On the *a*-theorem in the conformal window

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We show that for four-dimensional gauge theories in the conformal window, the anomaly, known as the a function, can be computed from a two-point function of the trace of the energy momentum tensor making it more amenable to lattice simulations. Concretely, we derive an expression for the a function as an integral over the renormalization scale of quantities related to two- and three-point functions of the trace of the energy momentum tensor. The crucial ingredients are that the square of the field strength tensor is an exactly marginal operator at the Gaussian fixed point and that the relevant three-point correlation function is finite when resummed to all orders. This allows us to define a scheme for which the three-point contribution vanishes, thereby explicitly establishing the strong version of the a theorem for this class of theories.

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I. INTRODUCTION

The conformal anomaly, first known as the central charge c of the Virasoro algebra, is key to the physics of conformal field theories (CFTs) as it is a measure of the number of degrees of freedom. The Weyl anomalies discovered in the 1970's (see [1] for a review of the topic) state that this central charge appears in the trace of the energy momentum tensor (TEMT) when there is a curved background, $\langle T^\rho{}_\rho \rangle_{\rm CFT} = -(\beta_c/(24\pi))R$, elevating the central charge to a β function; $\beta_c \equiv c$. The c theorem [2] can be stated in terms of Cardy's formula [3]

$$\Delta\beta_c^{2D} = \beta_c^{\rm UV} - \beta_c^{\rm IR} = 3\pi \int d^2x x^2 \langle \Theta(x)\Theta(0)\rangle_c \geq 0, \quad (1)$$

where $T^\rho{}_\rho|_{\rm flat} \to \Theta$ and $\langle \ldots \rangle_c$ stands for the connected component of the vacuum expectation value (VEV). It is assumed that the theory flows from an ultraviolet (UV) to an infrared (IR) fixed point (FP). The inequality, $\Delta\beta_c^{2D} \geq 0$, establishes the irreversibility of the renormalization group (RG) flow and follows from the positivity of the spectral representation and the finiteness of the correlator in (1).

In 4D, the situation is more involved as there are further terms in the TEMT,

$$\langle T^\rho{}_\rho(x) \rangle = -(\beta_a^{\rm IR} E_4 + \beta_b^{\rm IR} H^2 + \beta_c^{\rm IR} W^2) + 4\bar{b}^{\rm IR} \Box H. \quad (2)$$

Above $H \equiv R/(d-1)$ and as opposed to [4], we have omitted a cosmological constant term for brevity. In particular, we denote the coefficients of the geometric invariants by β functions except the $\Box R$ term which is a Weyl variation of the local R^2 term. In CFTs, $\beta_b = 0$ and β_a, β_b and $\bar{b}$ are the true conformal anomalies.

The a theorem, $\Delta\beta_a = \beta_a^{\rm UV} - \beta_a^{\rm IR} > 0$, was conjectured early on [5], and a proof in flat space uses the four-point function of the TEMT, anomaly matching and analyticity [6,7].[1] A stronger version of the theorem requires an interpolating function $\tilde{\beta}_a(\mu)$ that reduces to $\beta_a^{\rm UV,IR}$ at the respective FPs, satisfying monoticity, $d_{\ln\mu}\tilde{\beta}_a \geq 0$, along the RG flow. A perturbative argument was given in [9] by finding a function satisfying $d_{\ln\mu}\tilde{\beta}_a = (\chi_{AB} + \ldots)\beta^A\beta^B$, where the Zamolodchikov metric χ_{AB} is positive by unitarity at the Gaussian FP.[2]

A representation similar to (1) has been proposed involving two- and three-point functions [11],

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[1]In curved space, β_a can be assessed from a two-point function [8].

[2]This argument was generalized to conformal perturbations at interacting FPs in [10]. In both cases, the positivity is controlled by the smallness of perturbative corrections encoded in the dots. In 2D, the strong c theorem was proven in the original paper [2] without reference to perturbation theory.

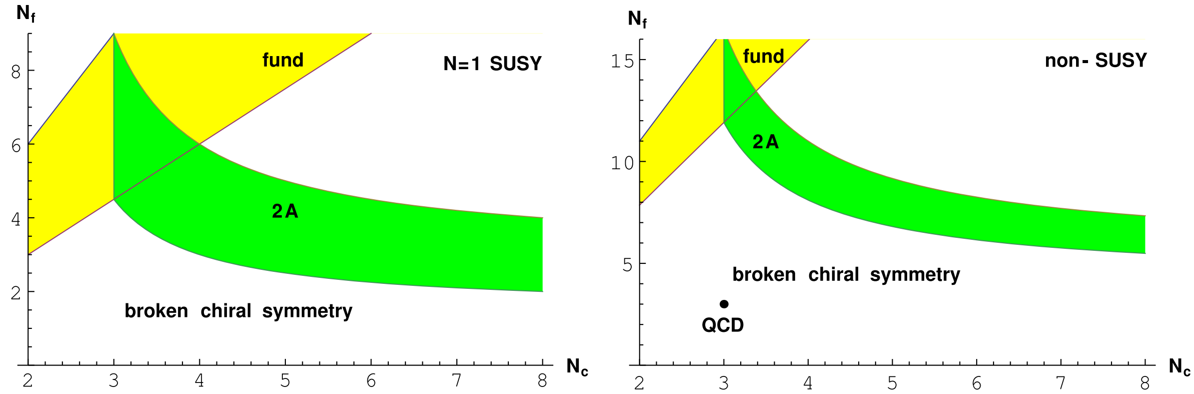

FIG. 1. The conformal window for supersymmetric (left) and nonsupersymmetric (right) gauge theories for quarks in the matter in the fundamental (yellow) and two index antisymmetric (green) representation of the $SU(N_c)$ gauge group. The upper boundaries are dictated by the loss of asymptotic freedom, and the lower boundaries are known in $\mathcal{N}=1$ supersymmetric gauge theories thanks to the electromagnetic duality [12], and for nonsupersymmetric gauge theories, they are debated in lattice simulations, and for the actual values, we have taken the boundaries given by Dyson-Schwinger equations [13]. Inside the yellow and green bands, the theory is expected to flow to an conformal IR FP. Below these regions, the chiral symmetry is spontaneously broken in the IR which is the case for quantum chromodynamics (QCD).

$$\Delta\beta_a = \frac{1}{3\cdot 2^8}\left(\int_x x^4 \langle \Theta(x)\Theta(0)\rangle_c - 2\int_x\int_y [(x\cdot y)^2 - x^2y^2]\langle\Theta(x)\Theta(y)\Theta(0)\rangle_c\right), \quad (3)$$

where $\int_x \equiv \int d^4x$. This expression is derived in Appendix B using conformal anomaly matching. As alluded above, this expression does not lend itself to positivity because of the presence of the three-point function. In this work, we will show that for gauge theories with gauge couplings only, the three-point function term drops for theories in the conformal window cf. Fig. 1. This establishes the positivity with Euclidean methods and makes the evaluation more amenable to lattice simulations.

A. Executive summary

In the remainder of this introduction, we give an executive summary of our work leaving the derivation of equations and definitions of schemes to the main part of the paper. Our assumptions are (i) that the TEMT assumes the form

$$\Theta \sim \beta^A[O_A] + \text{equation of motion terms} \quad (\Leftarrow \mathcal{L} = g_0^A O_A), \quad (4)$$

(summation over A implied) and (ii) that the beta functions $\beta^A \equiv \frac{d}{d\ln\mu} g^A$ vanish in the IR and UV.[3] Above operators with square brackets denote renormalized composite operators, e.g., $O_A \sim G^2$ in the case at hand, where $G^2 \equiv (G^a_{\mu\nu})^2$, is the standard field strength tensor squared known from quantum chromodynamics (QCD). Using these assumptions allows us to derive

$$\Delta\beta_a = \frac{1}{4}\int_{-\infty}^{\infty} (\chi^{\mathcal{R}}_{AB}(\mu')\beta^A\beta^B - \chi^{\mathcal{R}}_{ABC}(\mu')\beta^A\beta^B\beta^C) d\ln\mu', \quad (5)$$

from (3). The β functions are μ dependent through the couplings, and the χ's are the analogues of the Zamolodchikov metric (cf. Appendix A for definitions and notational conventions).[4] The expression (3) is derived in Sec. II and is a new result of this paper.

For our work, the crucial input is that the leading order correction to the noninteracting FP is

$$\chi^{\mathcal{R}}_{ABC} = \mathcal{O}((g^I)^2), \qquad \chi^{\mathcal{R}}_{AB} = \mathcal{O}((g^I)^0). \quad (6)$$

The main focus of this paper will be on asymptotically free QCD in the conformal window regime where (6) follows by using the conformal operator product expansion (OPE) (c.f. Appendix C) and is of course easily established by direct computation as well. Using (6) and our previous work on finiteness of two- and three-point functions, we are able to define a scheme, referred to as the $\mathcal{R}_{3\chi}$ scheme, for which $\chi^{\mathcal{R}_{3\chi}}_{ABC}(\mu) = 0$ along the flow. This establishes the main result of our paper

[3] For gauge theories with chiral symmetry breaking, the assumption $\Theta \sim \beta^A O_A$ breaks down since the goldstone bosons couple with a term $\Theta \sim \Box\pi^2$, which cannot be improved since it is in conflict with chiral symmetry [14,15] leading to subtleties for flow theorems [4,16].

[4] Whereas the χ's are dependent on a generic scheme $\mathcal{R}$, the two flow integrals themselves are scheme independent, cf. [4]. We will refer to χ_{ABC} as the three metric throughout in a loose analogy to the Zamolodchikov metric in two dimensions.

$$\Delta\beta_a = \frac{1}{4}\int_{-\infty}^{\infty}(\chi^{\mathcal{R}}_{AB}(\mu')\beta^A\beta^B)\,d\ln\mu' = \frac{1}{3\cdot 2^8}\int_x x^4\langle\Theta(x)\Theta(0)\rangle_c > 0, \tag{7}$$

valid for the assumptions specified above and satisfying (6). On a side note, this means that $\Delta\beta_a = 2\Delta\bar{b} \equiv 2(\bar{b}^{\mathrm{UV}} - \bar{b}^{\mathrm{IR}})$, for the same conditions, since the flow theorem for $\bar{b}$ can be expressed in terms of the same two-point function [4]. This relation was conjectured to hold for general classically conformal QFTs in [17]. In this work, we show under what conditions this relation holds. A case where it fails is when the theory contains scalar couplings and the three-point function does contribute.

The paper is organized as follows. The cornerstones, formula (5) and the scheme $\chi^{\mathcal{R}_\chi}_{ABC}(\mu) = 0$, are established in Secs. II and III, respectively. More precisely, in Sec. III. A, it is shown that $\chi^{\mathcal{R}_\chi}_{ABC}$ satisfies (6), used in Sec. III B to derive the finiteness of the counterterm which then allows the explicit construction of the scheme for which $\chi^{\mathcal{R}_\chi}_{ABC}(\mu) = 0$ in Sec. III C. Definitions, including notation from our previous work, are reviewed in Appendix A. The sum rule (3) is derived in Appendix B, and a more formal argument for the first equation in (6), underlying the above mentioned scheme, is given in Appendix C.

II. THE FLOW OF E_4 (OR β_a) AS AN INTEGRAL OVER THE RG SCALE

It is the aim of this section to derive (5). The presentation below is similar to the one given for the $\Box R$ flow in [4], where we have shown that[5]

$$\Delta\bar{b} = \frac{1}{3\cdot 2^7}\int_x x^4\langle\Theta(x)\Theta(0)\rangle_c = \frac{1}{8}\int_{-\infty}^{\infty}\chi^{\mathcal{R}}_{AB}\beta^A\beta^B\,d\ln\mu'. \tag{8}$$

The starting point is formula (3), which is derived in Appendix B. Using (8), one may write

$$\Delta\beta_a = 2\Delta\bar{b} - \frac{1}{4}\hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(p_x,p_y)|_{p_x=p_x=0}, \tag{9}$$

where

$$\Gamma_{\theta\theta\theta}(p_x,p_y) = \int_x\int_y e^{i(p_x\cdot x + p_y\cdot y)}\langle\Theta(x)\Theta(y)\Theta(0)\rangle_c, \tag{10}$$

and $\hat{P}_{\lambda_3}$ is defined in (B10).[6]

The transformation of the equation above into an integral representation over the RG scale, necessitates the discussion of the renormalization prescription. To regularize, we use dimensional regularization with $d = 4-2\epsilon$. The correlator is renormalized by a splitting of the bare function into a renormalized $\Gamma^{\mathcal{R}}$ and a counterterm $L^{\mathcal{R}} = \sum_{n\geq 1} L^{\mathcal{R}}_n\epsilon^{-n}$,

$$\Gamma_{\theta\theta\theta}(p_x,p_y) = \Gamma^{\mathcal{R}}_{\theta\theta\theta}(p_x,p_y,\mu) + L^{\mathbb{1},\mathcal{R}}_{(\theta)\theta\theta}(\mu)P_3 + L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}(\mu)\lambda_3, \tag{11}$$

which consists in a Laurent series. Above λ_3 (Källén function) and P_3 are the three- and two-point kinematic structures,

$$\lambda_3 \equiv p_x^4 + p_y^4 + p_z^4 - 2(p_x^2p_y^2 + p_x^2p_z^2 + p_y^2p_z^2),$$
$$P_3 \equiv p_x^4 + p_y^4 + p_z^4, \tag{12}$$

where momentum conservation, $p_z + p_x + p_y = 0$, is implied. The quantities $L^{\mathbb{1},\mathcal{R}}_{(\theta)\theta\theta}(\mu)$,$L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}(\mu)$ are Laurent series in ϵ depending on the running couplings of the theory. From (9), it is seen that $L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}$ is the key quantity which we analyze by its scale dependence

$$\chi^{\mathcal{R}}_{\theta\theta\theta}(\mu) = \left(2\epsilon - \frac{d}{d\ln\mu}\right)L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}(\mu) \overset{\epsilon\to 0}{=} -\frac{d}{d\ln\mu}L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}(\mu). \tag{13}$$

In the last equality, we used the result of [18] that $L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}$ is finite after resummation of divergences. In establishing the flow formula (5), we follow the logic of [4] and introduce the so-called momentum space subtraction (MOM) scheme, defined by

$$\chi^{\mathrm{MOM}}_{\theta\theta\theta} = -\frac{d}{d\ln p}\bigg|_{p=\mu}\hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(p,-p). \tag{14}$$

By solving the above ordinary differential equation, we arrive at

$$\hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(g^Q(p)) = \int_{\ln p/\mu_0}^{\infty}\chi^{\mathrm{MOM}}_{\theta\theta\theta}\,d\ln\mu' = \underbrace{\int_{\ln p/\mu_0}^{\ln\mu/\mu_0}\chi^{\mathrm{MOM}}_{\theta\theta\theta}\,d\ln\mu'}_{\hat{P}_{\lambda_3}\Gamma^{\mathrm{MOM}}_{\theta\theta\theta}(p/\mu,g^Q(\mu))} + \underbrace{\int_{\ln\mu/\mu_0}^{\infty}\chi^{\mathrm{MOM}}_{\theta\theta\theta}\,d\ln\mu'}_{L^{\mathbb{1},\mathrm{MOM}}_{\theta\theta\theta}(g^Q(\mu))}, \tag{15}$$

where the split on the second line is compatible with (13). In order to pass to the coupling coordinates, one uses the assumption (4) to write

[5]For remarks concerning adding a local term $\delta\mathcal{L} \sim w_0R^2$ to the bare action, with regards to Eqs. (3) and (8), cf. Appendix B.2 or our previous work [4].

[6]Assuming the IR limit p_x, $p_y \to 0$ to be regular, we can choose to approach 0 by taking, for example, $p_x = -p_y = p \to 0$ and define a function $f(p^2) \equiv \hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(p,-p)$.

$$\Gamma_{\theta\theta\theta} = \beta^A\beta^B\beta^C\Gamma_{ABC}(p_x, p_y), \tag{16}$$

with

$$\Gamma_{ABC}(p_x,p_y) = \int_x\int_y e^{i(p_x\cdot x+p_y\cdot y)}\langle[O_A(x)][O_B(y)][O_C(0)]\rangle_c. \tag{17}$$

Renormalization of the above correlator and further definitions [e.g., $\chi^{\mathcal{R}}_{ABC}$ in (A8)] are reviewed in Appendix A. It is now possible to define a MOM-scheme relation analogical to (14) for the correlator (17). In this MOM scheme, we can use the relation $\chi^{\rm MOM}_{\theta\theta\theta} = \beta^A\beta^B\beta^C\chi^{\rm MOM}_{ABC}$, which in turn follows from substituting (4) into (14).

The final expression for $\Delta\beta_a$ is obtained by taking $p \to 0$ limit of (15) and inserting the result to (9)

$$\Delta\beta_a = \frac{1}{4}\int_{-\infty}^{\infty}(\chi^{\rm MOM}_{AB}\beta^A\beta^B - \chi^{\rm MOM}_{ABC}\beta^A\beta^B\beta^C)d\ln\mu'. \tag{18}$$

Although as it stands, (18) is written in a specific scheme, just like for $\Delta\bar{b}$ [4], scheme independence followed by observing that a change from a scheme $\mathcal{R}_1$ to $\mathcal{R}_2$ is given by a cohomologically trivial term,

$$\delta\chi_{\theta\theta\theta} = \chi^{\mathcal{R}_2}_{\theta\theta\theta} - \chi^{\mathcal{R}_1}_{\theta\theta\theta} = \frac{d}{d\ln\mu}\omega, \tag{19}$$

where $\omega = (\beta^A\beta^B\beta^C\omega_{ABC})$ with ω_{ABC} parametrizing the change of scheme, cf. Eq. (A10). This establishes the representation (5) and completes the aim of this section. We have also checked that Eq. (18) is consistent with the MS-scheme formulas of [19] (namely 3.17b and 3.23 in this reference).

III. THE THREE METRIC χ_{ggg} IN GAUGE THEORIES

In this section, we restrict ourselves to QCD-like theories with one gauge coupling g and massless fermions. The generalization of the following result to multiple coupling theories satisfying (6) is straightforward. Before we proceed, let us establish the notation. The trace anomaly for gauge theories reads

$$\Theta = \frac{\beta}{2}[O_g], \tag{20}$$

where $\beta = \frac{d\ln g}{d\ln\mu}$ is the logarithmic beta function. The corresponding operator O_g is the field strength squared

$$[O_g] = \left[\frac{1}{g_0^2}G^2\right], \tag{21}$$

with the somewhat nonstandard treatment of the coupling constant (and G^2 has been defined previously). The mapping to the general expressions (5) and (7) is done by comparing (4) and (20), which gives $O_A \to O_g$ and $\beta^A \to \frac{1}{2}\beta$ omitting the superscript g on the β function for brevity.

In Sec. III A, it is shown that $\chi_{ggg} = \mathcal{O}(g^2)$, from where it is deduced, in Sec. III B, that the counterterm to the three-point function L_{ggg} is finite when summed to all orders for *all* points along the flow. Based on this, in Sec. III C, a scheme is defined for which $\chi^{\mathcal{R}_{3\chi}}_{ggg}(\mu) = 0$, leading to the main result of the paper.

A. The vanishing of χ_{ggg} at the UV fixed point

Technically, we will show that $\chi_{ggg} = \mathcal{O}(g^2)$. The three-point function can be computed at leading order directly in momentum space by evaluating the diagram in Fig. 2, which gives

$$\begin{aligned}\Gamma_{ggg}(p_x,p_y) &\equiv \frac{1}{g_0^6}\int_x\int_y e^{i(p_x\cdot x+p_y\cdot y)}\langle[G^2(x)][G^2(y)][G^2(0)]\rangle_c \\ &= \frac{1}{\pi^2}\frac{1}{\epsilon}(p_x^{4-2\epsilon}+p_y^{4-2\epsilon}+p_z^{4-2\epsilon}) \\ &\quad + \frac{1}{\pi^2}\left(-\frac{1}{2}\lambda_3 - P_3\right) + \mathcal{O}(\epsilon, g^2),\end{aligned} \tag{22}$$

where the two- and three-point kinematic structures P_3 and λ_3 are defined in (12). It is observed that at leading order in the coupling there is no divergent contribution to the three-point function kinematic structure λ_3; or more precisely, to the projector $\hat{P}_{\lambda_3}$ (B10) applied to the correlation function. The two-point function structure P_3 is not relevant for our work. From (11) and the definition of χ_{ggg} (A8), $\chi_{ggg} = \mathcal{O}(g^2)$ follows. In principle, this completes the task of this section, but we think it is instructive to add a few more comments.

First, as demonstrated in the Appendix C, this can also be understood from the fact that for an exactly marginal operator the λ_3 structure vanishes in a CFT [20]. In the language of [20], the structure P_3 is referred

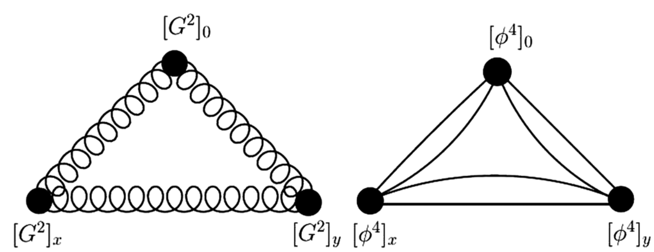

FIG. 2. (left) Leading order diagram contributing to Γ_{ggg} with no divergence in the three-point structure λ_3 after Fourier transformation. This is in accordance with (6). (right) Leading order diagram contributing to the correlator of ϕ^4 operators (23). In momentum space, this corresponds to a four loop graph and does lead to divergencies in the λ_3 structure from where one can infer that the ϕ^4 coupling acquires a RG running.

to as semilocal, and that is at least one delta function in coordinate space; $\delta(x)\frac{1}{y^{2d}}+\text{permutations}\leftrightarrow p^{4-2\epsilon}\frac{1}{\epsilon}+$ permutations, in the case (22). For noncoincident points, the correlation function is indeed proportional to $(d-4)$, cf. [21,22]. It is in particular instructive to consider a case where this fails. An example is a free conformally coupled scalar field for which ϕ^4 is an operator of scaling dimension four but since its perturbation $\delta\mathcal{L}\sim\lambda\phi^4$ induces an RG flow, namely $\beta_\lambda\neq 0$, it is not exactly marginal. In the explicit computation, one obtains in coordinate space

$$\langle[\phi^4(x)][\phi^4(y)][\phi^4(0)]\rangle_c=\frac{8}{x^4y^4(x-y)^4}, \qquad (23)$$

which is clearly not semilocal and will contribute to the λ_3 structure upon Fourier transformation. On a side note, it is a remarkable circumstance that from the evaluation (23), one can infer that the $\beta_\lambda\neq 0$, cf. discussion in Appendix C and [20].

Second, one might wonder whether something similar is possible for the two-point function. The answer is no for the following reasons. If it were possible to set $\chi_{AB}=0$ in some scheme then it would also imply that $\Theta=0$ by reflection positivity in (8) which is incompatible with a nontrivial flow. The explicit straightforward computation for the two-point function at leading order gives, $\langle[G^2(x)][G^2(0)]\rangle=96/x^8+\mathcal{O}(g^2)$, a noncontact term contribution, unlike (22), whose Fourier transform gives rise to $\ln\mu$-dependent term. Moreover, the formal argument given in Appendix C does not descend to two-point functions.

B. Finiteness of the three-point function

Following the analysis in [18], we study the finiteness of $L^{\mathbb{1},\mathcal{R}}_{ggg}$, the resummed Laurent series, after removing the regulator $\epsilon=(d-4)/2$. This serves as the basis for defining the $\mathcal{R}_{3\chi}$ scheme in the next section. The quantity $L^{\mathbb{1},\mathcal{R}}_{ggg}$ is defined in analogy to $L^{\mathbb{1},\mathcal{R}}_{\theta\theta\theta}$ in (11). In dimensional regularization, the renormalization group equation (A8) reads[7]

$$\chi^{\mathcal{R}}_{ggg}=(2\epsilon-\mathcal{L}_\beta)L^{\mathbb{1},\mathcal{R}}_{ggg}=(-2\hat\beta\partial_{\ln a_s}-6(\partial_{\ln a_s}\hat\beta)+2\epsilon)L^{\mathbb{1},\mathcal{R}}_{ggg}, \qquad (24)$$

where we used the d-dimensional logarithmic beta function $\hat\beta=-\epsilon+\beta$ and $a_s\equiv g^2/(4\pi)^2$. This equation allows for the integral solution

$$L^{\mathbb{1},\mathcal{R}}_{ggg}(a_s(\mu),\epsilon)|_{\mathrm{UV}}=\frac{\mu^{2\epsilon}}{\hat\beta^3}\int_{\ln\mu}^{\infty}\hat\beta^3(\mu')\chi^{\mathcal{R}}_{ggg}(\mu')\frac{d\ln\mu'}{\mu'^{2\epsilon}}. \qquad (25)$$

[7]The quantity $\chi^{\mathcal{R}}_{ggg}$ corresponds to $\bar\chi^a_{ggg}$ in the classic paper of Jack and Osborn [19].

This expression is well-defined for $\mu>0$, convergent for $\mu'\to\infty$, as we will show shortly for the asymptotically free and asymptotically safe case. Anticipating the result and removing the regulator ($\epsilon\to 0$), the expression becomes

$$\begin{aligned}L^{\mathbb{1},\mathcal{R}}_{ggg}(a_s(\mu),0)&=\frac{1}{\beta^3}\int_{\ln\mu}^{\infty}\beta^3(\mu')\chi^{\mathcal{R}}_{ggg}d\ln\mu'\\&=-\frac{1}{2\beta^3}\int_0^{a_s(\mu)}\beta^2(u)\chi^{\mathcal{R}}_{ggg}(u)\frac{du}{u},\end{aligned} \qquad (26)$$

where the second integral representation in (26) is useful for practical computation.

(i) In the asymptotically free case (Gaussian FP), the β function and the three metric behave as

$$\beta\sim a_s\overset{\mu\to\infty}{\sim}\frac{1}{\ln\mu},\qquad \chi^{\mathcal{R}}_{ggg}\sim a_s\sim\frac{1}{\ln\mu}, \qquad (27)$$

in accordance with (6). Inserting this back into the integral (26), we see that the solution behaves regularly near the UVFP

$$L^{\mathbb{1},\mathcal{R}}_{ggg}\overset{\mu\to\infty}{\sim}\mathcal{O}(1), \qquad (28)$$

since the $\ln\mu^{-3}$ from the integral is cancelled by the $1/\beta^3$ prefactor.

(ii) We can also apply similar arguments for RG flows in the vicinity of a nontrivial UVFP a_s^{UV}, which corresponds to the asymptotically safe scenario (see [23] for some recent discussion of this possibility). In this case, the UV behavior is

$$\beta\overset{\mu\to\infty}{\sim}\mu^{-\gamma^*},\qquad \chi^{\mathcal{R}}_{ggg}\overset{\mu\to\infty}{\sim}\chi^*_{ggg}, \qquad (29)$$

where $\chi^*_{ggg}\equiv\chi^{\mathcal{R}}_{ggg}(a_s^{\mathrm{UV}})$ and $\gamma^*=\partial_{\ln a_s}\beta|_{a_s^*}>0$ is the anomalous dimension of $[G^2]$ at the FP. One gets

$$\int_{\ln\mu}^{\infty}\beta^3(\mu')\chi^{\mathcal{R}}_{ggg}(\mu')d\ln\mu'\overset{\mu\to\infty}{\sim}\frac{1}{3\gamma^*}\chi^*_{ggg}\mu^{-3\gamma^*}. \qquad (30)$$

By inserting this back to (26), we find again finite UV behavior $L^{\mathbb{1},\mathcal{R}}_{ggg}\overset{\mu\to\infty}{\sim}\mathcal{O}(1)$. In fact, it is straightforward to see that provided $\gamma^*\neq 0$, for any FP $a_s{}^*$ the Eq. (24) always allows for a finite solution

$$L^{\mathbb{1},\mathcal{R}}_{ggg}(a_s{}^*,0)=-\frac{1}{6\gamma^*}\chi^*_{ggg}. \qquad (31)$$

Let us now turn to the issue of IR convergence. Clearly, the presence of $\frac{1}{\beta^3}$ in the solution (26) indicates additional problems when the IR limit $a_s\to a_s^{\mathrm{IR}}$ is taken. Note however, that near the IRFP another solution to (24) can be found

$$L^{\mathbb{1},\mathcal{R}}_{ggg}(a_s(\mu),\epsilon)|_{\mathrm{IR}}=-\frac{\mu^{2\epsilon}}{\hat\beta^3}\int_{-\infty}^{\ln\mu}\hat\beta^3(\mu')\chi^{\mathcal{R}}_{ggg}(\mu')\frac{d\ln\mu'}{(\mu')^{2\epsilon}}, \qquad (32)$$

well-defined for $\mu < \infty$. After taking $\epsilon \to 0$ [which is justified for the same reasons as (25)], one gets

$$L_{ggg}^{\mathbb{1},\mathcal{R}}(a_s(\mu),0) = -\frac{1}{\beta^3}\int_{-\infty}^{\ln\mu}\beta^3(\mu')\chi_{ggg}^{\mathcal{R}}(\mu')d\ln\mu' = -\frac{1}{2\beta^3}\int_{a_s^{\mathrm{IR}}}^{a_s(\mu)}\beta^2(u)\chi_{ggg}^{\mathcal{R}}(u)\frac{du}{u}. \quad (33)$$

By repeating the analysis leading to (27) and (28) near the IRFP, we conclude that (32) is well-defined in the vicinity of a_s^{IR}.

Assuming that the theory is free from any singularities in the coupling space, the solutions (25) and (32) should be compatible on overlapping domains. By subtracting (25) from (32) (and taking $\epsilon \to 0$ limit), we find a continuity condition

$$\left(\int_{-\infty}^{\ln\mu}+\int_{\ln\mu}^{\infty}\right)\beta^3(\mu')\chi_{ggg}^{\mathcal{R}}d\ln\mu' = \int_{-\infty}^{\infty}\beta^3(\mu')\chi_{ggg}^{\mathcal{R}}d\ln\mu' = 0. \quad (34)$$

This is consistent with the vanishing of the three-point contribution in (5), which can therefore be seen as the direct consequence of the finiteness and coupling continuity of $L_{ggg}^{\mathbb{1},\mathcal{R}}$. Indeed, in the next section, we will show how the above results can be used to construct a scheme, where the three metric vanishes.

C. Constructing the $\mathcal{R}_{3\chi}$ scheme for the three metric χ_{ggg}

A change of scheme, cf. Eq. (A10), is given by a finite shift $\omega_{ggg}(a_s)$ in the counterterms

$$L_{ggg}^{\mathbb{1},\mathcal{R}_2} = L_{ggg}^{\mathbb{1},\mathcal{R}_1} - \omega_{ggg}(a_s). \quad (35)$$

Using (24), we can deduce that under such a shift the three metric transforms as

$$\chi_{ggg}^{\mathcal{R}_2} = \chi_{ggg}^{\mathcal{R}_1} + (2\beta\partial_{\ln a_s} + 6(\partial_{\ln a_s}\beta))\omega_{ggg}. \quad (36)$$

Since the $\epsilon \to 0$ limit of $L_{ggg}^{\mathbb{1},\mathcal{R}_1}$ is uniform as shown in the previous section, we can choose

$$\omega_{ggg}(a_s) \equiv L_{ggg}^{\mathbb{1},\mathcal{R}}(a_s,\epsilon=0), \quad (37)$$

to define a new scheme $\mathcal{R}_{3\chi}$ for which

$$\chi_{ggg}^{\mathcal{R}_{3\chi}}(\mu) = 0 \quad (38)$$

is automatic. This scheme is new to this paper and not to be confused with the previously discussed MOM scheme.

By using (38) in the general scheme-independent expression (18), we finally arrive at the desired result

$$\Delta\beta_a = \frac{1}{16}\int_{-\infty}^{\infty}\chi_{ggg}^{\mathcal{R}}\beta^2 d\ln\mu'. \quad (39)$$

We would like to end this section by demonstrating how to construct such a scheme in perturbation theory. Using the two-loop formulas of [19], we extract

$$\chi_{ggg}^{\mathrm{MS}} = \frac{n_g}{4\pi^2}(-2\beta_0 a_s). \quad (40)$$

where β_0 is one-loop coefficient of the beta function $\beta = -\beta_0 a_s + \mathcal{O}(a_s{}^2)$, the gluons and N_f fermions are assumed to be in the adjoint and fundamental representation of an $\mathrm{SU}(N_c)$ gauge group, respectively ($n_g \equiv N_c^2 - 1$), and MS denotes the standard minimal subtraction scheme. By performing a scheme change (36) with

$$\omega_{ggg} = -\frac{1}{3}\frac{n_g}{4\pi^2}, \quad (41)$$

we achieve

$$\chi_{ggg}^{\mathcal{R}_{3\chi}} = 0 + \mathcal{O}(a_s{}^2), \quad (42)$$

as expected at this order in the perturbation theory. At the perturbative Banks-Zaks FP $a_s^{\mathrm{IR}} \propto \beta_0 \ll 1$. Since χ_{ggg} is absent, $\Delta\beta_a$ in this theory can be computed purely by substituting the known perturbative expressions for the beta function and χ_{gg} in (8) as was done up to five loops in [4] (see Eq. 60 and the discussion below in this reference).

To find $\chi_{ggg}^{\mathcal{R}_{3\chi}}$ at higher orders, one would need to use the two-loop beta function together with the three-loop expression of χ_{ggg}, which is not presently available to the authors. Nevertheless, using the formulas of this paper, some general predictions about the behavior of these higher order corrections can be made (cf. Appendix D).

Recently, χ_{ggg}^{MS} was computed to leading order in the large N_f expansion in [24] by resumming infinite number of bubble diagrams. The result reads

$$\chi_{ggg}^{\mathrm{MS}} = \frac{n_g}{4\pi^2}\frac{1}{3K}\frac{\partial}{\partial K}\left(K^2\bar{H}\left(\frac{2}{3}K\right)\right) + \mathcal{O}\left(\frac{1}{N_f}\right), \quad (43)$$

where $K = 2a_s N_f$ and

$$\bar{H}(x) = \frac{(80 - 60x + 13x^2 - x^3)x\Gamma(4-x)}{120(4-x)\Gamma(1+\frac{x}{2})\Gamma(2-\frac{x}{2})}. \quad (44)$$

Since to leading order in $\frac{1}{N_f}$ the β function is a one-loop exact $\beta = \frac{1}{3}K(1 + O(1/N_f))$, it is possible to use our formula (26) together with (37) to find the $\mathcal{R}_{3\chi}$ transformation corresponding to (43). The result reads

$$\omega_{ggg} = -\frac{1}{2}\frac{n_g}{4\pi^2}\frac{\bar{H}(\frac{2}{3}K)}{K} + \mathcal{O}\left(\frac{1}{N_f}\right). \quad (45)$$

The formula above represents an application of our result beyond perturbation theory. By reexpanding expression (45) in small K, we find that the leading term agrees with (41).

IV. CONCLUSIONS AND DISCUSSIONS

Our starting point was the derivation of a formula for the Euler anomaly or a function as an integral over the RG scale of a two- and three-point functions of the trace of the energy momentum tensor (5), valid for theories which are governed by β functions at both fixed points (4).

For gauge theories in the conformal window, the formula collapses to the two-point function (7). Our main assumption for the proof is that the ultraviolet and infrared solution (25) and (32), of the renormalization group equation (24), can be matched continuously. This allowed us to define an explicit prescription, the $\mathcal{R}_{3\chi}$ scheme (37), for which the three metric vanishes. In particular, our result means that for those theories, the Euler flow and the $\Box R$ flow are identical (8). The reason this works for gauge couplings, and not for generic couplings, is that for the former the three-point function collapses to a two-point function near the trivial FP (22). This is a consequence of the vanishing of the leading order contribution to the three metric (6) and can also be understood from the fact that the field strength tensor squared is an exactly marginal operator at zero coupling, cf. Appendix C. An example where this fails is a scalar free field theory for which ϕ^4 is not an exactly marginal operator and its nonzero β function induces a three-point function structure at leading order, prohibiting the use of the $\mathcal{R}_{3\chi}$ scheme.

In the light of the above remarks, one might wonder, whether the result can be applied to gauge theories with supersymmetry including scalar fields such as supersymmetric QCD (SQCD). The extension is possible owing the same form of the anomaly (4) in SQCD [25]. In supersymmetric gauge theories without a superpotential, the matter and gauge contributions are related through the Konishi anomaly [26], so that the trace anomaly can be expressed solely in terms of the field strength tensor squared up to equation of motion terms. We therefore expect that the main results of this paper should apply to $\mathcal{N} = 1$ SQCD in the conformal window.[8]

Another corollary of our analysis is that for the class of theories studied in this paper, the strong a theorem applies. One defines the off-critical quantity

[8]Some care has to be taken when passing from the dimensional regularization to SUSY-preserving schemes (see Appendix A of [27] for some details of how this is to be done).

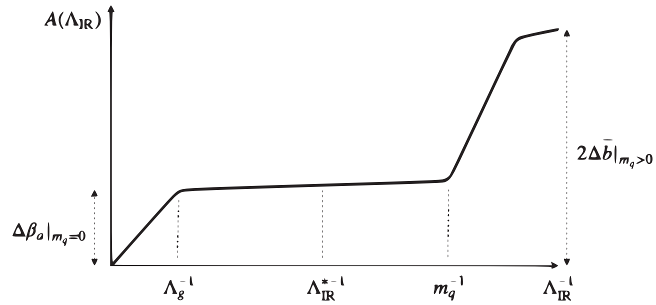

FIG. 3. The Euler flow, $\Delta\beta_a$ (7), as a function of the IR cutoff Λ_{IR} on the spacial integral (47). The scale Λ_g^{-1} is related to the running of the β function and can be expected to be of the order of the scale where the derivative of the β function changes sign [34,35]. The proposed formula is given in Eq. (48). Note that the asymptotic value $2\Delta\bar{b}|_{m_q>0}$ is a nontrivial quantity whose value is not yet understood [4]. The determination of the latter would therefore be an additional benefit of a lattice investigation.

$$\tilde{\beta}_a(\mu) = \beta_a^{\mathrm{UV}} - \frac{1}{16}\int_{\ln\mu}^{\infty} \chi_{gg}^{\mathrm{MOM}}\beta^2 d\ln\mu', \quad (46)$$

which reduces to β_a^{IR} in the limit $\mu \to 0$ by (39) and gives a (scheme dependent) interpolating function between the fixed points. The monotonicity of this function follows from the positivity of χ_{gg}^{MOM}, established in [4].

Moreover, the perspective of implementing the a theorem in the conformal window on the lattice have improved since it is related to a two-point function.[9] Supersymmetric lattice gauge theories [31] could be a particularly interesting test ground as the Euler anomaly is exactly known [32]. In practice though, lattice Monte Carlo simulation are done at a finite quark mass, which does not fall into our class of theories.[10] A pragmatic way to deal with this problem is to choose an infrared cutoff $\Lambda_{\mathrm{IR}}^{-1}$,

$$A(\Lambda_{\mathrm{IR}}, m_q, L) \equiv \frac{1}{3\cdot 2^8}\int_0^{\Lambda_{\mathrm{IR}}^{-1}} d^4x x^4 \langle\Theta(x)\Theta(0)\rangle_c, \quad (47)$$

on the integral (7). The function A is expected to plateau to $\Delta\beta_a|_{m_q=0}$ for $\Lambda_{\mathrm{IR}}^{-1}$ lower than the inverse quark mass m_q for which the theory behaves like a massless theory.[11]

[9]This requires the renormalization of the energy momentum tensor on the lattice which is a nontrivial task because of the breaking of the space-time symmetries [28]. See also [29,30] for some recent proposals using the gradient flow technique.

[10]The TEMT contains a term of a the form $\Theta \sim m(1+\gamma_m)\bar{q}q$ in addition to the β-function terms (4), where γ_m is the quark mass anomalous dimension. Thus, unless $\gamma_m^{\mathrm{IR}} = -1$, this does not correspond to a CFT in the IR.

[11]None of these scales should be confused with the lattice size L. In particular, $\Lambda_{\mathrm{IR}}^{-1} < L$ holds strictly by construction. To avoid finite size effects, one has to impose $m_q^{-1} \ll L$; as for the two-point function, the quark mass correction ought to be exponential $\exp(-m_H L)$, where $m_H \sim (m_q)^{\eta_H}$ with $\eta_H \equiv 1/(1+\gamma_m^{\mathrm{IR}})$ is a mass of a hadron [33].

More precisely, the flat region corresponds to the near-conformal behavior in the vicinity of the IRFP. Thus, one would expect $\Delta\beta_a$ to plateau to the massless case

$$\Delta\beta_a|_{m_q=0} \simeq A(\Lambda^*_{\rm IR}, m_q), \quad \Lambda_g^{-1} \ll (\Lambda^*_{\rm IR})^{-1} \ll m_q^{-1}, \tag{48}$$

for the above mentioned range, cf. Fig. 3 for a schematic illustration and an explanation about the scale Λ_g. It would be interesting to apply this procedure to the case where the IR phase is chirally broken and investigate the expected appearance of the $\ln m_q$ divergence induced by the goldstone bosons [4,16].

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APPENDIX A: CONVENTIONS FOR COMPUTING CORRELATORS

In this Appendix, we will give a brief review of the notation related to the renormalization of composite operator correlators adapted from [18].

We start with the two-point functions of classically marginal renormalized operators $[O_A]$

$$\Gamma_{AB}(p^2) = \int d^4x e^{ip\cdot x}\langle[O_A(x)][O_B(0)]\rangle_c = \Gamma^{\mathcal{R}}_{AB}(p^2,\mu) + L^{\mathbb{1},\mathcal{R}}_{AB}\mu^{-2\epsilon}p^4, \tag{A1}$$

where the subtraction constant $L^{\mathbb{1},\mathcal{R}}_{AB}$ is a function of couplings of the theory and it contains Laurent series in ϵ as well as a finite part and $\Gamma^{\mathcal{R}}_{AB}(p^2,\mu)$ is the finite renormalized correlator. A scheme $\mathcal{R}$ is determined by the choice of the finite part of $L^{\mathbb{1},\mathcal{R}}_{AB}$. The finite quantity called Zamolodchikov metric is obtained via

$$(2\epsilon - \mathcal{L}_\beta)L^{\mathbb{1},\mathcal{R}}_{AB} = \chi^{\mathcal{R}}_{AB}, \tag{A2}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative on a two tensor in coupling space

$$\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{AB} = \partial_A\hat\beta^C L^{\mathbb{1},\mathcal{R}}_{CB} + \partial_B\hat\beta^C L^{\mathbb{1},\mathcal{R}}_{AC} + \hat\beta^C\partial_C L^{\mathbb{1},\mathcal{R}}_{AB}. \tag{A3}$$

Since the bare correlator is scale independent, the Zamolodchikov metric can be also defined directly from the finite renormalized correlators, e.g., [36], as follows:

$$\left(-\frac{\partial}{\partial\ln\mu} + \mathcal{L}_\beta\right)\Gamma^{\mathcal{R}}_{AB}(p^2,\mu) = \chi^{\mathcal{R}}_{AB}p^4. \tag{A4}$$

A scheme change is implemented using a finite, coupling dependent constant ω_{AB} via

$$\Gamma^{\mathcal{R}}_{AB} \to \Gamma^{\mathcal{R}}_{AB} + \omega_{AB}p^4, \qquad L^{\mathbb{1},\mathcal{R}}_{AB} \to L^{\mathbb{1},\mathcal{R}}_{AB} - \omega_{AB}, \tag{A5}$$

leaving the bare correlator invariant. Under such change, we get a shift in the Zamolodchikov metric

$$\chi^{\mathcal{R}}_{AB} \to \chi^{\mathcal{R}}_{AB} + \mathcal{L}_\beta\omega_{AB}. \tag{A6}$$

The three-point functions can be defined in a similar manner

$$\Gamma_{ABC}(p_x,p_y) = \int d^4xd^4y e^{i(p_x\cdot x+p_y\cdot y)}\langle[O_A(x)][O_B(y)][O_C(0)]\rangle_c = \Gamma^{\mathcal{R}}_{ABC} + L^{\mathbb{1},\mathcal{R}}_{(A)BC}p_x^4 + L^{\mathbb{1},\mathcal{R}}_{ABC}P_{yz} + \text{cyclic}, \tag{A7}$$

where cyclic permutation over the pairs (A,x), (B,y), and (C,z) is implied. Furthermore, $p_x+p_y+p_z=0$ and $P_{yz} = p_x^4 - p_x^2(p_y^2+p_z^2)$ are kinematic structures vanishing whenever *any* of the three external momenta $p_{x,y,z}$ is set to zero. Just as above, $L^{\mathbb{1},\mathcal{R}}_{(A)BC}$ and $L^{\mathbb{1},\mathcal{R}}_{ABC}$ are subtraction constants containing Laurent series and finite parts. It follows that the $L^{\mathbb{1},\mathcal{R}}_{(A)BC}$ coefficients can be determined from the two-point functions information (see [18] for the exact definition).

The new, purely three-point information is encoded in the $L^{\mathbb{1},\mathcal{R}}_{ABC}$ tensor. Again the scale derivative,

$$(2\epsilon - \mathcal{L}_\beta)L^{\mathbb{1},\mathcal{R}}_{ABC} = \chi^{\mathcal{R}}_{ABC}, \tag{A8}$$

proves useful. Above $\mathcal{L}_\beta$ denotes the Lie derivative, acting on a three-tensor,

$$\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{ABC} = \partial_A\hat\beta^D L^{\mathbb{1},\mathcal{R}}_{DBC} + \partial_B\hat\beta^D L^{\mathbb{1},\mathcal{R}}_{ADC} + \partial_C\hat\beta^D L^{\mathbb{1},\mathcal{R}}_{ABD} + \hat\beta^D\partial_D L^{\mathbb{1},\mathcal{R}}_{ABC}. \tag{A9}$$

We can also define $\chi^{\mathcal{R}}_{ABC}$ through the finite renormalized correlator $\Gamma^{\mathcal{R}}_{ABC}$ by using a relation analogical to (A4) and projecting onto P_{yz} etc.

The scheme change in $\chi^{\mathcal{R}}_{ABC}$ is implemented via constant ω_{ABC}, which is now *independent* of ω_{AB} from (A6) (this follows since the structure $L^{\mathbb{1},\mathcal{R}}_{ABC}$ is independent of the two-point function). Under such scheme change, we have $L^{\mathbb{1},\mathcal{R}}_{ABC} \to L^{\mathbb{1},\mathcal{R}}_{ABC} - \omega_{ABC}$, and therefore,

$$\chi^{\mathcal{R}}_{ABC} \to \chi^{\mathcal{R}}_{ABC} + \mathcal{L}_\beta\omega_{ABC}. \tag{A10}$$

APPENDIX B: DERIVATION OF THE THREE-POINT SUM RULE

In this Appendix, we provide a derivation of the three-point sum rule (3) used in Sec. II to derive a RG-scale integral representation for the Euler flow $\Delta\beta_a$. Using the quantum action principle, a constraint on a gravity counterterm is worked out in Sec. B. 1, which is then used in the anomaly matching argument in Sec. B. 2 to derive the sum rule.

1. Renormalization in curved space and $\beta_a^{\rm UV}$

In an external gravitational field, one needs to add counterterms,

$$\mathcal{L}_{\rm gravity} = -(a_0 E_4 + c_0 W^2 + b_0 H^2), \tag{B1}$$

to the action to renormalize the theory [37]. The bare couplings are defined as $a_0 = \mu^{d-4}(a^{\mathcal{R}}(\mu) + L_a^{\mathcal{R}}(\mu))$ etc., and geometric quantities are the same as in (2) except for $\Box R$, which being a total derivative does not contribute to the action. The main idea is that the quantum action principle (differentiation with respect to sources) leads to finite quantities and thus, to constraints on the counterterms. Concretely, a triple Weyl variation $\delta s(x)$ ($g_{\mu\nu} \to e^{-2s(x)} g_{\mu\nu}$) leads to

$$\int_x \int_y e^{i(p_x\cdot x + p_y \cdot y)} \frac{\delta^3}{\delta s(x)\delta s(y)\delta s(0)} \ln \mathcal{Z} = (2k_\epsilon a_0 - 8b_0)\lambda_3 + \Gamma_{\theta\theta\theta}(p_x,p_y) = [\text{finite}], \tag{B2}$$

where the abbreviation $k_\epsilon \equiv (d-4)(d-3)(d-2)$ is introduced and we used (10) to include the dynamical contribution. By using (11), we conclude that the finiteness of $L_{\theta\theta\theta}^{\mathbb{1},\mathcal{R}}$ ensures finiteness of the combination $(2k_\epsilon a_0 - 8b_0)$ in (B2). Since b_0 has been shown to be finite [18], it is to be concluded that the quantity $k_\epsilon a_0$ is finite. In particular, this means that the $\epsilon \to 0$ limit $k_\epsilon a_0$ is meaningful

$$\lim_{\epsilon\to 0} k_\epsilon a_0 \equiv \lim_{\epsilon\to 0} k_\epsilon (L_a^{\rm UV} + a^{\rm UV}) = -2\beta_a^{\rm UV}. \tag{B3}$$

In the last step, we used that $a^{\rm UV}$ is finite and that $L_a^{\rm UV} = \frac{\beta_a^{\rm UV}}{2\epsilon}$. The latter follows from $\beta_a = -(\frac{d}{d\ln\mu} - 2\epsilon)L_a$ and the stationarity property $\frac{d}{d\ln\mu}L_a^{\rm UV} = 0$ at FPs [which can be seen by writing $L_a \sim x_1 + x_2(g^I - g^{I,\rm UV})$ with $x_{1,2}$ constants and using $\beta^{I,\rm UV} = 0$]. Equation (B3) is a relevant observation as this implies finiteness of the corresponding term in the dilaton effective action.

2. Sum rule from the dilaton effective action

In the three-point sum rule (3), the Euler flow β_a arises, in dimensional regularization, from an evanescent operator. This can be seen by writing the d-dimensional Euler term as a sum of a four dimensional and an evanescent term

$$\sqrt{g}E_d = \sqrt{g}E_4 - k_\epsilon e^{2\epsilon s}(-2\Box s(\partial s)^2 + (\partial s)^4 - 2\epsilon(\partial s)^4), \tag{B4}$$

where we have assumed the conformally flat metric $g_{\alpha\beta} = e^{-2s(x)}\delta_{\alpha\beta}$ and $k_\epsilon \sim \epsilon$ is defined below (B2). The $\sqrt{g}E_4$ term is a total derivative,

$$\sqrt{g}E_4 = \partial O = -4(d-3)(d-2)\Big[\frac{1}{2}\Box(e^{2\epsilon s}(\partial s)^2) + \partial(e^{2\epsilon s}\partial s((1-\epsilon)(\partial s)^2 - \Box s))\Big], \tag{B5}$$

characteristic of topological terms. The evanescent part of the gravitational counterterms (B1) becomes the Wess-Zumino term of the dilaton effective action in [6]

$$\begin{aligned}\mathcal{L}_{gravity} &\supset a_0 \int d^dx\sqrt{g}(E_d - E_4) \\ &= -k_\epsilon a_0 \int d^dx(-2\Box s(\partial s)^2 + (\partial s)^4 - 2\epsilon(\partial s)^4) \\ &\xrightarrow{\epsilon\to 0} 2\beta_a^{\rm UV}\int d^4x(-2\Box s(\partial s)^2 + (\partial s)^4) = 2\beta_a^{\rm UV} S_{WZ},\end{aligned} \tag{B6}$$

where we have used (B3). In the preceding argument, the finiteness of $k_\epsilon a_0$ (and b_0) was essential to ensure UV finiteness of the dilaton effective action and match the bare coefficient of the Wess-Zumino term to the Euler anomaly $\beta_a^{\rm UV}$.

Similarly, the IR effective action contains the term $2\beta_a^{\rm IR} S_{WZ}$, which contributes at $\mathcal{O}(s^3)$

$$\ln\mathcal{Z} = -4\bar{b}^{\rm IR}\int_x (\Box s)^2 - (4\beta_a^{\rm IR} - 8\bar{b}^{\rm IR})\int_x (\partial s)^2 \Box s + \cdots. \tag{B7}$$

We are now ready to put all the pieces together. By Fourier transforming the third functional derivative with respect to s of (B7), we see that at low momenta, the lhs of (B2) behaves as

$$-(4\beta_a^{\rm IR} - 8\bar{b}^{\rm IR})\lambda_3 + \cdots, \tag{B8}$$

where the dots stand for nonlocal contributions subleading in the momentum expansion. Assuming momentum conservation, $p_z = -(p_x + p_y)$, λ_3 (12) assumes the form

$$\lambda_3 = 4[(p_x\cdot p_y)^2 - p_x^2 p_y^2], \tag{B9}$$

with the associated projector $\hat{P}_{\lambda_3}\lambda_3 = 1$ being

$$\hat{P}_{\lambda_3} = \frac{1}{96}[(\partial_{p_x} \cdot \partial_{p_y})^2 - \partial^2_{p_x}\partial^2_{p_y}], \tag{B10}$$

for which the P_3 structure automatically vanishes ($\hat{P}_{\lambda_3} P_3 = 0$). Applying $\hat{P}_{\lambda_3}$ to the right-hand side of (B2), one gets

$$-\hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(p_x, p_y)\Big|_{p_x=p_x=0} - (4\beta_a^{\rm UV} - 8\bar{b}^{\rm UV}) = -(4\beta_a^{\rm IR} - 8\bar{b}^{\rm IR}), \tag{B11}$$

where we used that $(2k_\epsilon a_0 - 8b_0) \to -(4\beta_a^{\rm UV} - 8\bar{b}^{\rm UV})$ for $\epsilon \to 0$. The three-point sum rule in momentum space follows

$$\Delta\beta_a = 2\Delta\bar{b} - \frac{1}{4}\hat{P}_{\lambda_3}\Gamma_{\theta\theta\theta}(p_x, p_y)\Big|_{p_x=p_y=0}, \tag{B12}$$

which in position space assumes the form

$$\Delta\beta_a = \frac{1}{3\cdot 2^8}\Bigg(\underbrace{\int_x x^4 \langle \Theta(x)\Theta(0)\rangle_c}_{3\cdot 2^9\cdot\Delta\bar{b}} - 2\int_x\int_y [(x\cdot y)^2 - x^2 y^2]\langle\Theta(x)\Theta(y)\Theta(0)\rangle_c\Bigg). \tag{B13}$$

The Euler flow formula (B13) is invariant under the addition of the local $\delta\mathcal{L} \sim \omega_0 R^2$ term unlike the sum rule for $\Delta\bar{b}$ (8) which needs to be amended. Such a scheme change (denoted as "R^2 scheme" in [4]) should be viewed as independent of (A6) and (A10).[12] More concretely, the contribution of such a shift precisely cancels between two- and three-point parts in (B13). The reason this has to happen is that β_a is well-defined at each FP and not only as a difference, like $\Delta\bar{b}$. At last, we would like to mention that Eq. (B13) itself has been derived by Anselmi [11] using different methods.

APPENDIX C: VANISHING OF χ_{ggg} AT THE UV FIXED POINT—FORMAL ARGUMENT

In this Appendix, we will demonstrate how $\chi_{ggg} = 0$ in the free theory can be derived by using standard OPE arguments [38]. We start the discussion by considering a general perturbation,

$$\delta S = \lambda \int_x O_\lambda(x), \tag{C1}$$

for some coupling constant λ that can be set to 1 without loss of generality. We now deform the constant $\lambda \to \lambda + \delta\lambda$; the corrections to a generic correlator $\langle\ldots\rangle$ in the perturbed theory read

$$\langle\ldots\rangle = \langle\ldots\rangle_{\delta\lambda=0} + \delta\lambda\int_x \langle O_\lambda(x)\ldots\rangle_{\delta\lambda=0} + \frac{1}{2}\delta\lambda^2\int_x\int_y \langle O_\lambda(x)O_\lambda(y)\ldots\rangle_{\delta\lambda=0} + \mathcal{O}(\delta\lambda^3). \tag{C2}$$

The $\delta\lambda^2$ term in (C2) can be obtained by using the OPE

$$O_\lambda(x)O_\lambda(y) = \frac{C^\lambda_{\lambda\lambda}}{|x-y|^4}O_\lambda(x) + \cdots, \tag{C3}$$

where dots encompass terms irrelevant for the calculation. By inserting this expression back into (C2) and evaluating the $\int_y$ integral with a UV cutoff Λ, one finds that a logarithmic divergence appears

$$\sim C^\lambda_{\lambda\lambda}\delta\lambda^2 \ln\Lambda \int_x \langle O_\lambda(x)\ldots\rangle_{\delta\lambda=0}. \tag{C4}$$

This divergence can be removed by adding a counterterm of the form

$$\delta\lambda^2 C^\lambda_{\lambda\lambda}\ln(\Lambda/\mu)\int_x O_\lambda(x). \tag{C5}$$

However, adding such a term amounts to introducing a β function

$$\beta_\lambda \sim C^\lambda_{\lambda\lambda}\delta\lambda^2 + \mathcal{O}(\delta\lambda^3). \tag{C6}$$

Hence, the nonvanishing of the β_λ function and the OPE coefficient $C^\lambda_{\lambda\lambda}$ are directly related.

We restrict our attention to the case at hand where $O_\lambda \equiv O_g = G^2$ in the free-field theory. Since the value of δg only affects the normalization of the kinetic term, it is clear that the theory remains free (and therefore a CFT) for any value of δg. Thus, $\beta = 0$ and in the free theory,

$$C^g_{gg} = 0. \tag{C7}$$

Using the fact that in a CFT the three-point function of an operator O_λ is proportional to $C^\lambda_{\lambda\lambda}$, we conclude that also the three-point function of O_g has to vanish, which directly implies that

$$\chi^{\rm free}_{ggg} = 0. \tag{C8}$$

Note that the above argument shows that O_g is exactly marginal at the Gaussian fixed point. In general, an operator is called exactly marginal if its beta function vanishes, which is equivalent to the vanishing of the corresponding three-point function as shown in [20,38].[13]

[12] For a more thorough discussion of various classes of schemes, we refer the reader to [4], namely Secs. 2.3.2 and 2.3.3 of this reference.

[13] Note that the nonzero QCD β function should be understood as a consequence of coupling to fermions and gluons rather than a deformation by G^2.

APPENDIX D: THE $\mathcal{R}_{3\chi}$ SCHEME IN PERTURBATION THEORY

In this Appendix, we will construct explicitly the solutions (26) and (32) for a theory with a trivial UVFP and a Banks-Zaks FP in the IR.

We start with the analysis near the trivial UVFP. The scheme (37) means that given a $\chi^{\mathcal{R}}_{ggg}$, we should be able to obtain ω_{ggg} to any given order in a_s through (26). We demonstrate how this works for the first two nonvanishing orders in perturbation theory. Introducing the notation,

$$\chi^{\mathcal{R}}_{ggg} = \chi^{(1)}_{ggg} a_s + \chi^{(2)}_{ggg} {a_s}^2 + \mathcal{O}({a_s}^3), \tag{D1}$$

the scheme change ω_{ggg} to $\mathcal{O}({a_s}^2)$ is given by performing the integral on the right-hand side of (26) and expanding the result in a_s,

$$\omega_{ggg}\|_{\mathrm{UV}} = \frac{\chi^{(1)}_{ggg}}{6\beta_0} + \frac{(2\beta_1\chi^{(1)}_{ggg} - \beta_0\chi^{(2)}_{ggg})a_s}{2\beta_0^2} - \frac{7(2\beta_1^2\chi^{(1)}_{ggg} - \beta_0\beta_1\chi^{(2)}_{ggg}){a_s}^2}{4\beta_0^3} + \mathcal{O}({a_s}^3), \tag{D2}$$

where the two-loop β function is parameterized by $\beta = -\beta_0 a_s - \beta_1 {a_s}^2$. It is easily verified that

$$\chi^{\mathcal{R}_{3\chi}}_{ggg} = \chi^{\mathcal{R}}_{ggg} + (2\beta\partial_{\ln a_s} + 6(\partial_{\ln a_s}\beta))\omega_{ggg} = 0 + \mathcal{O}({a_s}^3). \tag{D3}$$

Note that (D2) is $\mathcal{O}(1)$ and therefore, nonzero at the UVFP even though $\chi^{\mathcal{R}}_{ggg}$ itself vanishes there.

In the IR, we assume a (Banks-Zaks) FP at $a_s^{\mathrm{IR}} = -\frac{\beta_0}{\beta_1} \ll 1$, which exists for an asymptotically free theory with $\beta_0 > 0, \beta_1 < 0$. The three metric (D1) expands to

$$\chi^*_{ggg} \equiv \chi^{\mathcal{R}}_{ggg}(a_s^{\mathrm{IR}}) = \frac{\beta_0^2}{\beta_1}\left(-r + \frac{\chi^{(2)}_{ggg}}{\beta_1}\right) + \mathcal{O}(\beta_0^3), \tag{D4}$$

where we used that $\chi^{(1)}_{ggg} = \beta_0 r$ for some finite constant r [c.f. (40)]. Close to this FP, (32) admits a perturbative solution in $\Delta a_s \equiv a_s - a_s^{\mathrm{IR}}$,

$$\omega_{ggg}|_{\mathrm{IR}} = \frac{\chi^*_{ggg}}{6\gamma^*} - \frac{(2\beta_1 r - \chi^{(2)}_{ggg})}{4\beta_0}\Delta a_s + \frac{7(2\beta_1^2 r - \beta_1\chi^{(2)}_{ggg})}{20\beta_0^2}(\Delta a_s)^2 + \mathcal{O}((\Delta a_s)^3). \tag{D5}$$

Above we used that $\gamma^* = -\partial_{\ln a_s}\beta|_{a_s^*} = \frac{\beta_0^2}{\beta_1}$ is the anomalous dimension of O_g (field strength tensor squared). In the limit $a_s \to a_s^{\mathrm{IR}}$, we get the expected dependence (31). Direct computation shows that (D5) is compatible with (D3). The leading $\mathcal{O}(1)$ parts of the IR solution (D5) and the UV solution (D2) match, up to $\mathcal{O}(\beta_0)$ corrections, provided that

$$-2\beta_1 r + \chi^{(2)}_{ggg} = \mathcal{O}(\beta_0). \tag{D6}$$

The above should be regarded as the necessary condition for continuity of ω_{ggg}, equivalent to (34).

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