Local free-fall temperatures of charged BTZ black holes in massive gravity

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We obtain a $(3+3)$-dimensional global flat embedding of the generalized $(2+1)$ charged Bañados-Teitelboim-Zanelli black holes in massive gravity. We also study the local free-fall temperatures for freely falling observers starting from rest and investigate the effect of the charge and graviton mass in free-fall temperatures.

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I. INTRODUCTION

As discovered by Hawking [1], an observer located at asymptotic infinity sees the Hawking temperature T_H of a black hole that emits characteristic thermal radiation. On the other hand, a fiducial observer at a finite distance from a black hole sees a local temperature described by the Tolman temperature [2]

$$T_{\rm FID} = \frac{T_H}{\sqrt{g_{\mu\nu}\xi^\mu\xi^\nu}}, \tag{1.1}$$

where ξ^μ is a timelike Killing vector. Later, Unruh [3] showed that a uniformly accelerating observer in a flat spacetime, with a proper acceleration a, detects thermal radiation at the Unruh temperature

$$T_U = \frac{a}{2\pi}. \tag{1.2}$$

These two effects are related; i.e., the Hawking effect for a fiducial observer in a black hole spacetime can be considered as the Unruh effect for a uniformly accelerated observer in a higher-dimensional global embedding Minkowski spacetime (GEMS). These ideas and their corresponding GEMSs are studied through the analysis of de Sitter [4] and anti-de Sitter (AdS) spacetimes [5,6]. Furthermore, Deser and Levin [7] have shown that the GEMS approach provides a unified derivation of temperature for uncharged Bañados-Teitelboim-Zanelli (BTZ) [8,9], Schwarzschild-AdS, and Reissner-Nordström (RN) spacetimes. After these works, we have constructed GEMSs for the charged BTZ [10] and RN-AdS [11] spacetimes according to this approach. Since then, there have been many works on a variety of curved spacetimes [12–25]. Furthermore, several years ago, Brynjolfsson and Thorlacius [26] used the GEMS approach to define a local temperature for a freely falling observer outside Schwarzschild(-AdS) and RN spacetimes, showing that freely falling temperatures remain finite at event horizons while they approach the Hawking temperatures at asymptotic infinities. Here, a freely falling local temperature is defined at special turning points of radial geodesics where a freely falling observer is momentarily at rest with respect to a black hole. After the work, we have extended the results to RN-AdS [27], Gibbons-Maeda-Garfinkle-Horowitz-Strominger black holes [28], a modified Schwarzschild black hole in rainbow spacetime [29], and a Schwarzschild-Tangherlini-AdS black hole [30]. However, up to now, all these studies of finding freely falling temperatures have been mainly restricted to massless graviton cases. By the way, it is known that massive gravitons in general relativity [31,32] can be introduced by various channels, one of which is breaking the Lorentz symmetry of the system [33]. The modification to the behavior of a black hole by including a graviton mass has also been considered in the extended phase space in order to study the phase transition of black holes [34]. Moreover, the graviton mass terms have been exploited to investigate many interesting models such as, for instance, Gauss-Bonnet massive gravity [35]. It has been noticed that the massive gravitons can yield interesting modification of black hole thermodynamics.

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On the other hand, since the pioneering work in 1992, the $(2+1)$-dimensional BTZ black hole in massless gravity [8,9] has become a useful model for realistic black hole physics [36]. Moreover, significant interest in this model has recently increased with the discovery that the thermodynamics of higher-dimensional black holes can often be interpreted in terms of the BTZ solution [37]. It is therefore interesting to study the geometry of $(2+1)$-dimensional black holes and their thermodynamics through further investigation. Moreover, it is possible to construct a BTZ black hole in massive gravity [38–42]. In fact, an asymptotically AdS charged BTZ black hole has been constructed in a massive theory of gravity, and various different aspects of such a solution have been studied. In these works, they have considered three-dimensional massive gravity with an Abelian U(1) gauge field and negative cosmological constant of which the action is of the form

$$S=-\frac{1}{16}\int d^3x\sqrt{g}\left[\mathcal{R}-2\Lambda+\mathcal{L}(\mathcal{F})+\tilde{M}^2\sum_{i=1}^{4}c_i\mathcal{U}_i(g,f)\right], \tag{1.3}$$

where $\mathcal{R}$ is the scalar curvature, $\Lambda(=-1/l^2)$ is the cosmological constant, $\mathcal{L}(\mathcal{F})$ is the Lagrangian for the vector gauge field, $\tilde{M}$ is the massive parameter, and f is the reference metric. $\mathcal{F}(=F_{\mu\nu}F^{\mu\nu})$ is the Maxwell invariant in which $F_{\mu\nu}(=\partial_\mu A_\nu-\partial_\nu A_\mu)$ is the Faraday tensor and A_μ is the U(1) gauge potential. c_i are the constants for massive gravity, and $\mathcal{U}_i$ are the symmetric polynomials of eigenvalues. Here, they take an ansatz that $\mathcal{U}_1=c/r$, $\mathcal{U}_2=\mathcal{U}_3=\mathcal{U}_4=0$, where c is a positive constant.

In this paper, we will generalize the Unruh, Hawking, and freely falling temperatures of the charged BTZ black hole in the massless case[1] to those in the massive gravity (1.3) with the ansatz in terms of the GEMS approach. In Sec. II, we will briefly summarize the GEMS embedding of the charged BTZ black hole in the massless gravity [10] and then newly obtain desired temperatures of the black holes as measured by freely falling observers. In Secs. III and IV, we will derive the GEMS embeddings of the uncharged and charged BTZ black holes in the massive gravity and then evaluate local temperatures of the black holes as measured by freely falling observers, respectively. In particular, in Sec. IV, we discuss the effect of charge and massive gravitons on the Hawking temperature in the charged BTZ black hole in the massive gravity. Finally, our conclusions are drawn in Sec. V.

II. CHARGED BTZ BLACK HOLE IN MASSLESS GRAVITY

A. GEMS of charged BTZ black hole

We consider the $(2+1)$-dimensional charged BTZ black hole in the massless gravity described by the 3-metric

$$ds^2=N^2dt^2-N^{-2}dr^2-r^2d\phi^2 \tag{2.1}$$

with the lapse function

$$N^2=-m+\frac{r^2}{l^2}-2q^2\ln\frac{r}{l}, \tag{2.2}$$

where $m=8M$ and $q=2Q$ with M and Q being the mass and electric charge of the BTZ black hole, respectively. Now, the mass m can be written in terms of the event horizon r_H as

$$m=\frac{r_H^2}{l^2}-2q^2\ln\frac{r_H}{l}, \tag{2.3}$$

and the Hawking-Bekenstein horizon surface gravity is given by [43]

$$k_H=-\frac{1}{2}(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu)|_{r\to r_H}=\frac{r_H}{l^2}-\frac{q^2}{r_H}. \tag{2.4}$$

Then, according to the GEMS approach, a $(3+3)$-dimensional AdS GEMS,

$$ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2+(dz^3)^2-(dz^4)^2+(dz^5)^2, \tag{2.5}$$

is given by the coordinate transformations for $r\geq r_H$ [10],

$$\begin{aligned}
z^0&=k_H^{-1}\left(\frac{r^2-r_H^2}{l^2}-2q^2\ln\frac{r}{r_H}\right)^{1/2}\sinh k_Ht,\\
z^1&=k_H^{-1}\left(\frac{r^2-r_H^2}{l^2}-2q^2\ln\frac{r}{r_H}\right)^{1/2}\cosh k_Ht,\\
z^2&=\frac{l}{r_H}r\sinh\frac{r_H}{l}\phi,\\
z^3&=\frac{l}{r_H}r\cosh\frac{r_H}{l}\phi,\\
z^4&=k_H^{-1}\int dr\frac{q^2l[r^2+r_H^2+2r^2g(r)]^{1/2}}{r_H^2r\left[1-\frac{q^2l^2}{r_H^2}g(r)\right]^{1/2}},\\
z^5&=k_H^{-1}\int dr\frac{q\left[2r_H^2+\frac{r_H^4+q^4l^4}{r_H^2}g(r)\right]^{1/2}}{r_H^2\left[1-\frac{q^2l^2}{r_H^2}g(r)\right]^{1/2}},
\end{aligned} \tag{2.6}$$

where

$$g(r)=\frac{2r_H^2}{r^2-r_H^2}\ln\frac{r}{r_H}. \tag{2.7}$$

Here, one notes that, due to l'Hôpital's rule, $g(r)$ approaches unity as r goes to $r=r_H$.

For the trajectories, which follow the Killing vector $\xi=\partial_t$, we can obtain a constant 3-acceleration,

[1] In this work, we will call it massless when $\tilde{M}$ is zero.

$$a_3 = \frac{\frac{r}{l^2} - \frac{q^2}{r}}{\left(\frac{r^2 - r_H^2}{l^2} - 2q^2 \ln \frac{r}{r_H}\right)^{1/2}}. \tag{2.8}$$

In static detectors (ϕ, $r = \text{const}$) described by a fixed point in the (z^2, z^3, z^4, z^5) plane, an observer, who is uniformly accelerated in the $(3+3)$-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0,z^1) described by a proper acceleration a_6 as follows:

$$a_6^{-2} = (z^1)^2 - (z^0)^2 = \frac{l^2\left(r^2 - r_H^2 - 2q^2 l^2 \ln \frac{r}{r_H}\right)}{\left(r_H - \frac{q^2 l^2}{r_H}\right)^2}. \tag{2.9}$$

Here, we have the relation with a constant acceleration a_3,

$$a_6^2 - a_3^2 = -\frac{1}{l^2} + \frac{\frac{q^4 l^2}{r^2 r_H^2} - \frac{q^2}{r_H^2} g(r)}{1 - \frac{q^2 l^2}{r_H^2} g(r)}. \tag{2.10}$$

One notes that, in the limit of $q = 0$, $a_6^2 - a_3^2$ becomes $-1/l^2$ as expected [7,10].

As was shown by Unruh [3], the Unruh temperature for a uniformly accelerated observer in the $(3+3)$-dimensional flat spacetime can be read as

$$T_U = \frac{a_6}{2\pi}, \tag{2.11}$$

so we can obtain

$$T_U = \frac{r_H - \frac{q^2 l^2}{r_H}}{2\pi l (r^2 - r_H^2 - 2q^2 l^2 \ln \frac{r}{r_H})^{1/2}}. \tag{2.12}$$

This is exactly the same with the local temperature measured by a fiducial observer staying at a finite distance from the black hole, the so-called fiducial temperature,

$$T_{\text{FID}} = \frac{T_H}{\sqrt{g_{00}}}, \tag{2.13}$$

where the Hawking temperature T_H is measured by an asymptotic observer,

$$T_H = \frac{1}{2\pi}\left(\frac{r_H}{l^2} - \frac{q^2}{r_H}\right). \tag{2.14}$$

Next, by introducing dimensionless variables

$$x = \frac{r_H}{r}, \qquad a = \frac{l}{r_H}, \qquad b = q^2, \tag{2.15}$$

we can rewrite the Hawking temperature in Eq. (2.14) as follows:

$$T_H \cdot r_H = \frac{1}{2\pi}\left(\frac{1}{a^2} - b\right). \tag{2.16}$$

B. Free-fall temperature of charged BTZ black hole

Now, we assume that an observer at rest is freely falling from the radial position $r = r_0$ at $\tau = 0$ [26–30]. The equations of motion for the orbit of the observer are given as

$$\frac{dt}{d\tau} = \frac{N(r_0)}{N^2(r)},$$
$$\frac{dr}{d\tau} = -[N^2(r_0) - N^2(r)]^{1/2}. \tag{2.17}$$

Exploiting Eqs. (2.6) and (2.17), we obtain the freely falling acceleration $\bar{a}_6$ in the $(3+3)$-dimensional GEMS embedded spacetime

$$\bar{a}_6 = \frac{1}{l}\left[\frac{(1 + \frac{q^2 l^2}{r r_H})(1 - \frac{q^2 l^2}{r r_H})}{1 - \frac{q^2 l^2}{r_H^2} g(r)}\right]^{1/2}, \tag{2.18}$$

which gives us the temperature measured by the freely falling observer at rest (FFAR)

$$T_{\text{FFAR}} = \frac{\bar{a}_6}{2\pi} = \frac{1}{2\pi l}\left[\frac{(1 + \frac{q^2 l^2}{r r_H})(1 - \frac{q^2 l^2}{r r_H})}{1 - \frac{q^2 l^2}{r_H^2} g(r)}\right]^{1/2}. \tag{2.19}$$

It is appropriate to comment that in the limit of $q = 0$ free-fall temperature for an uncharged BTZ black hole seen by the freely falling observer is reduced to

$$T_{\text{FFAR}} \cdot r_H = \frac{r_H}{2\pi l} = \frac{1}{2\pi a}. \tag{2.20}$$

Making use of the dimensionless variables introduced in Eq. (2.15), we can rewrite the free-fall temperature in Eq. (2.19) as

$$T_{\text{FFAR}} \cdot r_H = \frac{1}{2\pi a}\left[\frac{(1 + a^2 b x)(1 - a^2 b x)}{1 + \frac{2a^2 b x^2}{1 - x^2} \ln x}\right]^{1/2}, \tag{2.21}$$

where the relevant range of x is given by $0 \le x \le 1$.

In the limit of $x = 0$, we obtain

$$T_{\text{FFAR}} \cdot r_H(x = 0) = \frac{1}{2\pi a}, \tag{2.22}$$

where we have used the identity $\lim_{x\to 0} x^2 \ln x = 0$. Here, we note that the above value in Eq. (2.22) is independent of the parameters b. This means that, as $r \to \infty$, the charge of the charged BTZ black hole does not affect the free-fall temperature T_{FFAR}. In other words, the free-fall temperature T_{FFAR} as $r \to \infty$ in Eq. (2.22) is the same as that of the uncharged BTZ black hole in Eq. (2.20).

At $x = 1$, namely, at the event horizon $r = r_H$, we end up with

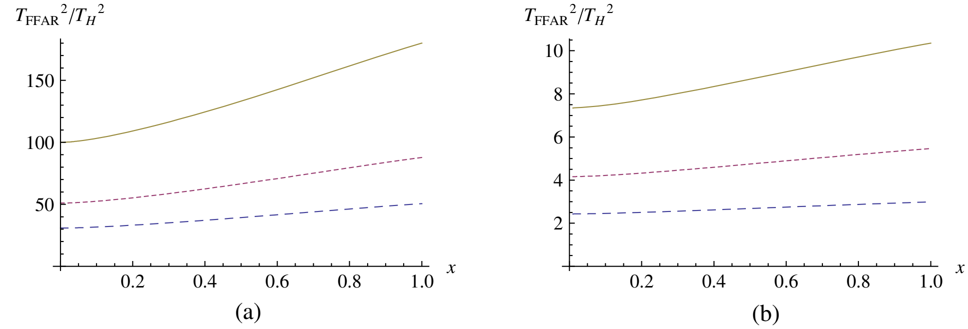

FIG. 1. Free-fall temperature for the charged BTZ black hole in the massless gravity: (a) for a fixed $a = 2$ with $b = 0.16, 0.18, 0.20$; (b) for a fixed $b = 0.16$ with $a = 1.2, 1.4, 1.6$ from bottom to top, respectively.

$$T_{\rm FFAR} \cdot r_H(x = 1) = \frac{1}{2\pi a}(1 + a^2 b)^{1/2}, \quad (2.23)$$

where we have exploited the relation $\lim_{x\to 1} \frac{\ln x}{1-x^2} = -\frac{1}{2}$. The above result in Eq. (2.23) shows that there exists no singularity of $T_{\rm FFAR}$ at $x = 1$. Moreover, we note that

$$T_{\rm FFAR}(x = 1) > T_{\rm FFAR}(x = 0), \quad (2.24)$$

which means that the free-fall temperature at the event horizon $r = r_H$ is greater than that at the infinity, as expected. These are summarized in Fig. 1 by plotting $T^2_{\rm FFAR}$ in units of T^2_H. Here, Fig. 1(a) was drawn by changing the charge b, while Fig. 1(b) was drawn by changing the cosmological constant a.

III. UNCHARGED BTZ BLACK HOLE IN MASSIVE GRAVITY

A. GEMS of uncharged BTZ black hole in massive gravity

Now, let us consider a $(2 + 1)$-dimensional uncharged BTZ black hole in the massive gravity described by the 3-metric in Eq. (2.1) with the lapse function

$$N^2 = -m + \frac{r^2}{l^2} + 2Rr. \quad (3.1)$$

Here, the notation R related to the mass term in Eq. (1.3) is given by

$$R = \frac{1}{2}\tilde{M}^2 c c_1. \quad (3.2)$$

Now, the mass m of the uncharge BTZ black hole in the massive gravity can be given in terms of the event horizon r_H as

$$m = \frac{r_H^2}{l^2} + 2Rr_H, \quad (3.3)$$

and the Hawking-Bekenstein horizon surface gravity is of the form

$$k_H = \frac{r_H}{l^2} + R. \quad (3.4)$$

Exploiting the GEMS approach, we obtain a $(3 + 2)$-dimensional AdS GEMS,

$$ds^2 = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 + (dz^3)^2 - (dz^4)^2, \quad (3.5)$$

given by the coordinate transformations for $r \geq r_H$ and $R > 0$ as

$$\begin{aligned}
z^0 &= k_H^{-1}\left[\frac{r^2 - r_H^2}{l^2} + 2R(r - r_H)\right]^{1/2} \sinh k_H t,\\
z^1 &= k_H^{-1}\left[\frac{r^2 - r_H^2}{l^2} + 2R(r - r_H)\right]^{1/2} \cosh k_H t,\\
z^2 &= \frac{l}{r_H} r \sinh \frac{r_H}{l}\phi,\\
z^3 &= \frac{l}{r_H} r \cosh \frac{r_H}{l}\phi,\\
z^4 &= k_H^{-1}\int dr \frac{\left[R^2 l^2 r_H^2 r^2\left(1 + \frac{4r_H}{r+r_H}\right) + Rl^4 f(r)\right]^{1/2}}{r_H^2 r\left[1 + \frac{2Rl^2}{r+r_H}\right]^{1/2}}, \quad (3.6)
\end{aligned}$$

where

$$f = \frac{2r_H^3 r^2}{l^4} + \frac{2r^2 R^2 r_H^2}{r + r_H}. \quad (3.7)$$

For the trajectories, which follow the Killing vector $\xi = \partial_t$, we can obtain a constant 3-acceleration,

$$a_3 = \frac{\frac{r}{l^2} + R}{\left[\frac{r^2 - r_H^2}{l^2} + 2R(r - r_H)\right]^{1/2}}. \quad (3.8)$$

In static detectors (ϕ, $r =$ const) described by a fixed point in the (z^2, z^3, z^4) plane, an observer, who is uniformly accelerated in the $(3 + 2)$-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0,z^1) described by a proper acceleration a_5 as follows:

$$a_5^{-2} = (z^1)^2 - (z^0)^2 = \frac{l^2[r^2 - r_H^2 + 2Rl^2(r - r_H)]}{(r_H + Rl^2)^2}. \quad (3.9)$$

Here, we have the relation with a constant acceleration a_3,

$$a_5^2 - a_3^2 = -\frac{1}{l^2}, \quad (3.10)$$

which is the same as the result for the uncharged BTZ black hole in the massless gravity [7,10].

Exploiting the relation in Eq. (2.11), we arrive at the Unruh temperature for a uniformly accelerated observer in the $(3+2)$-dimensional flat spacetime:

$$T_U = \frac{r_H + Rl^2}{2\pi l[r^2 - r_H^2 + 2Rl^2(r - r_H)]^{1/2}}. \quad (3.11)$$

This is exactly the same with the fiducial temperature $T_{\rm FID}$ in Eq. (2.13) with T_H being the Hawking temperature measured by an asymptotic observer,

$$T_H = \frac{1}{2\pi}\left(\frac{r_H}{l^2} + R\right). \quad (3.12)$$

As a result, one can say that the Hawking effect for a fiducial observer in the black hole spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensional flat spacetime. Next, introducing a new additional dimensionless variable,

$$d = Rr_H, \quad (3.13)$$

together with the other dimensionless variables in Eq. (2.15), we rewrite the Hawking temperature in Eq. (3.12) as follows:

$$T_H \cdot r_H = \frac{1}{2\pi}\left(\frac{1}{a^2} + d\right). \quad (3.14)$$

B. Free-fall temperature of uncharged BTZ black hole in massive gravity

Now, we assume that an observer at rest is freely falling from the radial position $r = r_0$ at $\tau = 0$ [26–30]. The equations of motion for the orbit of the observer are given by Eq. (2.17). Exploiting Eqs. (3.6) and (2.17), we obtain the freely falling acceleration $\bar{a}_5$ in the $(3+2)$-dimensional GEMS embedded spacetime

$$\bar{a}_5 = \frac{1}{l}, \quad (3.15)$$

which gives us the temperature measured by the freely falling observer at rest as follows:

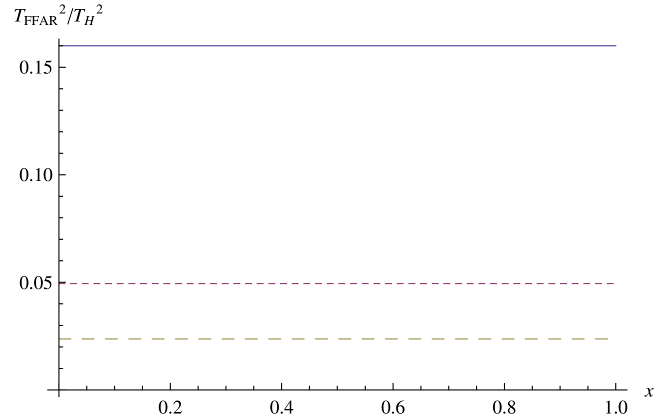

FIG. 2. Free-fall temperature for the uncharged BTZ black hole in the massive gravity for a fixed $a = 2$ with $d = 1, 2, 3$ from top to bottom.

$$T_{\rm FFAR} \cdot r_H = \frac{1}{2\pi a}. \quad (3.16)$$

This is exactly the same as the result for the uncharged BTZ black hole in the massless gravity. The squared free-fall temperature $T_{\rm FFAR}^2$ is depicted in Fig. 2 in units of T_H^2.

IV. CHARGED BTZ BLACK HOLE IN MASSIVE GRAVITY

A. GEMS of charged BTZ black hole in massive gravity

A $(2+1)$-dimensional charged BTZ black hole in massive gravity is described by the 3-metric in Eq. (2.1) with the lapse function [38,42]

$$N^2 = -m + \frac{r^2}{l^2} - 2q^2 \ln\frac{r}{l} + 2Rr, \quad (4.1)$$

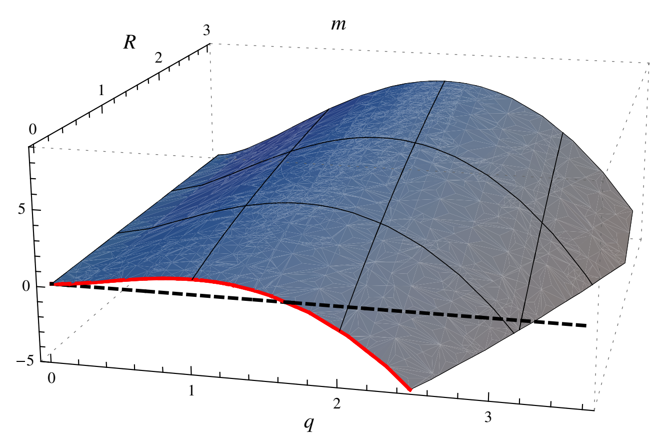

FIG. 3. Upper region of the surface in the mass-charge relation in which charged BTZ black holes in massive gravity can exist.

where R is given by Eq. (3.2). First of all, it is appropriate to comment on the metric function (4.1), which goes to positive infinities if $r \to 0$ and $r \to \infty$ so that there is a minimum at

$$r_{\rm min} = \frac{-Rl^2 + l\sqrt{4q^2 + R^2 l^2}}{2}. \tag{4.2}$$

Thus, the metric function has the value of

$$N^2(r)|_{r\to r_{\rm min}} = -m + q^2 - \frac{1}{2}R^2 l^2 + \frac{1}{2}Rl\sqrt{4q^2 + R^2 l^2} - 2q^2 \ln\left(\frac{-Rl + \sqrt{4q^2 + R^2 l^2}}{2}\right). \tag{4.3}$$

One can easily see that when $N^2(r_{\rm min}) < 0$ there are two roots of r_+ and r_- and when $N^2(r_{\rm min}) = 0$ the two roots coincide and one has an extreme black hole. This is depicted in Fig. 3, in which black holes with two horizons exist over the (q, R) surface, an extremal black hole exists at the (q, R) surface, and black holes cannot exist below the (q, R) surface. On the figure, the red curve is drawn for $R = 0$, in which case one has some difficulties [44] such as a logarithmic divergent boundary term at $r \to \infty$ and a cosmic censorship problem due to having arbitrarily negative values of m. Reference [45] has studied how to circumvent the problems. For a similar reason, the mass function in Eq. (4.3) would have the same problems, which may be addressed elsewhere. Here, we will obtain the GEMS embedding only for $r > r_+$ and nonextremal cases.

Now, the mass m in Eq. (4.1) can be given in terms of the event horizon r_H as

$$m = \frac{r_H^2}{l^2} - 2q^2 \ln\frac{r_H}{l} + 2Rr_H, \tag{4.4}$$

and the Hawking-Bekenstein horizon surface gravity is of the form

$$k_H = \frac{r_H}{l^2} - \frac{q^2}{r_H} + R. \tag{4.5}$$

Exploiting the GEMS approach, we obtain a $(3+3)$-dimensional AdS GEMS,

$$ds^2 = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 + (dz^3)^2 - (dz^4)^2 + (dz^5)^2, \tag{4.6}$$

given by the coordinate transformations for $r \geq r_H$ as

$$\begin{aligned}
z^0 &= k_H^{-1}\left[\frac{r^2 - r_H^2}{l^2} - 2q^2 \ln\frac{r}{r_H} + 2R(r - r_H)\right]^{1/2} \sinh k_H t,\\
z^1 &= k_H^{-1}\left[\frac{r^2 - r_H^2}{l^2} - 2q^2 \ln\frac{r}{r_H} + 2R(r - r_H)\right]^{1/2} \cosh k_H t,\\
z^2 &= \frac{l}{r_H} r \sinh\frac{r_H}{l}\phi,\\
z^3 &= \frac{l}{r_H} r \cosh\frac{r_H}{l}\phi,\\
z^4 &= k_H^{-1}\int dr \frac{\left[q^4 l^2(r^2 + r_H^2 + 2r^2 g(r)) + R^2 l^2 r_H^2 r^2\left(1 + \frac{4r_H}{r+r_H}\right) + Rl^4 f_1(r)\right]^{1/2}}{r_H^2 r\left[1 - \frac{q^2 l^2}{r_H^2} g(r) + \frac{2Rl^2}{r+r_H}\right]^{1/2}},\\
z^5 &= k_H^{-1}\int dr \frac{\left[q^2 r^2\left(2r_H^2 + \frac{r_H^4 + q^4 l^4}{r_H^2} g(r)\right) + R^2 q^2 l^4 r^2\left(g(r) + \frac{4r_H}{r+r_H}\right) + Rl^4 f_2(r)\right]^{1/2}}{r_H^2 r\left[1 - \frac{q^2 l^2}{r_H^2} g(r) + \frac{2Rl^2}{r+r_H}\right]^{1/2}},
\end{aligned} \tag{4.7}$$

where

$$f_1 = \frac{2r_H^3 r^2}{l^4} + \frac{2q^4 r^2}{r_H} g(r) + \frac{2r^2}{r + r_H}(q^4 + R^2 r_H^2), \qquad f_2 = \frac{2q^2 r_H r^2}{l^2}[1 + g(r)] + \frac{2q^2 r_H^2 r(2r + r_H)}{l^2(r + r_H)}, \tag{4.8}$$

and $g(r)$ is given by Eq. (2.7). In the limit of $q = 0$, the coordinate transformations in Eq. (4.7) are reduced to those in Eq. (3.6) with $f_1 \to f$ and $f_2 \to 0$. Here, note that the dimensionality $(3+3)$ becomes $(3+2)$ since z^5 disappears in this limit.

For the trajectories, which follow the Killing vector $\xi = \partial_t$, we can obtain a constant 3-acceleration:

$$a_3 = \frac{\frac{r}{l^2} - \frac{q^2}{r} + R}{\left[\frac{r^2 - r_H^2}{l^2} - 2q^2 \ln\frac{r}{r_H} + 2R(r - r_H)\right]^{1/2}}. \quad (4.9)$$

In static detectors (ϕ, $r = \text{const}$) described by a fixed point in the (z^2, z^3, z^4, z^5) plane, an observer, who is uniformly accelerated in the $(3+3)$-dimensional flat spacetime, follows a hyperbolic trajectory in (z^0, z^1) described by a proper acceleration a_6 as follows:

$$a_6^{-2} = (z^1)^2 - (z^0)^2 = \frac{l^2[r^2 - r_H^2 - 2q^2 l^2 \ln\frac{r}{r_H} + 2Rl^2(r - r_H)]}{(r_H - \frac{q^2 l^2}{r_H} + Rl^2)^2}. \quad (4.10)$$

Here, we have the relation with a constant acceleration a_3,

$$a_6^2 - a_3^2 = -\frac{1}{l^2} + \frac{\frac{q^4 l^2}{r^2 r_H^2} - \frac{q^2}{r_H^2} g(r) - \frac{2Rq^2 l^2}{r r_H (r + r_H)}}{1 - \frac{q^2 l^2}{r_H^2} g(r) + \frac{2Rl^2}{r + r_H}}, \quad (4.11)$$

which becomes Eq. (2.10) in the limit of $R = 0$ and Eq. (3.10) in the limit of $q = 0$.

Exploiting the relation in Eq. (2.11), we arrive at the Unruh temperature for a uniformly accelerated observer in the $(3+3)$-dimensional flat spacetime:

$$T_U = \frac{r_H - \frac{q^2 l^2}{r_H} + Rl^2}{2\pi l[r^2 - r_H^2 - 2q^2 l^2 \ln\frac{r}{r_H} + 2Rl^2(r - r_H)]^{1/2}}. \quad (4.12)$$

This is exactly the same with the fiducial temperature $T_{\rm FID}$ in Eq. (2.13) with T_H being the Hawking temperature measured by an asymptotic observer,

$$T_H = \frac{1}{2\pi}\left(\frac{r_H}{l^2} - \frac{q^2}{r_H} + R\right). \quad (4.13)$$

As a result, one can say that the Hawking effect for a fiducial observer in the black hole spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensional flat spacetime.

In terms of the dimensionless variables, the Hawking temperature in Eq. (4.13) can be rewritten as

$$T_H \cdot r_H = \frac{1}{2\pi}\left(\frac{1}{a^2} - b + d\right). \quad (4.14)$$

In Fig. 4(a), we have drawn the Hawking temperatures $T_H^{\rm CBTZ,m}/T_H^{\rm BTZ,m}$ of the charged/uncharged BTZ black hole in the massive gravity compared to $T_H^{\rm CBTZ}/T_H^{\rm BTZ}$ of the charged/uncharged BTZ black hole in the massless gravity. One can see that the massive gravitons in the BTZ black holes only make the Hawking temperatures shift parallelly. As $r_H \to \infty$, all the Hawking temperatures are proportional to r_H, while they are being curved near the event horizon when either q is large or r_H is small. In Fig. 4(b), one can understand the relative roles of charge and massive gravitons in the charged BTZ black hole in the massive gravity that when $q^2 > Rr_H$, $T_H^{\rm BTZ} > T_H^{\rm BTZ,m}$, while when $q^2 < Rr_H$, $T_H^{\rm BTZ} < T_H^{\rm BTZ,m}$.

B. Free-fall temperature of charged BTZ black hole in massive gravity

Now, we assume that an observer at rest is freely falling from the radial position $r = r_0$ at $\tau = 0$ [26–30]. The equations of motion for the orbit of the observer are given by Eq. (2.17). Exploiting Eqs. (4.7) and (2.17), we obtain the freely falling acceleration $\bar{a}_6$ in the $(3+3)$-dimensional GEMS embedded spacetime,

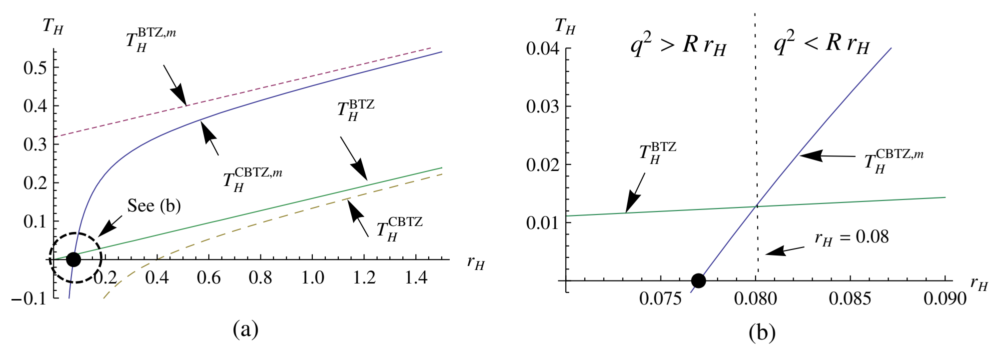

FIG. 4. (a) Hawking temperatures for the charged/uncharged BTZ black hole in the massive gravity $T_H^{\rm CBTZ,m}/T_H^{\rm BTZ,m}$ and the charged/uncharged BTZ black hole in the massless gravity $T_H^{\rm CBTZ}/T_H^{\rm BTZ}$. Here, we have chosen $q = 0.4$, $R = 2$, and $l = 1$ for the charged BTZ black hole in the massive gravity. (b) The relative effect of charge and massive gravitons in the charged BTZ black hole in the massive gravity: $T_H^{\rm BTZ} > T_H^{\rm BTZ,m}$ when $q^2 > Rr_H$ and $T_H^{\rm BTZ} < T_H^{\rm BTZ,m}$ when $q^2 < Rr_H$.

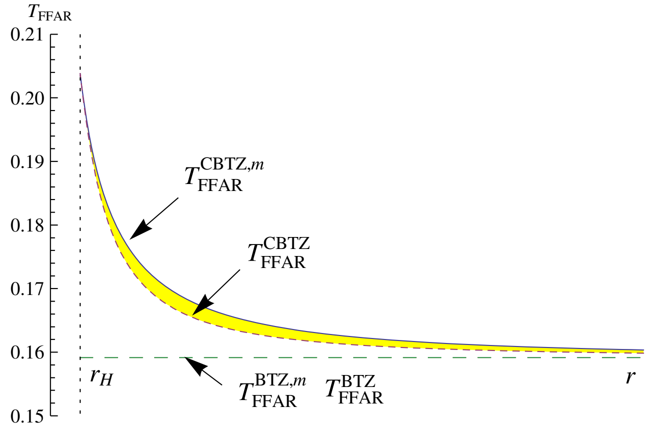

FIG. 5. Free-fall temperatures for the charged/uncharged BTZ black hole in the massive/massless gravity. The gap between the charged BTZ black holes in the massive/massless gravity comes from the gravitons in the massive gravity. The free-fall temperatures for the uncharged BTZ black holes in the massive/massless gravity are constant all over $r \geq r_H$. Here, we have chosen $r_H = 0.5$, $q = 0.4$, $R = 2$, and $l = 1$ for the charged BTZ black hole in the massive gravity.

$$\bar{a}_6 = \frac{1}{l}\left[\frac{\left(1+\frac{q^2l^2}{rr_H}\right)\left(1-\frac{q^2l^2}{rr_H}+\frac{2Rl^2}{r+r_H}\right)}{1-\frac{q^2l^2}{r_H^2}g(r)+\frac{2Rl^2}{r+r_H}}\right]^{1/2}, \tag{4.15}$$

which gives us the temperature measured by the freely falling observer at rest,

$$T_{\rm FFAR} = \frac{1}{2\pi l}\left[\frac{\left(1+\frac{q^2l^2}{rr_H}\right)\left(1-\frac{q^2l^2}{rr_H}+\frac{2Rl^2}{r+r_H}\right)}{1-\frac{q^2l^2}{r_H^2}g(r)+\frac{2Rl^2}{r+r_H}}\right]^{1/2}. \tag{4.16}$$

It is appropriate to comment that in the limit of $R = 0$ the free-fall temperature in Eq. (4.16) is reduced to Eq. (2.19), as expected. Similarly, one can readily check that in the $R = 0$ limit the physical results possessing R terms obtained in this section are reduced to those for the charged BTZ black hole in the massless gravity. On the other hand, in the $q = 0$ limit, the physical results possessing q terms obtained in this section are also reduced to those for the uncharged BTZ black hole in the massive gravity discussed in the previous section.

In Fig. 5, we have drawn the free-fall temperatures for the charged/uncharged BTZ black hole in the massive/massless gravity. Here, one can see the gap between the charged BTZ black holes in the massive/massless gravity, while they are the same at both r_H and $r \to \infty$. This gap arises from the massive gravitons of the charged BTZ black holes in the massive gravity. On the other hand, the free-fall temperatures of the uncharged BTZ black holes in the massive/massless gravity remain constant all over $r \geq r_H$. This means that freely falling observers feel the temperature insensitive to the mass term if they freely fall in an uncharged BTZ black hole.

Now, introducing the dimensionless variables in Eqs. (2.15) and (3.13), we rewrite the free-fall temperature in Eq. (4.16) as

$$T_{\rm FFAR}\cdot r_H = \frac{1}{2\pi a}\left[\frac{(1+a^2bx)\left(1-a^2bx+\frac{2a^2dx}{1+x}\right)}{1+\frac{2a^2bx^2}{1-x^2}\ln x+\frac{2a^2dx}{1+x}}\right]^{1/2}. \tag{4.17}$$

Here, the relevant range of x is given by $0 \leq x \leq 1$. In the limit of $x = 0$, we obtain

$$T_{\rm FFAR}\cdot r_H(x=0) = \frac{1}{2\pi a}, \tag{4.18}$$

as in Eq. (2.22). Here, we note that the above value in Eq. (4.18) does not depend on the parameters b and d. This means that, as $r \to \infty$, the charge and mass term of the charged BTZ black hole in the massive gravity do not contribute to the free-fall temperature of $T_{\rm FFAR}$ as shown in Fig. 5. Moreover, all the free-fall temperature $T_{\rm FFAR}$ as $r \to \infty$ is the same as that of the uncharged BTZ black hole in the massless gravity in Eq. (2.20).

On the other hand, at $x = 1$ corresponding to $r = r_H$, we are left with

$$T_{\rm FFAR}\cdot r_H(x=1) = \frac{1}{2\pi a}(1+a^2b)^{1/2}. \tag{4.19}$$

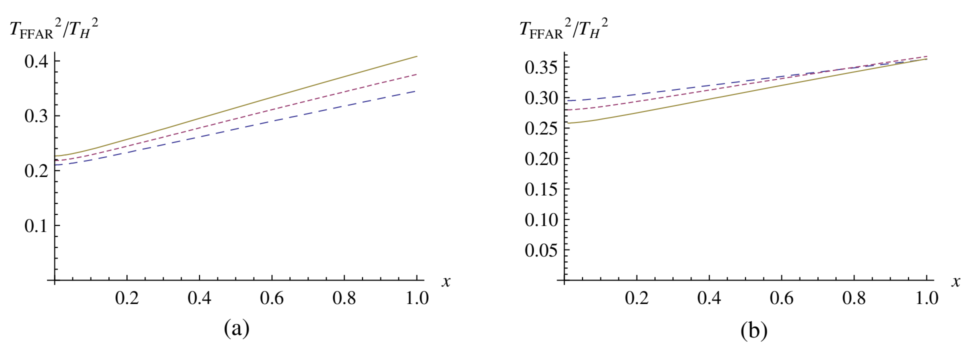

FIG. 6. Free-fall temperature for the charged BTZ black hole in the massive gravity: (a) for a fixed $d = 1$ with $a = 2$ and $b = 0.16$, 0.18, 0.20 from bottom to top; (b) for a fixed $d = 1$ with $b = 0.16$ and $a = 1.2$, 1.4, 1.6 from top to bottom.

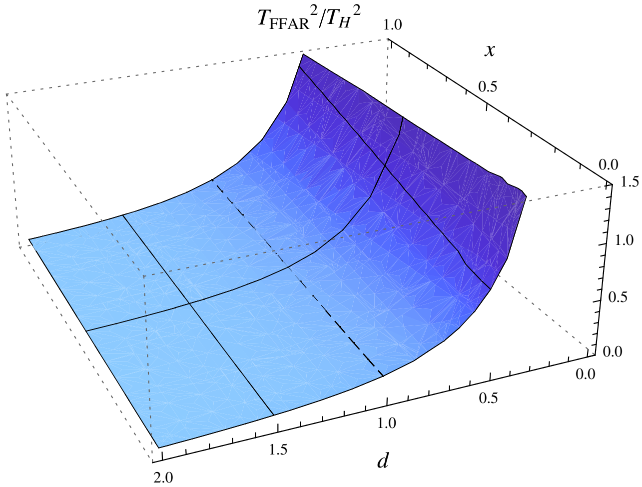

FIG. 7. Free-fall temperature for the charged BTZ black hole in the massive gravity with varying d for a fixed $a = 2$ and $b = 0.18$. The dashed line represents the curve of $d = 1$ with $a = 2$ and $b = 0.18$ in Fig. 6(a).

This free-fall temperature in Eq. (4.19) is the same as that of the charged BTZ black hole in the massless gravity in Eq. (2.23). Here, we note that at event horizon $r = r_H$ there exists no dependence on the parameter d in the above free-fall temperature in Eq. (4.19), even though we consider the massive gravity effect of the charged BTZ black hole case.

We have depicted in Fig. 6 the free-fall temperature T^2_{FFAR} in units of T^2_H for a fixed $d = 1$. Figure 6(a) was drawn by changing the charge b, while Fig. 6(b) was drawn by changing the cosmological constant. The free-fall temperature for varying d is depicted in Fig. 7, where it shows that if d is small the whole variation of T^2_{FFAR} in T^2_H is large in the range of $0 \le x \le 1$.

V. DISCUSSION

In summary, we have globally embedded a charged BTZ black hole in the massive/massless gravity into a $(3+3)$-dimensional flat spacetime, while having embedded an uncharged BTZ black hole in the massless/massive gravity into a $(2+2)/(3+2)$-dimensional flat spacetime as shown in the Table I.

Making use of the embedded coordinates, we have directly obtained the Unruh, Hawking, and freely falling temperatures in the (un)charged BTZ black hole in the massive/massless gravity and shown that the Hawking effect for a fiducial observer in a curved spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensionally embedded flat spacetime. Moreover, we have evaluated all the free-fall temperatures of the (un)charged BTZ black hole in the massive/massless gravity measured by observers freely falling into a black hole. As a result, we have found that all the free-fall temperatures given by $\frac{1}{2\pi l}$ at the infinity end up hotter but finite at the horizon when black holes are charged, while remaining the same as they start when black holes are uncharged like in Table II.

TABLE I. Various GEMS embedding dimensions.

q	R	Black holes	Embedding dimensions
$q = 0$	$R = 0$	Uncharged BTZ black hole in massless gravity	$(2+2)$
$q \neq 0$	$R = 0$	Charged BTZ black hole in massless gravity	$(3+3)$
$q = 0$	$R \neq 0$	Uncharged BTZ black hole in massive gravity	$(3+2)$
$q \neq 0$	$R \neq 0$	Charged BTZ black hole in massive gravity	$(3+3)$

TABLE II. Free-fall temperature at the infinity and event horizon.

BTZ black holes	$r \to \infty$	$r \to r_H$
Uncharged BTZ black hole in massless/massive gravity	$\frac{1}{2\pi l}$	$\frac{1}{2\pi l}$
Charged BTZ black hole in massless/massive gravity	$\frac{1}{2\pi l}$	$\frac{1}{2\pi l}\left(1+\frac{q^2 l^2}{r_H^2}\right)^{1/2}$

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