Wormholes with $\rho(R, R')$ matter in $f(R, T)$ gravity

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Models of static wormholes are investigated in the framework of $f(R,T)$ gravity (where R is the curvature scalar and T is the trace of the energy-momentum tensor). An attempt to link the energy density of the matter component to the Ricci scalar is made, which for the Morris and Thorne wormhole metric with constant redshift function yields $R(r) = 2b'(r)/r^2$. Exact wormhole solutions are obtained for three particular cases when $f(R,T) = R + 2\lambda T$: $\rho(r) = \alpha R(r) + \beta R'(r)$, $\rho(r) = \alpha R^2(r) + \beta R'(r)$, and $\rho(r) = \alpha R(r) + \beta R^2(r)$. Additionally, traversable wormhole models are obtained for the two first cases. However, when the wormhole matter energy density is of the third type, only solutions with constant shape correspond to traversable wormholes. Exact wormhole solutions possessing the same properties can be constructed when $\rho = \alpha R(r) + \beta R^{-2}(r)$, $\rho = \alpha R(r) + \beta r R^2(r)$, $\rho = \alpha R(r) + \beta r^{-1} R^2(r)$, $\rho = \alpha R(r) + \beta r^2 R^2(r)$, $\rho = \alpha R(r) + \beta r^3 R^2(r)$, and $\rho = \alpha r^m R(r) \log(\beta R(r))$, as well. On the other hand, for $f(R,T) = R + \gamma R^2 + 2\lambda T$ gravity, two wormhole models are constructed, assuming that the energy density of the wormhole matter is $\rho(r) = \alpha R(r) + \beta R^2(r)$ and $\rho(r) = \alpha R(r) + \beta r^3 R^2(r)$, respectively. In this case, the functional form of the shape function is taken to be $b(r) = \sqrt{\hat{r}_0 r}$ (where $\hat{r}_0$ is a constant) and the possible existence of appropriate static traversable wormhole configurations is proven. The explicit forms of the pressures P_r and P_l leading to this result are found in both cases. As a general feature, the parameter space can be divided into several regions according to which of the energy conditions are valid. These results can be viewed as an initial step towards using specific properties of the new exact wormhole solutions to propose new functional forms for describing the matter content of wormholes.

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I. INTRODUCTION

A well-known and quite interesting solution in general relativity (GR) is a geometrical bridge connecting two distant regions of the Universe. It may also be possible for this type of bridge to connect two different universes. Weyl was the first to discuss this concept of a wormhole or bridge in 1921 [1]. After that, the now famous example of a static wormhole appeared, now known as an Einstein-Rosen bridge [2]. According to the discussions in the more recent literature, a traversable wormhole admits superluminal travel as a global effect of spacetime topology, making the object a very interesting concept in modern theoretical physics (see, e.g., Refs. [3–31]). In general, a wormhole may be visualized as a tunnel with two mouths or ends, through which observers may safely travel. A wormhole can be described in terms of a metric with several constraints, which any solution must satisfy in order to qualify as a wormhole. The metric of a static wormhole can be written as [32]

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ and $V(r) = 1 - b(r)/r$. The function $b(r)$ in Eq. (1) is called the shape function, since it represents the spatial shape of the wormhole. The redshift function $U(r)$ and the shape function $b(r)$ are bound to obey the following conditions [32]:

(1) The radial coordinate r lies between $r_0 \leq r < \infty$, where r_0 is the radius of the throat. The throat is the minimal surface area of the attachment.

(2) At the throat, $r = r_0$, $b(r_0) = r_0$, and for the region outside of the throat $1 - b(r)/r > 0$.

(3) $b'(r_0) < 1$ (with the $'$ meaning a derivative with respect to r), i.e., it should obey the flaring-out condition at the throat.

(4) $b(r)/r \to 0$ as $|r| \to \infty$, to ensure the asymptotic flatness of the space-time geometry.

(5) $U(r)$ must be finite and nonvanishing at the throat r_0.

However, in theory, it could be possible that the wormhole solution is not asymptotically flat, i.e., that the $b(r)/r \to 0$ condition is not satisfied and the wormhole is nontraversable. It is known from studies in the recent literature that to make the wormhole traversable in these cases one can effectively glue an exterior flat geometry into the interior geometry at some junction radius and thus get a useful result. Below, for some of the exact wormhole models to be considered in this paper, we will see that they actually are nontraversable wormholes, and this is the reason why this procedure could become potentially important for us. But, on the other hand, since the study of such models would be cumbersome, and it lies beyond the scope of the present paper, we will omit this treatment here. Another interesting aspect concerning traversable wormholes is their possible existence due to exotic matter at the throat, thus violating the null energy condition (see, for instance, Refs. [32–35]). This simply implies that the exotic matter either induces very strong negative pressures, or that the energy density is negative, as seen by static observers.

An important point is that the link between the existence of matter with negative pressure (in order to construct the wormhole configuration) and the explanation of the recently discovered accelerated expansion of the Universe has generated renewed interest in wormholes. It is well known that, in the case of GR, it is necessary to have an energy source generating a negative pressure in order to accelerate the expansion of the Universe (see Refs. [36–66] and references therein). The same source could in principle be used to construct a wormhole configuration for distant travel. There are actually different dark energy models, including some fluid models such as the Chaplygin and van der Waals gasses, with nonlinear equations of state. In the recent literature, various ways of representing dark energy have been proposed, some of which have a lot in common with the models discussed here. On the other hand, in order to make a specific dark energy model work competitively well, one needs to include additional ideas like a nongravitational interaction between dark energy and dark matter. A nongravitational interaction can be useful for solving the cosmological coincidence problem as well, as has been discussed in various papers using phase-space analyses. However, a nongravitational interaction can also suppress or generate future time singularities. Detailed discussions of some of these topics can be found in the references at the end of this paper.

An alternative way to avoid dark energy and nongravitational interactions of any sort when explaining the observational data is to consider modified theories of gravity. In the recent literature, several well-motivated modifications of GR have been used to construct wormholes, black holes, gravastars, and other kinds of star models. The advantage of a modification is the possibility of avoiding the need to introduce any sort of dark energy (see, e.g., Refs. [67–85]), making it very attractive for different applications. More precisely, a generic modification will add a term to the field equations which, in comparison to the field equation for GR, will be interpreted as dark energy. A modified theory can be constructed by changing either the geometric or the matter part of the theory. In other words, each modification comes with a particular interpretation of the energy content of the Universe. which is responsible for its dynamics and physics.

On the other hand, considering extra material contributions can give rise to a viable modified theory of gravity, as in the case of $f(R,T)$ gravity, where T is the trace of the energy-momentum tensor, given by the following form of the total action [67]:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R,T) + \int d^4x\sqrt{-g}L_m, \quad (2)$$

where $f(R,T)$ is an arbitrary function of the Ricci scalar R and the trace of the energy-momentum tensor T, while g is the metric determinant, and L_m is the matter Lagrangian density, which is related to the energy-momentum tensor as

$$T_{ij} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}L_m)}{\partial g^{ij}} - \frac{\partial}{\partial x^k}\frac{\partial(\sqrt{-g}L_m)}{\partial(\partial g^{ij}/\partial x^k)}\right]. \quad (3)$$

We would *a priori* expect that the material corrections yielding this $f(R,T)$ gravity are due to the existence of imperfect fluids. On the other hand, quantum effects such as particle production can also become a motivation to consider matter-content-modified theories of gravity. However, each of these specific modifications (changes in the classical matter part of the theory) must be dealt with carefully, in order to avoid misleading interpretations of the results' physical meaning. Actually, $f(R,T)$ gravity seems well suited to address wormhole construction issues and (being free from any misleading aspects) has been considered extensively in the recent literature. On the other hand, wormholes have not been detected yet, and thus our final aim in the study of these solutions is only to improve our theoretical knowledge. The growing number of papers that address different aspects of wormholes aim at clarifying their physical nature, and this forces us to make different assumptions about their matter content, some of which can make the field equations too complicated to be treated analytically. Fortunately, various interesting exact wormhole models have already been obtained for GR and some modified theories of gravity. The models of the present paper will also be dealt with analytically, and will provide a new class of wormhole solutions not reported elsewhere.

In particular, we are interested in finding new exact static wormhole models by assuming different forms for their matter content, in the frame of $f(R,T)$ gravity with the action given by Eq. (2). In other words, we will construct exact wormhole models by assuming that the energy density of the wormhole matter can be described by either $\rho(r)=\alpha R(r)+\beta R'(r)$, $\rho(r)=\alpha R^2(r)+\beta R'(r)$, or $\rho(r)=\alpha R(r)+\beta R^2(r)$, and with $f(R,T)=R+2\lambda T$. By studying a particular wormhole solution corresponding to $\rho(r)=\alpha R(r)+\beta R'(r)$, we conclude that, for appropriate values of the parameters of the model, we can expect violations of the null energy condition (NEC) (in terms of the pressure P_r) and the dominant energy condition (DEC) (in terms of P_l) at the throat. On the other hand, $\rho\geq 0$, and the validity of the NEC and DEC in terms of the pressures P_l and P_r is assured everywhere, including at the throat of the wormhole. Therefore, we also report a violation of the weak energy condition (WEC) in terms of the pressure P_r, while it will be satisfied in terms of the other pressure P_l. Moreover, for the same model, we also conclude that in the case $\beta>0$ we would mainly observe regions where the violation of the NEC in terms of P_r causes a violation of the DEC in terms of P_l. However, if we consider a domain where $\beta<0$, then we find regions where both energy conditions in terms of both pressures are simultaneously valid. We must also mention that the validity of the WEC in this case will also be observed owing to the fact that $\rho\geq 0$ is satisfied. A similar situation is obtained when considering the impact of the parameter α on the validity of the energy conditions. A detailed analysis of the energy conditions for other models can be found in the appropriate subsections below.

In the second part of the paper, corresponding to a different choice for $f(R,T)=R+\gamma R^2+2\lambda T$ gravity, in addition to the form of the matter-energy density, we also specify the functional form of the shape function and establish the possible existence of appropriate static wormhole configurations, i.e., we find the forms of the pressures P_r and P_l yielding a static traversable wormhole solution. In particular, we assume the two energy density profiles $\rho(r)=\alpha R(r)+\beta R^2(r)$ and $\rho(r)=\alpha R(r)+\beta r^3R^2(r)$ to describe the matter content of the wormhole. Further, we take the functional form of the shape function to be $b(r)=\sqrt{\hat{r}_0 r}$ (where $\hat{r}_0$ is a constant). Studying the model with $\rho(r)=\alpha R(r)+\beta R^2(r)$ leads to the result that there is a region where both energy conditions, i.e., the NEC ($\rho+P_r\geq 0$ and $\rho+P_l\geq 0$) and DEC ($\rho-P_r\geq 0$ and $\rho-P_l\geq 0$), in terms of both pressures are valid. In all cases, the Ricci scalar has the form

$$R(r)=\frac{2b'(r)}{r^2}, \tag{4}$$

which is obtained directly from the wormhole metric (1). Hereafter, we will omit the argument r and write R instead of $R(r)$.

The paper is organized as follows. In Sec. II we present a detailed form of the field equations to be solved for $f(R,T)=R+2\lambda T$ and $f(R,T)=R+\gamma R^2+2\lambda T$ gravity. In Sec. III we discuss three exact wormhole solutions, assuming that the matter content of the wormhole can be described by one of the energy density profiles $\rho(r)=\alpha R(r)+\beta R'(r)$, $\rho(r)=\alpha R^2(r)+\beta R'(r)$, or $\rho(r)=\alpha R(r)+\beta R^2(r)$, when $f(R,T)=R+2\lambda T$. In Sec. IV we obtain two wormhole solutions by assuming that $f(R,T)=R+\gamma R^2+2\lambda T$ and that the profile of the wormhole matter is either $\rho(r)=\alpha R(r)+\beta R^2(r)$ or $\rho(r)=\alpha R(r)+\beta r^3R^2(r)$. In all cases, we perform a detailed study of the validity of the energy conditions. Finally, Sec. V is devoted to a discussion and conclusions, with indications of possible future directions to pursue in order to complete this research project.

II. FIELD EQUATIONS

In this section, we address some issues that are crucial for constructing exact traversable wormhole solutions. Following Refs. [30,31], we consider the case $U(r)=1$. Moreover, we provide the explicit form of the field equations for both gravities. To proceed, let us assume that L_m only depends on the metric components, which means that

$$T_{ij}=g_{ij}L_m-2\frac{\partial L_m}{\partial g^{ij}}. \tag{5}$$

Varying the action (2) with respect to the metric g_{ij} provides the field equations

$$\begin{aligned} &f_R(R,T)\left(R_{ij}-\frac{1}{3}Rg_{ij}\right)+\frac{1}{6}f(R,T)g_{ij}\\ &=8\pi G\left(T_{ij}-\frac{1}{3}Tg_{ij}\right)-f_T(R,T)\left(T_{ij}-\frac{1}{3}Tg_{ij}\right)\\ &\quad-f_T(R,T)\left(\theta_{ij}-\frac{1}{3}\theta g_{ij}\right)+\nabla_i\nabla_j f_R(R,T), \end{aligned} \tag{6}$$

with $f_R(R,T)=\frac{\partial f(R,T)}{\partial R}$, $f_T(R,T)=\frac{\partial f(R,T)}{\partial T}$, and

$$\theta_{ij}=g^{ij}\frac{\partial T_{ij}}{\partial g^{ij}}. \tag{7}$$

To obtain wormhole solutions, we make the further assumption that $L_m=-\rho$ in order not to imply the vanishing of the extra force. Now, if we take into account that $f(R,T)=R+2f(T)$ with $f(T)=\lambda T$ (where λ is a constant), we can rewrite the above equations as

$$G_{ij}=(8\pi+2\lambda)T_{ij}+\lambda(2\rho+T)g_{ij}, \tag{8}$$

where G_{ij} is the usual Einstein tensor. After some algebra, for three of the components of the field equations (8) we get [19]

$$\frac{b'(r)}{r^2} = (8\pi + \lambda)\rho - \lambda(P_r + 2P_l), \tag{9}$$

$$-\frac{b(r)}{r^3} = \lambda\rho + (8\pi + 3\lambda)P_r + 2\lambda P_l, \tag{10}$$

$$\frac{b(r) - b'(r)r}{2r^3} = \lambda\rho + \lambda P_r + (8\pi + 4\lambda)P_l, \tag{11}$$

where we have used the static wormhole metric given by Eq. (1). In deriving the above equations we have considered an anisotropic fluid with matter content of the form $T^i_j = \mathrm{diag}(-\rho, P_r, P_l, P_l)$, where $\rho = \rho(r)$ $[P_r = P_r(r)$ and $P_l = P_l(r)]$ is the energy density, while P_r and P_l are the radial and lateral pressures, respectively. They are measured perpendicularly to the radial direction. The trace T of the energy-momentum tensor reads $T = -\rho + P_r + 2P_l$. Moreover, Eqs. (9)–(11) admit the solutions

$$\rho = \frac{b'(r)}{r^2(8\pi + 2\lambda)}, \tag{12}$$

$$P_r = -\frac{b(r)}{r^3(8\pi + 2\lambda)}, \tag{13}$$

and

$$P_l = \frac{b(r) - b'(r)r}{2r^3(8\pi + 2\lambda)}. \tag{14}$$

It is obvious that when we choose a form for the energy density the shape function $b(r)$ is obtained by direct integration of Eq. (12).

To finish this section, we recall some aspects concerning the above calculations that will yield the equations required to construct wormhole solutions in

$$f(R, T) = R + \gamma R^2 + 2f(T) \tag{15}$$

gravity. In particular, it is easy to see that for the wormhole metric (1), for $\Box f_R$, we have

$$\Box f_R = \left(1 - \frac{b(r)}{r}\right)\left(\frac{f'_R}{r} + f''_R + \frac{f'_R(b(r) - b'(r)r)}{2r^2(1 - b(r)/r)}\right), \tag{16}$$

while

$$\nabla_1\nabla_1 f_R = \frac{f'_R(b(r) - b'(r)r)}{2r^2(1 - b(r)/r)} + f''_R, \tag{17}$$

$$\nabla_2\nabla_2 f_R = r\left(1 - \frac{b(r)}{r}\right)f'_R, \tag{18}$$

$\nabla_0\nabla_0 f_R = 0$, and $\nabla_3\nabla_3 f_R = r(1 - \frac{b(r)}{r})f'_R \sin^2\theta$. Therefore, after some algebra, for the field equations we obtain

$$\frac{b'(r)}{r^2} = 8\pi\rho - \frac{\gamma}{2}R^2 - \lambda T + \Box f_R, \tag{19}$$

$$-\frac{b(r)}{r^3} = 8\pi P_r + 2\lambda(P_r + \rho) + \frac{\gamma}{2}R^2 + \lambda T + 2\gamma R\left(\frac{b(r) - b'(r)r}{r^3}\right) + \frac{b(r) - b'(r)r}{2r^2}f'_R + \left(1 - \frac{b(r)}{r}\right)f''_R - \Box f_R, \tag{20}$$

and

$$\frac{b(r) - b'(r)r}{2r^3} = 8\pi P_l + 2\lambda(P_l + \rho) - \gamma R\frac{b(r) + b'(r)r}{r^3} + \frac{\gamma}{2}R^2 + \lambda T + \frac{1}{r}\left(1 - \frac{b(r)}{r}\right)f'_R - \Box f_R. \tag{21}$$

III. MODELS IN $f(R,T) = R + 2\lambda T$ GRAVITY

In this section we perform an analysis of three different exact static wormhole models, taking into account that $f(R, T) = R + 2\lambda T$.

A. Matter with $\rho(r) = \alpha R(r) + \beta R'(r)$

Let us study wormhole formation in the presence of matter when its energy density is given by

$$\rho(r) = \alpha R(r) + \beta R'(r), \tag{22}$$

where the prime denotes a derivative with respect to r, while $R(r)$ is the Ricci scalar given by Eq. (4). With such an assumption, the direct integration of Eq. (12) gives a wormhole solution described by the following shape function:

$$b(r) = c_2 - \frac{4\beta c_1(\lambda + 4\pi)e^{\frac{-Ar}{4\beta\lambda + 16\pi\beta}}(32\beta^2(\lambda + 4\pi)^2 + A^2r^2 + 8\beta(\lambda + 4\pi)Ar)}{A^3}, \tag{23}$$

where $A = 4\alpha\lambda + 16\pi\alpha - 1$. Despite the long and complicated form of the shape function (23), the derivative has the very simple form

$$b'(r) = c_1 r^2 e^{\frac{-Ar}{4\beta\lambda + 16\pi\beta}}. \tag{24}$$

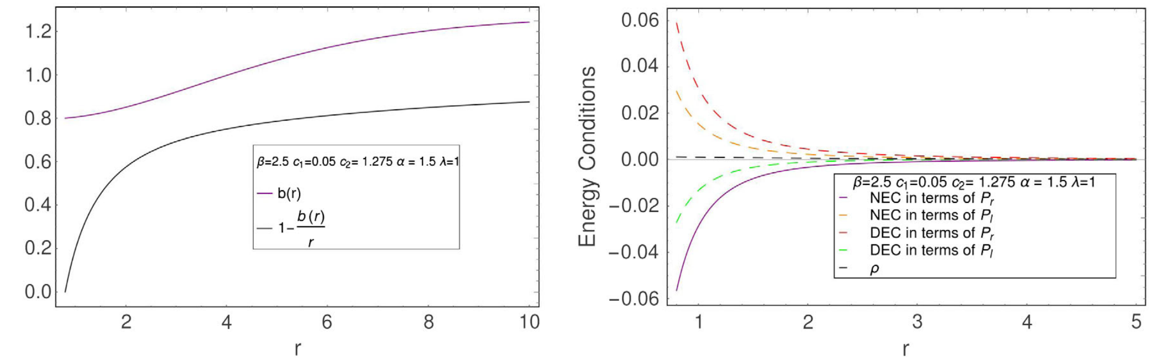

FIG. 1. Behavior of the shape function $b(r)$ for the model described by Eq. (22) (left panel). We see that the solution for $b(r)$ satisfies $1 - b(r)/r > 0$, for $r > r_0$. The right panel shows that the DEC in terms of P_r and the NEC in terms of P_l are valid everywhere, and that $\rho \geq 0$ also holds everywhere. On the other hand, the same plot demonstrates that the NEC and the DEC in terms of P_r and P_l, respectively, are not valid at the throat of the wormhole. This particular wormhole model was obtained for $c_1 = 0.05$, $c_2 = 1.275$, $\alpha = 1.5$, $\beta = 2.5$, and $\lambda = 1$.

Therefore, after some algebra, the explicit form of the energy density is

$$\rho = \frac{c_1}{2\lambda + 8\pi} e^{\frac{-Ar}{4\beta\lambda + 16\pi\beta}}, \tag{25}$$

and it is not hard to find the explicit forms of the pressures P_r and P_l from Eqs. (13) and (14), respectively. After some algebra, we obtain

$$P_r = -\frac{c_2}{2r^3(\lambda + 4\pi)} + \frac{4\beta c_1 e^{r\left(\frac{1}{4\beta\lambda + 16\pi\beta} - \frac{\alpha}{\beta}\right)}(32\beta^2(\lambda + 4\pi)^2 + 8\beta(\lambda + 4\pi)Ar + A^2 r^2)}{A^3} \tag{26}$$

and

$$P_l = \frac{c_2}{4(\lambda + 4\pi)r^3} - \frac{c_1 e^{\frac{-Ar}{4\beta\lambda + 16\pi\beta}}(4\beta(\lambda + 4\pi) + Ar)(32\beta^2(\lambda + 4\pi)^2 + A^2 r^2)}{4A^3(\lambda + 4\pi)r^3}, \tag{27}$$

respectively. Now, let us discuss a particular wormhole solution described by ρ, P_r, and P_l given by Eqs. (25), (26), and (27), respectively.

A particular wormhole solution can be found, for instance, if we consider $c_1 = 0.05$, $c_2 = 1.275$, $\alpha = 1.5$, $\beta = 2.5$, and $\lambda = 1$. The throat of this wormhole model occurs at $r_0 \approx 0.8$ and $b'(r_0) \approx 0.02$. The behaviors of the shape function and $1 - b(r)/r$ are presented in the left plot of Fig. 1. On the other hand, the behavior of the energy conditions can be found in the right plot of the same figure. In particular, for this specific wormhole solution we should expect a violation of the NEC in terms of P_r and of the DEC in terms of P_l at the throat. On the other hand, $\rho \geq 0$, and the validity of the NEC and DEC in terms of the pressures P_l and P_r can be observed everywhere, including at the throat of the wormhole. Therefore, we also observe the violation of the WEC in terms of P_r, while it remains valid in terms of P_l.

In general, our study shows that for the case $\beta > 0$ we mainly get regions where the violation of the NEC in terms of P_r causes a violation of the DEC in terms of P_l. However, if we consider $\beta < 0$ regions, then we can find regions where both energy conditions in terms of both pressures are satisfied at the same time. The plots in Fig. 2 correspond to an example of one of these valid regions, where both the NEC and the DEC in terms of both pressures are fulfilled, for $c_2 = 1.5$, $c_1 = 0.5$, $\alpha = 0.5$, $\lambda = -10$, and different values of the parameter β (< 0). Also, it should be mentioned that the validity of the WEC in this case follows from the fact that $\rho \geq 0$ is also satisfied. A similar situation was reached when we studied the impact of α on the validity of the energy conditions.

B. Matter with $\rho(r) = \alpha R^2(r) + \beta R'(r)$

Now we concentrate our attention on another exact static wormhole model, which can be described by the following shape function:

$$b(r) = \frac{\beta(8\beta(\lambda + 4\pi)(r\mathrm{Li}_2(A_1) - 4\beta(\lambda + 4\pi)\mathrm{Li}_3(A_1)) + r^2 \log(1 - A_1))}{2\alpha} + c_4, \tag{28}$$

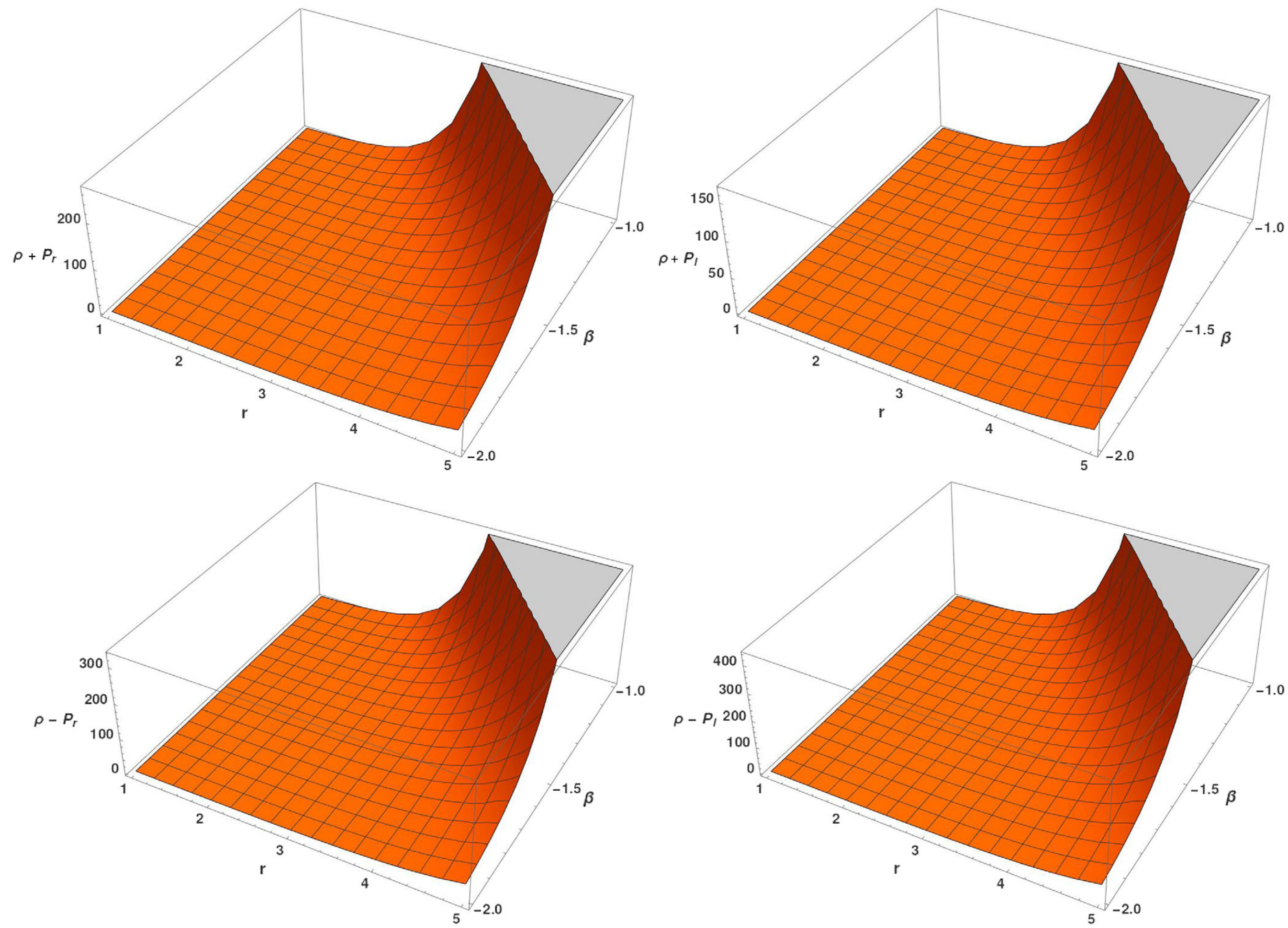

FIG. 2. Behavior of the NEC in terms of the pressures P_r and P_l, respectively (top panels). The NEC in terms of P_r is given by the top-left plot, while the top-right plot represents the NEC in terms of P_l. The bottom panels show the behavior of the DEC in terms of both pressures. In particular, the bottom-left and bottom-right plots correspond to the behavior of the DEC in terms of P_r and P_l, respectively. The model is described by Eq. (22) and the behavior of the energy conditions is for $c_2 = 1.5$, $c_1 = 0.5$, $\alpha = 0.5$, $\lambda = -10$, and different negative values of β.

where $A_1 = -\frac{8e^{\frac{r}{4\lambda\beta+16\pi\beta}}\alpha(\lambda+4\pi)}{c_3}$, while Li_2 and Li_3 are the polylogarithm functions of orders 2 and 3, respectively. We obtained this solution for the shape function $b(r)$ by assuming that the matter content is described by the following energy density:

$$\rho = \alpha R(r)^2 + \beta R'(r). \tag{29}$$

On the other hand, as

$$b'(r) = \frac{r^2}{8\alpha(\lambda+4\pi) + c_3 e^{-\frac{r}{4\beta\lambda+16\pi\beta}}}, \tag{30}$$

for ρ, P_r, and P_l (similarly to the previous case) we get

$$\rho = \frac{1}{16\alpha(\lambda+4\pi)^2 + 2c_3(\lambda+4\pi)e^{-\frac{r}{4\beta\lambda+16\pi\beta}}}, \tag{31}$$

$$P_r = -\frac{c_4}{2(\lambda+4\pi)r^3} + \frac{\beta(8\beta(\lambda+4\pi)(r\mathrm{Li}_2(A_1) - 4\beta(\lambda+4\pi)\mathrm{Li}_3(A_1)) + r^2\log(1-A_1))}{4\alpha(\lambda+4\pi)r^3}, \tag{32}$$

and

$$P_l = \frac{1}{4(\lambda+4\pi)r^3}\left(c_4 - \frac{r^3}{8\alpha(\lambda+4\pi) + c_3 e^{-\frac{r}{4\beta\lambda+16\pi\beta}}} + 2(2(\lambda+4\pi)r^3 P_r + c_4)\right), \tag{33}$$

respectively.

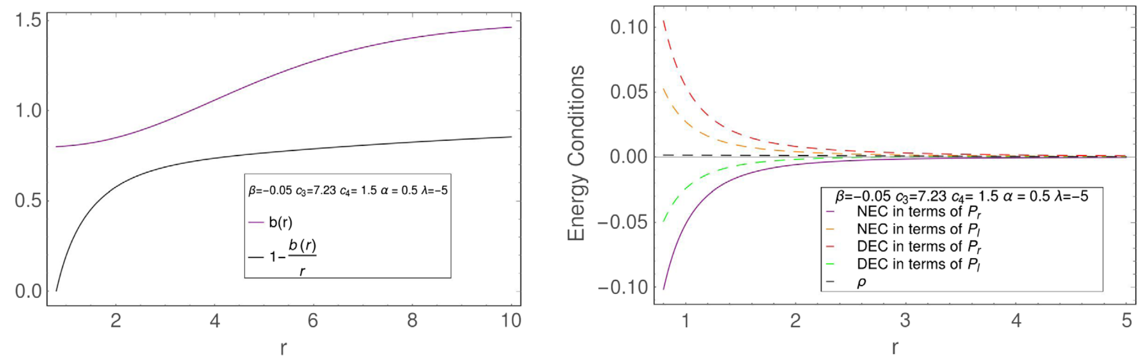

FIG. 3. A plot of the shape function $b(r)$ for the model described by Eq. (29) is presented in the left panel. It readily shows that the solution for $b(r)$ satisfies $1 - b(r)/r > 0$, for $r > r_0$. The right panel proves that the DEC in terms of P_r and the NEC in terms of P_l are valid everywhere, and that $\rho \geq 0$ also holds everywhere. On the other hand, the same plot also shows that the NEC and the DEC in terms of the pressures P_r and P_l, respectively, are not valid at the throat of the wormhole. This specific wormhole model is for $c_3 = 7.23$, $c_4 = 1.5$, $\alpha = 0.5$, $\beta = -0.05$, and $\lambda = -5$. The throat occurs at $r_0 = 0.8$ and $b'(r_0) \approx 0.015$.

The behaviors of the shape function and the energy conditions for the specific wormhole model corresponding to $c_3 = 7.23$, $c_4 = 1.5$, $\alpha = 0.5$, $\beta = -0.05$, and $\lambda = -5$ can be found in Fig. 3. The throat of this specific wormhole occurs at $r_0 = 0.8$ and $b'(r_0) \approx 0.0154$. The study of this particular case shows that the energy conditions have the same qualitative behavior as in the case for the model with the energy density described by Eq. (22). Therefore, we will omit further discussion of this issue to save space; rather, we would like to concentrate our attention on some region where both the NEC and WEC in terms of P_r are fulfilled, since $\rho \geq 0$. We can see this in Fig. 4. The behavior of the NEC and ρ depicted there correspond to $c_3 = 1.23$, $c_4 = 0.5$, $\alpha = 0.5$, $\lambda = 5$, and some negative values of β. Moreover, we also see that for the same case the NEC in terms of P_l is valid too, but only for $\beta \in [-0.1, 0.0]$. On the other hand, the DEC in terms of P_r is not valid at all, while the DEC in terms of P_l is fulfilled. The parameter space can always be divided in such a way that, in each region, some group of energy conditions are valid. If in future analyses we are able to constrain the equation of state of the wormhole matter content, it will be possible to identify the relevant region of the model parameter space.

In the next subsection we consider other exact static wormhole solutions which, in order to be to be traversable, should be described by a constant shape function, despite the complex form of the matter energy density and the exact form of the shape function, which is not a constant. In other words, the solutions obtained will describe nontraversable exact static wormhole models, i.e., in these cases, when the shape function is not constant, we will obtain nontraversable wormhole solutions characterized by the impossibility of satisfying the asymptotic flatness requirement. In summary,

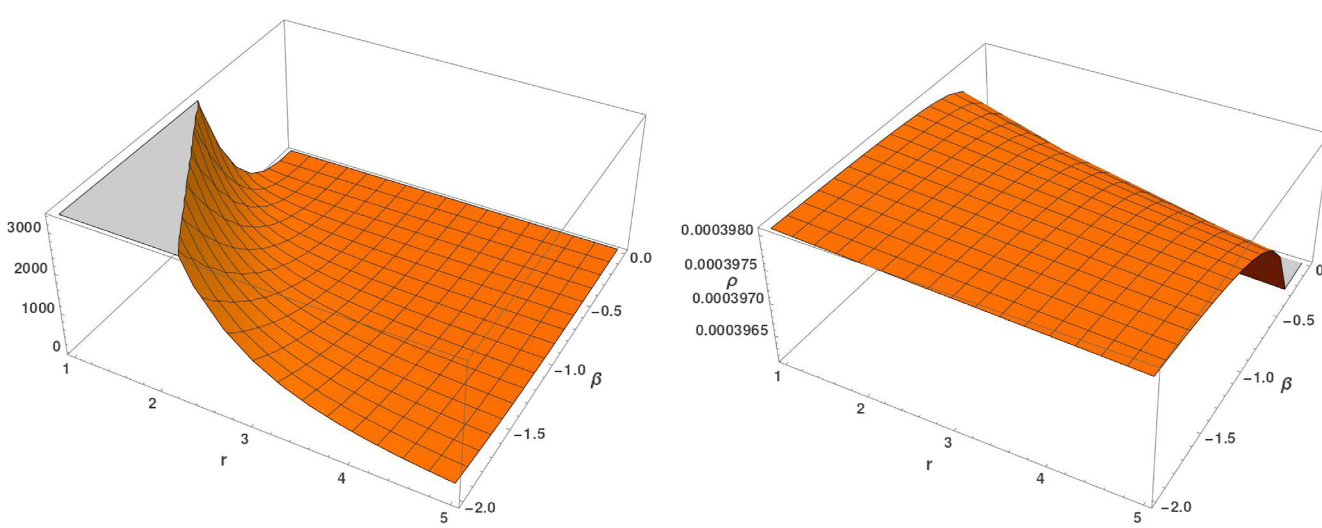

FIG. 4. The behavior of NEC in terms of P_r is shown in the left panel. The right panel represents the behavior of ρ. The model is described by Eq. (29) and the behavior is for $c_3 = 1.23$, $c_4 = 0.5$, $\alpha = 0.5$, $\lambda = 5$, and different negative values of β.

the study of these new models shows that their parameters are such that the shape functions turn out to be constant, in order to describe viable traversable wormholes.

C. Matter with $\rho(r) = \alpha R(r) + \beta R^2(r)$

If we assume that the matter content of the wormhole is

$$\rho = \alpha R(r) + \beta R^2(r), \tag{34}$$

then we obtain two wormhole solutions. One solution describes a wormhole with a constant shape function, i.e., $b(r) = \text{const}$ (which means that we have a traversable wormhole solution), while the second solution describes a wormhole with

$$b(r) = c_5 - \frac{r^3(4\alpha\lambda + 16\pi\alpha - 1)}{24\beta(\lambda + 4\pi)}, \tag{35}$$

where c_5 is an integration constant. Now, let us concentrate our attention on the last case, that is, when the shape function is given by Eq. (35). In particular, for ρ, P_r, and P_l, we get

$$\rho = \frac{1 - 4\alpha(\lambda + 4\pi)}{16\beta(\lambda + 4\pi)^2}, \tag{36}$$

$$P_r = \frac{1}{48(\lambda + 4\pi)^2}\left(\frac{4\alpha(\lambda + 4\pi) - 1}{\beta} - \frac{24c_5(\lambda + 4\pi)}{r^3}\right), \tag{37}$$

and

$$P_l = \frac{1}{48(\lambda + 4\pi)^2}\left(\frac{4\alpha(\lambda + 4\pi) - 1}{\beta} + \frac{12c_5(\lambda + 4\pi)}{r^3}\right), \tag{38}$$

respectively, using Eqs. (12), (13), and (14). Moreover, the NEC in terms of P_r and P_l, reads

$$\rho + P_r = \frac{1}{24(\lambda + 4\pi)^2}\left(\frac{1 - 4\alpha(\lambda + 4\pi)}{\beta} - \frac{12c_5(\lambda + 4\pi)}{r^3}\right) \tag{39}$$

and

$$\rho + P_l = \frac{6\beta c_5(\lambda + 4\pi) + r^3(1 - 4\alpha(\lambda + 4\pi))}{24\beta(\lambda + 4\pi)^2 r^3}, \tag{40}$$

respectively. On the other hand, the DEC in terms of P_r and P_l reads

$$\rho - P_r = \frac{6\beta c_5(\lambda + 4\pi) + r^3(1 - 4\alpha(\lambda + 4\pi))}{12\beta(\lambda + 4\pi)^2 r^3} \tag{41}$$

and

$$\rho - P_l = \frac{r^3(1 - 4\alpha(\lambda + 4\pi)) - 3\beta c_5(\lambda + 4\pi)}{12\beta(\lambda + 4\pi)^2 r^3}, \tag{42}$$

respectively. However, further study shows that only the solutions with a constant shape function represent traversable wormholes. In other cases, we will have nontraversable wormhole models. Moreover, exact wormhole solutions with the same properties can be constructed with $\rho = \alpha R(r) + \beta R^{-2}(r)$, $\rho = \alpha R(r) + \beta r R^2(r)$, $\rho = \alpha R(r) + \beta r^{-1} R^2(r)$, $\rho = \alpha R(r) + \beta r^2 R^2(r)$, and $\rho = \alpha R(r) + \beta r^3 R^2(r)$ as well. In other words, in these cases the values of the model parameters that ensure that the shape function satisfies the required constraints (including the asymptotic flatness requirement) only allow wormhole models with a constant shape function: the values of the model parameters force the shape function $b(r)$ to be constant. On the other hand, similarly to the two previous cases, the NEC and DEC in terms of P_r and P_l are not valid at the throat of the wormhole: they are only valid far from the throat. Another family of exact wormholes of the same nature can be constructed when the matter energy density has the following form:

$$\rho = \alpha r^m R(r)\log(\beta R(r)). \tag{43}$$

We already mentioned that, in theory, we can glue an exterior flat geometry into the interior geometry at some junction radius, making these solutions represent traversable wormholes. However, an interesting question relevant to the models presented above arises, namely, what is the role of $R'(r)$ in the traversable wormhole formation process? On the other hand, another interesting question is: does the assumption $L_m = -\rho$, with the matter energy density considered in this subsection, prevent the formation of traversable wormholes? This should be answered as well. We hope to discuss these issues in a forthcoming paper.

IV. SOME MODELS IN $f(R,T) = R + \gamma R^2 + 2\lambda T$ GRAVITY

In this section we want to address another interesting question concerning the models obtained in Sec. III C, i.e., the models that yield nontraversable wormholes for $f(R,T) = R + 2\lambda T$ gravity. Here we will consider these models from another viewpoint: if the reason for nontraversability is the considered form of $f(R,T)$ gravity, then we can change this and consider, for instance, gravity of the form $f(R,T) = R + \gamma R^2 + 2\lambda T$. On the other hand, in order to construct exact wormhole models, we take advantage of the following shape function:

$$b(r) = \sqrt{\hat{r}_0 r}. \tag{44}$$

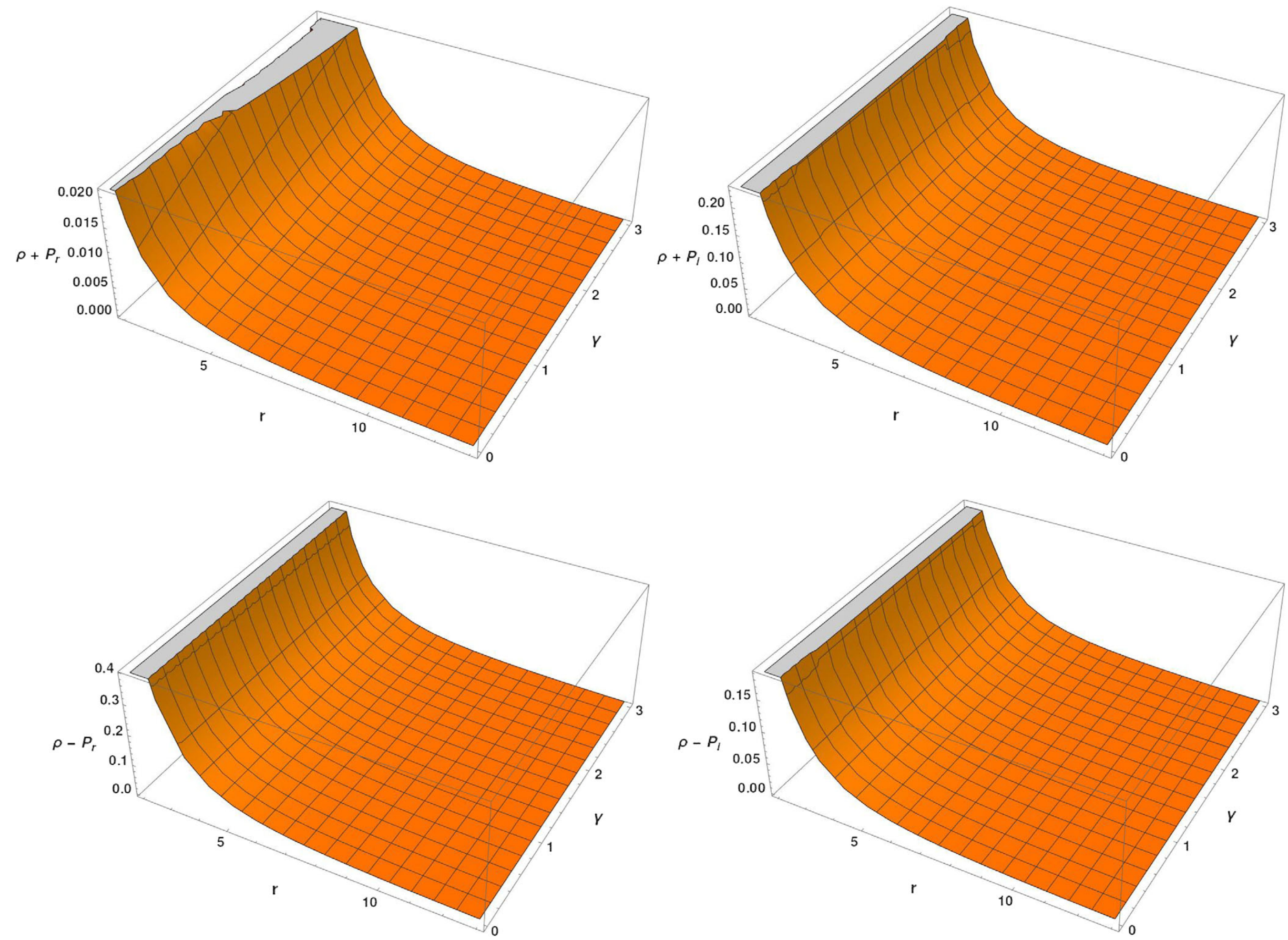

FIG. 5. The top-left and top-right panels show the NEC in terms of P_r and P_l, respectively, while the bottom-left and bottom-right panels show the behavior of the DEC in terms of P_r and P_l, respectively. The model is described by Eq. (34) and the behavior for the energy conditions is for the values $\hat{r}_0 = 2$, $\lambda = -15$, $\beta = 0.7$, $\alpha = 1.5$, and different values of the parameter γ, which is responsible for the R^2 contribution to gravity.

It is easy to see that it describes a traversable wormhole. In this case, according to the structure of the equations, we only need to reconstruct the forms of P_r and P_l, from two algebraic equations.

As an example, we study the model described by Eq. (34). After some algebra, both pressures can be written in the following way:

$$P_r = -\frac{2r^2(4\alpha(\lambda+4\pi)+1)\sqrt{r\hat{r}_0} + 70\gamma\sqrt{r\hat{r}_0} + 8\beta(\lambda+4\pi)\hat{r}_0 - 71\gamma\hat{r}_0}{8(\lambda+4\pi)r^5}, \tag{45}$$

and

$$P_l = \frac{256\pi^2(\alpha r^2\sqrt{r\hat{r}_0} + \beta\hat{r}_0) + B + C}{16\lambda(\lambda+4\pi)r^5}, \tag{46}$$

where

$$B = 8\pi(2r^2(8\alpha\lambda - 1)\sqrt{r\hat{r}_0} + 50\gamma\sqrt{r\hat{r}_0} + 16\beta\lambda\hat{r}_0 - 57\gamma\hat{r}_0),$$

and

$$C = \lambda(2r^2(8\alpha\lambda - 1)\sqrt{r\hat{r}_0} + 170\gamma\sqrt{r\hat{r}_0} + 16\beta\lambda\hat{r}_0 - 185\gamma\hat{r}_0).$$

This means that we have a traversable wormhole model whose shape function is given by Eq. (44), the matter content is described by Eq. (34) for the energy density, while P_r and P_l are given by Eqs. (45) and (46), respectively. We plot the behaviors of the NEC and DEC in terms of both pressures P_r and P_l in Fig. 5 for $\hat{r}_0 = 2$, $\lambda = -15$, $\beta = 0.7$, $\alpha = 1.5$, and different values of the parameter γ (which is responsible for the R^2 contribution to gravity). We see that there is a region where both energy conditions in terms of both pressures are fulfilled. Moreover, we also checked that the SEC is valid for the same region presented in Fig. 5, and since $\rho \geq 0$ we also expect that the WEC is valid. However, another interesting situation that deserves our attention is observed for $\hat{r_0} = 2$, $\lambda = 15$, $\beta = 0.7$, $\alpha = -1.5$, and different values of γ. In particular, we observe that for small r and for the considered $\gamma \in [0, 3]$, the NEC in terms of both pressures is not valid. Moreover, fulfillment in both cases is achieved for the same r and γ. The validity of the DEC in terms of P_l is observed in the whole considered region, while the DEC in terms of P_r for small r is not satisfied. This is the same behavior we observed for the other models in this paper; however, this particular model is interesting because further analysis shows that, for small r, we have $\rho < 0$, which means that the WEC is also violated. Now, in the previous model the WEC was violated due to the nonvalidity of $\rho + P \geq 0$, and we always we have $\rho \geq$; but here this violation comes from, the violation of $\rho + P \geq 0$, and $\rho \geq 0$. We also see that for this case the parameter γ does not play a role in the validity of the energy conditions. The family of wormholes presented here definitely requires further study to be better understood.

Another candidate for a traversable wormhole is a model described by the matter content

$$\rho(r) = \alpha R(r) + \beta r^3 R^2(r), \tag{47}$$

with

$$P_r = -\frac{8\beta(\lambda+4\pi)r^3\hat{r}_0 + 2r^2(4\alpha(\lambda+4\pi)+1)\sqrt{r\hat{r}_0} + 70\gamma\sqrt{r\hat{r}_0} - 71\gamma\hat{r}_0}{8(\lambda+4\pi)r^5}, \tag{48}$$

and

$$P_l = \frac{256\pi^2 r^2(\alpha\sqrt{r\hat{r}_0} + \beta r\hat{r}_0) + B_1 + C_1}{16\lambda(\lambda+4\pi)r^5}, \tag{49}$$

where

$$B_1 = \lambda(16\beta\lambda r^3\hat{r}_0 + 2r^2(8\alpha\lambda - 1)\sqrt{r\hat{r}_0} + 5\gamma(34\sqrt{r\hat{r}_0} - 37\hat{r}_0)),$$

and

$$C_1 = 8\pi(16\beta\lambda r^3\hat{r}_0 + 2r^2(8\alpha\lambda - 1)\sqrt{r\hat{r}_0} + \gamma(50\sqrt{r\hat{r}_0} - 57\hat{r}_0)).$$

The behaviors of the energy conditions in this model are qualitatively the same.

V. DISCUSSION AND CONCLUSIONS

In this paper we have constructed a number of wormhole models corresponding to the family $f(R,T)$ of extended theories of gravity. We have restricted ourselves to the two cases $f(R,T) = R + \lambda T$ and $f(R,T) = R + \gamma R^2 \lambda T$, (where $T = \rho + P_r + 2P_l$ is the trace of the energy-momentum tensor). In the first case we investigated three different wormhole models, assuming that the energy density profile of wormhole matter can be parametrized by the Ricci scalar. Specifically, we considered the following three possibilities for ρ to describe the wormhole matter: $\rho(r) = \alpha R(r) + \beta R'(r)$, $\rho(r) = \alpha R^2(r) + \beta R'(r)$, and $\rho(r) = \alpha R(r) + \beta R^2(r)$. In the first two cases, we have proven the possibility of traversable wormhole formation. Moreover, we studied a particular wormhole solution described by, e.g., $\rho(r) = \alpha R(r) + \beta R'(r)$ and showed that for appropriate values of the parameters of the model one can expect a violation of the NEC in terms of P_r and of the DEC in terms of P_l at the throat. On the other hand, $\rho \geq 0$, and the validity of both the NEC and DEC in terms of P_l and P_r, respectively, can be checked everywhere, including at the wormhole throat. Therefore, we also observed a violation of the WEC in terms of P_r, while it is still valid in terms of P_l. Moreover, for the same model our study also showed that, generically, for $\beta > 0$, regions will be found where the violation of the NEC in terms of P_r will induce a violation of the DEC in terms of P_l. However, when we considered $\beta < 0$ regions we encountered domains where both energy conditions in terms of both pressures are valid at the same time.

On the other hand, for the second model, described by $\rho(r) = \alpha R^2(r) + \beta R'(r)$, we found a particular traversable wormhole solution provided we take $c_3 = 7.23$, $c_4 = 1.5$ (both of which are integration constants), $\alpha = 0.5$, $\beta = -0.05$, and $\lambda = -5$. The throat of this specific wormhole occurs at $r_0 = 0.8$ and $b'(r_0) \approx 0.0154$. The study of this particular case showed that the energy conditions have the same qualitative behavior as in the previous model. However, further analysis showed the existence of regions where the NEC and WEC in terms of P_r can be valid, since

$\rho \geq 0$, and this can be achieved for $c_3 = 1.23$, $c_4 = 0.5$, $\alpha = 0.5$, $\lambda = 5$, and some negative values of the parameter β. Moreover, we also saw that, for the same case, the NEC in terms of P_l will be valid only for $\beta \in [-0.1, 0.0]$. On the other hand, the DEC in terms of P_r will not be valid at all, while the DEC in terms of P_l will be valid.

In summary, we can claim that in all cases the parameter space can be split in such a way that some set of the energy conditions are indeed fulfilled. In the future, if we are able to constrain the equation of state of the wormhole matter, it might be possible to clearly identify the relevant region of the model parameter space.

We now briefly mention other important aspects of our study, i.e., when the energy density profile of the wormhole matter is given by $\rho(r) = \alpha R(r) + \beta R^2(r)$. In this case our analysis has led to two solutions for the shape function. The constant solution describes a traversable wormhole. However, nonconstant shape functions describe nontraversable wormhole solutions. Moreover, we also showed that nontraversable exact wormhole solutions can be found for the cases when $\rho = \alpha R(r) + \beta R^{-2}(r)$, $\rho = \alpha R(r) + \beta r R^2(r)$, $\rho = \alpha R(r) + \beta r^{-1} R^2(r)$, $\rho = \alpha R(r) + \beta r^2 R^2(r)$, $\rho = \alpha R(r) + \beta r^3 R^2(r)$, and $\rho = \alpha r^m R(r) \log(\beta R(r))$. In theory, we know that in the case of a nontraversable wormhole solution we can glue an exterior flat geometry into the interior geometry at some junction radius, making these solutions represent traversable wormholes. This procedure can be put to work here. On the other hand, a sequence of interesting questions relevant to the models discussed above have yet to be answered: What is the main role of the term $R'(r)$ in the formation of traversable wormholes? Does the assumption $L_m = -\rho$ with the matter energy density considered in this paper radically prevent the formation of a traversable wormhole? Is there a different reason for this? We expect to be able to answer these questions in a forthcoming paper.

Additionally, in the second part of the paper we dealt with two particular exact traversable wormhole models for $f(R, T) = R + \gamma R^2 \lambda T$ gravity. In particular, we considered the two matter profiles given by $\rho(r) = \alpha R(r) + \beta R^2(r)$, and $\rho(r) = \alpha R(r) + \beta r^3 R^2(r)$. Both of them describe wormhole models with the shape function $b(r) = \sqrt{\hat{r}_0 r}$ (where $\hat{r}_0$ is a constant). For this case, we also studied the validity of the energy conditions and, specifically, in the case of the model with $\rho(r) = \alpha R(r) + \beta R^2(r)$ we observed that there is a region where both energy conditions [i.e., the NEC ($\rho + P_r \geq 0$ and $\rho + P_l \geq 0$) and DEC ($\rho - P_r \geq 0$ and $\rho - P_l \geq 0$)] in terms of both pressures are fulfilled.

To finish, we have obtained a number of new, exact wormhole solutions, which in several cases are also traversable. The new models were derived by assuming specific forms of the wormhole matter profile, parametrized by the Ricci scalar. Our study of the validity of the energy conditions reveals that the solutions constructed exhibit a rich behavior, and it is always possible to find some regions in which the NEC and the DEC in terms of both pressures are simultaneously fulfilled. On the other hand, the regions where some of the energy conditions are violated may also prove to be very useful in the future when astronomical data on wormholes and their matter content begins to accumulate. The study carried out here is merely an initial step towards a deeper investigation of these new types of wormholes. In particular, we still need to understand the main role of the $R'(r)$ term for traversable wormhole formation. Another interesting question is whether the assumption $L_m = -\rho$ with the matter energy density considered in this paper is crucial in order to prevent the formation of traversable wormholes. Is there another reason for this? These issues are relevant in view of the situation observed in Sec. III C. We expect to clarify them in forthcoming papers involving, in particular, the study of the shadows and gravitational lensing properties of the new wormholes.

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