4}{4}e^{-4t}}(d\tilde{x}_1 d\tilde{x}_2 + a_0^2 e^{-2t} d\tilde{x}_1\, dt), \quad (104)$$

$$\tilde{B} = \frac{\tilde{x}_2}{\tilde{x}_2^2 - \frac{a_0^4}{4}e^{-4t}}(d\tilde{x}_1 \wedge d\tilde{x}_2 + a_0^2 e^{-2t} d\tilde{x}_1 \wedge dt). \quad (105)$$

The dilaton field that supports the dual background is obtained in the following form

$$\tilde{\phi} = \vartheta_0 + \vartheta_1 \tilde{x}_3 - \log\left(\frac{a_0^2 + 2\tilde{x}_2 e^{2t}}{a_0^2 - 2\tilde{x}_2 e^{2t}}\right), \quad (106)$$

where ϑ_0, ϑ_1 are some constants.

IV. CONFORMAL INVARIANCE OF THE T-DUAL MODELS UP TO TWO-LOOP ORDER (FIRST ORDER IN α')

So far, we have been concerned with the conformal invariance of the T-dual models up to one-loop order (zeroth order in α'). As mentioned in Sec. III, the conditions for conformal invariance of the σ-model with action (3) can be interpreted as field equations for $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ of the string effective action [24,25]. These equations to the first order in α' take the following form [26]

$$\mathcal{R}_{\mu\nu} - (H^2)_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \frac{1}{2}\alpha' \Big[\mathcal{R}_{\mu\rho\sigma\lambda}\mathcal{R}_\nu{}^{\rho\sigma\lambda} + 2\mathcal{R}_{\mu\rho\sigma\nu}(H^2)^{\rho\sigma} + 2\mathcal{R}_{\rho\sigma\lambda(\mu}H_{\nu)}{}^{\lambda\delta}H^{\rho\sigma}{}_\delta + \frac{1}{3}(\nabla_\mu H_{\rho\sigma\lambda})(\nabla_\nu H^{\rho\sigma\lambda})$$
$$- (\nabla_\lambda H_{\rho\sigma\mu})(\nabla^\lambda H^{\rho\sigma}{}_\nu) + 2H_{\mu\rho\sigma}H_{\nu\lambda\delta}H^{\eta\delta\sigma}H_\eta{}^{\lambda\rho} + 2H_{\mu\rho\sigma}H_{\nu\lambda}{}^\sigma(H^2)^{\lambda\rho}\Big] + \mathcal{O}(\alpha'^2) = 0, \quad (107a)$$

$$\nabla^\lambda H_{\lambda\mu\nu} - (\nabla^\lambda \phi')H_{\mu\nu\lambda} + \alpha'[\nabla^\lambda H^{\rho\sigma}{}_{[\mu}\mathcal{R}_{\nu]\lambda\rho\sigma} - (\nabla_\lambda H_{\rho\mu\nu})(H^2)^{\lambda\rho} - 2(\nabla^\lambda H^{\rho\sigma}{}_{[\mu})H_{\nu]\rho\delta}H_{\lambda\sigma}{}^\delta] + \mathcal{O}(\alpha'^2) = 0, \quad (107b)$$

$$2\Lambda + \nabla^2\phi' - (\nabla\phi')^2 + \frac{2}{3}H^2 - \alpha'\Big[\frac{1}{4}\mathcal{R}_{\mu\rho\sigma\lambda}\mathcal{R}^{\mu\rho\sigma\lambda} - \frac{1}{3}(\nabla_\lambda H_{\mu\nu\rho})(\nabla^\lambda H^{\mu\nu\rho}) - \frac{1}{2}H^{\mu\nu}{}_\lambda H^{\rho\sigma\lambda}\mathcal{R}_{\mu\nu\rho\sigma}$$
$$- \mathcal{R}_{\mu\nu}(H^2)^{\mu\nu} + \frac{3}{2}(H^2)_{\mu\nu}(H^2)^{\mu\nu} + \frac{5}{6}H_{\mu\nu\rho}H^\mu{}_{\sigma\lambda}H^{\nu\sigma}{}_\delta H^{\rho\lambda\delta}\Big] + \mathcal{O}(\alpha'^2) = 0, \quad (107c)$$

where $\phi' = \phi + \alpha' q H^2$ for some coefficient q [26], $(H^2)^{\mu\nu} = H^{\mu\rho\sigma}H_{\rho\sigma}{}^\nu$ and $\mathcal{R}_{\mu\rho\sigma\lambda}$ is the Riemann tensor field. We note that round brackets denote the symmetric part on the indicated indices whereas square brackets denote the antisymmetric part. Below using the above equations we check the conformal invariance conditions of the T-dual models up to two-loop order (first order in α'). In fact, we introduce new solutions for two-loop B-function equations of the σ-model with a nonvanishing field strength H and the dilaton field in both cases of the absence and presence of a cosmological constant Λ.

(i) As shown in subsection A of Sec. III, the background of the original σ-model (43) is given by the formulas (44) and (45) so that this model is equivalent to the H_4 WZW model. Therefore, it should be conformally invariant. In the case of this model, the only nonvanishing component of the Riemann tensor is $\mathcal{R}_{x_1y_1x_2y_1} = -(e^{x_1+y_1})/4$. Moreover, the only nonvanishing component of $(H^2)^{\mu\nu}$ is $(H^2)^{y_2y_2} = -1/2$ and all components of $\nabla_\lambda H_{\mu\nu\sigma}$ vanish. Hence using these results, the field equations (107a)–(107c) are satisfied for the metric (44) and the tensor field (45) together with the dilaton field (49) and zero cosmological constant as this was expected.

(ii) In order to investigate the conformal invariance conditions of the dual model to the H_4 WZW (the σ-model (53) up to the first order in α', we first find that the only nonvanishing components of $(\tilde{H}^2)_{\mu\nu}$ and $(\tilde{H}^2)^{\mu\nu}$ are $(\tilde{H}^2)_{y_1y_1} = -(e^{y_1} + \tilde{x}_2)^2/2(e^{y_1} - \tilde{x}_2)^2$ and $(\tilde{H}^2)^{y_2y_2} = (\tilde{H}^2)_{y_1y_1}$, respectively. Also, the only nonvanishing components of $\tilde{\nabla}_\lambda \tilde{H}_{\mu\mu\rho}$ may be expressed as

$$\tilde{\nabla}_{\tilde{x}_2}\tilde{H}_{\tilde{x}_1\tilde{x}_2y_1} = \frac{e^{2y_1}}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3},$$
$$\tilde{\nabla}_{y_1}\tilde{H}_{\tilde{x}_1\tilde{x}_2y_1} = -\frac{\tilde{x}_2 e^{2y_1}}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3}. \quad (108)$$

Using these results together with the given data for this model in subsection A of Sec. III, one verifies the field Eqs. (107a) and (107b) for the metric (54) and the tensor field (55) along with the dilaton field (58). The Eq. (107c) is also satisfied with $\tilde{\Lambda} = 0$.

(iii) Under the coordinate transformation (75), the background of the original σ-model (70) was represented by (76) and (77). It was shown that resulting background as an exact CFT satisfies the vanishing of the one-loop B-functions equations (48a)–(48c) with the dilaton field (78). Using the expressions (76) and (77) for the background fields one may verify that the only nonvanishing components of Riemann tensor are $\mathcal{R}_{t\varphi t\varphi} = r^4/l^2$, $\mathcal{R}_{trtr} = 1/l^2$ and $\mathcal{R}_{\varphi r\varphi r} = -1$; consequently, the Kretschmann scalar is computed to be $K = 12$. Moreover, we get that the only nonvanishing components of $(H^2)^{\mu\nu}$ are $(H^2)^{tt} = (2l^2)/r^2$, $(H^2)^{\varphi\varphi} = -2/r^2$ and $(H^2)^{rr} = -2r^2$, and all components of $\nabla_\lambda H_{\mu\nu\sigma}$ vanish. Putting these pieces together, one verifies equations (107a) and (107b) with the dilaton field (78). It is then interesting to note that in this case the field equation (107c) is satisfied if the following relation is held between the constants ${\zeta_1}^2$, Λ, b and α':

$$\alpha' = -\frac{1}{4}\left(2 + \frac{{\zeta_1}^2}{2b} - \Lambda\right). \tag{109}$$

(iv) As mentioned in the preceding section, the dual model of the $GL(2,\mathbb{R})$ WZW [Eqs. (92)–(94)] does satisfy the vanishing of the one-loop B-functions equations. Unfortunately, this background does not satisfy the equations for the two-loop B-functions. One can show that for this background all Eqs. (107a)–(107c) are satisfied except for the components of $B^G_{ii}, (i = t, x, r)$, B^B_{tx} and B^Φ.

V. SUMMARY AND CONCLUDING REMARKS

Using the PL T-duality approach in the presence of spectators we have constructed some non-Abelian T-dualizable σ-models on $2+2$-dimensional target manifolds $M \approx O \times \mathbf{G}$ and $\tilde{M} \approx O \times \tilde{\mathbf{G}}$, where $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are two-dimensional real non-Abelian and Abelian Lie groups, respectively. We have shown that the original σ- models are equivalent to the H_4 and $GL(2,\mathbb{R})$ WZW models. In this way, we could obtain some new T-dual backgrounds for these WZW models. The most interesting feature of our results is the invariance of the H_4 WZW model under the non-Abelian T-duality. We have shown that the $GL(2,\mathbb{R})$ WZW model as a T-dualizable σ- model is equivalent to $\mathrm{AdS}_3 \times \mathbb{R}$ space and has no horizon and no curvature singularity, while the dual spacetime of the $GL(2,\mathbb{R})$ WZW model is stationary and asymptotically flat and has a single horizon and a curvature singularity. Moreover, it was shown that for the line element (76), the Killing vectors $\partial/\partial t$ and $(1/b)\partial/\partial z$ with $b < 0$ are timelike. Analogously, one can show that the dual line element (92) possesses three independent Killing vectors $\sqrt{3}\partial/\partial t$, $3/2(\sqrt{3}\partial/\partial t - \partial/\partial x)$ and $(1/b)\partial/\partial z$. The first two Killing vectors become timelike for the ranges $r > 2$ and $r > 4/3$, respectively. The last Killing vector stays everywhere timelike for $b < 0$. Hence, the duality has involved the timelike directions. In summary, in the case of the effect of the non-Abelian T-duality (here as the PL T-duality on a semi-Abelian double) on the $GL(2,\mathbb{R})$ WZW model three points have been highlighted.

(1) The non-Abelian T-duality transformation has changed the asymptotic behavior of solutions from $\mathrm{AdS}_3 \times \mathbb{R}$ to flat space.
(2) This transformation has related a solution with no horizon and no curvature singularity to a solution with a single horizon and a curvature singularity.
(3) The duality has involved the timelike directions.

We have also obtained the noncritical Bianchi type III string cosmological model with a nonvanishing field strength from a T-dualizable σ-model and have shown that this model describes an exact CFT. Most importantly, we have discussed the conformal invariance of the T-dual σ-models such that the duals of the H_4 WZW model are conformally invariant up to the first order in α', while the conformal invariance condition for the dual spacetime of the $GL(2,\mathbb{R})$ WZW model has only been satisfied up to zeroth order in α'.

As we have shown, all our models satisfy the vanishing of the one-loop Beta-functions equations. Therefore, each pair of them consists of two canonically equivalent models. Among these models, only (43) and (60) and their dual pairs (53) and (62), respectively, satisfy the equations for two-loop B-functions.

The findings of our study showed that $2+2$-dimensional manifold $M \approx O \times \mathbf{G}$ with two-dimensional real non-Abelian Lie group $\mathbf{G} = A_2$ is wealthy. In addition to PL symmetric backgrounds constructed out in this paper one can obtain other string and gravitational backgrounds from mutually T-dualizable σ- models on manifold $M \approx O \times \mathbf{G}$ with $\mathbf{G} = A_2$ when the dual manifold is $\tilde{M} \approx O \times \tilde{\mathbf{G}}$ with $\tilde{\mathbf{G}} = 2A_1$. In this regard, the following further developments come to mind.

(i) Plane-parallel (pp-)wave: Homogenous plane wave is generally defined by the metric of the following form [32]

$$ds^2 = 2du dv - A_{\mu\nu}(u) X^\mu X^\nu du^2 + dX^2, \tag{110}$$

where dX^2 is the standard metric on Euclidean space E^d and $X \in E^d$. A special case of isotropic homogenous plane wave metric can be chosen by $A_{\mu\nu}(u) = \lambda(u)\delta_{\mu\nu}$. Furthermore, for special choice of $\lambda(u) = k/u^2$, the metric becomes [37]

$$ds^2 = 2du dv - \frac{k}{u^2}(x^2 + y^2) du^2 + dx^2 + dy^2, \tag{111}$$

where k is an arbitrary real constant. The metric (111) does satisfy the conformal invariance conditions equations up to the first order in α', Eqs. (107a)–(107c), with zero field strength. The cosmological constant Λ in this case vanishes and dilaton field is obtained to be [37]

$$\phi = \gamma_0 + \gamma_1 u + 2k \log u, \tag{112}$$

where γ_0 and γ_1 are the constants of integration. Since the field strength H is zero, one can easily consider explicit expressions for the field B in such a way that the terms concerning B-field in action of σ-model contribute to the Lagrangian as the total derivatives, which can be ignored. To obtain the non-Abelian T-dual geometry of pp-wave background in the PL T-duality approach with spectators, we first construct the original σ-model corresponding to the pp-wave metric (111). In this case, a convenient choice of the spectator-dependent matrices may be expressed as

$$E^{+}_{0\,ab} = \begin{pmatrix} -k({y_1}^2 + {y_2}^2) & 1 \\ 1 & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$F^{+^{(1)}}_{\alpha\beta} = 0, \qquad F^{+^{(2)}}_{\alpha b} = 0. \tag{113}$$

Inserting (113) into Eqs. (13)–(16) and noting that $\Pi^{ab}(g)$ is zero, the action of original σ-model (27) yields

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^- [\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + e^{x_1}(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1) - k({y_1}^2 + {y_2}^2)\partial_+ x_1 \partial_- x_1]. \tag{114}$$

Carrying out the coordinates transformation $x_1 \to \ln u$, $x_2 \to v$, $y_1 \to x$, $y_2 \to y$ one arrives at the pp-wave metric (111) from action (114). Thus, inserting (41) and (113) into Eqs. (32)–(35) and then using (31) one can obtain a non-Abelian T-dual σ-model to (114).

(ii) Gödel and Gödel-type metrics: Among the known exact solutions of Einstein field equations gravity, the Gödel and Gödel-type metrics [38] play a special role. It was shown within the usual general relativity that these solutions describe rotating universes, and allow for the existence of closed timelike curves. These metrics are compatible with incoherent matter distribution at rest and can be described by the line element looking like

$$ds^2 = l^2\left[-dt^2 + (\beta - 1)r^2 d\varphi^2 - 2r dt d\varphi + \frac{dr^2}{r^2} + dz^2\right], \tag{115}$$

for some constants l, β. The metric is a direct product of $\mathbb{R}$ associated with the coordinate z and the three-dimensional metric of (t, φ, r). The original Gödel metric [38] is recovered when we take $\beta = 1/2$. In Ref. [39] it has been shown that the Gödel metric can be considered as exact solutions in string theory for the full $\mathcal{O}(\alpha')$ action including both dilaton field ϕ and field strength H. Following Ref. [39] we assume that the dilaton field depends only on the z coordinate, so $\phi(z) = \phi_0 + fz$ for some constants ϕ_0, f. With this assumption and taking the zero field strength, $H = 0$, the field equations (107a) and (107b) are satisfied for the metric (115) in such a way that the inverse string tension α' has to satisfy relation $\alpha' = 4l^2\beta$ only with $\beta = 1$ or $\beta = 3/4$. Finally, the field equation (107c) is satisfied if the following relation is held between the constants f, Λ and l:

$$f^2 = \begin{cases} -\frac{3}{4} + 2\Lambda l^2 & \text{if } \beta = 1 \\ -\frac{2}{3} + 2\Lambda l^2 & \text{if } \beta = \frac{3}{4} \end{cases}.$$

In addition, one can check that the metric (115) for $\beta = 1/2$ (the original Gödel metric) along with the respective B-field and dilaton field

$$B = \frac{l^2}{2} r d\varphi \wedge dt,$$
$$\phi = \phi_0 + fz,$$

satisfy the field equations (107a)–(107c) provided that $\alpha' = -l^2/2$ and $f^2 = 1/2 + 2\Lambda l^2$. Now, by using the above results and by choosing the appropriate spectator-depended background matrices we can construct a σ-model including the Gödel (Gödel-type) metric in the form (115) and the given B-fields. In this way, one can study the non-Abelian T-duality of the Gödel (Gödel-type) metric. We intend to address this problem in the future.

(iii) Ads_4 metric: AdS_4 metric with radius l and constant negative scalar curvature $\Lambda = -12/l^2$ is one of the maximal symmetric four-dimensional spacetimes. A simple form of this metric in coordinates (z, x^+, x^-, ρ) is given by

$$ds^2 = \frac{l^2}{z^2}(dz^2 - dx^+ dx^- + d\rho^2). \tag{116}$$

Solving the field equations (107a)–(107c) for metric (116) one should be so lucky to obtain an appropriate field strength along with a dilaton field. Then he/she can study the non-Abelian T-duality of the AdS_4 metric.

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APPENDIX: THE WZW MODELS BASED ON THE H_4 AND $GL(2,\mathbb{R})$ LIE GROUPS

In this Appendix, we construct the WZW models based on the H_4 and $GL(2,\mathbb{R})$ Lie groups. To define a WZW model, in general, given a Lie algebra with generators T_a and structure constants $f^c{}_{ab}$, one needs a nondegenerate ad-invariant symmetric bilinear form $\Omega_{ab} = \langle T_a, T_b\rangle$ on Lie algebra $\mathcal{G}$ so that it satisfies the following relation [33]

$$f^d{}_{ab}\Omega_{dc} + f^d{}_{ac}\Omega_{db} = 0. \tag{A1}$$

The WZW model based on a Lie group $\mathbf{G}$ is defined on a Riemannian surface Σ as a world sheet by the following action [33]

$$\begin{aligned} I(g) = &\frac{1}{2}\int_\Sigma d\sigma^+ d\sigma^- \Omega_{ab} L^a_+ L^b_- \\ &+ \frac{1}{12}\int_B d^3\sigma \varepsilon^{\gamma\alpha\beta} L^a_\gamma L^b_\alpha L^c_\beta \Omega_{ad} f^d{}_{bc}, \end{aligned}\tag{A2}$$

where B is a three-manifold bounded by world sheet Σ, and the components of the left invariant one-forms L^a_α's are defined via $g^{-1}\partial_\alpha g = L^a_\alpha T_a$ in which $g\colon \Sigma \to \mathbf{G}$ is an element of Lie group $\mathbf{G}$.

1. The H_4 WZW model

Before proceeding to construct the model, let us first introduce the oscillator Lie algebra h_4 of the Lie group H_4. The Lie algebra h_4 is generated by the generators $\{N, A_+, A_-, M\}$ with the following nonzero Lie brackets

$$[N, A_+] = A_+, [N, A_-] = -A_-, \qquad [A_-, A_+] = M. \tag{A3}$$

Using (A1) and (A3) one can simply get a nondegenerate ad-invariant bilinear form Ω_{ab} on h_4 as

$$\Omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{A4}$$

In order to calculate the L^a_α's on the Lie group H_4 we parametrize an element of H_4 as

$$g = e^{mM}\, e^{a_- A_-}\, e^{nN}\, e^{a_+ A_+}, \tag{A5}$$

Finally, the WZW action on the H_4 Lie group is worked out to be of the form [16]

$$\begin{aligned} I(g) = \frac{1}{2}\int d\sigma^+ d\sigma^- [&-\partial_+ n\partial_- m - \partial_+ m\partial_- n \\ &+ e^n(\partial_+ a_+ \partial_- a_- + \partial_+ a_- \partial_- a_+) \\ &+ a_+ e^n(\partial_+ a_- \partial_- n - \partial_+ n \partial_- a_-)]. \end{aligned}\tag{A6}$$

2. The $GL(2,\mathbb{R})$ WZW model

The $gl(2,\mathbb{R})$ Lie algebra is spanned by the generators $\{J_3, J_+, J_-, I\}$ which obey the following commutation rules

$$\begin{aligned} &[J_3, J_+] = 2J_+, \qquad [J_3, J_-] = -2J_-, \\ &[J_+, J_-] = J_3, \qquad [I, .] = 0. \end{aligned}\tag{A7}$$

where I is the central generator. We notice that $gl(2,\mathbb{R}) = sl(2,\mathbb{R}) \oplus u(1)$. Using (A7), a nondegenerate solution to (A1) is obtained to be of the form

$$\Omega_{ab} = \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \tag{A8}$$

for some nonzero constants a, b. In order to construct the WZW model based on the $GL(2,\mathbb{R})$ Lie group we parameterize the $GL(2,\mathbb{R})$ with coordinates $\{\theta_3, \theta_+, \theta_-, \theta\}$ so that its elements can be written as

$$g = e^{\theta_+ J_+} e^{\theta_3 J_3} e^{\theta_- J_-} e^{\theta I}. \tag{A9}$$

Using (A9), we then obtain

$$\begin{aligned} L^{J_3}_\pm &= \theta_- e^{-2\theta_3}\partial_\pm\theta_+ + \partial_\pm\theta_3, \\ L^{J_+}_\pm &= e^{-2\theta_3}\partial_\pm\theta_+, \\ L^{J_-}_\pm &= -\theta_-^2 e^{-2\theta_3}\partial_\pm\theta_+ - 2\theta_-\partial_\pm\theta_3 + \partial_\pm\theta_-, \\ L^I_\pm &= \partial_\pm\theta. \end{aligned}\tag{A10}$$

Finally, the $GL(2,\mathbb{R})$ WZW action looks like

$$\begin{aligned} I(g) = \frac{1}{2}\int d\sigma^+ d\sigma^- [&b\partial_+\theta\partial_-\theta \\ &+ 2a(\partial_+\theta_3\partial_-\theta_3 + e^{-2\theta_3}\partial_+\theta_-\partial_-\theta_+)]. \end{aligned}\tag{A11}$$

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