Finite energy but infinite entropy production from moving mirrors

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Accelerating mirrors provide a simple conceptual laboratory for studying particle production and the relation between trajectory and particle, energy, and entropy fluxes. We focus on the relation between energy and entropy, studying some special cases with finite total energy but infinite integrated entropy (though the entropy flux may be finite at any particular moment). We present a new asymptotically static moving mirror trajectory with solvable beta Bogolyubov coefficients, total energy, and fully relativistic particle count. The integrated entropy diverges despite finite global radiative particle and energy emission. By comparing closely related trajectories, we point out some general principles (e.g., the asymptotic time dependence of energy flux and entropy flux for different convergence and divergence behaviors) but also how subtle distinctions can affect the physics and its relation to black hole end states. Another class of models includes exponentially accelerated mirrors in proper time; one of its unexpected behaviors is finite energy emission but divergent entropy. We compare mirrors exponentially accelerated in other coordinates as well, showing their close relation and an interesting duality property.

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I. INTRODUCTION

Particle production from vacua in spacetime, e.g., [1–3], is a fascinating aspect of quantum field theory, connecting dynamics, energy flux, and information. One of the simplest systems for investigating these concepts is the accelerating mirror in $1+1$-dimensional spacetime with scalar particle production [4–6]. Since few analytic solutions are known, new ones can give useful insights into the relations between these quantities.

Of particular interest is entropy and its connection to information [7–11]. Accelerating mirrors can generate analog black hole solutions, e.g., [12–15], allowing exploration of the formation, emission, and perseverance or decay of black holes, and the associated information content of the particles produced. We focus here on the special situations where not only energy flux, at a given moment, but the total energy emitted is finite, yet the integrated entropy is infinite (and even the entropy flux may diverge).

This has a twofold purpose. First, few solutions with finite total energy, and fewer still with analytically calculable energy and total finite particle production, are known [16–18], so new solutions can help reveal the similarities and differences between their innate properties. Second, examining the relation between the energy and particle flux [19], and their integrated quantities, and the entropy flux and integrated entropy, at the level of major disparity such as one being finite and the other infinite, offers a "stress test" to simple assumptions about their connection. Indeed, interesting recent work has revealed that "information"[1] need not be associated with any energy transport [20]. Ideally, these steps can eventually provide some further clarity on the fundamental relation between particle production and information.

In Sec. II, we present a new analytically solvable, asymptotically static mirror with interesting physical properties, and study its particle, energy, and entropy production. We contrast this with finite radiation, asymptotically null and drifting mirrors based on exponential acceleration in Sec. III. Section IV summarizes the diversity of behaviors in entropy despite the similarity of characteristics in energy flux or total energy. We discuss some future prospects and conclude in Sec. V. We use units $c = \hbar = 1$.

II. FINITE RADIATION, ASYMPTOTICALLY STATIC SOLUTION

Asymptotically static mirrors are of particular interest because they should have finite total energy production.[2] However, this class is somewhat difficult to explore because the literature has only three solved[3] asymptotic

[1]In the form of correlations with thermal emissions.

[2]We restrict interest to trajectories that are not oscillating with infinitely increasing frequency of oscillation.

[3]Here, "solved" means only those trajectories where the beta Bogolyubov coefficients are analytically known, allowing calculation of particle flux and total number of particles.

static mirrors: the Walker-Davies (1982) [16], Arctx (2013) [17], and the self-dual solution (2017) [18].

Here, we present a new solution that is also asymptotically static and with finite radiation, but simpler, more tractable, and more general in some respects than the first two previously known solutions. It also has significantly different, time-asymmetric dynamics than the aforementioned recent third solution.

Note there is, as yet, no known exactly one-to-one analytically demonstrated correspondence[4] to black hole particle production for asymptotically static trajectories which solve the soft particle production problem (e.g., [22]) and represent complete evaporation[5] with no left-over remnant [23], so it is worthwhile exploring such cases further.

A. New trajectory: betaK

Asymptotically static mirrors have useful physical properties so it is worthwhile attempting new solutions [18]. Notably, they have total finite particle emission, avoiding the soft particle [22] production problem. We have found a new solution we call betaK, due to its exactly solvable beta Bogolyubov coefficients that take the form of a modified Bessel function. In addition, it has some other advantages over the previous studied motions. The betaK trajectory is given by

$$z(t) = -\frac{v}{\kappa}\sinh^{-1}(\kappa t) = -\frac{v}{\kappa}\ln\left(\sqrt{\kappa^2 t^2 + 1} + \kappa t\right) \quad (1)$$

$$\dot{z} = \frac{-v}{\sqrt{\kappa^2 t^2 + 1}}, \quad (2)$$

where z is the spatial coordinate, t the time coordinate, κ a scaling parameter that in the black hole case would be related to the surface gravity[6] $\kappa \equiv (4M)^{-1}$, and v is the maximum velocity of the mirror, occurring at $t = 0$. One can readily see that at asymptotically large times, past and future, the mirror becomes asymptotically static, $\dot{z} \to 0$. Moreover the velocity is time symmetric (and the trajectory is time asymmetric).

The advantages include the ability to numerically solve for $N(v)$ particle count, in contrast to Arctx's non-functional particle count [17]. In addition, $z(t)$ is manifestly invertible, in contrast to the Walker-Davies mirror trajectory [16], $t(z)$, which is not transcendentally invertible for $z(t)$. Moreover, this trajectory is found to be much more numerically tractable for all its interesting quantities than either Walker-Davies or Arctx.

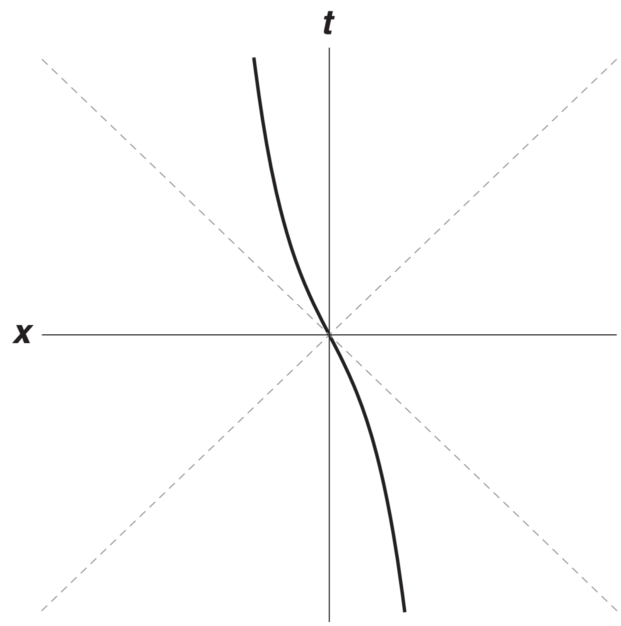

FIG. 1. The trajectory for betaK is asymmetric in time, and asymptotically static with finite energy and finite particle count. Here the maximum speed $v = 1/2$ and $\kappa = 1$.

1. Trajectory

The spacetime diagrams for the trajectory Eq. (1) are illustrated in Fig. 1 with a standard spacetime diagram and Fig. 2 with a conformal or Penrose diagram. The symmetries and asymptotically static character are reasonably evident in both (e.g., the mirror approaches time-like future infinity, i^+, along the vertical axis).

This immediately classifies the dynamics of this solution with those "future and past asymptotically static trajectories" (see Refs. [16–18]). This motion is distinct from the typically infinite energy producing, late-time thermal trajectories of the "asymptotically null" solutions (see the black mirror [12–15] or the thermal mirror [24–26]). Moreover, it exhibits regularizing behavior distinct even from the asymptotically inertial, soft particle producing trajectories of asymptotically drifting solutions (see e.g., Refs. [27–30]).

2. Energy flux

The energy flux of particles produced[7] is related to the trajectory by [6] (see also Ref. [17]),

$$F(t) = -\frac{1}{12\pi}\frac{\dddot{z}(1-\dot{z}^2) + 3\dot{z}\ddot{z}^2}{(1-\dot{z})^4(1+\dot{z})^2}. \quad (3)$$

[4]The correspondence exists for an asymptotically null trajectory [12–15]. Explicit derivations of the collapsing shell stress tensor in different vacuums can be found in [21].

[5]Complete evaporation is evident from the form of the late-time field modes which evolve into their early-time form, indicating no redshifting influence or soft-particle producing remnant.

[6]The parameter κ is not the acceleration of the moving mirror, as it is for the uniformly accelerated observer in the thermal Unruh effect [3]. The thermal moving mirror [24] has proper acceleration $\alpha(\tau) = \tau^{-1}$, independent of κ-scale [25].

[7]Notice Eq. 4.1 in [6] is not normalized by a factor of 4π.

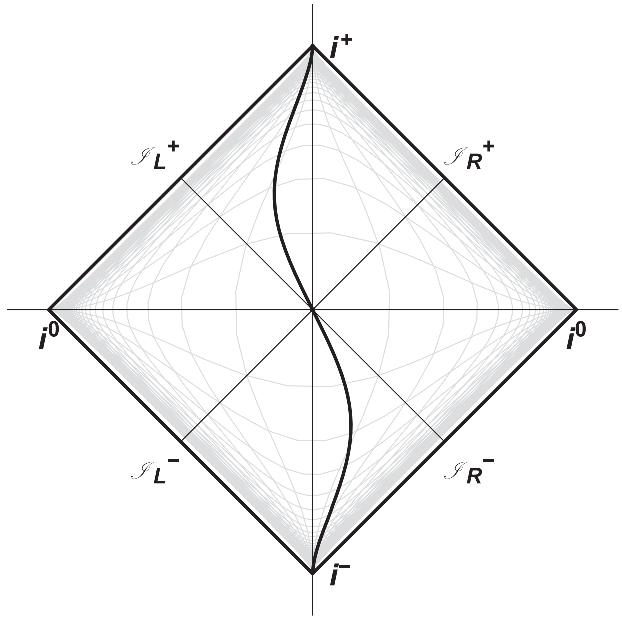

FIG. 2. The trajectory for betaK, as in Fig. 1, but plotted in a conformal diagram.

Computation of the flux emitted to the right of the mirror (by convention) from the stress tensor observed by measurement at future-null infinity, $\mathcal{I}_R^+$, is straightforward,

$$F(t) = \frac{\kappa^2}{12\pi} \frac{v\sqrt{\kappa^2 t^2 + 1}(2\kappa^2 t^2 + v^2 - 1)}{(v - \sqrt{\kappa^2 t^2 + 1})^2 (v + \sqrt{\kappa^2 t^2 + 1})^4}. \quad (4)$$

The apparent divergence at $\kappa^2 t^2 = v^2 - 1$ is avoided by the restriction $0 < v < 1$ and real time. The flux has a central valley[8] of negative energy flux (NEF). As a well-known [31] feature of the moving mirror model, the appearance of NEF here is of no particular surprise.[9] The flux does have positive energy emission approaching the $t \to \pm\infty$ asymptotes, with total positive energy as seen in the next subsection.

3. Total energy

Due to the asymptotically static character, the total energy is finite. The total energy emitted from the right side of the accelerating mirror is analytically calculable,

$$E_R = \frac{\kappa}{96\pi} \frac{\gamma^2}{v^2} [\gamma(6 - 8v^2)\sin^{-1} v + \pi\gamma v^4 + 4v^3 - 6v], \quad (5)$$

where $\gamma \equiv (1 - v^2)^{-1/2}$ is the Lorentz factor (again, v is the maximum speed of the mirror). Accounting for both sides, the total energy $E_T = E_R + E_L$ takes the remarkably simple expression

$$E_T = \frac{\kappa\gamma^3 v^2}{48}, \quad (6)$$

demonstrating immediately three physical results: (1) zero maximum speed gives zero energy (no particle production), (2) the total energy is positive, and (3) an arbitrarily fast mirror, $v \to 1$, gives divergence of energy production due to the Lorentz factor.

4. Particle flux

Quite unusually, the beta Bogolyubov coefficients describing particle emission can be solved for, with the fairly simple result,

$$\beta_R(\omega, \omega') = -\frac{2v\sqrt{\omega\omega'}}{\pi\omega_p} e^{-\frac{\pi}{2} v\omega_n} K_{iv\omega_n}(\omega_p), \quad (7)$$

where $K_n(z)$ is a modified Bessel function of the second kind, $\omega_p \equiv \omega + \omega'$ and $\omega_n \equiv \omega - \omega'$, where ω' and ω are the in- and outgoing mode frequencies, respectively, and $\kappa = 1$ for convenience. See Appendix C for detail in the derivation.

The particle spectrum per mode per mode (modulus squared) is

$$|\beta_R|^2 = \frac{4v^2\omega\omega'}{\pi^2\omega_p^2} e^{-\pi v\omega_n} |K_{iv\omega_n}(\omega_p)|^2, \quad (8)$$

and accounting for both sides, $|\beta_T|^2 = |\beta_R|^2 + |\beta_L|^2$, gives

$$|\beta_T|^2 = \frac{8v^2\omega\omega'}{\pi^2\omega_p^2} \cosh(\pi v\omega_n) |K_{iv\omega_n}(\omega_p)|^2. \quad (9)$$

As a crosscheck, the total energy can be retrieved using the particles through globally summing quanta,

$$E_T = \int_0^\infty \int_0^\infty \omega \cdot |\beta_T|^2 d\omega d\omega', \quad (10)$$

which gives, reinstating the scale κ,

$$E_T = \frac{\kappa\gamma^3 v^2}{48}, \quad (11)$$

demonstrating consistency of the solution with Eq. (6).

5. Particle spectrum

The spectrum, N_ω, or particle count per mode, detected at $\mathcal{I}_R^+$ is found by inserting Eq. (8) into

[8]The central valley of NEF is in contrast to many asymptotic drifting mirrors which have off-set late-time "death gasps" of NEF; see e.g., Ref. [10].

[9]Of course, NEF is not a universal occurrence in the moving mirror model, even for finite energy emission mirrors, as will be explicitly demonstrated by Eq. (24) which has finite energy emission but no negative energy flux at all.

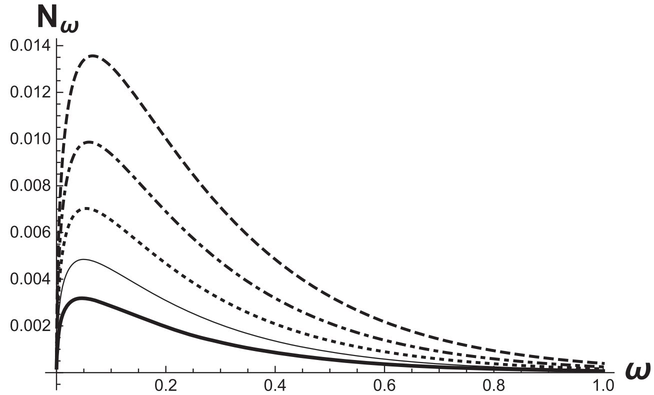

FIG. 3. The spectrum, N_ω, or particle count per mode detected at the right Cauchy surface, of the asymptotically static time-asymmetric mirror betaK, is plotted vs frequency in units of κ. The different lines correspond to different maximum speed trajectories: $v = 0.5, 0.6, 0.7, 0.8, 0.9$ from bottom to top, for thick solid, thin solid, dotted, dot-dashed, and dashed, respectively. This asymptotically static solution has no infrared divergence (soft particles) suffered by mirrors that are asymptotically drifting.

$$N_\omega = \int_0^\infty |\beta_R|^2 d\omega'. \tag{12}$$

Figure 3 illustrates the results for different maximum mirror speeds.

6. Total particles

The particle count $N(v)$ from the mirror with maximum speed v comes from summing over the spectrum,

$$N(v) = \int_0^\infty \int_0^\infty |\beta_T|^2 d\omega d\omega'. \tag{13}$$

This total particle count is finite. While it is unusual for moving mirror solutions to have finite particle count, our solution is not only finite, but numerically tractable for any choice of maximum speed, $0 \le v < 1$. In this case, there are no soft particles; i.e., all the massless scalar particles are "hard." The total number of particles from accelerating mirrors is commonly infinite, due to soft particles, even when the total energy is finite (even for asymptotically inertial-drifting mirrors). Figure 4 shows the behavior of $N(v)$, Eq. (13). The number is small, $N(v) < 1$ (recall particle number is dimensionless and so the result is independent of the dimensional scale-parameter κ), and increases monotonically with the chosen maximum speed v.

7. Entropy flux

It is noteworthy that emission of von Neumann entanglement entropy does not change sign for this solution, Eq. (1). The entropy flux (see e.g., Refs. [8,10,27,32] and references therein),

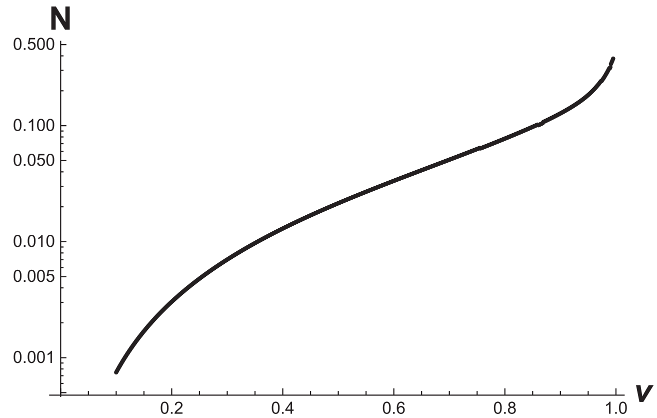

FIG. 4. The time-asymmetric mirror betaK, being asymptotically static, has finite particle count. The total particle emission count is plotted for trajectories with relativistic maximum speeds v.

$$S(t) = -\frac{1}{6}\tanh^{-1}\dot{z}(t), \tag{14}$$

radiated to the right (left) is always positive (negative), with

$$S(t)_{R,L} = \pm\frac{1}{6}\tanh^{-1}\left(\frac{v}{\sqrt{\kappa^2 t^2 + 1}}\right). \tag{15}$$

Figure 5 illustrates this entropy flux. Note that in the far past and future, where $\kappa|t| \gg 1$, the entropy flux $S(t) \sim 1/|t|$. This property will be important for the next subsection, and the general comparison of energy characteristics to entropy characteristics.

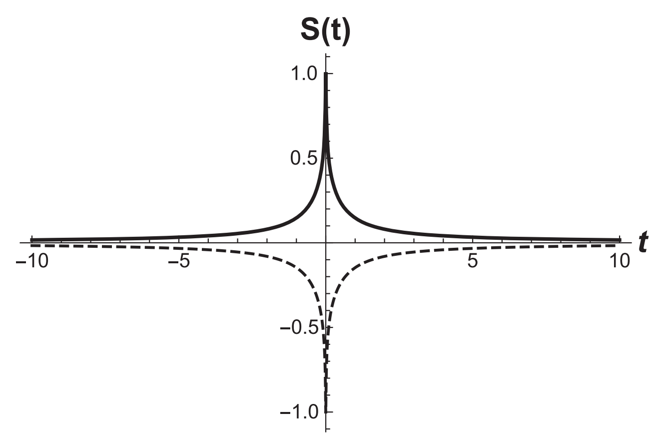

FIG. 5. The time-asymmetric mirror betaK from Eq. (1), being asymptotically static with finite energy and finite particle count, might be expected to have finite integrated entropy, however this is not the case. The entropy flux is plotted as the solid line (dashed line) and is always positive (negative) as emitted from the right (left) side of the mirror. Here the flux $S(t)$ from Eq. (15) is shown for the choice of maximum speed $v = 0.99999$ and $\kappa = 1$. See the text for discussion of the integrated entropy integral and its divergence.

8. Integrated entropy

Despite the finite energy, and finite particle emission, and the preservation of unitarity, the integrated amount of entropy diverges. This is given by (integrating over u at right future null infinity, $\mathcal{I}_R^+$),

$$S_I = \int_{-\infty}^{\infty} S(t)(1-\dot{z})dt, \tag{16}$$

and since $\dot{z} \to 0$ at large times, we see the divergence arises from $S(t)$ itself.

We can understand the divergence mathematically by noting that, in the previous subsection, we saw that the entropy flux only dies as $1/t$ for large $|t|$ and, hence, the integrated entropy has a logarithmic divergence. More generally, for an asymptotically static mirror (where $\dot{z} \to 0$), if $\dot{z} \sim t^{-n}$ at late times with $n > 0$, then the proper acceleration $\alpha \equiv \gamma^3 \ddot{z} \sim t^{-n-1}$ and the rapidity $\eta = \tanh^{-1} \dot{z}$ and entropy flux $S(t)$ have late-time contributions going as t^{-n}. Note that unitarity is preserved for $n > 0$. Then the integrated entropy $S_I \sim t^{-n+1}$ and, hence, diverges if $n < 1$ (for $n = 1$, S_I diverges logarithmically—this is precisely the betaK behavior). We can also calculate that the energy flux will die off as t^{-n-2} and so the total energy gets a late-time contribution going as t^{-n-1}. Thus betaK (which has $n = 1$) represents the "boundary" case between the mirror giving both finite energy and finite integrated entropy and the one producing finite energy but infinite integrated entropy.

The physical interpretation of this is less clear. The mirror has a finite particle count and energy. It may be that it has an infinite number of particle states with infinitesimal mean occupation, so N is finite but since S_I counts the number of states it is divergent.

B. Comparison to related finite energy mirrors

We identified above the asymptotic behavior of $\dot{z} \sim t^{-1}$ as a key ingredient for finite energy but infinite integrated entropy. Let us explore this further by comparing the betaK case to two other examples with the same approach to the asymptotic static limit.

One is the self-dual solution of [18]. This has

$$\dot{z} = \frac{2v\kappa t}{\kappa^2 t^2 + 1}, \tag{17}$$

which indeed has the same behaviors of $\dot{z} \sim t^{-1}$, $\alpha \sim t^{-2}$, $F \sim t^{-3}$, and $S(t) \sim t^{-1}$. However, because $\dot{z}$ and hence η and $S(t)$ are odd in time, the logarithmic divergence in the integral for the total entropy cancels and the total integrated entropy was found in [18] to be finite. So overall time asymmetry vs symmetry of the particular *a priori* chosen dynamics play an important role, of course, since a generic chosen function is neither even nor odd. The dynamic, Eq. (1), with solvable betas Eq. (7), being time-asymmetric, compliments the time-symmetric dynamic solved in [18].

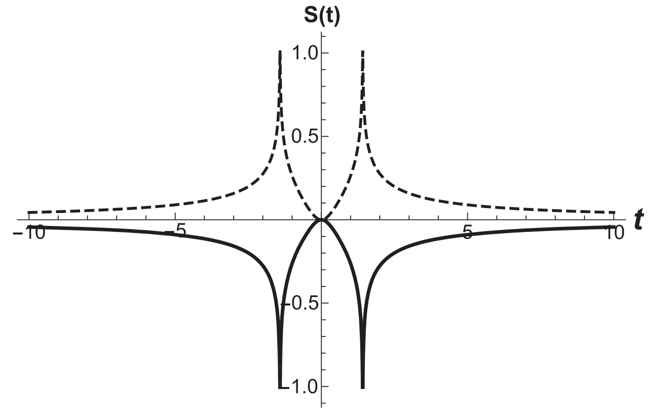

FIG. 6. The entropy for the Event^{-1} mirror with trajectory Eq. (18), with $\kappa = 1$. The trajectory is asymptotically static with finite energy but divergent integrated entropy, similar to betaK. The solid (dashed) line is the entropy to the right (left).

For direct comparison to betaK we therefore we need to study an even function $\dot{z}$ that still asymptotes as $\dot{z} \sim |t|^{-1}$. The simplest instance of this after betaK is a model we call Even t^{-1},

$$\dot{z} = v\sqrt{\frac{27}{4}} \frac{\kappa^2 t^2}{(\kappa^2 t^2 + 1)^{3/2}} \tag{18}$$

$$z = \frac{v}{\kappa}\sqrt{\frac{27}{4}}\left[\frac{-\kappa t}{\sqrt{\kappa^2 t^2 + 1}} + \ln\left(\kappa t + \sqrt{\kappa^2 t^2 + 1}\right)\right]. \tag{19}$$

This has maximum velocity v at $\kappa^2 t^2 = 2$, and is asymptotically static with $\dot{z} \sim t^{-1}$. Again, the total energy is finite, but since $\dot{z}$ and, hence, $S(t)$ are even, the integrated entropy is again infinite.

Figure 6 shows the entropy flux as a function of time, which can be compared to Fig. 5. Both die off as $1/t$ for large times. While betaK has its maximum velocity for $t = 0$, and hence a spike in entropy flux there, the Event^{-1} model has maximum velocity and entropy spikes at $\kappa^2 t^2 = 2$.

III. FINITE RADIATION, ASYMPTOTICALLY NULL AND DRIFTING DYNAMICS

As a counterpoint to the previous section on asymptotically static mirrors, their finite energy, asymptotically vanishing entropy flux, and infinite integrated entropy, we consider an asymptotically null and then a drifting mirror.

Asymptotically null mirrors have no guarantee of finite energy production but we develop a new solution that does, and has interesting thermodynamic properties, allowing us to study further the relation between energy and entropy.

The solution employs exponential acceleration in proper time and can be viewed as a new trajectory in the series:

(i) $\alpha(u) \sim e^u$ for Carlitz-Willey [24–26]
(ii) $\alpha(t) \sim e^t$ for Hotta-Shino-Yoshimura [17,25,33]
(iii) $\alpha(x) \sim e^x$ for Davies-Fulling [6,25].

In this section, we investigate an exponentially accelerated trajectory for the mirror in proper time,

$$\alpha(\tau) = -\kappa e^{\kappa\tau}, \tag{20}$$

where the negative sign is by convention to send the mirror accelerating to the left.

Based on the infinite total energy results of the aforementioned exponentially asymptotically accelerated null mirrors, one might expect an infinite total energy for trajectory Eq. (20). A system which feels an ever increasing acceleration might likewise produce an ever increasing energy. However, this is not the case with this equation of motion, Eq. (20).

A better intuition is found by considering that $\alpha(\tau) \sim e^\tau$ is increasingly gentle relative to $\alpha(t) \sim e^t$, due to time dilation. Therefore, compared to the HSY trajectory, the exponential accelerated proper time trajectory will have diminished energy flux. Of course, the HSY trajectory has infinite total energy production overall, but $\alpha(t) = -\kappa e^{\kappa t}/2 \leftrightarrow \alpha(\tau) = \kappa \mathrm{csch} \kappa\tau$, which has a finite proper time divergence ($\tau \to 0$). Since there is no finite proper time divergence in Eq. (20), one might plausibly anticipate that the energy flux, will also not diverge. However, when it comes to the total energy we shall see that a finite value depends on just how sufficiently diminished the flux dies off and not the presence of an asymptotic divergence in proper time.

A. Trajectory dynamics

The dynamical trajectory functions are

$$\eta(\tau) = -e^{\kappa\tau} \qquad \gamma(\tau) = \cosh(e^{\kappa\tau}) \tag{21}$$

$$w(\tau) = -\sinh(e^{\kappa\tau}) \qquad \mathrm{v}(\tau) = -\tanh(e^{\kappa\tau}) \tag{22}$$

$$z(\tau) = -\frac{1}{\kappa}\mathrm{Shi}(e^{\kappa\tau}) \qquad t(\tau) = \frac{1}{\kappa}\mathrm{Chi}(e^{\kappa\tau}), \tag{23}$$

where the rapidity, η, can be found by a proper time derivative of the acceleration, $\eta'(\tau) = \alpha(\tau)$. Elsewhere, the Lorentz factor is $\gamma = \cosh\eta$, celerity (proper velocity) $w = dz/d\tau = \sinh\eta$, velocity $v = dz/dt = \tanh\eta$, and Shi (Chi) is the hyperbolic sine (cosine) integral. We plot the trajectory $z(t)$ function in Fig. 7. Note that for large t (or τ), $z \sim t$. The mirror velocity asymptotically approaches the speed of light, as to be expected.

B. Energy flux

To calculate the energy flux produced by this exponentially accelerating mirror, we use the energy flux relation $12\pi F(\tau) = -\alpha'(\tau)e^{2\eta(\tau)}$ [25] to find

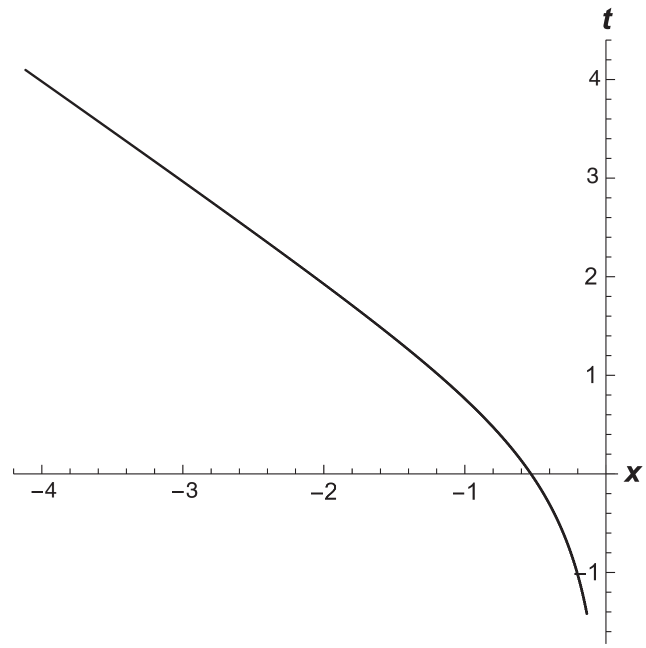

FIG. 7. The exponential accelerating trajectory $z(\tau)$ from Eq. (23) is plotted in a coordinate time t spacetime diagram, with $\kappa = 1$.

$$F(\tau) = \frac{\kappa^2}{12\pi} e^{\kappa\tau - 2e^{\kappa\tau}}. \tag{24}$$

The energy flux is plotted in Fig. 8. Note the emission is always positive—there is no negative energy flux (NEF). This is a particularly interesting case because unlike the no NEF solutions of Carlitz-Willey [24], the black mirror [12], or Hotta-Shino-Yoshimura (HSY)[10] [17,25,33,34], for example, in the far past and future the energy flux asymptotes to zero, despite $\alpha(\tau) \sim e^\tau$. This indicates that the radiation process completely terminates (as far as energy evaporation is concerned); for a black hole analog this would correspond to evaporation with an asymptotically infinite Doppler-shifting remnant (a 'super-remnant', if you will) consistent with the conservation of energy without backreaction [35].

C. Total energy

Recall our criteria from Sec. II A 8 for finite entropy flux and integrated entropy. For the asymptotically static mirror we wanted $\dot{z}$ to die off quicker than $1/t$, giving the acceleration dying quicker than $1/t^2$ and the flux dying quicker than $1/t^3$ in order to get both finite total energy and integrated entropy. Here, however, we have $\dot{z}$ going to a constant (the speed of light), acceleration exponentially increasing, but energy flux dying off rapidly.

[10] Exponential acceleration in coordinate time, the HSY trajectory, is also referred to as the Arcx trajectory; see, for instance, [17,25].

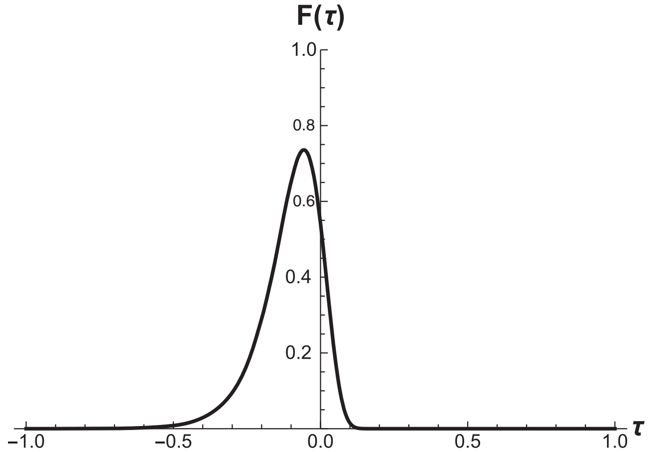

FIG. 8. The energy flux to the right of the exponential accelerating mirror contains no negative energy flux and demonstrates terminal evaporation. Here $\kappa = \sqrt{48\pi}$, normalized so that thermal energy flux would be $F = 1$.

To continue the investigation, we calculate the total energy for an observer at $\mathcal{I}_R^+$,

$$E = \int F(\tau) du, \tag{25}$$

where $u = t - z$ is the null coordinate. This integral can be done and has a simple form. Since $du/d\tau = \cosh\eta - \sinh\eta = e^{-\eta} = e^{e^{\kappa t}}$ then

$$E = \frac{\kappa}{12\pi}\int_{-\infty}^{\infty} \kappa e^{\kappa\tau - 2e^{\kappa\tau}} e^{e^{\kappa\tau}} d\tau. \tag{26}$$

The result of the integral is unity, and so the total energy emitted to the observer at $\mathcal{I}_R^+$ is finite, with

$$E = \frac{\kappa}{12\pi}. \tag{27}$$

Thus, the condition for finite energy seems to depend only the energy flux dying away sufficiently quickly, and not on the asymptotic behavior of individual trajectory dynamics quantities such as $\dot{z}$ or acceleration *per se* (though in combinations they do determine the flux).

This result from the exponentially accelerating mirror has a drifting mirror counterpart [36], where the acceleration asymptotically approaches zero in the far future and the mirror can coast at the speed of light. There $E = \kappa/(96\pi)$ found in [17].

The new trajectory with exponential acceleration in proper time, Eq. (20), is the only one of the exponential forms mentioned to possess finite energy. It is surprising, without yet considering the entropy, that despite an ever increasing asymptotically infinite acceleration the system radiates a finite total energy.

D. Entropy flux and integrated entropy

The entropy flux, $S(\tau)$, is found from the rapidity $\eta(\tau) = -6S(\tau)$ [18,27], so that

$$S(\tau) = \frac{1}{6} e^{\kappa\tau}. \tag{28}$$

It clearly diverges at late times, in stark contrast with the rapidly vanishing energy flux, Eq. (24). The integrated entropy is not saved by integration over u at $\mathcal{I}_R^+$, as the integral

$$S_I = \frac{1}{6}\int_{-\infty}^{\infty} e^{\kappa\tau} e^{e^{\kappa\tau}} d\tau \tag{29}$$

also diverges. This demonstrates a loosening between the information content and energy content carried by the radiation. Despite the finite energy production, unitarity is lost because the entropy flux $S(\tau)$ does not asymptote to a constant, but diverges as $\tau \to +\infty$.

Note that the entropy and the proper acceleration $\alpha = \kappa\eta$ simply scale together for the new exponential mirror, with

$$\alpha = -6\kappa S. \tag{30}$$

This is in contrast to the other exponential forms: Carlitz-Willey, Hotta-Shino-Yoshimura, and Davies-Fulling, respectively, have $\eta = -\kappa u/2$, $\eta = \kappa x$, and $\eta = -\kappa t$, so the entropy involves inverse hyperbolic trig functions of the acceleration.

We comment that the regime of applicability of $\eta(\tau) = -6S(\tau)$ is reliant on the assumption that $p'(u) \to 1$ for $u \to -\infty$ as used [32]. In terms of the trajectory, $z(t)$, this is the requirement that the mirror starts asymptotically static in the far past (not the far future). The trajectory, Eq. (20) is asymptotically static in the far past.

Furthermore, it is sufficient but not necessary that finite energy will result if $S'(u) \to 0$ as $u \to \pm\infty$ (see Ref. [32]). Interestingly, our exponentially accelerated trajectory in proper time, Eq. (20), is a case where the energy is finite, even though $S'(u) \to \infty$ as $u \to +\infty$.

E. Exponential in τ multiplicatively shifted

We can use the technique of multiplicatively shifting the mirror trajectory, i.e., $\dot{z} \to v\dot{z}$, to regularize the infinite asymptotic acceleration [18]. This takes the asymptotically null mirror to an asymptotically drifting one. As we just saw, if $\dot{z} = dz/dt = -\tanh(e^{\kappa\tau})$ then $\alpha(\tau) = -\kappa e^{\kappa\tau}$. This gave zero flux at late times but infinite entropy. But if we multiplicatively shift to

$$\dot{z} = v \tanh(e^{\kappa\tau}), \tag{31}$$

which has asymptotically constant velocity less than the speed of light, then the proper acceleration

TABLE I. The energy and entropy properties are summarized for the models discussed in this article. The flux behaviors listed are those in the asymptotic future. All these models have finite total energy but differing entropy behaviors. Note the self-dual solution avoids infinite total entropy through its self-dual nature (symmetry in time).

Model	Energy flux	Total energy	Entropy flux	Integrated entropy
betaK [Eq. (1)]	$\sim t^{-3}$	$\kappa v^2\gamma^2/48$	$\sim t^{-1}$	Log divergent
Self-dual [18]	$\sim t^{-3}$	$\kappa v^2\gamma(\gamma^2+3)/48$	$\sim t^{-1}$	Finite
Event^{-1} [Eq. (18)]	$\sim t^{-3}$	Finite	$\sim t^{-1}$	Log divergent
Exptau [Eq. (20)]	$\to 0$	$\kappa/(12\pi)$	Diverges	Infinite
Exptau(v) [Eq. (31)]	$\to 0$	Finite	Finite	Infinite

$$\alpha(\tau) = \frac{v\kappa e^{\kappa\tau}}{\cosh^2(e^{\kappa\tau}) - v^2\sinh^2(e^{\kappa\tau})}, \tag{32}$$

which for any $v < 1$ goes to zero for large τ. So at large times, energy flux and acceleration goes to 0, and the rapidity η and entropy flux $S(\tau)$ are finite, while the integrated entropy still diverges.

As a broad principle relevant to the several cases discussed, for entropy there is a straightforward relation to the velocity $\dot{z}$, along the lines of the criterion in Sec. II A 8. Recall $S(\tau) = -(1/6)\eta = -(1/6)\tanh^{-1}\dot{z}$. When asymptotically $\dot{z} \to 0$ then the entropy flux goes to zero, and if this proceeds quickly enough then the integrated entropy stays finite. For the exponentially accelerating mirror cases, $\dot{z} \to \text{const}$ and so integrated entropy is infinite. In the drifting mirror subcase (i.e., exponential acceleration regularized to approach zero), with $v < 1$, $S(\tau)$ stays finite while for the nonregularized $v = 1$ case above we have $S(\tau) \sim \tanh^{-1}(1) \to \infty$.

IV. SUMMARY OF ENTROPY RESULTS

In Table I, we summarize trajectories considered in this paper, showing how there can be a diversity of behaviors in entropy even with the same characteristics in energy flux or total energy.

The first three mirrors are closely related in their properties, showing the "boundary" case of energy flux dying off as t^{-3} and entropy flux diminishing as t^{-1}. This leads to a logarithmic divergence in integrated entropy—except for the self-dual mirror which is saved by its time symmetry (i.e., self-dual nature). If the flux fades more rapidly then the integrated entropy would be finite. The betaK and Event^{-1} mirrors are new, with the betaK case of particular interest due to its solvable beta Bogolyubov coefficients and tractable particle production characteristics.

The last two mirrors add to the exponential mirror family (which is completed in Appendix A), with Exptau being a new solution on a par with well known mirrors—with the added attractions of having no negative energy flux, flux asymptoting to zero, and a particularly simple linear relation between proper acceleration and entropy. The Exptau(v) case is the drifting mirror sibling that regularizes the acceleration from infinity at large times to zero and keeps the entropy flux finite.

V. CONCLUSIONS

Particle production from accelerating mirrors by itself is a fascinating physics phenomenon, but its relation to entropy and information brings unexpected depths to the study of moving mirrors. We presented four new trajectories, comparing and contrasting their particle production, energy flux, entropy flux, and integrated entropy characteristics.

Looking for a time-asymmetric finite particle creation solution, we found the betaK mirror which is only the fourth solved asymptotically static mirror, and has beta Bogolyubov coefficients of the form of a modified Bessel function. It has a simple expression for its finite total energy, and calculable finite total particle count, but infinite integrated entropy. This raises interesting questions regarding the exact relation between particle and energy production and information. A close relative is the Event^{-1} mirror, slightly more complex and with different patterns of entropy flux though the same asymptotic behavior. We also compared these to the self-dual mirror introduced in [18], which again has the same asymptotic energy and entropy flux behaviors but a finite integrated entropy due to its time symmetry.

We presented general guidelines to the asymptotic behaviors in velocity, proper acceleration, energy flux, and entropy flux; in particular we identified a "boundary" behavior where when the velocity asymptotically vanishes more rapidly than t^{-1}, and hence the other three quantities asymptotically vanish more rapidly than t^{-2}, t^{-3}, and t^{-1}, respectively, the integrated entropy would remain finite.

Moving from asymptotically static to asymptotically null and drifting mirrors, we studied Exptau, a new mirror in the exponential acceleration family (that includes the Davies-Fulling and Carlitz-Willey mirrors), this one exponential in proper time. It has no negative energy flux at any time, and the flux rapidly vanishes asymptotically, analogous to concluded evaporation (energy emission ends) of a black hole. Interestingly, the entropy is directly proportional to the acceleration, and becomes infinite. This seems to imply a disconnect, in this case at least and asymptotically, between information (presumably related to entropy) and the state of the black hole (which has evaporated). We also introduced a regularized variant, Exptau(v), that asymptotically drifts at less than the speed of light and has vanishing asymptotic acceleration. Its entropy flux remains finite, though its integrated entropy diverges. In the Appendix we also completed the exponential family by investigating acceleration in advanced time v, and identifying interesting "duality"-like relations.

Considering future directions, as we have seen from investigating proper time exponential acceleration in Eq. (20), it could be useful to work with proper time in more general contexts, such as for the energy-entropy flux relations, which are easy to express in terms of both null time and proper time. In terms of null time u, $\eta(u) = -6S(u)$ and we can write Eq. (3) as

$$F(u) = \frac{1}{2\pi}[6S'(u)^2 + S''(u)], \tag{33}$$

and in terms of proper time we can use the relation $\eta(\tau) = -6S(\tau)$ to write

$$F(\tau) = \frac{1}{2\pi}S''(\tau)e^{-12S(\tau)}. \tag{34}$$

This result demonstrates a direct relationship between negative energy flux and entanglement entropy: It is the sign of $S''(\tau)$ that determines the emission of NEF.

The possible concavity of the entropy found here (see Ref. [7] for a relation in terms of correlations) indicates the connection to the locally negative energy which emerges in the usual analysis of the static Casimir effect and of vacuum polarization near black hole horizons, yet in this moving mirror case, the negative energy is radiated.

The simplicity of Eq. (34) contains the deeper underlying symmetry of the model [35], namely the Möbius transformations of $SL(2,\mathbb{R})$,

$$p(u) \to \frac{ap(u)+b}{cp(u)+d}, \qquad ad - bc = 1, \tag{35}$$

in the Schwarzian derivative

$$-24\pi F(u) = \{p,u\} \equiv \frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2, \tag{36}$$

of the trajectory dynamics as encapsulated in the null-coordinate function, $p(u)$ (the v position of the mirror as function of u). We intend to explore this symmetry as connected to the Sachdev-Ye-Kitaev (SYK) model (see e.g., Ref. [37] and references therein) whose action also has this emergent conformal symmetry (in the IR, large N limit), as a consequence of the two special properties: conformal flatness and conformal invariance [38].

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APPENDIX A: EXPONENTIAL ACCELERATION IN ADVANCED TIME v

In Sec. III, we added an important new exponential mirror solution with interesting properties. This leaves only one "exponential" unconsidered; in addition to exponential acceleration in τ, u, t and x, for completeness we now investigate the only clock not yet used for exponential acceleration: advanced time v (not to be confused with velocity).

For advanced time $v = t + z$, the proper acceleration behavior $\alpha(v) = \kappa e^{\kappa v}$ implies $\eta = \kappa v$. Since

$$\frac{d\eta}{d\tau} = e^{\eta}\frac{d\eta}{dv} = \kappa e^{\kappa v}, \tag{A1}$$

then $\kappa\tau = -e^{-\eta}$ and we have the interesting property that the acceleration is scale independent, i.e., $\alpha(\tau) = -1/\tau$. Recall that in [25] such scale independence—but with a positive sign—was shown to give eternal thermality of the radiation.

This raises a second interesting aspect: from Eq. A19 of [25] we had obtained eternal thermality from $\alpha(v) = -(1/2)\sqrt{\kappa/|v|}$. That is effectively the back side of the Carlitz-Willey eternally thermal moving mirror [17,24,26]. Both seem to be solutions, hinting at a potential "duality" (not in a strict mathematical sense) in the representation. Pursuing this further, Eq. A18 of [25] showed that exponential acceleration in a u clock is also thermal $\alpha(\tau) \sim 1/\tau$. We have verified that $\alpha(u) = (1/2)\sqrt{\kappa/|u|}$ gives the same $\alpha(\tau) = 1/\tau$ solution as exponential in u. So $v \leftrightarrow e^{-v}$, and similar for u, seem to be related for these forms.

Table II summarizes the complete family of mirrors with acceleration exponential in the various time variables. The expressions for energy flux have similarities to each other, with the exponential in u having constant, thermal flux

TABLE II. Summary of acceleration, rapidity, and energy flux properties for the exponential acceleration family.

Observable	Coordinate				
	t	x	u	v	τ
Proper acceleration, α	$-\kappa e^{\kappa t}$	$-\kappa e^{-\kappa x}$	$-\frac{\kappa}{2} e^{\frac{\kappa u}{2}}$	$\frac{\kappa}{2} e^{\frac{\kappa v}{2}}$	$-\kappa e^{\kappa \tau}$
Rapidity, η	κx	$-\kappa t$	$-\frac{\kappa u}{2}$	$\frac{\kappa v}{2}$	$-e^{\kappa \tau}$
Energy flux, F	$\frac{\kappa^2}{48\pi}(1 - e^{4\eta})$	$\frac{\kappa^2}{48\pi}(1 - e^{4\eta})$	$\frac{\kappa^2}{48\pi}$	$-\frac{\kappa^2}{48\pi} e^{4\eta}$	$-\frac{\kappa^2}{12\pi}\eta e^{2\eta}$

(Carlitz-Willey). The entropy flux, proportional to η, will in all these cases asymptotically diverge in contrast to the finite energy of the proper time exponentially accelerated mirror of Sec. III.

APPENDIX B: ENERGY INTEGRAL

The total energy result, Eq. (6), perhaps can be obtained most easily as follows. First, an integration by parts of Eq. (3), with the correct Jacobian, that exploits the ability to ignore the negligible boundary terms. The same can be done for the left side by considering the right side again but for the trajectory reflection. Summing leads to the simple integral,

$$E_T = \frac{1}{6\pi}\int_{-\infty}^{\infty} \frac{\ddot{z}^2}{(1-\dot{z}^2)^3} dt, \tag{B1}$$

where substitution of derivatives of the asymptotically static trajectory, Eq. (1), yields Eq. (6).

APPENDIX C: BETA COEFFICIENT INTEGRAL

The beta coefficient result, Eq. (7), is perhaps most easily obtained by integrating with respect to laboratory time, t. The unnormalized integral (Eq. (2.25) of [17]) is

$$\beta_{\omega\omega'} = \int_{-\infty}^{\infty} e^{i(\omega_n z - \omega_p t)}(\dot{z}\omega_p - \omega_n) dt. \tag{C1}$$

It is convenient to work in units of κ, restrict v, ω, ω' to positive reals and $v < 1$. Expanding $\omega_p \equiv \omega + \omega'$, and $\omega_n \equiv \omega - \omega'$, integrating, and normalizing by dividing by $4\pi\sqrt{\omega\omega'}$ gives the result Eq. (7).

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