Tidal Love numbers of black holes and neutron stars in the presence of higher dimensions: Implications of GW170817

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We calculate the tidal Love numbers of black holes and neutron stars in the presence of higher dimensions. The perturbation equations around an arbitrary static and spherically symmetric metric for the even-parity modes are presented in the context of an *effective* four-dimensional theory on the brane. This subsequently leads to the desired expression for the tidal Love number for black holes in the presence of extra spatial dimensions. Surprisingly, these numbers are nonzero and (more importantly) negative. We extend our method to determine the tidal Love number of neutron stars in a spacetime inheriting extra dimensions and show that, in the context of effective gravitational theory on the brane, they are smaller than in general relativity. Finally, we explicitly demonstrate that earlier constraints on the parameters inherited from higher dimensions are consistent with the bound on the tidal deformability parameter from the GW170817 event as well.

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I. INTRODUCTION AND MOTIVATION

The recent success of gravitational-wave (GW) detectors in detecting GW from mergers of binary black holes (BHs) and neutron stars (NSs) has been a feather in the cap of scientific and technological advancement [1–7]. The increasing sensitivity of GW detectors will enable us to make more and more specific statements with regard to different astronomical aspects of GWs. Thus, with GW astronomy established on a firm footing, we can now turn our attention to answer some of the more fundamental questions of nature. These include: (a) the subtle relations regarding black hole physics (e.g., the no-hair theorem, the area increase theorem, and the existence of black hole horizons themselves) [8–13]; (b) evidence of theories beyond general relativity, in particular, the presence of higher-curvature terms in the Einstein-Hilbert action and theories involving specific curvature scalar couplings (known as Horndeski theories), among others [14–22]; and finally (c) the fundamental structure of spacetime itself [23–29]. An interesting line of thought in understanding the fundamental structure of spacetime happens to be the question of the presence (or absence) of extra spatial dimensions in addition to our usual four-dimensional spacetime.

Incidentally, the idea of the existence of spatial extra dimensions arose for a completely different reason. Originally, these extra dimensions have appeared quite naturally in string theory, whose existence requires ten or more dimensions. However, later the existence of extra dimensions returned to the community in order to solve the "gauge hierarchy problem" [30–36]. This has to do with the fact that the scale of physics appears very much uncorrelated and hierarchical in nature. This is because the scale of electroweak symmetry breaking, which is $\sim 10^3$ GeV, appears to be completely disconnected from the Planck scale at $\sim 10^{18}$ GeV. Another and probably more practical reason to worry about the gauge hierarchy problem has to do with the running of the mass of the Higgs boson. In order to get the observed value of the Higgs mass at the Large Hadron Collider, one needs to fine-tune the counterterm arising out of renormalization to one part in 10^{15}. This large fine-tuning is another reason for the existence of the gauge hierarchy problem [35–37].

In the context of extra dimensions it is indeed possible to cure the gauge hierarchy problem. Broadly speaking, there are two possible ways to achieve this. First, one can introduce large extra dimensions such that even though the four-dimensional Planck's constant is large enough, the fundamental higher-dimensional Planck's constant is on the TeV scale due to volume suppression [30–32,35]. The above model does not incorporate gravitational dynamics and hence is not of much significance from the GW point of view. On the other hand, it is also possible to provide a

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gravitational resolution to the gauge hierarchy problem. One of the many possible ways to achieve this is by introducing warped anti–de Sitter geometry in the bulk, resulting in an exponential suppression of physical scales on the brane [38–41]. We will mainly work with the second possibility, where gravity itself comes to the rescue.

However, since all of the observations regarding GWs are made in our four-dimensional spacetime it is legitimate to ask for an effective four-dimensional theory starting from the original higher-dimensional one [38,42–51]. As such, we have modeled the extra dimension in a simple manner, i.e., with one additional spacelike dimension. By imposing orbifold symmetry on the extra dimension one can determine the effective gravitational field equation inheriting additional corrections from the bulk geometry [45,46,52,53]. Such effective gravitational field equations have been derived in the context of general relativity in Ref. [52], which subsequently was generalized to many other situations as well [54–56]. In this work we will content ourselves with the general-relativistic situation, and with the advent of GW astronomy we would like to test the theory using the strong-field regime of gravity.

The presence of extra dimensions would leave very specific signatures on the GWs emitted by merging BHs or NSs. In principle, these signatures will be present for the entire duration of the GW signal, often separated into three distinct regimes, commonly known in GW terminology as the inspiral, merger, and ringdown phases. The inspiral and merger phases would require a detailed numerical analysis, which is presently unavailable, while the ringdown phase can be understood analytically, since it involves the computation of quasinormal modes (QNMs) from perturbed BHs in these theories [57–64]. For both the higher-dimensional black holes or black holes in the effective gravitational theory one can indeed obtain distinct signatures of extra dimensions in the QNMs, which recently have been investigated in some detail in the literature [28,65] (also see Ref. [66]).

In this paper, we explore yet another observational window put forward by the recent GW observations, namely, the modifications to the tidal Love number due to the presence of higher dimensions (for some recent works regarding the tidal Love number in other theories beyond general relativity, see Refs. [67,68]). The tidal Love number for a neutron star or black hole essentially corresponds to the deformation in the respective objects caused by an external tidal field [69–80]. Interestingly, for black holes in general relativity the tidal Love number identically vanishes, and hence determining the tidal Love number for black holes in theories beyond general relativity is of significant importance. If the associated tidal Love number for black holes are nonzero, they may provide a crucial hint about these theories beyond general relativity [67]. On the other hand, the tidal Love number for neutron stars are also of sufficient importance in the context of the equation of state (EoS) of the material forming the neutron star. Since the composition of neutron stars is not very well known, tight constraints on the EoS parameter do not exist either. Given the recent detection of an NS-NS merger by the Advanced LIGO detectors, it is desirable to understand the EoS parameter of NSs either by appropriate theoretical modeling or using numerical simulations. The best way to get an understanding of the tidal Love number is from the early regime of the inspiral phase, as the signal is very clean. However, the influence of tidal effects is through the phase of the waveform and is only a small correction. Thus, to detect this one may invoke the matched-filtering technique by integrating the measured waveform, such that the accumulated phase shift due to the tidal corrections becomes larger. The influence of the internal structure of the neutron star on the gravitational-wave phase is characterized by the ratio of the induced quadrupole moment to the perturbing external tidal field and is denoted by λ. This is related to the dimensionless tidal Love number k_2 through the relation $(3/2)(G_4\lambda/R^5)$, where G_4 is the four-dimensional gravitational constant and R is the radius of the neutron star. A systematic study of the determination of the tidal Love number from perturbations of Einstein's gravitational field equations was presented in Refs. [70,71] and later used extensively in the GW literature, where the ideas were both refined and broadened [72,73]. In this paper, we would like to understand the modifications to the tidal Love number of both black holes and neutron stars due to the presence of extra dimensions. This is evidently the first step in trying to understand the complete set of modifications to the GW signal which may arise in the strong-field regime of gravity.

The paper is organized as follows. In Sec. II we discuss the general framework of gravitational perturbations outside a compact object, which could be either a black hole or a neutron star. The formalism derived here is subsequently applied in Sec. III to discuss the tidal Love number for a black hole in four-dimensional spacetime in the presence of higher dimensions. Finally, we also demonstrate the modifications to the tidal Love number associated with neutron stars pertaining to the existence of higher dimensions. Implications of the results derived here for current and future GW merger events are explored in Sec. V. We finally conclude with a discussion of our results. Some relevant computations are presented in the Appendix.

Notations and conventions: Throughout the paper we assume $\hbar = 1 = c$. Greek indices are used to represent four-dimensional quantities and we work with the mostly positive signature convention. By and large, in this paper by the phrase "tidal Love number" we essentially mean the dimensionful tidal Love number λ. Whenever the dimensionless tidal Love number is used it will be mentioned explicitly.

II. PERTURBATION OUTSIDE A COMPACT OBJECT IN THE PRESENCE OF EXTRA DIMENSIONS: GENERAL ANALYSIS

In this section we determine the master equation satisfied by the even-parity gravitational perturbations—which is essential for the computation of the electric-type tidal Love number—in the exterior region of a compact object, which could be either a black hole or a neutron star. In the case of a black hole, the exterior equation alone is sufficient to determine the tidal Love number. This will help us to determine whether the electric-type tidal Love number can have a nonzero value in a black hole background in the presence of extra dimensions. On the other hand, for neutron stars (unlike the case for black holes) understanding the gravitational perturbation in the exterior region is not sufficient to determine the tidal Love number: one needs to solve the gravitational perturbation in the interior of the star as well, which we will address in the subsequent sections. Since the structure of the gravitational field equations in the exterior region is of importance, irrespective of the nature of the compact object, we will first characterize the background spacetime around which perturbations will be considered in some detail, before deriving the associated master equation for gravitational perturbation around this background with full generality.

A. Setting up the background spacetime

The spacetime under consideration inherits one additional spatial dimension and we are interested in its effect on the gravitational field equations associated with the four-dimensional brane hypersurface we are living in. This can be achieved by starting from the five-dimensional bulk Einstein's equations and then projecting onto the lower-dimensional brane hypersurface, using appropriate normal vectors. Since there is no matter field present in the exterior region, the sole contribution will come from gravitational effects. In such a vacuum exterior spacetime, the projected field equation describing the dynamics of gravity can be written as [53]

$$^{(4)}G_{\mu\nu} + E_{\mu\nu} = 0. \tag{1}$$

Here $E_{\mu\nu}$ corresponds to the electric part of the bulk Weyl tensor and $^{(4)}G_{\mu\nu}$ is the Einstein tensor projected onto the brane starting from the bulk. Interestingly, the derivation of the above equation uses minimal information about the nature of the extra dimension; in particular, it only requires the brane to be located at an orbifold fixed point associated with the extra dimension. Thus, the result presented in this work will be general and possibly will hold for a large class of extra dimensions. This fact is reflected in the determination of $E_{\mu\nu}$, which requires information about the bulk spacetime that in general is not available. Thus, following Refs. [45,46,53,81], we will assume that $E_{\mu\nu}$ can be represented by a perfect fluid, defined by the energy density U and pressure P. The exact correspondence between the structure of $E_{\mu\nu}$ and that of the bulk spacetime can be understood as and when we get a handle on the nature of physical theories near the Planck scale.

Since by definition the computation of the tidal Love number involves an equilibrium configuration, we will consider a static and spherically symmetric spacetime for the background geometry. The line element for such a spacetime can always be cast as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2, \tag{2}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions of the radial coordinate alone. Therefore, the associated field equations simplify to

$$e^{-2\lambda(r)}\left(\frac{1}{r^2} - \frac{2\lambda'}{r}\right) - \frac{1}{r^2} = -24\pi\tilde{U}(r), \tag{3}$$

$$e^{-2\lambda(r)}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi(\tilde{U} + 2\tilde{P}), \tag{4}$$

$$e^{-2\lambda}\left\{\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{r}(\nu' - \lambda')\right\} = 8\pi(\tilde{U} - \tilde{P}). \tag{5}$$

Here we have defined

$$\tilde{U} = \frac{2G_4}{(8\pi G_4)^2\lambda_{\rm b}}U, \qquad \tilde{P} = \frac{2G_4}{(8\pi G_4)^2\lambda_{\rm b}}P, \tag{6}$$

where G_4 is the four-dimensional Newton's constant and $\lambda_{\rm b}$ is the brane tension [45,46,53]. As is evident from the above equations, the quantities $\tilde{U}$ and $\tilde{P}$ (or U and P) encode all of the higher-dimensional effects and hence are completely determined by $E_{\mu\nu}$. In particular, the "dark radiation" term U is related to $E_{\mu\nu}$ as $U = -(G_4/G_5)^2 E_{\mu\nu}u^\mu u^\nu$, where G_5 is the five-dimensional gravitational constant, while the "dark pressure" term P essentially originates from the spatially trace-free and symmetric part of $E_{\mu\nu}$. One can also verify the correctness of the above equations from dimensional analysis as well. Thus, the above set of equations can be thought of as the background equations in the presence of an anisotropic fluid, such that $\rho = 3\tilde{U}$, $p_r = \tilde{U} + 2\tilde{P}$, and $p_\perp = \tilde{U} - \tilde{P}$. Given this structure, we would like to determine the master equation governing the even-parity gravitational perturbation, which we elaborate in the next section.

B. Equations governing even-parity perturbations

Having described the background gravitational field equations, depending on the metric functions $\nu(r)$ and $\mu(r)$ along with the effect of extra dimensions (encoded in $\tilde{U}$ and $\tilde{P}$), let us concentrate on the structure of the perturbed equations. These are obtained from the first order variation of the background metric presented in Eq. (1); i.e., one considers a modified metric $g_{\mu\nu}^{\rm mod} = g_{\mu\nu} + h_{\mu\nu}$ and hence computes the differential equations that $h_{\mu\nu}$ satisfies to its leading order. The perturbation $h_{\mu\nu}$ can be further subdivided into two classes, namely, even and odd, depending on its transformation under parity. The electric-type tidal Love numbers are governed by the even-parity modes, while the magnetic-type tidal Love numbers are dictated by the odd-parity modes. Since the even-parity modes have direct observational consequences for GW physics, in this work we will concentrate on the electric-type tidal Love numbers with the hope of returning to the discussion of magnetic-type tidal Love numbers elsewhere. Thus, we will exclusively work with even-parity gravitational perturbations. These even-parity perturbations are characterized by the three functions $H_0(r)$, $H_2(r)$, and $K(r)$ as follows:

$$h^{\rm even}_{\mu\nu;\ell m} = \mathrm{diag}[e^{-2\nu(r)}H_0, e^{2\lambda(r)}H_2(r), r^2K(r), r^2\sin^2\theta K(r)]Y_{\ell m}(\theta,\phi). \tag{7}$$

Here we are working with fixed values for ℓ and m, which will suffice as the determination of the tidal Love number requires $\ell = 2$ but is independent of the choice of m. For the sake of generality, we will keep ℓ arbitrary for the moment, but will set $\ell = 2$ as and when necessary.

In general, the ten components of the gravitational field equations will lead to ten perturbed equations. On the other hand, except for the (r,θ) component, all of the other off-diagonal entries in the perturbed equations are trivially satisfied. Among others, the right-hand sides of the angular components of the perturbation equations are identical, and hence $\delta G^\theta_\theta - \delta G^\phi_\phi = 0$, which yields $H_2 = H_0 \equiv H(r)$. Moreover, $\delta G^r_\theta = 8\pi G_4 \delta T^r_\theta$ results in $K' = H' + 2\nu' H$. Thus, one need not consider both H_0 and H_2 as two independent perturbation components; rather, one can concentrate on the differential equation satisfied by H alone. This will also help to determine the nature of the remaining perturbation component $K(r)$ through the relation $K' = H' + 2\nu' H$ introduced earlier. To proceed further, we need to take care of the addition of the angular components as well as the temporal and radial components of the perturbation equation.

First, the addition of the angular components, i.e., the equation $\delta G^\theta_\theta + \delta G^\phi_\phi = 8\pi(\delta T^\theta_\theta + \delta T^\phi_\phi)$, yields

$$\begin{aligned} &e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'\{2 + r(\nu' - \lambda')\} - e^{-2\lambda}r^2H'' \\ &\quad - e^{-2\lambda}rH'(3r\nu' - r\lambda' + 2) \\ &\quad - 16\pi r^2\delta p_\perp - 16\pi r^2 H p_\perp = 0. \end{aligned} \tag{8}$$

Here $p_\perp = \tilde{U} - \tilde{P}$, as defined earlier. Thus, the above equation depends on double derivatives of both $H(r)$ and $K(r)$. Proceeding further, we can use the radial component of the perturbation equation, which reads $\delta G^r_r = 8\pi\delta T^r_r$. Expressing this in terms of derivatives of the perturbed and unperturbed metric components, we obtain

$$\begin{aligned} &e^{-2\lambda}(1 + r\nu')rK' - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - e^{-2\lambda}rH' \\ &\quad + \left\{\frac{1}{2}\ell(\ell+1) - 1 - 8\pi r^2 p_r\right\}H - 8\pi r^2\delta p_r = 0. \end{aligned} \tag{9}$$

Here the radial pressure p_r is defined as $\tilde{U} + 2\tilde{P}$, and inherits the contributions from the existence of higher dimensions. Finally, the remaining perturbation equation $\delta G^t_t = 8\pi\delta T^t_t$ can be expressed in terms of metric perturbations as

$$\begin{aligned} &e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'(3 - r\lambda') - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - re^{-2\lambda}H' \\ &\quad - \left\{\frac{1}{2}\ell(\ell+1) + 1 - 8\pi r^2\rho\right\}H + 8\pi r^2\delta\rho = 0. \end{aligned} \tag{10}$$

The density ρ defined above (similarly to p_r and $p_\perp$) is expressible in terms of $\tilde{U}$ and $\tilde{P}$, such that $\rho \equiv 3\tilde{U}$.

Having written down all of the equations governing the perturbations, the aim is to arrive at another equation that does not involve any perturbation in the matter sector. In other words, we want to eliminate $\delta\tilde{U}$ and $\delta\tilde{P}$ from the above equations. As a first step in that direction, we consider the subtraction of the (t,t) and (r,r) components of the perturbation equation, i.e., the subtraction of Eq. (9) from Eq. (10), which yields

$$\begin{aligned} &e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'\{2 - r(\lambda' + \nu')\} - \ell(\ell+1)H \\ &\quad + 16\pi r^2(2\tilde{U} + \tilde{P})H + 16\pi r^2\left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\}\delta\tilde{P} = 0. \end{aligned} \tag{11}$$

In the above expression we have written ρ and p_r in terms of $\tilde{U}$ and $\tilde{P}$, and used $\delta\tilde{U} = (\partial\tilde{U}/\partial\tilde{P})\delta\tilde{P}$. In an identical fashion, we can also rewrite Eq. (8) in terms of $\tilde{U}$ and $\tilde{P}$, which will involve a $\delta\tilde{P}$ term. Thus, one can use it to eliminate the $\delta\tilde{P}$ term appearing in Eq. (11), which results in

$$\begin{aligned} &\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)e^{-2\lambda}r^2K'' + \left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)e^{-2\lambda}rK'\{2 - r(\lambda' + \nu')\} - \left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)\ell(\ell+1)H + 16\pi r^2(2\tilde{U} + \tilde{P})H\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right) \\ &\quad + \left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\}[e^{-2\lambda}r^2K'' - e^{-2\lambda}r^2H'' + e^{-2\lambda}\{2 + (\nu' - \lambda')\}rK' - e^{-2\lambda}(3r\nu' - r\lambda' + 2)rH' - 16\pi r^2H(\tilde{U} - \tilde{P})] = 0. \end{aligned} \tag{12}$$

As desired, the above equation involves no reference to the $\delta\tilde{U}$ or $\delta\tilde{P}$ term whatsoever. Even though the above equation depends on both H and K, a close inspection reveals that it depends on K only through its derivative. Thus, one may use the result $K' = H' + 2\nu' H$ to transform the above differential equation into one that depends solely on $H(r)$. This results in the following differential equation for $H(r)$:

$$\left(3\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)e^{-2\lambda}r^2(H'' + 2\nu' H' + 2\nu'' H) + e^{-2\lambda}r(H' + 2\nu' H)\left\{(2 - r\lambda')\left(3\frac{\partial\tilde{U}}{\partial\tilde{P}}\right) + r\nu'\left(2 + \frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\}$$
$$- e^{-2\lambda}(3r\nu' - r\lambda' + 2)rH'\left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\} + 16\pi r^2(2\tilde{U} + \tilde{P})H\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)$$
$$- 16\pi r^2 H(\tilde{U} - \tilde{P})\left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\} - \left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\}e^{-2\lambda}r^2 H'' - \left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)\ell(\ell + 1)H = 0. \tag{13}$$

This explicitly depicts how one may manipulate all of the perturbation equations so as to eliminate any term depending on $\delta\tilde{P}$ and $\delta\tilde{U}$ along with any remaining gravitational perturbation components in order to arrive at a single equation for the gravitational perturbation H. In the next section, we will manipulate these terms to express the above equation in a more tractable form, which can subsequently be used to determine the tidal Love number.

C. Master equation outside a compact object in the presence of higher dimensions

In this section, we will determine the master equation satisfied by the even-parity gravitational perturbation [introduced in Eq. (7)] outside a compact object in the presence of an extra dimension. For that purpose, one may start by computing the coefficients of H'', H', and H in the equation for the single gravitational perturbation $H(r)$ presented in Eq. (13). It turns out that all of these coefficients are greatly simplified, ultimately resulting in the following structure:

$$e^{-2\lambda}r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)H'' + re^{-2\lambda}\left\{\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right\}(2 - r\lambda' + r\nu')H' + e^{-2\lambda}r^2\left(6\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)(\nu'' - \nu'\lambda' + \nu'^2)H$$
$$- 4\nu'^2 e^{-2\lambda}r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)H + 12re^{-\lambda}\nu'\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)H + 4\nu'^2 e^{-2\lambda}r^2 H - \left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)\ell(\ell + 1)H$$
$$+ 16\pi r^2(2\tilde{U} + \tilde{P})H\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right) - 16\pi r^2 H(\tilde{U} - \tilde{P})\left\{1 + 2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\right\} = 0. \tag{14}$$

The coefficients of all of the derivatives of H can be easily read off from the above expression. In particular, the structural similarity between the coefficients of H'' and H' suggests dividing the above equation by $r^2 e^{-2\lambda}\{(\partial\tilde{U}/\partial\tilde{P}) - 1\}$, which yields

$$H'' + \left(\frac{2}{r} - \lambda' + \nu'\right)H' + 6\frac{(\partial\tilde{U}/\partial\tilde{P})}{(\partial\tilde{U}/\partial\tilde{P}) - 1}\left(\nu'' - \nu'\lambda' + \nu'^2 + \frac{2\nu'}{r}\right)H - 4\nu'^2 H$$
$$- e^{2\lambda}\frac{\ell(\ell + 1)}{r^2}H + 16\pi e^{2\lambda}(2\tilde{U} + \tilde{P})H - 16\pi e^{2\lambda}H(\tilde{U} - \tilde{P})\frac{\{1 + 2(\partial\tilde{U}/\partial\tilde{P})\}}{(\partial\tilde{U}/\partial\tilde{P}) - 1} = 0. \tag{15}$$

In order to arrive at the above expression, we have manipulated various terms appearing in the coefficient of $H(r)$. The above expression can be written in a more suggestive form if we keep in mind that so far we have not used the background field equations. In particular, we can use the background field equations to replace $\nu(r)$ and $\lambda(r)$ by a more suitable expression. First of all, one can integrate Eq. (3) in order to yield

$$e^{-2\lambda} = 1 - \frac{2\tilde{m}(r)}{r}, \qquad \tilde{m}(r) \equiv G_4 M + 12\pi\int dr r^2\tilde{U}(r), \tag{16}$$

where M is the mass of the central compact object. One can further use Eq. (3) as well as Eq. (4) in order to arrive at

$$r(\nu' - \lambda') = 8\pi r^2 e^{2\lambda}(\tilde{P} - \tilde{U}) + \frac{2\tilde{m}(r)}{r} e^{2\lambda}, \quad (17)$$

where Eq. (16) has also been used. Another significant relation can be derived by using Eq. (5):

$$\nu'' + \nu'^2 - \nu'\lambda' = 8\pi e^{2\lambda}(\tilde{U} - \tilde{P}) - \frac{1}{r}(\nu' - \lambda'). \quad (18)$$

Note that the last term can again be written in terms of $\tilde{U}$ and $\tilde{P}$ by using Eq. (17). Thus, in Eq. (15) we can use both Eq. (16) and Eq. (18) to write down all of the background quantities in terms of $\tilde{U}$, $\tilde{P}$, and the derivative $(\partial\tilde{U}/\partial\tilde{P})$. This results in the following structure of the master equation for even-parity gravitational perturbation:

$$H'' + \left\{\frac{2}{r} + 8\pi r e^{2\lambda}(\tilde{P} - \tilde{U}) + \frac{2\tilde{m}(r)}{r^2} e^{2\lambda}\right\}H' + 6\frac{(\partial\tilde{U}/\partial\tilde{P})}{(\partial\tilde{U}/\partial\tilde{P}) - 1}\{24\pi e^{2\lambda}\tilde{U}\}H\left\{-4\nu'^2 - e^{2\lambda}\frac{\ell(\ell+1)}{r^2} + 16\pi e^{2\lambda}(2\tilde{U} + \tilde{P})\right\}H$$
$$- 16\pi e^{2\lambda}H(\tilde{U} - \tilde{P})\frac{\{1 + 2(\partial\tilde{U}/\partial\tilde{P})\}}{(\partial\tilde{U}/\partial\tilde{P}) - 1} = 0. \quad (19)$$

As is evident, the only information about the background spacetime comes from the $e^{2\lambda}$ and ν'^2 terms; the rest of the terms have been converted into some combination of the dark radiation and dark pressure terms, which carry imprints of the presence of higher dimensions. The above expression can be further simplified by appropriately grouping the various terms that appear in the coefficient of $H(r)$. In particular, by expressing $(\partial\tilde{U}/\partial\tilde{P}) = \{(\partial\tilde{U}/\partial\tilde{P}) - 1\} + 1$ we can write down the final compact expression for the differential equation satisfied by the perturbation $H(r)$ as

$$H'' + \left\{\frac{2}{r} + 8\pi r e^{2\lambda}(\tilde{P} - \tilde{U}) + \frac{2\tilde{m}(r)}{r^2} e^{2\lambda}\right\}H' + \left\{-4\nu'^2 - e^{2\lambda}\frac{\ell(\ell+1)}{r^2} + 16\pi e^{2\lambda}(9\tilde{U} + 3\tilde{P})\right\}H$$
$$+ \frac{16\pi e^{2\lambda}(6\tilde{U} + 3\tilde{P})}{(\partial\tilde{U}/\partial\tilde{P}) - 1}H = 0, \quad (20)$$

which is our desired result. Note that this is a single differential equation for $H(r)$ (one of the perturbation components of the even-parity metric perturbation) and hence depicts the master equation that one must solve. An understanding of $H(r)$ will also lead to an understanding of the other metric perturbation components. However, for arbitrary choices of $\tilde{U}$ and $\tilde{P}$ this is as far as we can go; to proceed further we need to have a relation between $\tilde{U}$ and $\tilde{P}$. Only then can we explicitly compute the solution to the above equation, either analytically or by numerical techniques.

III. COMPUTATION OF THE TIDAL LOVE NUMBER FOR A BRANEWORLD BLACK HOLE

We have already developed the basic equation governing even-parity gravitational perturbations in the exterior region of a compact object. If the compact object depicts a neutron star, we have to write down the corresponding equations in its interior as well before obtaining the associated tidal Love number. However, for black holes the situation is much simpler: we just have to solve the exterior solution and use some suitable boundary conditions requiring regularity at the black hole horizon. Keeping this in mind, in this section we discuss the tidal Love number for braneworld black holes. As mentioned earlier, this requires some choices for the dark radiation term $\tilde{U}$ and dark pressure term $\tilde{P}$, for which we will use the most favored equation of state for the Weyl fluid (given by $E_{\mu\nu}$). This will help us to explicitly write down the background solution and extract information about the equation-of-state parameter. This in turn will help us present Eq. (20) in a more appropriate form satisfied by the gravitational perturbation $H(r)$, which we will subsequently solve to find the electric-type tidal Love number.

A. Background spacetime

In this section we present the background spacetime, given an appropriate equation-of-state parameter, depicting a black hole solution on the brane [45]. As is evident from Eq. (20), in the context of a Weyl fluid induced from higher-dimensional spacetime, the most interesting equation-of-state parameter that greatly simplifies the background and perturbation equations corresponds to $2\tilde{U} + \tilde{P} = 0$ [45,81]. Thus, the equation-of-state parameter is $(\partial\tilde{U}/\partial\tilde{P}) = -(1/2)$. With this choice, the differential equation for the background spacetime, presented in Eqs. (3)–(5), can be explicitly solved with the following structure for the Weyl fluid:

$$\tilde{U} = -\frac{\tilde{P}_0}{2r^4}, \qquad \tilde{P} = \frac{\tilde{P}_0}{r^4}, \quad (21)$$

where $\tilde{P}_0$ is a constant dependent on the nature of the bulk spacetime, i.e., it provides the signature of the existence of higher dimensions. The metric elements, on the other hand, take the following form:

$$e^{2\nu} = e^{-2\lambda} = 1 - \frac{2G_4M}{r} - \frac{12\pi\tilde{P}_0}{r^2}. \qquad (22)$$

This is similar to a Reissner-Nordström black hole, with one crucial difference: in the context of a Reissner-Nordström black hole coefficient of r^{-2} must be positive, while in the present context it can be positive or negative depending on the sign of $\tilde{P}_0$. In what follows we will assume $\tilde{P}_0 > 0$, which will be the only nontrivial possibility when we consider the case of the exterior of a compact object in the next section. Keeping this in mind, in the context of a black hole as well we will consider the case in which the coefficient of the r^{-2} term is negative. Given the above structure of the metric elements associated with the background spacetime, it immediately follows that

$$2\nu' = \left(\frac{2G_4M}{r^2} + \frac{24\pi\tilde{P}_0}{r^3}\right)\left(1 - \frac{2G_4M}{r} - \frac{12\pi\tilde{P}_0}{r^2}\right)^{-1} = -2\lambda', \qquad (23)$$

and in this particular case we have $\tilde{m}(r) = G_4M + (6\pi\tilde{P}_0/r)$. These expressions will be used extensively in later sections when we explicitly compute the tidal Love number for the above black hole solution located on the brane.

Before we delve into the computation of the tidal Love number, let us briefly discuss how to even define the tidal Love number in the present context. For this purpose, suppose that the above system is placed in a static, external quadrupolar field $\mathcal{E}_{ij}$, where i and j are spatial indices. In response to the above, the compact object will develop a quadrupole moment Q_{ij} and in the asymptotic rest frame the temporal components of the metric element, in the present context, can be written as [70]

$$\frac{1}{2}(1 + g_{tt}) = \frac{G_4M}{r} + \frac{\beta}{2}\left(\frac{G_4M}{r}\right)^2 + \frac{3G_4Q_{ij}}{2r^3}\left(n^in^j - \frac{1}{3}\delta^{ij}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) - \frac{1}{2}\mathcal{E}_{ij}n^in^jr^2 + \mathcal{O}(r^3), \qquad (24)$$

where we have introduced a dimensionless constant $\beta \equiv (12\pi\tilde{P}_0/G_4^2M^2)$ which has been inherited from the extra dimension. As is evident, in the $\beta \to 0$ limit we recover the expansion of g_{tt} as in general relativity. Further, the quantity $n^i \equiv x^i/r$ and the above expansion defines the quadrupole moment Q_{ij} of the compact object and the external quadrupolar field $\mathcal{E}_{ij}$. The proportionality constant between them corresponds to the tidal Love number λ, defined as $Q_{ij} = -\lambda\mathcal{E}_{ij}$. Further, we can expand both Q_{ij} and $\mathcal{E}_{ij}$ in terms of spherical harmonics, such that

$$\mathcal{E}_{ij} = \sum_{m=-2}^{2}\mathcal{E}_mY_{(2m)ij}, \qquad Q_{ij} = \sum_{m=-2}^{2}Q_mY_{(2m)ij}, \qquad (25)$$

where $Y_{(2m)ij}$ are traceless tensors related to the $\ell = 2$ spherical harmonics $Y_{2m}(\theta,\phi)$ by the relation $Y_{2m} = Y_{(2m)ij}n^in^j$. Thus, using the expressions for $\mathcal{E}_{ij}$ and Q_{ij} presented in Eq. (25), the expansion of the temporal component of the metric becomes

$$\frac{1}{2}(1 + g_{tt}) = \frac{G_4M}{r} + \frac{\beta}{2}\left(\frac{G_4M}{r}\right)^2 + \frac{3G_4}{2r^3}\sum_{m=-2}^{2}Q_mY_{2m} + \mathcal{O}\left(\frac{1}{r^4}\right) - \frac{1}{2}\sum_{m=-2}^{2}\mathcal{E}_mY_{2m}r^2 + \mathcal{O}(r^3). \qquad (26)$$

Since the tidal Love number does not depend on the specific value of m, it can still be determined from the relation $Q_m = -\lambda\mathcal{E}_m$. Thus, the tidal Love number is derived as follows: (a) using the background spacetime geometry presented in Eqs. (21)–(22), one can rewrite the perturbation equation presented in Eq. (20); (b) this can be subsequently solved to determine $H(r)$ and, hence, $h_{00} = e^{-2\nu}H_0(r)$; then, (c) one can use the expansion of the temporal component of the metric element [presented in Eq. (26)] to determine Q_m and $\mathcal{E}_m$, which in turn determines the tidal Love number. In the next section, we give the final form of the perturbation equation in the present context.

B. Derivation of the final form of the master equation

Having spelled out the structure of the background spacetime along with the notion of the tidal Love number, let us concentrate on the derivation of the final form of the master equation. For this we rewrite Eq. (20) using the background metric elements presented in Eqs. (21)–(22). This yields the following differential equation for $H(r)$ in the background of a braneworld black hole:

$$\left(1 - \frac{2G_4M}{r} - \frac{12\pi\tilde{P}_0}{r^2}\right)H'' + \left\{\frac{2}{r}\left(1 - \frac{2G_4M}{r} - \frac{12\pi\tilde{P}_0}{r^2}\right) + 8\pi r\left(\frac{3\tilde{P}_0}{2r^4}\right) + \frac{2}{r^2}\left(G_4M + \frac{6\pi\tilde{P}_0}{r}\right)\right\}H' + \left\{-\left(\frac{2G_4M}{r^2} + \frac{24\pi\tilde{P}_0}{r^3}\right)^2\left(1 - \frac{2G_4M}{r} - \frac{12\pi\tilde{P}_0}{r^2}\right)^{-1} - \frac{\ell(\ell+1)}{r^2} + 16\pi\left(-\frac{3\tilde{P}_0}{2r^4}\right)\right\}H = 0, \qquad (27)$$

where both sides of the original equation have been multiplied by $e^{-2\lambda}$. It is always advantageous to rewrite any equation in terms of dimensionless quantities, which prompts us to introduce a new dimensionless variable x and replace the radial coordinate r, such that $x = (r/G_4M) - 1$. This results in the transformation of H' and H'', such that H' gets scaled by $(G_4M)^{-1}$, while H'' gets scaled by $(G_4M)^{-2}$. Finally, by multiplying Eq. (27) throughout by r^2

and subsequently introducing the variable x in appropriate places while removing the radial coordinate, we obtain

$$\{x^2-1-\beta\}\partial_x^2 H+2x\partial_x H+\left\{-\ell(\ell+1)-\frac{4(x+1+\beta)^2}{(1+x)^2(x^2-1-\beta)}-\frac{2\beta}{(1+x)^2}\right\}H=0. \tag{28}$$

The above simple structure of the gravitational perturbation equation is obtained by again introducing the dimensionless quantity $\beta \equiv (12\pi\tilde{P}_0/G_4^2M^2)$. Note that for $\beta=0$, i.e., when extra-dimensional effects are absent the above equation reduces to

$$\{x^2-1\}\partial_x^2 H+2x\partial_x H+\left\{-\ell(\ell+1)-\frac{4}{(x^2-1)}\right\}H=0, \tag{29}$$

exactly coinciding with the result derived in Ref. [70] in the context of general relativity. This acts as an acid test of the formalism developed above since it explicitly demonstrates the correctness of our result by reproducing the general-relativistic result in an appropriate limit. We will now discuss the solution of the above equation in the asymptotic limit in order to determine the associated tidal Love number. For this purpose we will be using Eq. (26), the asymptotic expansion of the temporal component of the perturbed metric.

C. Tidal Love numbers of braneworld black holes

In this section we explicitly compute the tidal Love number for the above black hole solution in the presence of an extra dimension. As an aside, we also demonstrate why the tidal Love number for general-relativistic black holes must vanish, while they can be nonzero in the present context.[1] However, unlike the case of general relativity, where an exact solution to the perturbation equation presented in Eq. (29) is possible in terms of Legendre polynomials, in the present context a general analytic solution seems difficult. Before commenting on the possibility of getting an analytic solution, let us write down Eq. (28) in a more suggestive form. This can be achieved by introducing a new variable y, related to the old one by $x=\sqrt{1+\beta}y$. Thus, the terms involving $\partial_x H$ and $\partial_x^2 H$ will get modified by the introduction of $(1+\beta)^{-1/2}$ and $(1+\beta)^{-1}$, respectively, such that the master equation for $H(x)$ will now be converted into a master equation for $H(y)$, which reads

$$\{y^2-1\}\partial_y^2 H+2y\partial_y H-\left\{\ell(\ell+1)+\frac{4}{(y^2-1)}\frac{(y+\sqrt{1+\beta})^2}{(\sqrt{1+\beta}y+1)^2}+\frac{2\beta}{(\sqrt{1+\beta}y+1)^2}\right\}H=0. \tag{30}$$

As is evident, in terms of the new variable y the coefficients of $\partial_y^2 H$ and $\partial_y H$ are identical to the corresponding differential equation for general relativity; however, the term proportional to $H(y)$ differs significantly. Due to the complicated nature of the coefficient of $H(y)$, this differential equation (unlike the $\beta=0$ case) does not have a general analytic solution. But in order to determine the tidal Love number it is sufficient that we understand the asymptotic limit, and (as we will demonstrate below) an analytic solution can indeed be obtained. Since $y=(1+\beta)^{-1/2}\{(r/M)-1\}$, the asymptotic (i.e., large-r) limit implies $y\to\infty$. Hence, the above differential equation simplifies significantly and we obtain

$$\{y^2-1\}\partial_y^2 H+2y\partial_y H-\left\{\ell(\ell+1)+\frac{4}{y^2-1}\left(\frac{1+\frac{\beta}{2}}{1+\beta}\right)\right\}H=0. \tag{31}$$

The fact that the general-relativistic result is reproduced in the $\beta\to 0$ limit is apparent from the above differential equation. It turns out that the above equation admits analytic solutions in terms of associated Legendre polynomials. The details of the solution and asymptotic limits of the associated Legendre polynomial are given in the Appendix. For our purpose, we can take a cue from the Appendix [in particular, see Eq. (A26)] and write down the asymptotic solution of the above differential equation with $\ell=2$ as

$$H(y)=\left\{\frac{3A_1\sqrt{\pi}}{\Gamma(3-\bar{\beta})}\right\}y^2+\left\{-\frac{A_1}{15\Gamma(-2-\bar{\beta})}+\frac{B_1\Gamma(3+\bar{\beta})e^{i\pi\bar{\beta}}}{15}\right\}y^{-3}, \tag{32}$$

where we defined $\bar{\beta}^2=4\{1+(\beta/2)\}\{1+\beta\}^{-1}$, and A_1 and B_1 are arbitrary constants of integration. At this stage, in order to compare with the asymptotic expansion of the metric elements it is essential to reintroduce the radial coordinate r through the relation $y=x(1+\beta)^{-1/2}$, with $x=(r/G_4M)-1$. In the asymptotic limit, we obtain the following structure for the metric perturbation:

[1]Of course, the fact that the tidal Love number for black holes in general relativity must vanish holds when the expansion of the temporal component of the metric element is truncated as presented in Eq. (26). If higher-order corrections are taken into account, the tidal Love number for black holes in general relativity may turn out to be nonzero.

$$H(\beta;r)=\left\{\frac{3A_1\sqrt{\pi}}{\Gamma(3-\bar{\beta})(1+\beta)}\right\}\times\left(\frac{r}{G_4M}\right)^2+(1+\beta)^{3/2}\left\{-\frac{A_1}{15\Gamma(-2-\bar{\beta})}+\frac{B_1\Gamma(3+\bar{\beta})e^{i\pi\bar{\beta}}}{15}\right\}\left(\frac{G_4M}{r}\right)^3. \tag{33}$$

To ensure the correctness of the above result, we must demonstrate that it is consistent with general relativity, which follows from considering the $\beta\to 0$ limit of Eq. (33), which yields

$$H_{\mathrm{GR}}(r)\equiv H(\beta=0;r)=\left\{\frac{3A_1\sqrt{\pi}}{\Gamma(1)}\right\}\left(\frac{r}{G_4M}\right)^2+\left\{\frac{8B_1}{5}\right\}\left(\frac{G_4M}{r}\right)^3, \tag{34}$$

where we have used the fact that $\bar{\beta}\to 2$ as the parameter β vanishes. Further, since by definition $\Gamma(1)=1$, it follows from the identity $\Gamma(1)=0\times\Gamma(0)$ that $\Gamma(0)$ diverges. Thus, given that $\Gamma(-4)=(-1/4)(-1/3)(-1/2)(-1)\Gamma(0)$, we can immediately argue that $\Gamma(-4)$ diverges. The fact that $\Gamma(-4)$ diverges has been used in order to arrive at Eq. (34). As is evident, the solution for the metric perturbation, presented in Eq. (34), exactly matches the general-relativistic result presented in [70]. This suggests using the same constant in the general case as well, and we also introduce $B=B_1\exp(i\pi\mu)$. Thus, in terms of the newly defined arbitrary constants A and B, the solution to the metric perturbation in the background of a braneworld black hole takes the form

$$H(r)=\left\{\frac{3A}{\Gamma(3-\bar{\beta})(1+\beta)}\right\}\times\left(\frac{r}{G_4M}\right)^2+(1+\beta)^{3/2}\left\{-\frac{A}{15\sqrt{\pi}\Gamma(-2-\bar{\beta})}+\frac{B\Gamma(3+\bar{\beta})}{15}\right\}\left(\frac{G_4M}{r}\right)^3, \tag{35}$$

which has the correct general-relativistic limit. Thus, at both the differential-equation and asymptotic-solution levels we have explicitly verified that the general-relativistic result can be reproduced in the appropriate limit. Now that we know the exact form of the asymptotic solution, we can compute the tidal Love number. This will essentially follow from Eq. (24). In the presence of the perturbation, the temporal component of the metric involving the perturbation is $g_{tt}^{\mathrm{mod}}=-e^{2\nu}-e^{-\nu}H(r)Y_{2m}$, where $e^{2\nu}$ is given by Eq. (22) and $H(r)$ is given by Eq. (35). Hence, using the expressions for $e^{2\nu}$ and $H(r)$, we obtain the following relation in the asymptotic limit:

$$\begin{aligned}&\frac{G_4M}{r}+\frac{\beta}{2}\left(\frac{G_4M}{r}\right)^2+\frac{3G_4}{2r^3}\sum_{m=-2}^{2}Q_mY_{2m}+\mathcal{O}\left(\frac{1}{r^3}\right)-\frac{1}{2}\sum_{m=-2}^{2}\mathcal{E}_mY_{2m}r^2+\mathcal{O}(r^3)\\&=\frac{G_4M}{r}+\frac{\beta}{2}\left(\frac{G_4M}{r}\right)^2-\frac{1}{2}\sum_{m=-2}^{2}\left\{\frac{3A}{\Gamma(3-\bar{\beta})(1+\beta)}\right\}\left(\frac{r}{G_4M}\right)^2Y_{2m}\\&\quad-\frac{1}{2}(1+\beta)^{3/2}\sum_{m=-2}^{2}\left\{-\frac{A}{15\sqrt{\pi}\Gamma(-2-\bar{\beta})}+\frac{B\Gamma(3+\bar{\beta})}{15}\right\}\left(\frac{r}{G_4M}\right)^{-3}Y_{2m}.\end{aligned} \tag{36}$$

As it turns out, the unperturbed components exactly cancel out, and a subsequent matching of the powers of r on both sides of the above equation yields

$$\left\{\frac{3A}{\Gamma(3-\bar{\beta})(1+\beta)}\right\}=(G_4M)^2\mathcal{E}_m,\qquad(1+\beta)^{3/2}\left\{-\frac{A}{15\sqrt{\pi}\Gamma(-2-\bar{\beta})}+\frac{B\Gamma(3+\bar{\beta})}{15}\right\}=-3\frac{G_4Q_m}{(G_4M)^3}. \tag{37}$$

Given the above relations, the tidal Love number can be easily determined by first taking the ratio of the above equations, and then using the definition $Q_m=-\lambda\mathcal{E}_m$. Performing the above, the tidal Love number is

$$G_4\lambda\equiv\frac{1}{3}(1+\beta)^{5/2}\left\{-\frac{\Gamma(3-\bar{\beta})}{45\sqrt{\pi}\Gamma(-2-\bar{\beta})}+\frac{B\Gamma(3+\bar{\beta})\Gamma(3-\bar{\beta})}{45A}\right\}(G_4M)^5, \tag{38}$$

where we used a single m value to determine the tidal Love number. Note that one can compute another dimensionless number using λ that is independent of the radial distance, which is simply given by $G_4\lambda/(G_4M)^5$. We also call this the dimensionless tidal Love number, which is denoted by Λ. Whether the dimensionless tidal Love number under consideration is k_2 or Λ should be clear from the context.

To provide another independent derivation of the same, we also compute the dimensionless tidal Love number k_2 in the asymptotic limit, which reads

$$k_2 = \frac{1}{2}\left(\frac{2H - rH'}{3H + rH'}\right). \tag{39}$$

The value for $H(r)$ in the asymptotic limit has already been computed, whose derivative is also simple enough to determine. Hence, one can immediately determine the combinations $2H - rH'$ and $3H + rH'$. Substituting both of these expressions into Eq. (39) leads to the following expression for k_2:

$$k_2 = \frac{1}{2}(1+\beta)^{5/2}\left\{-\frac{\Gamma(3-\bar{\beta})}{45\sqrt{\pi}\Gamma(-2-\bar{\beta})} + \frac{B}{45A}\Gamma(3+\bar{\beta})\Gamma(3-\bar{\beta})\right\}\left(\frac{G_4 M}{r}\right)^5. \tag{40}$$

This exactly coincides with Eq. (38), with the identification $k_2 = (3/2)G_4\lambda r^{-5}$. Thus, the above result provides yet another verification of the derivation of the tidal Love number for braneworld black holes.

However, in order to numerically estimate the dimensionless tidal Love number, we need to determine the unknown constants A and B appearing in Eqs. (38) and (40). These unknown constants can be obtained by using the gravitational perturbation $H(r)$ and its derivative $H'(r)$ at some fixed radius R. For black holes this would correspond to the event horizon, while for a neutron star this is the radius that determines the extent of the neutron star. Since for general relativity an exact solution of the perturbation equation is known, it is possible to explicitly determine the tidal Love number in terms of these quantities. However, unlike the situation in general relativity, in the present context we do not have a handle on the analytic solution for the gravitational perturbation $H(r)$ at an arbitrary radius. Thus, we cannot compute A and B using $H(r)$ and $rH'(r)$ at some finite radius R. But some physical insight may be gained by instead looking at the expression for the dimensionless tidal Love number Λ, for which the following expression can be determined from Eq. (38):

$$\Lambda = \frac{B}{135A}\{(1+\beta)^{5/2}\Gamma(3+\bar{\beta})\Gamma(3-\bar{\beta})\} - \left\{\frac{\Gamma(3-\bar{\beta})}{\Gamma(-2-\bar{\beta})}\frac{(1+\beta)^{5/2}}{135\sqrt{\pi}}\right\}. \tag{41}$$

As is evident, the vanishing of β implies $\bar{\beta} \to 2$ and hence a $\Gamma(-4)$ term (which diverges) appears in the denominator of the second quantity in the above expression. Thus, we indeed have $\Lambda \to \Lambda_{\rm GR} = (8B/45A)$ as β vanishes. Hence, we can always arrive at the desired general-relativistic limit. However, like for k_2, it is difficult to analytically determine the nature of Λ due to the presence of the arbitrary constants A and B. As is evident from Eq. (41), in the presence of higher dimensions it is most likely that even for black holes the dimensionless tidal Love number Λ will be nonzero. Furthermore, depending on the sign of the ratio (B/A), the dimensionless tidal Love number Λ can also become negative. If correct, such a nonzero *but* negative value of the dimensionless tidal Love number Λ may serve as a distinct signature of the existence of higher dimensions.

At this stage it is worthwhile to point out the implications of the above result for k_2, the other dimensionless tidal Love number presented in Eq. (40). Since the dimensionless tidal Love number k_2 varies with radius as r^{-5}, it follows that asymptotically it should vanish. In the context of general relativity, on the other hand, k_2 identically vanishes everywhere (not just asymptotically), while in the presence of extra dimensions (following the above discussion) one could possibly argue that the quantity (B/A) may result in a negative contribution to the dimensionless tidal Love number k_2, which ultimately asymptotes to zero.

The analytical understanding of the dimensionless tidal Love number k_2 or Λ presented above can also be verified by numerically solving the general differential equation presented in Eq. (30). The result of such a numerical analysis is shown in Fig. 1, which depicts the variation of the dimensionless tidal Love number k_2 with the radial distance from black hole. Similarly, in Fig. 2 numerical estimates of the dimensionless tidal Love number k_2 are plotted against the parameter β and the black hole mass. The plots presented in both Figs. 1 and 2 clearly reveal a negative value for the dimensionless tidal Love number k_2, further bolstering our claim. This has been achieved by

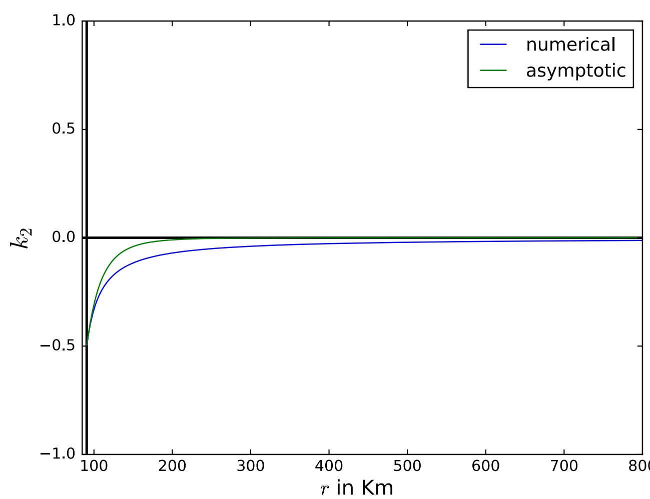

FIG. 1. This figure compares the variation of the dimensionless tidal Love number k_2 with the radial distance from the black hole derived from numerical and theoretical analyses, for a black hole with mass $M_{\rm BH} = 10\ M_\odot$. As is evident from both the theoretical and numerical analyses, the dimensionless tidal Love number k_2 is nonzero and negative for any finite r, unlike in the case for a general-relativistic black hole, where it always vanishes. Further, the difference between the theoretical and numerical analyses is very small in the large-radius limit; however, in the near-horizon region there is a moderate difference between them.

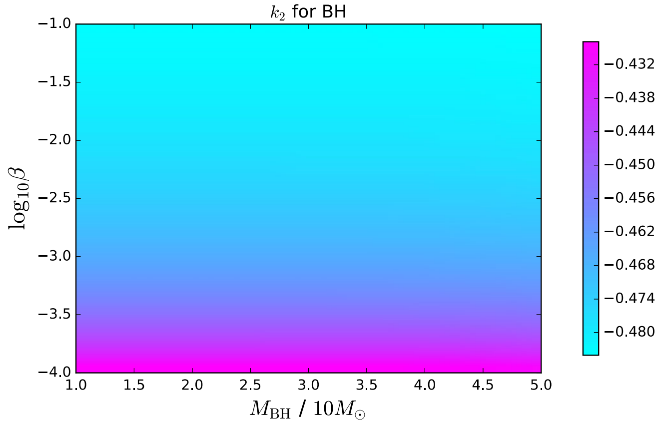

FIG. 2. This figure depicts the variation of the dimensionless tidal Love number k_2 with the black hole mass as well as the parameter β inherited from the extra dimensions from numerical analysis. The black hole mass $M_{\rm BH}$ has been normalized to $(M_{\rm BH}/10\ M_\odot)$. As is evident from the panel on the right, the dimensionless tidal Love number k_2 is nonzero and negative, which is expected from our theoretical analysis as well. As is evident, there is very little variation of k_2 with mass but it varies considerably with β. As β changes from 10^{-4} to 0.1 the dimensionless tidal Love number changes from −0.430 to −0.480. See text for more discussion.

computing $y(\partial_y H/H)$ starting from the differential equation presented in Eq. (30), as y approaches the asymptotic limit. The numerical computation reveals that for all values of y, even when it reaches some asymptotic limit, $y(\partial_y H/H)$ remains above 2 and hence from Eq. (39) it immediately follows that k_2 should be negative (see Fig. 1 for a clear illustration of this fact). Further, the correctness of the theoretical techniques demonstrated above is also evident from the comparison of the theoretical estimates with the numerical analysis, as presented in Fig. 1. Since the match is almost exact at large radial distance, we have computed k_2 at $r = 400$ km in the plot presented in Fig. 2. Thus, the negativity and nonzero nature of the dimensionless tidal Love number in this situation is borne out by our numerical simulations as well. (see Fig. 3 for variation of the other dimensionless tidal Love number Λ with β).

Before concluding this section, let us briefly mention that even though these results were derived in the context of black holes, they remain equally applicable in the context of neutron stars as well, as long as the Weyl stress tensor outside of the neutron star is still given by Eq. (21). In that case, Eq. (40) still provides the tidal Love number, but the unknown constants need to be determined on the surface of the star. But that requires an understanding of the gravitational perturbation in the interior of the neutron star. In the next section we explicitly demonstrate how to achieve this understanding.

IV. TIDAL LOVE NUMBERS FOR A NEUTRON STAR ON THE BRANE

In this section we compute the tidal Love number associated with a neutron star located on the brane. The results derived in Secs. II and III are still useful, albeit outside of the neutron star. In particular, if we assume that the dark radiation and dark pressure have the same form outside of the neutron star, then the asymptotic behavior of the metric perturbation remains the same. Hence, Eq. (40) will hold outside of the neutron star as well. However, the unknown constants can only be determined if the gravitational perturbation in the interior of the neutron star is known. This must be done in a separate manner, as the matter equation-of-state parameter also comes into play.

A. Background equations in the interior of the neutron star

Let us discuss the background spacetime in the interior of the neutron star in the presence of an extra dimension. The presence of matter in the interior of the neutron star complicates the situation significantly by introducing linear as well as quadratic terms depending on the matter energy-momentum tensor. In particular, the effective gravitational field equations in this context take the following form [53]:

$$G_{\mu\nu} + E_{\mu\nu} = 8\pi G_4 T_{\mu\nu} + (8\pi G_5)^2\left(-\frac{1}{4}T_{\mu\alpha}T^{\alpha}_{\nu} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}T^2 g_{\mu\nu}\right), \tag{42}$$

where G_5 is the five-dimensional gravitational constant, $T_{\mu\nu}$ is the matter energy-momentum tensor, and $E_{\mu\nu}$ is the projection of the Weyl tensor introduced above. To proceed further, we assume that $E_{\mu\nu}$ can be represented as in Sec. II, while $T_{\mu\nu}$ depicts a perfect fluid with some energy density ρ and pressure p. Thus, for a static and spherically symmetric background spacetime whose line element is given by Eq. (2), the associated field equations take the following form:

$$e^{-2\lambda(r)}\left(\frac{1}{r^2} - \frac{2\lambda'}{r}\right) - \frac{1}{r^2} = -8\pi G_4\rho\left(1 + \frac{\rho}{2\lambda_{\rm b}}\right) - 24\pi\tilde{U}(r), \tag{43}$$

$$e^{-2\lambda(r)}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi G_4\left\{p + \frac{\rho}{2\lambda_{\rm b}}(\rho + 2p)\right\} + 8\pi(\tilde{U} + 2\tilde{P}), \tag{44}$$

$$e^{-2\lambda}\left\{\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{r}(\nu' - \lambda')\right\} = 16\pi G_4\left\{p + \frac{\rho}{2\lambda_b}(\rho + 2p)\right\} + 16\pi(\tilde{U} - \tilde{P}). \tag{45}$$

Here we have defined $\tilde{U}$ and $\tilde{P}$ as in Eq. (6) with G_4 being the four-dimensional Newton's constant and λ_b is the brane tension. Thus, the above set of equations can be thought of as the background equations associated with an anisotropic fluid with two additional components: one coming from the bulk Weyl tensor (i.e., the dark radiation and dark pressure components) characterized by $\tilde{U}$ and $\tilde{P}$, and one coming from the matter energy-momentum tensor and its quadratic combination. In this case, the effective energy density $\rho_{\rm eff}$ and effective pressure $p_{\rm eff}$ are

$$\rho_{\rm eff} = \rho\left(1 + \frac{\rho}{2\lambda_b}\right), \qquad p_{\rm eff} = p + \frac{\rho}{2\lambda_b}(\rho + 2p). \tag{46}$$

Hence, the above set of equations depicts an anisotropic two-fluid system as the background. Thus, when computing the gravitational perturbation around this background, we have to consider not only perturbations of $\tilde{U}$ and $\tilde{P}$, but of ρ and p as well. As we will see, this will immediately lead to problems by complicating the scenario quite a bit. As it turns out, even then we can make an educated guess about the nature of the Weyl fluid in the interior of the neutron star to avoid the issues of the two-fluid system.

B. Perturbation equations in the interior of the neutron star

The gravitational perturbation equations outside of the neutron star were already discussed in the earlier sections. In this section we concentrate on the gravitational perturbation in the interior of the neutron star. The modifications to the gravitational field equations due to the presence of matter were already demonstrated in Eq. (42). The gravitational perturbations inside the neutron star in a static and spherically symmetric background obeys Eqs. (43)–(45) and involves both the bulk Weyl tensor and linear and quadratic contributions from the matter sector.

In this context as well, the fact that perturbations must satisfy the symmetries of the background spacetime forces the even-parity perturbation to take the form advocated in Eq. (7). Since the angular component of the energy-momentum tensor of the matter fluid and the contribution from bulk Weyl tensor are identical, it follows that $\delta G^\theta_\theta - \delta G^\phi_\phi = 0$. Hence, as in Sec. II B, in this case we also have $H_0 = H_2 \equiv H(r)$. Similarly, the result $\delta G^r_\theta = 8\pi G_4 \delta T^r_\theta$ will lead to $K' = H' + 2\nu' H$, which is identical to that obtained in Sec. II B. Thus, the fact that only two perturbation components $[H(r)$ and $K(r)]$ are necessary to characterize the even-parity gravitational perturbation and that they are connected by a differential equation holds both inside and outside the neutron star. On the other hand, all of the remaining relations connecting the components of the perturbed metric with perturbations in the matter sector will lead to a different set of equations. Let us start with the addition of the perturbation equations in the angular directions, i.e., we concentrate on the equation $\delta G^\theta_\theta + \delta G^\phi_\phi = 8\pi G_4(\delta T^\theta_\theta + \delta T^\phi_\phi)$. By expanding the above equation, we have

$$\begin{aligned}&e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'\{2 + r(\nu' - \lambda')\} - e^{-2\lambda}r^2H'' - e^{-2\lambda}rH'(3r\nu' - r\lambda' + 2)\\&\quad - 16\pi r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}} - 1\right)\delta\tilde{P} - 16\pi r^2 H(\tilde{U} - \tilde{P}) - 16\pi r^2 H\tilde{p}_{\rm eff} - 16\pi r^2\delta\tilde{p}_{\rm eff} = 0.\end{aligned} \tag{47}$$

Here $\tilde{U}$ and $\tilde{P}$ have the usual expressions, and $\tilde{\rho}_{\rm eff} = G_4\rho_{\rm eff}$ and $\tilde{p}_{\rm eff} = G_4 p_{\rm eff}$. Subsequently, the radial perturbation equation, namely, $\delta G^r_r = 8\pi G_4\delta T^r_r$, yields

$$\begin{aligned}&e^{-2\lambda}(1 + r\nu')rK' - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - e^{-2\lambda}rH' + \left\{\frac{1}{2}\ell(\ell+1) - 1 - 8\pi r^2(\tilde{p}_{\rm eff} + \tilde{U} + 2\tilde{P})\right\}H\\&\quad - 8\pi r^2\delta\tilde{p}_{\rm eff} - 8\pi r^2\left(2 + \frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\delta\tilde{P} = 0.\end{aligned} \tag{48}$$

Finally, the remaining equation corresponding to the temporal part of the perturbation equation, namely, $\delta G^t_t = 8\pi G_4\delta T^t_t$, yields

$$\begin{aligned}&e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'(3 - r\lambda') - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - re^{-2\lambda}H' - \left\{\frac{1}{2}\ell(\ell+1) + 1 - 8\pi r^2(\tilde{\rho}_{\rm eff} + 3\tilde{U})\right\}H\\&\quad + 8\pi r^2\left(\frac{\partial\tilde{\rho}_{\rm eff}}{\partial\tilde{p}_{\rm eff}}\right)\delta\tilde{p}_{\rm eff} + 8\pi r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\delta\tilde{P} = 0.\end{aligned} \tag{49}$$

The difficulty associated with the above equations should be apparent by now. Unlike for the perturbation equations outside of the neutron star, here we have to eliminate both $\delta\tilde{P}$ and $\delta\tilde{p}_{\rm eff}$. As we will see, this will result in terms that depend on K. Hence, in the most general setting it will not be possible to arrive at a single master equation governing all of the even-parity gravitational perturbations (for a similar situation, see Ref. [82]).

However, for the moment let us press on and see how far we can go. We start by solving for $\delta\tilde{p}_{\rm eff}$, which can be done using Eq. (47),

$$\begin{aligned}16\pi r^2\delta\tilde{p}_{\rm eff} &= e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'\{2+r(\nu'-\lambda')\} - e^{-2\lambda}r^2H'' - e^{-2\lambda}rH'(3r\nu'-r\lambda'+2)\\ &\quad - 16\pi r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}-1\right)\delta\tilde{P} - 16\pi r^2H(\tilde{U}-\tilde{P}) - 16\pi r^2H\tilde{p}_{\rm eff}.\end{aligned} \tag{50}$$

Thus, by using Eq. (50) we can replace $\delta\tilde{p}_{\rm eff}$ in the other two equations. However, they will both depend on $\delta\tilde{P}$. Hence, to eliminate $\delta\tilde{P}$ we focus on Eq. (48). First of all, we substitute $\delta\tilde{p}_{\rm eff}$ from the above equation into Eq. (48), which results in the following expression:

$$\begin{aligned}&e^{-2\lambda}(1+r\nu')rK' - \left\{\frac{1}{2}\ell(\ell+1)-1\right\}K - e^{-2\lambda}rH' + \left\{\frac{1}{2}\ell(\ell+1)-1-8\pi r^2(\tilde{p}_{\rm eff}+\tilde{U}+2\tilde{P})\right\}H\\ &\quad - 8\pi r^2\left(2+\frac{\partial\tilde{U}}{\partial\tilde{P}}\right)\delta\tilde{P} = \frac{1}{2}e^{-2\lambda}r^2K'' + \frac{1}{2}e^{-2\lambda}rK'\{2+r(\nu'-\lambda')\} - \frac{1}{2}e^{-2\lambda}r^2H'' - \frac{1}{2}e^{-2\lambda}rH'(3r\nu'-r\lambda'+2)\\ &\quad - 8\pi r^2\left(\frac{\partial\tilde{U}}{\partial\tilde{P}}-1\right)\delta\tilde{P} - 8\pi r^2H(\tilde{U}-\tilde{P}) - 8\pi r^2H\tilde{p}_{\rm eff}.\end{aligned} \tag{51}$$

After simplifying the above equation further, one can immediately solve for the quantity $\delta\tilde{P}$,

$$\begin{aligned}48\pi r^2\delta\tilde{P} &= -48\pi r^2H\tilde{P} - 2\left\{\frac{1}{2}\ell(\ell+1)-1\right\}K + 2\left\{\frac{1}{2}\ell(\ell+1)-1\right\}H\\ &\quad - e^{-2\lambda}r^2K'' + e^{-2\lambda}r^2K'(\nu'+\lambda') + e^{-2\lambda}r^2H'' - e^{-2\lambda}rH'(3r\nu'-r\lambda').\end{aligned} \tag{52}$$

In the final step one has to use both Eqs. (50) and (52) to eliminate both $\delta\tilde{p}$ and $\delta\tilde{P}$ in Eq. (49), which can be done in a straightforward manner. However, the differential equation so obtained will depend on $K(r)$ and its derivative. Thus, unlike the exterior scenario, even if one uses the relation $K' = H' + 2\nu' H$ all of the dependences on the angular part of the gravitational perturbation cannot be eliminated. This suggests that in the interior one needs to solve a set of coupled differential equations. This makes handling the interior structure of the neutron star in the presence of extra dimensions more difficult.

However, one can avoid this problem and arrive at interesting scenarios with exact solutions to the above problem if some suitable assumptions are made. Before going into the details, note that the problem of getting a closed-form solution is mainly associated with the fact that in the interior of the neutron star we have a two-fluid system. If we can set the extra-dimensional contribution coming from the dark radiation and dark pressure to zero, we may be able to circumvent the problem. In particular, the quantity of significant interest in this context corresponds to the continuity of matter and the metric across the surface of the neutron star. This continuity equation in the presence of an extra dimension reads [83]

$$\tilde{p}_{\rm eff} + \tilde{U}_- + 2\tilde{P}_- = \tilde{U}_+ + 2\tilde{P}_+ = \frac{3\tilde{P}_0}{2R^4}, \tag{53}$$

where R is the radius of the neutron star. Here, the $+$ sign corresponds to a configuration outside of the neutron star, while the $-$ sign is corresponds to the interior configuration. Thus, the most economic way to get an analytic handle on the perturbation equation is by assuming that $\tilde{U}_- = 0 = \tilde{P}_-$, i.e., the effects from the extra dimension due to the Weyl stress tensor identically vanish in the interior of the neutron star. This significantly simplifies the analysis presented above and, more importantly, reduces the above system of equations representing the gravitational perturbation in the interior of the neutron star to a single-fluid system. In particular, the field equations presented in Eqs. (47)–(49) can be rewritten in the following manner:

$$e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'\{2 + r(\nu' - \lambda')\} - e^{-2\lambda}r^2H'' - e^{-2\lambda}rH'(3r\nu' - r\lambda' + 2) - 16\pi r^2 H\tilde{p}_{\rm eff} - 16\pi r^2\delta\tilde{p}_{\rm eff} = 0, \tag{54}$$

$$e^{-2\lambda}(1 + r\nu')rK' - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - e^{-2\lambda}rH' + \left\{\frac{1}{2}\ell(\ell+1) - 1 - 8\pi r^2\tilde{p}_{\rm eff}\right\}H - 8\pi r^2\delta\tilde{p}_{\rm eff} = 0, \tag{55}$$

$$e^{-2\lambda}r^2K'' + e^{-2\lambda}rK'(3 - r\lambda') - \left\{\frac{1}{2}\ell(\ell+1) - 1\right\}K - \left\{\frac{1}{2}\ell(\ell+1) + 1 - 8\pi r^2\tilde{\rho}_{\rm eff}\right\}H - re^{-2\lambda}H' + 8\pi r^2\left(\frac{\partial\tilde{\rho}_{\rm eff}}{\partial\tilde{p}_{\rm eff}}\right)\delta\tilde{p}_{\rm eff} = 0. \tag{56}$$

As emphasized earlier, the gravitational perturbation equations now involve only a single matter field with energy density $\tilde{\rho}_{\rm eff}$ and isotropic pressure $\tilde{p}_{\rm eff}$ along with their perturbations. Note that even though the influence of the Weyl fluid vanishes in the interior, the effect of the extra dimension is still present through the effective matter energy-momentum tensor. Since the above equations reduce to the familiar form as in Sec. II, we can employ the same strategy here as well, i.e., we first eliminate any term involving $\delta\tilde{p}_{\rm eff}$, and then eliminate all of the references to the K term appearing in these equations with the help of the relation $K' = H' + 2\nu' H$. Finally, by combining all of these equations in an appropriate manner we obtain a single differential equation for the master variable inside the neutron star,

$$H'' + \left[\frac{2}{r} + e^{2\lambda}\left\{\frac{2\tilde{m}_{\rm eff}(r)}{r^2} + 4\pi r(\tilde{p}_{\rm eff} - \tilde{\rho}_{\rm eff})\right\}\right]H' + H\left[-\ell(\ell+1)\frac{e^{2\lambda}}{r^2} + 4\pi e^{2\lambda}\left\{5\tilde{\rho}_{\rm eff} + 9\tilde{p}_{\rm eff} + \frac{\tilde{\rho}_{\rm eff} + \tilde{p}_{\rm eff}}{d\tilde{p}_{\rm eff}/d\tilde{\rho}_{\rm eff}}\right\} - 4\nu'^2\right] = 0. \tag{57}$$

This is essentially a general relativity problem, but with a different energy density and pressure, which must match the exterior solution having a nonzero Weyl stress tensor. The important point about the above differential equation is that the energy density and pressure appearing in it are not just the matter energy density and pressure; they also have contributions from the higher dimensions through the brane tension $\lambda_{\rm b}$. In particular, the equation-of-state parameter for the effective fluid becomes

$$\frac{\partial\tilde{\rho}_{\rm eff}}{\partial\tilde{p}_{\rm eff}} = \frac{\partial\rho_{\rm eff}}{\partial p_{\rm eff}} = \frac{\partial\rho_{\rm eff}}{\partial p}\left(\frac{\partial p_{\rm eff}}{\partial p}\right)^{-1} = \left(1 + \frac{\rho}{\lambda_{\rm b}}\right)\frac{\partial\rho}{\partial p}\left(1 + \frac{\rho}{\lambda_{\rm b}}\frac{\partial\rho}{\partial p} + \frac{\rho}{\lambda_{\rm b}} + \frac{p}{\lambda_{\rm b}}\frac{\partial\rho}{\partial p}\right)^{-1}. \tag{58}$$

Thus, the equation-of-state parameter is indeed modified in the presence of an extra dimension through the brane tension $\lambda_{\rm b}$, such that in the limit $\lambda_{\rm b} \to 0$ we get back the correct equation-of-state parameter.

The above analysis explicitly demonstrates that in the context of neutron stars there will be two sources of modifications to the tidal Love number in the presence of higher dimensions. The first such modification comes from the interior of the neutron star, as the master equation for the perturbation is different due to the presence of additional terms in the matter sector that depend on the brane tension, along with a modified boundary condition at the surface of the star. This will lead to different values for the perturbation $H(r)$ and its derivative $H'(r)$ at the stellar surface. The second modification comes from the fact that the differential equation for the gravitational perturbation outside of the neutron star due to the presence of the Weyl stress tensor will be different. Thus, an estimation of the tidal Love number with a modified differential equation and with modified boundary conditions will certainly differ from that in the general-relativistic situation.

Since in this particular context it is not possible to provide an analytical estimation of the tidal Love number (or, better, the dimensionless tidal Love number), we have to resort to numerical methods. However, taking a cue from our earlier discussion regarding the tidal Love number for BHs, we can argue that extra dimensions could possibly decrease the numerical value of the dimensionless tidal Love number from that in general relativity. If this is indeed the case, there will be interesting consequences. For example, given a central density, the dimensionless tidal Love number Λ will be smaller in the presence of higher dimensions i.e., in situations with finite $\lambda_{\rm b}$. Thus, if GW experiments rule out some model with a given central density, they can come into the picture again if extra dimensions are taken into account. This may have observational ramifications, as we will discuss below.

Keeping these intriguing possibilities in mind, we perform a numerical analysis by solving the differential equation governing the behavior of the even-parity gravitational perturbation in the interior of the neutron star, which provides the numerical estimations for the

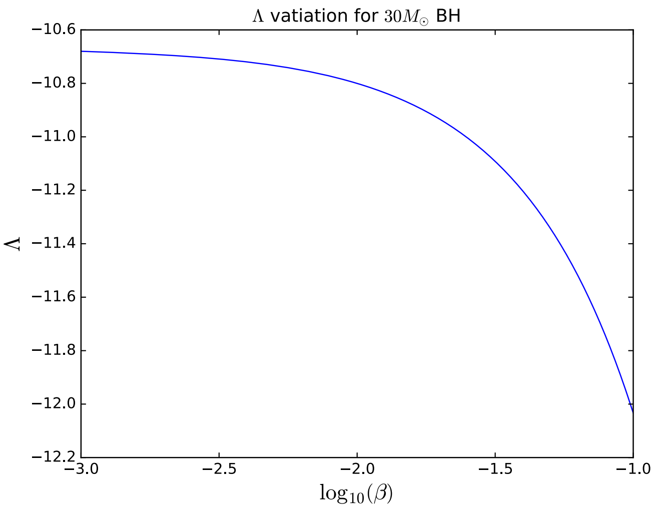

FIG. 3. This figure depicts the variation of the dimensionless tidal Love number Λ with the parameter β inherited from the extra dimensions using numerical analysis. The black hole mass M_{BH} has been taken to be $\sim 30\ M_\odot$. As is evident, like for k_2, the other dimensionless tidal Love number Λ is also nonzero and negative, which is expected from our theoretical analysis as well. There is little variation of Λ with β, which is $\sim 20\%$. As β changes from 10^{-3} to 0.1, the dimensionless tidal Love number Λ changes from -10.6 to -12.2.

gravitational perturbation and its derivative at the surface of the NS. These are then used as the boundary conditions to solve for the gravitational perturbation in the exterior region, leading to a determination of the dimensionless tidal Love number Λ. Of course, such estimations for the tidal Love number will depend on the choice of the EoS parameter for the material forming the neutron star. To illustrate this, we plot these numerical estimates of Λ against the brane tension λ_{b} for a tabulated EoS [84] and polytropic EoS in Fig. 4. Further numerical estimations of Λ are also presented for various choices of the brane tension λ_{b} and the central charge density of the neutron star in Fig. 5. As is evident from Fig. 5, as the central density increases the dimensionless tidal Love number decreases, since it becomes more difficult to deform the NS by applying an external tidal field. Similarly, it is clear from Fig. 5 (and also from Fig. 4) that the dimensionless tidal Love number Λ attains smaller and smaller values as the brane tension λ_{b} decreases, i.e., as the system departs further from general relativity. This is completely consistent with our earlier theoretical consideration. This explicitly demonstrates the consistency of the theoretical framework used in this work with the numerical analysis performed to estimate the dimensionless tidal Love number. In the next section we discuss the implications for GW170817 and possible observability in future GW experiments.

V. IMPLICATIONS OF GW170817 AND FUTURE MERGER EVENTS

In this section we comment on the possible implications of the results derived above in the context of recent GW observation from NS-NS (e.g., GW170817) and BBH mergers. For GW170817, Advanced LIGO provides a constraint on the dimensionless tidal Love number $\Lambda < 800$ [6]. This constraint in turn provides bounds on the parameter space of the EoS of the neutron star by ruling out

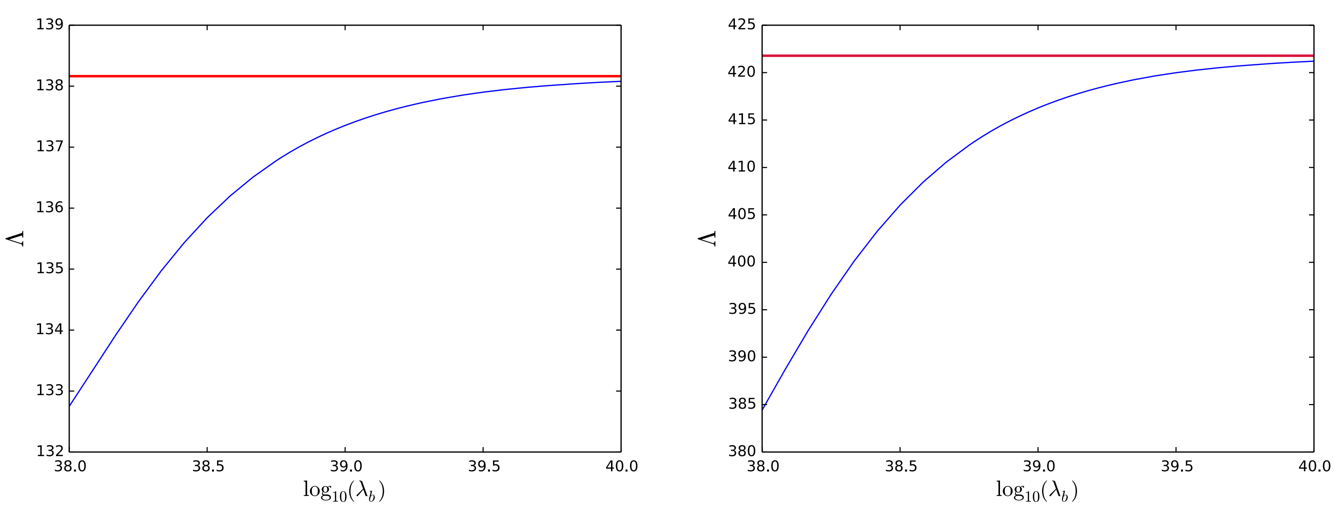

FIG. 4. This figure depicts the variation of the dimensionless tidal Love number Λ with the brane tension λ_{b} from our numerical analysis for two choices of the equation of state of the neutron star. The curve on the left is for a tabulated equation-of-state parameter [84], while the one on the right is due to a polytropic equation of state, with $\Gamma = 5/3$. In both panels the red thick line depicts the asymptotic value of the dimensionless tidal Love number Λ, which corresponds to the general-relativistic limit. As is evident, the dimensionless tidal Love number Λ is nonzero and is smaller than the general-relativistic value for a finite λ_{b}, signaling the possible existence of extra dimensions. Note that the small-scale tests of the gravitational interaction forbid one from numerically estimating the brane tension λ_{b} below 10^{38} (in units of mass density) [85].

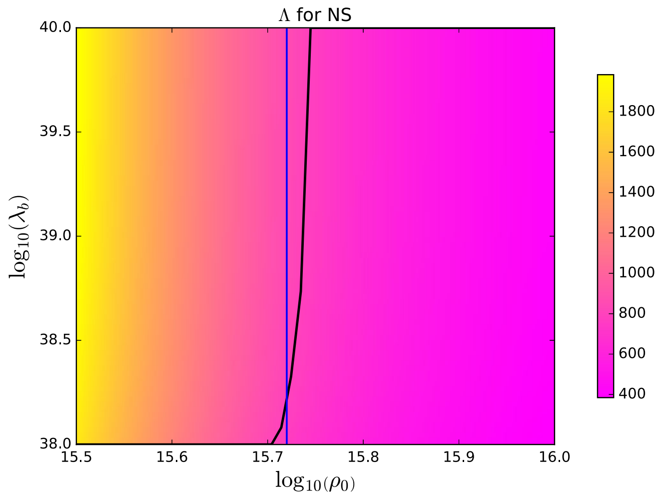

FIG. 5. This figure depicts the variation of the dimensionless tidal Love number Λ with the central density ρ_0 of a $\Gamma = 5/3$ polytropic neutron star as well as the brane tension λ_b, as computed from our numerical analysis. The general-relativistic result is obtained by taking the limit $\lambda_b \to \infty$; thus, any finite value for λ_b will denote a departure from general relativity. As is evident from the panel on the right, Λ is nonzero, and increases with decreasing central density, as it should. The black curve is the $\Lambda = 800$ line, such that points on the left are ruled out by the GW170817 event, while those on the right remain viable options [6,7]. While the blue vertical line presents a certain central density, taken to be $10^{15.72}$ in $\mathrm{gm\,cm^{-3}}$. However, the value of Λ changes with the central density and hence the $\Lambda = 800$ curve (being mass dependent) provides a crude estimation of the bound from the GW170817 event, which will suffice for our purpose. Moreover, the range of values for the brane tension λ_b is consistent with the small-scale tests of Newton's law.

several EoSs leading to a high central density of the neutron star. However, this analysis was completely within the context of general relativity. As our numerical analysis depicts, the presence of extra dimensions reduces the estimations of the dimensionless tidal Love number Λ. Hence, it follows that a significant portion of the EoS parameter space (earlier ruled out by general relativity estimations) will become viable again. This is clearly illustrated in Fig. 5. The quantity calculated is the dimensionless tidal Love number Λ for a polytropic EoS with $\Gamma = 5/3$ as a function of both the stellar central density and the brane tension λ_b. The thick red line in Fig. 5 represents the $\Lambda = 800$ curve and (as is evident from the figure) the regions to the left of the $\Lambda = 800$ curve are ruled out by the recent GW170817 measurement, while those on the right are still viable (assuming general relativity). However, the presence of an extra dimension leads to a smaller value for Λ. Thus, one may consider a certain central density (depicted by the blue vertical line in Fig. 5), which is ruled out by the GW170817 measurement if general relativity is assumed to be correct theory, but comes into existence as one considers smaller values of brane tension signalling possible presence of extra dimension. Further, we note that the parameter space for the brane tension considered in this work is completely consistent with the small-scale tests of Newton's law [86–88]. Thus, if the tidal Love number is more accurately measured in future observations, that in turn will provide stringent bounds on the brane tension λ_b, which will be more accurate compared to the small-scale tests of Newton's law and hence we will really be probing some of the microscopic features of spacetime through GW experiments. This is a direct consequence of the presence of extra dimensions on the NS-NS merger.

On the other hand, for observations related to BBH mergers the situation is more subtle, but also more predictable. The dimensionless tidal Love number Λ values associated with BHs in the presence of an extra dimension are computed numerically by asymptotic matching of the other dimensionless tidal Love number k_2. As is evident from Fig. 3, numerical estimates for the dimensionless tidal Love number Λ are in the range ~ -10 to -20. Further, from Fig. 3 we can also see the variation of the dimensionless tidal Love number Λ for a braneworld BH with the parameter β, which is also not very large ($\sim$20%). Given the present sensitivity of Advanced LIGO's detectors and their associated errors, a negative Λ value of a few hundred would be easily detected, but a value of a few tens (as is the case here) would probably be lost in noise. On a brighter note, the upcoming Einstein Telescope (or LISA) will increase the sensitivity of detection by an order of magnitude, meaning that lower values of the dimensionless tidal Love number Λ could be detected if they turn out to be negative, which may act as a very good test bed for higher-dimensional theories.

VI. CONCLUSIONS

Understanding the possible implications of theories beyond general relativity has become a topic of significant importance in recent years, thanks to the detection of GWs from binary BHs and NSs which have provided first-hand experience of the strong-gravity regime. Among various other possibilities, the existence of extra dimensions and their implications and observability in the context of GWs are of significant interest. Following this trend, in this work we explored the effect of higher spatial dimensions on the tidal Love number of BHs and NSs. For this purpose, we started with an understanding of the modifications to the static, even-parity gravitational perturbation in the exterior region of an NS or BH due to the existence of extra dimensions. These modifications to the differential equation satisfied by the gravitational perturbation also leads to possible modifications to the tidal Love number.

In particular, using both theoretical and numerical techniques we explicitly demonstrated that the presence of an extra dimension will, beyond a doubt, make the tidal

Love numbers nonzero and (more importantly) *negative* for braneworld BHs. On the other hand, it is well known that for BHs in general relativity the tidal Love number vanishes. Incidentally, the general relativity result can also be derived by taking the appropriate limit of our higher-dimensional result. We would like to emphasize that even though the idea of nonzero tidal Love numbers for BHs is not new, for the specific case of extra dimensions we found that the tidal Love numbers are negative, unlike other scenarios. The negativity of tidal Love number for BHs is an additional distinguishing feature of extra dimensions, and can possibly be exploited using real data in the future for further confirmation.

We further demonstrated the relevant modifications to the differential equations governing gravitational perturbations in the interior of a neutron star as well. In general, this will lead to a two-fluid system consisting of the Weyl fluid (inherited from the higher dimension) and the fluid inside the neutron star. However, due to its difficult nature we have considered a simpler situation in which the Weyl fluid is nonexistent inside the neutron star but is certainly present outside, which preserves the continuity of the physical quantities on the surface of the NS. In this particular context the essential modifications to the equations satisfied by the gravitational perturbations in the interior of the neutron star involve the term ρ^2/λ_b, where ρ is the energy density of the fluid filling the interior of the NS and λ_b is the brane tension. As $\lambda_b \to \infty$ the general relativity result can be obtained. As we have explicitly demonstrated by numerically solving these equations, in the case of NS-NS or NS-BH binaries the presence of extra dimensions will induce deviations in the dimensionless tidal Love number Λ from its *expected* general-relativistic behavior, which essentially decreases the numerical estimations of the tidal Love number. This results in an interesting possibility, namely, a certain parameter space associated with the central density of the NS (which was ruled out earlier by GW170817 using general-relativistic methods) may be viable for a finite λ_b, signaling the possible existence of extra dimensions. Moreover, future GW observations will constrain the dimensionless tidal Love number Λ to a greater accuracy, which in turn may lead to further stringent constraints (better than the existing ones) on the brane tension λ_b. Further, given these modifications to the dimensionless tidal Love number Λ, there may be a very good chance of detection in future GW detectors because of the low signal-to-noise ratio of these measurements. Thus, using both theoretical and numerical techniques we have explicitly demonstrated a modification to the dimensionless tidal Love number Λ (or k_2) for the specific scenario of compactified extra spatial dimensions. The techniques employed here are quite general, and can be equivalently applied to understand the nature of the tidal Love number in various other theories of gravity beyond general relativity as well. We leave these questions for future studies.

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APPENDIX: SOLVING THE DIFFERENTIAL EQUATION FOR EVEN-PARITY GRAVITATIONAL PERTURBATIONS

Here we solve for Eq. (31). For this purpose, it will be helpful to define the new quantity $\bar{\beta}^2 = 4\{1+(\beta/2)\}\{1+\beta\}^{-1}$. Thus, the differential equation takes the following form:

$$\{y^2-1\}\partial_y^2 H + 2y\partial_y H - \left\{\ell(\ell+1) + \frac{\bar{\beta}^2}{y^2-1}\right\}H = 0. \tag{A1}$$

This differential equation is exactly the same as the differential equation satisfied by the associated Legendre polynomials $P_m^\ell(y)$ and $Q_m^\ell(y)$, but with the exception that $m = \bar{\beta}$ is now a fraction. However, in the context of general relativity $\beta = 0$ and we have the usual case with $m = 2$. The solution of this equation (found in Ref. [89]) is given by

$$H(\ell, m; y) = A_1 P_\ell^m(y) + B_1 Q_\ell^m(y), \tag{A2}$$

where $P_\ell^m(y)$ and $Q_\ell^m(y)$ are the Legendre functions of the first and second kind, respectively. Further, the arbitrary constants that arise from solving the above second-order differential equations are A_1 and B_1, respectively. Since we are interested in the asymptotic behavior, we would like to write down the above functions as polynomials in y. This can be achieved by first expanding them in terms of hypergeometric functions and then employing a power-series expansion. When written in terms of the confluent hypergeometric functions ${}_2F_1$ they are

$$Q_\ell^m(y) = e^{im\pi}2^{-\ell-1}\sqrt{\pi}\frac{\Gamma(\ell+m+1)}{\Gamma(\ell+\frac{3}{2})}y^{-\ell-m-1}(y^2-1)^{m/2}{}_2F_1\left(1+\frac{\ell+m}{2},\frac{1+\ell+m}{2},\ell+\frac{3}{2};\frac{1}{y^2}\right), \tag{A3}$$

$$P_\ell^m(y) = 2^{-\ell-1}\pi^{-1/2}\frac{\Gamma(-\frac{1}{2}-\ell)}{\Gamma(-\ell-m)}y^{-\ell+m-1)}(y^2-1)^{-m/2}{}_2F_1\left(\frac{1}{2}+\frac{\ell-m}{2},1+\frac{\ell-m}{2},\ell+\frac{3}{2};\frac{1}{y^2}\right),$$
$$+2^\ell\frac{\Gamma(\frac{1}{2}+\ell)}{\Gamma(1+\ell-m)}y^{\ell+m}(y^2-1)^{-m/2}{}_2F_1\left(-\frac{\ell+m}{2},\frac{1}{2}-\frac{\ell+m}{2},\frac{1}{2}-\ell;\frac{1}{y^2}\right). \tag{A4}$$

For arguments smaller than unity, the hypergeometric functions have the following polynomial expansion:

$$_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)}\frac{x^j}{j!}. \tag{A5}$$

Using this expression for the confluent hypergeometric series, we can rewrite the associated Legendre polynomial of the second kind Q_ℓ^m from Eq. (A3) as

$$Q_\ell^m(y) = e^{im\pi}2^{-\ell-1}\sqrt{\pi}\frac{\Gamma(\ell+m+1)}{\Gamma(\ell+\frac{3}{2})}y^{-\ell-m-1}(y^2-1)^{m/2}\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(1+\frac{\ell+m}{2})\Gamma(\frac{1+\ell+m}{2})}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1+\ell+m}{2}+j)\Gamma(1+\frac{\ell+m}{2}+j)}{\Gamma(\ell+\frac{3}{2}+j)}\frac{y^{-2j}}{j!}. \tag{A6}$$

It is worthwhile to define

$$\alpha(\ell,m) = e^{im\pi}2^{-\ell-1}\sqrt{\pi}\frac{\Gamma(\ell+m+1)}{\Gamma(1+\frac{\ell+m}{2})\Gamma(\frac{1+\ell+m}{2})}, \tag{A7}$$

$$\beta_j(\ell,m) = \frac{\Gamma(\frac{1+\ell+m}{2}+j)\Gamma(1+\frac{\ell+m}{2}+j)}{\Gamma(\ell+\frac{3}{2}+j)}, \tag{A8}$$

in terms of which $Q_\ell^m(y)$ has the following expression:

$$Q_\ell^m(y) = \alpha(\ell,m)y^{-\ell-m-1}(y^2-1)^{m/2}\sum_{n=0}^{\infty}\beta_n(\ell,m)\frac{y^{-2n}}{n!}. \tag{A9}$$

Since we are interested in the asymptotic limit, we can expand the above in powers of y^{-1}, immediately leading to,

$$Q_\ell^m(y) = \alpha(\ell,m)y^{-\ell-1}(1-y^{-2})^{m/2}\sum_{j=0}^{\infty}\beta_j(\ell,m)\frac{y^{-2j}}{j!}$$
$$= \alpha(\ell,m)y^{-\ell-1}\left\{\sum_{k=0}^{\infty}(-1)^k\frac{(\frac{m}{2})!}{k!(\frac{m}{2}-k)!}y^{-2k}\right\}\left\{\sum_{j=0}^{\infty}\beta_j(\ell,m)\frac{y^{-2j}}{j!}\right\} \tag{A10}$$

By expanding this series we obtain the first three nontrivial terms in the expression for Q_ℓ^m:

$$Q_\ell^m(y) \simeq \alpha(\ell,m)y^{-\ell-1}\left[\beta_0(\ell,m)+\left\{-\left(\frac{m}{2}\right)\beta_0(\ell,m)+\beta_1(\ell,m)\right\}y^{-2}\right.$$
$$\left.+\left\{\frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)\beta_0(\ell,m)-\left(\frac{m}{2}\right)\beta_1(\ell,m)+\frac{1}{2}\beta_2(\ell,m)\right\}y^{-4}\right]. \tag{A11}$$

Let us concentrate on the associated Legendre polynomial of the first kind P_ℓ^m given by Eq. (A4). Using the series expansion of the hypergeometric function, we obtain

$$
\begin{aligned}
P_\ell^m(y) = {} & 2^{-\ell-1}\pi^{-1/2}\frac{\Gamma(-\frac{1}{2}-\ell)}{\Gamma(-\ell-m)}y^{-\ell+m-1}(y^2-1)^{-m/2} \\
& \times\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\frac{1}{2}+\frac{\ell-m}{2})\Gamma(1+\frac{\ell-m}{2})}\sum_{j=0}^{\infty}\frac{\Gamma(1+\frac{\ell-m}{2}+j)\Gamma(\frac{1}{2}+\frac{\ell-m}{2}+j)}{\Gamma(\ell+\frac{3}{2}+j)}\frac{y^{-2j}}{j!} \\
& + 2^{\ell}\frac{\Gamma(\frac{1}{2}+\ell)}{\Gamma(1+\ell-m)}y^{\ell+m}(y^2-1)^{-m/2} \\
& \times\frac{\Gamma(\frac{1}{2}-\ell)}{\Gamma(-\frac{\ell+m}{2})\Gamma(\frac{1}{2}-\frac{\ell+m}{2})}\sum_{j=0}^{\infty}\frac{\Gamma(-\frac{\ell+m}{2}+j)\Gamma(\frac{1}{2}-\frac{\ell+m}{2}+j)}{\Gamma(\frac{1}{2}-\ell+j)}\frac{y^{-2j}}{j!}.
\end{aligned}
\tag{A12}
$$

For convenience, in this case we introduce the four quantities

$$
\gamma(\ell,m) = 2^{-\ell-1}\pi^{-1/2}\frac{\Gamma(-\frac{1}{2}-\ell)}{\Gamma(-\ell-m)}\frac{\Gamma(\ell+\frac{3}{2})}{\Gamma(\frac{1}{2}+\frac{\ell-m}{2})\Gamma(1+\frac{\ell-m}{2})},
\tag{A13}
$$

$$
\sigma_j(\ell,m) = \frac{\Gamma(1+\frac{\ell-m}{2}+j)\Gamma(\frac{1}{2}+\frac{\ell-m}{2}+j)}{\Gamma(\ell+\frac{3}{2}+j)},
\tag{A14}
$$

$$
\chi(\ell,m) = 2^{\ell}\frac{\Gamma(\frac{1}{2}+\ell)}{\Gamma(1+\ell-m)}\frac{\Gamma(\frac{1}{2}-\ell)}{\Gamma(-\frac{\ell+m}{2})\Gamma(\frac{1}{2}-\frac{\ell+m}{2})},
\tag{A15}
$$

$$
\Pi_j(\ell,m) = \frac{\Gamma(-\frac{\ell+m}{2}+j)\Gamma(\frac{1}{2}-\frac{\ell+m}{2}+j)}{\Gamma(\frac{1}{2}-\ell+j)},
\tag{A16}
$$

and hence the associated Legendre polynomial can be written as

$$
P_\ell^m(y) = \gamma(\ell,m)y^{-\ell+m-1}(y^2-1)^{-m/2}\sum_{j=0}^{\infty}\sigma_j(\ell,m)\frac{y^{-2j}}{j!} + \chi(\ell,m)y^{\ell+m}(y^2-1)^{-m/2}\sum_{j=0}^{\infty}\Pi_j(\ell,m)\frac{y^{-2j}}{j!}.
\tag{A17}
$$

Expanding the above expression for large values of y, we obtain

$$
\begin{aligned}
P_\ell^m(y) &= \gamma(\ell,m)y^{-\ell-1}(1-y^{-2})^{-m/2}\sum_{j=0}^{\infty}\sigma_j(\ell,m)\frac{y^{-2j}}{j!} + \chi(\ell,m)y^{\ell}(1-y^{-2})^{-m/2}\sum_{j=0}^{\infty}\Pi_j(\ell,m)\frac{y^{-2j}}{j!} \\
&= \gamma(\ell,m)y^{-\ell-1}\sum_{k=0}^{\infty}\frac{(\frac{m}{2}+k-1)!}{k!(\frac{m}{2}-1)!}y^{-2k}\sum_{j=0}^{\infty}\sigma_j(\ell,m)\frac{y^{-2j}}{j!} + \chi(\ell,m)y^{\ell}\sum_{k=0}^{\infty}\frac{(\frac{m}{2}+k-1)!}{k!(\frac{m}{2}-1)!}y^{-2k}\sum_{j=0}^{\infty}\Pi_j(\ell,m)\frac{y^{-2j}}{j!}.
\end{aligned}
\tag{A18}
$$

Keeping the first few nontrivial terms, the above expression yields

$$
\begin{aligned}
P_\ell^m(y) \simeq {} & \gamma(\ell,m)y^{-\ell-1}\bigg[\sigma_0(\ell,m) + \bigg\{\left(\frac{m}{2}\right)\sigma_0(\ell,m) + \sigma_1(\ell,m)\bigg\}y^{-2} \\
& + \bigg\{\frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right)\sigma_0(\ell,m) + \left(\frac{m}{2}\right)\sigma_1(\ell,m) + \frac{1}{2}\sigma_2(\ell,m)\bigg\}y^{-4}\bigg] \\
& + \chi(\ell,m)y^{\ell}\bigg[\Pi_0(\ell,m) + \bigg\{\left(\frac{m}{2}\right)\Pi_0(\ell,m) + \Pi_1(\ell,m)\bigg\}y^{-2} \\
& + \bigg\{\frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right)\Pi_0(\ell,m) + \left(\frac{m}{2}\right)\Pi_1(\ell,m) + \frac{1}{2}\Pi_2(\ell,m)\bigg\}y^{-4}\bigg].
\end{aligned}
\tag{A19}
$$

This completes our discussion of the general solution and the first few nontrivial terms in its asymptotic expansion.

However, keeping future applications in mind, let us concentrate on the $\ell = 2$ situation and the leading-order contribution, which yields

$$H(2,m;y) \simeq A_1\{\gamma(2,m)\sigma_0(2,m)y^{-3} + \chi(2,m)\Pi_0(2,m)y^2\} + B_1\alpha(2,m)\beta_0(2,m)y^{-3}$$
$$= \{\gamma(2,m)\sigma_0(2,m)A_1 + \alpha(2,m)\beta_0(2,m)B_1\}y^{-3} + \chi(2,m)\Pi_0(2,m)A_1y^2. \tag{A20}$$

Given the results in Eqs. (A7)–(A8) and Eqs. (A13)–(A16), we finally obtain

$$\alpha(2,m) = \frac{\sqrt{\pi}}{8}e^{im\pi}\frac{\Gamma(3+m)}{\Gamma(2+\frac{m}{2})\Gamma(\frac{3}{2}+\frac{m}{2})}, \qquad \beta_0(2,m) = \frac{\Gamma(2+\frac{m}{2})\Gamma(\frac{3}{2}+\frac{m}{2})}{\Gamma(\frac{7}{2})}, \tag{A21}$$

$$\gamma(2,m) = \frac{1}{8\sqrt{\pi}}\frac{\Gamma(-\frac{5}{2})}{\Gamma(-2-m)}\frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{3}{2}-\frac{m}{2})\Gamma(2-\frac{m}{2})}, \qquad \sigma_0(2,m) = \frac{\Gamma(\frac{3}{2}-\frac{m}{2})\Gamma(2-\frac{m}{2})}{\Gamma(\frac{7}{2})}, \tag{A22}$$

$$\chi(2,m) = 4\frac{\Gamma(\frac{5}{2})}{\Gamma(3-m)}\frac{\Gamma(-\frac{3}{2})}{\Gamma(-1-\frac{m}{2})\Gamma(-\frac{1}{2}-\frac{m}{2})}, \qquad \Pi_0(2,m) = \frac{\Gamma(-1-\frac{m}{2})\Gamma(-\frac{1}{2}-\frac{m}{2})}{\Gamma(-\frac{3}{2})}. \tag{A23}$$

Using the above results, we finally obtain

$$\alpha(2,m)\beta_0(2,m) = \frac{\Gamma(3+m)}{15}e^{im\pi}, \qquad \gamma(2,m)\sigma_0(2,m) = \frac{1}{8\sqrt{\pi}}\frac{\Gamma(-\frac{5}{2})}{\Gamma(-2-m)}, \qquad \chi(2,m)\Pi_0(2,m) = \frac{4\Gamma(\frac{5}{2})}{\Gamma(3-m)}. \tag{A24}$$

Further, the result $\Gamma(n+1) = n\Gamma(n)$ can be used to obtain

$$\Gamma\left(-\frac{5}{2}\right) = \left(-\frac{2}{5}\right)\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{2}{5}\right)\left(-\frac{2}{3}\right)\Gamma\left(-\frac{1}{2}\right) = -\frac{8\sqrt{\pi}}{15}, \tag{A25}$$

as well as $\Gamma(5/2) = (3/4)\sqrt{\pi}$. Thus, the solution becomes

$$H(2,m;y) = \left\{\frac{3A_1\sqrt{\pi}}{\Gamma(3-m)}\right\}y^2 + \left\{-\frac{A_1}{15\Gamma(-2-m)} + \frac{B_1\Gamma(3+m)e^{i\pi\bar{\beta}}}{15}\right\}y^{-3}. \tag{A26}$$

This is the result we use in the main text.

[1] B. P. Abbott *et al.* (VIRGO and LIGO Scientific Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017).

[2] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), Binary Black Hole Mergers in the First Advanced LIGO Observing Run, Phys. Rev. X **6**, 041015 (2016).

[3] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[4] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), Tests of General Relativity with GW150914, Phys. Rev. Lett. **116**, 221101 (2016).

[5] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[6] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[7] B. P. Abbott *et al.* (Virgo and LIGO Scientific Collaborations), GW170817: Measurements of Neutron Star Radii and Equation of State, Phys. Rev. Lett. **121**, 161101 (2018).

[8] A. E. Broderick, T. Johannsen, A. Loeb, and D. Psaltis, Testing the no-hair theorem with event horizon telescope observations of sagittarius A*, Astrophys. J. **784**, 7 (2014).

[9] M. Cabero, C. D. Capano, O. Fischer-Birnholtz, B. Krishnan, A. B. Nielsen, A. H. Nitz, and C. M. Biwer, Observational

tests of the black hole area increase law, Phys. Rev. D **97**, 124069 (2018).

[10] V. Cardoso, E. Franzin, and P. Pani, Is the Gravitational-Wave Ringdown a Probe of the Event Horizon?, Phys. Rev. Lett. **116**, 171101 (2016); Erratum, Phys. Rev. Lett. **117**, 089902(E) (2016).

[11] J. Abedi, H. Dykaar, and N. Afshordi, Echoes from the Abyss: Tentative evidence for Planck-scale structure at black hole horizons, Phys. Rev. D **96**, 082004 (2017).

[12] M. Wade, J. D. E. Creighton, E. Ochsner, and A. B. Nielsen, Advanced LIGOs ability to detect apparent violations of the cosmic censorship conjecture and the no-hair theorem through compact binary coalescence detections, Phys. Rev. D **88**, 083002 (2013).

[13] V. Cardoso and P. Pani, Tests for the existence of black holes through gravitational wave echoes, Nat. Astron. **1**, 586 (2017).

[14] E. Berti *et al.*, Testing general relativity with present and future astrophysical observations, Classical Quantum Gravity **32**, 243001 (2015).

[15] E. Babichev, K. Koyama, D. Langlois, R. Saito, and J. Sakstein, Relativistic stars in beyond Horndeski theories, Classical Quantum Gravity **33**, 235014 (2016).

[16] S. Bhattacharya and S. Chakraborty, Constraining some Horndeski gravity theories, Phys. Rev. D **95**, 044037 (2017).

[17] P. Creminelli and F. Vernizzi, Dark Energy After GW170817 and GRB170817A, Phys. Rev. Lett. **119**, 251302 (2017).

[18] R. Kase and S. Tsujikawa, Dark energy in Horndeski theories after GW170817: A review, arXiv:1809.08735.

[19] S. Bhattacharyya and S. Shankaranarayanan, Quasinormal modes as a distinguisher between general relativity and $f(\mathcal{R})$ gravity, Phys. Rev. D **96**, 064044 (2017).

[20] G. Pratten, Covariant perturbations of $f(\mathcal{R})$ black holes: The Weyl terms, Classical Quantum Gravity **32**, 165018 (2015).

[21] S. Mukherjee and S. Chakraborty, Horndeski theories confront the Gravity Probe B experiment, Phys. Rev. D **97**, 124007 (2018).

[22] I. Banerjee, S. Chakraborty, and S. SenGupta, Excavating black hole continuum spectrum: Possible signatures of scalar hairs and of higher dimensions, Phys. Rev. D **96**, 084035 (2017).

[23] S. S. Seahra, C. Clarkson, and R. Maartens, Detecting Extra Dimensions with Gravity Wave Spectroscopy: The Black String Brane-World, Phys. Rev. Lett. **94**, 121302 (2005).

[24] C. Clarkson and R. Maartens, Gravity-wave detectors as probes of extra dimensions, Gen. Relativ. Gravit. **37**, 1681 (2005); Int. J. Mod. Phys. D **14**, 2347 (2005).

[25] S. S. Seahra, C. Clarkson, and R. Maartens, Delocalization of brane gravity by a bulk black hole, Classical Quantum Gravity **22**, L91 (2005).

[26] S. S. Seahra and C. Clarkson, Gravitational waves in the black string braneworld, Classical Quantum Gravity **26**, 245004 (2009).

[27] H. Witek, Numerical relativity in higher-dimensional spacetimes, Int. J. Mod. Phys. A **28**, 1340017 (2013).

[28] S. Chakraborty, K. Chakravarti, S. Bose, and S. SenGupta, Signatures of extra dimensions in gravitational waves from black hole quasinormal modes, Phys. Rev. D **97**, 104053 (2018).

[29] D. Andriot and G. Lucena Gmez, Signatures of extra dimensions in gravitational waves, J. Cosmol. Astropart. Phys. 06 (2017) 048.

[30] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, The Hierarchy problem and new dimensions at a millimeter, Phys. Lett. B **429**, 263 (1998).

[31] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, New dimensions at a millimeter to a Fermi and superstrings at a TeV, Phys. Lett. B **436**, 257 (1998).

[32] I. Antoniadis, A Possible new dimension at a few TeV, Phys. Lett. B **246**, 377 (1990).

[33] V. Rubakov and M. Shaposhnikov, Extra space-time dimensions: Towards a solution to the cosmological constant problem, Phys. Lett. **125B**, 139 (1983).

[34] V. Rubakov and M. Shaposhnikov, Do we live inside a domain wall?, Phys. Lett. **125B**, 136 (1983).

[35] A. Perez-Lorenzana, An introduction to extra dimensions, J. Phys. Conf. Ser. **18**, 224 (2005).

[36] C. Csaki, TASI lectures on extra dimensions and branes, arXiv:hep-ph/0404096.

[37] R. Sundrum, Tasi 2004 lectures: To the fifth dimension and back, arXiv:hep-th/0508134.

[38] L. Randall and R. Sundrum, A Large Mass Hierarchy from a Small Extra Dimension, Phys. Rev. Lett. **83**, 3370 (1999).

[39] W. D. Goldberger and M. B. Wise, Modulus Stabilization with Bulk Fields, Phys. Rev. Lett. **83**, 4922 (1999).

[40] L. Randall and R. Sundrum, An Alternative to Compactification, Phys. Rev. Lett. **83**, 4690 (1999).

[41] H. Davoudiasl, J. Hewett, and T. Rizzo, Bulk gauge fields in the Randall-Sundrum model, Phys. Lett. B **473**, 43 (2000).

[42] J. Garriga and T. Tanaka, Gravity in the Brane World, Phys. Rev. Lett. **84**, 2778 (2000).

[43] A. Chamblin, S. Hawking, and H. Reall, Brane world black holes, Phys. Rev. D **61**, 065007 (2000).

[44] H. Davoudiasl, J. Hewett, and T. Rizzo, Experimental probes of localized gravity: On and off the wall, Phys. Rev. D **63**, 075004 (2001).

[45] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezania, Black holes on the brane, Phys. Lett. B **487**, 1 (2000).

[46] R. Maartens, Geometry and dynamics of the brane world, arXiv:gr-qc/0101059.

[47] R. Gregory, Braneworld black holes, Lect. Notes Phys. **769**, 259 (2009).

[48] R. Emparan, G. T. Horowitz, and R. C. Myers, Exact description of black holes on branes, J. High Energy Phys. 01 (2000) 007.

[49] R. Gregory, The Gregory-Laflamme instability, arXiv:1107.5821.

[50] L. Lehner and F. Pretorius, Final state of Gregory-Laflamme instability, arXiv:1106.5184.

[51] V. P. Frolov and A. A. Shoom, Gregory-Laflamme instability of 5D electrically charged black strings, Phys. Rev. D **79**, 104002 (2009).

[52] T. Shiromizu, K.-i. Maeda, and M. Sasaki, The Einstein equation on the 3-brane world, Phys. Rev. D **62**, 024012 (2000).

[53] R. Maartens, Brane world gravity, Living Rev. Relativity **7**, 7 (2004).

[54] S. Chakraborty and S. SenGupta, Spherically symmetric brane spacetime with bulk $f(\mathcal{R})$ gravity, Eur. Phys. J. C **75**, 11 (2015).
[55] S. Chakraborty and S. SenGupta, Spherically symmetric brane in a bulk of $f(\mathcal{R})$ and Gauss-Bonnet gravity, Classical Quantum Gravity **33**, 225001 (2016).
[56] S. Chakraborty and S. SenGupta, Effective gravitational field equations on m-brane embedded in n-dimensional bulk of Einstein and $f(\mathcal{R})$ gravity, Eur. Phys. J. C **75**, 538 (2015).
[57] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon, Oxford, 1992), p. 646.
[58] H.-P. Nollert, Topical review: Quasinormal modes: The characteristic 'sound' of black holes and neutron stars, Classical Quantum Gravity **16**, R159 (1999).
[59] K. D. Kokkotas and B. G. Schmidt, Quasinormal modes of stars and black holes, Living Rev. Relativity **2**, 2 (1999).
[60] E. Berti, V. Cardoso, and A. O. Starinets, Quasinormal modes of black holes and black branes, Classical Quantum Gravity **26**, 163001 (2009).
[61] R. A. Konoplya and A. Zhidenko, Quasinormal modes of black holes: From astrophysics to string theory, Rev. Mod. Phys. **83**, 793 (2011).
[62] R. A. Konoplya, K. Murata, J. Soda, and A. Zhidenko, Looking at the Gregory-Laflamme instability through quasinormal modes, Phys. Rev. D **78**, 084012 (2008).
[63] R. A. Konoplya and A. Zhidenko, Stability and quasinormal modes of the massive scalar field around Kerr black holes, Phys. Rev. D **73**, 124040 (2006).
[64] P. Kanti and R. A. Konoplya, Quasi-normal modes of brane-localised standard model fields, Phys. Rev. D **73**, 044002 (2006).
[65] B. Toshmatov, Z. Stuchlik, J. Schee, and B. Ahmedov, Quasinormal frequencies of black hole in the braneworld, Phys. Rev. D **93**, 124017 (2016).
[66] S. Aneesh, S. Bose, and S. Kar, Gravitational waves from quasinormal modes of a class of Lorentzian wormholes, Phys. Rev. D **97**, 124004 (2018).
[67] V. Cardoso, E. Franzin, A. Maselli, P. Pani, and G. Raposo, Testing strong-field gravity with tidal Love numbers, Phys. Rev. D **95**, 084014 (2017); Publisher's Note, Phys. Rev. D **95**, 089901(A) (2017).
[68] S. S. Yazadjiev, D. D. Doneva, and K. D. Kokkotas, Tidal Love numbers of neutron stars in $f(R)$ gravity, Eur. Phys. J. C **78**, 818 (2018).
[69] J. M. Lattimer and M. Prakash, The physics of neutron stars, Science **304**, 536 (2004).
[70] T. Hinderer, Tidal Love numbers of neutron stars, Astrophys. J. **677**, 1216 (2008).
[71] E. E. Flanagan and T. Hinderer, Constraining neutron star tidal Love numbers with gravitational wave detectors, Phys. Rev. D **77**, 021502 (2008).
[72] T. Binnington and E. Poisson, Relativistic theory of tidal Love numbers, Phys. Rev. D **80**, 084018 (2009).
[73] T. Damour and A. Nagar, Relativistic tidal properties of neutron stars, Phys. Rev. D **80**, 084035 (2009).
[74] T. Hinderer, B. D. Lackey, R. N. Lang, and J. S. Read, Tidal deformability of neutron stars with realistic equations of state and their gravitational wave signatures in binary inspiral, Phys. Rev. D **81**, 123016 (2010).
[75] W. Del Pozzo, T. G. F. Li, M. Agathos, C. Van Den Broeck, and S. Vitale, Demonstrating the Feasibility of Probing the Neutron Star Equation of State with Second-Generation Gravitational Wave Detectors, Phys. Rev. Lett. **111**, 071101 (2013).
[76] E. Poisson, Tidal deformation of a slowly rotating black hole, Phys. Rev. D **91**, 044004 (2015).
[77] R. A. Porto, The tune of Love and the Nature(ness) of spacetime, Fortschr. Phys. **64**, 723 (2016).
[78] K. Yagi and N. Yunes, I-Love-Q relations: From compact stars to black holes, Classical Quantum Gravity **33**, 095005 (2016).
[79] E. Barausse, R. Brito, V. Cardoso, I. Dvorkin, and P. Pani, The stochastic gravitational-wave background in the absence of horizons, Classical Quantum Gravity **35**, 20LT01 (2018).
[80] K. Chakravarti *et al.*, Systematic effects from black hole-neutron star waveform model uncertainties on the neutron star equation of state, arXiv:1809.04349.
[81] T. Harko and M. Mak, Vacuum solutions of the gravitational field equations in the brane world model, Phys. Rev. D **69**, 064020 (2004).
[82] P. Char and S. Datta, Relativistic tidal properties of superfluid neutron stars, Phys. Rev. D **98**, 084010 (2018).
[83] C. Germani and R. Maartens, Stars in the brane world, Phys. Rev. D **64**, 124010 (2001).
[84] M. Alford, M. Braby, M. W. Paris, and S. Reddy, Hybrid stars that masquerade as neutron stars, Astrophys. J. **629**, 969 (2005).
[85] C. Germani and R. Maartens, Stars in the brane world, Phys. Rev. D **64**, 124010 (2001).
[86] J. C. Long, H. W. Chan, A. B. Churnside, E. A. Gulbis, M. C. M. Varney, and J. C. Price, Upper limits to submillimeter-range forces from extra space-time dimensions, Nature (London) **421**, 922 (2003).
[87] C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt, and H. E. Swanson, Submillimeter tests of the gravitational inverse-square law, Phys. Rev. D **70**, 042004 (2004).
[88] S. J. Smullin, A. A. Geraci, D. M. Weld, J. Chiaverini, S. P. Holmes, and A. Kapitulnik, New constraints on Yukawa-type deviations from Newtonian gravity at 20 microns, Phys. Rev. D **72**, 122001 (2005); Erratum, Phys. Rev. D **72**, 129901(E) (2005).
[89] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, Illinois, 1965).