Effect of the self-gravity of shells on a high energy collision in a rotating Bañados-Teitelboim-Zanelli spacetime

Kota Ogasawara[1,*] and Naoki Tsukamoto[2,†]

[1]*Department of Physics, Rikkyo University, Tokyo 171-8501, Japan*
[2]*Frontier Research Institute for Interdisciplinary Sciences & Department of Physics, Tohoku University, Sendai, Miyagi 980-8578, Japan*

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We consider a collision of two dust thin shells with a high center-of-mass (CM) energy including their self-gravity in a Bañados-Teitelboim-Zanelli (BTZ) spacetime. The shells divide the BTZ spacetime into three domains and the domains are matched by the Darmois-Israel junction conditions. We treat only the collision of two shells which corotate with a background BTZ spacetime because of the junction conditions. The counterpart of the corotating shell collision is a collision of two particles with vanishing angular momenta. We compare the dust thin shell collision and the particle collision in order to investigate the effects of the self-gravity of colliding objects on the high CM energy collision. We show that the self-gravity of the shells affects the position of an event horizon and it covers the high-energy collisional event. Therefore, we conclude that the self-gravity of colliding objects suppresses its CM energy and that any observer who stands outside of the event horizon cannot observe the collision with an arbitrary high CM energy.

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I. INTRODUCTION

In 1977, Piran and Shaham [1] discussed a collisional Penrose process which is a kind of an energy extraction [2] from a Kerr black hole and they found that the center-of-mass (CM) energy of two particles can be arbitrarily large in a near-horizon limit in the extremal Kerr spacetime. In 2009, Bañados, Silk, and West (BSW) rediscovered the arbitrarily high CM energy of the particle collision and they pointed out that rotating black holes can act as particle accelerators [3]. The process is often called BSW collision or BSW process after BSW's work. See Harada and Kimura [4] for a brief review on the BSW process.

In 2016, the LIGO Scientific and the Virgo Collaborations have reported the first detection of gravitational waves and they ensured the existence of astrophysical black holes [5]. Physics in strong gravitational fields of black holes and compact objects such as BSW process would be more interest among researchers not only in general relativity but also in astronomy and astrophysics.

Several critical comments on the BSW process were given in Refs. [6–8]. It is well known that there is the upper bound of the angular momentum of the Kerr black hole in an astrophysical situation [9]. It needs arbitrarily long proper time of either of two particles with the infinite CM energy to reach the event horizon for a maximally rotating Kerr black hole. If the self-gravity of the colliding particles is strong, the self-gravity will weaken the CM energy. We should keep in mind that an observer distant from a black hole may not see the products of the collision with high energy and/or very massive and the observed products must be highly red-sifted even if the CM energy is very large [8].

The details of the BSW collision have been investigated after stimulation by the criticism. Patil *et al.* [10] considered a finite CM energy of a collision of two particles with a finite proper time. The collision of particle with an innermost stable circular orbit [11,12], off-equatorial-plane collisions [13], a collision in a weak electromagnetic field [14], and the BSW collision in the near-horizon geometry [15] have been investigated. Nongeodesic particle collisions have been considered in [16,17]. A close relation between the BSW collision and the Aretakis instability which is a test-field instability of an extremal horizon [18–25] was pointed out [26]. The details of the collisional Penrose process were also investigated by several authors [27–36].

Particle collisions with high CM energy occur not only in a Kerr black hole spacetime but also in a Kerr naked singularity [37], a Kerr-Newmann [38], a Kerr-(anti-)de Sitter [39], lower-dimensional [40–45], and higher-dimensional spacetimes [26,46–48]. A particle collision with an unbounded CM energy in an extremal Reissner-Nordström spacetime [49] and a higher-dimensional Reissner-Nordström spacetime [26] have been investigated as the electromagnetic

*k.ogasawara@rikkyo.ac.jp
†tsukamoto@rikkyo.ac.jp

counterpart of the BSW collision.[1] The BSW collision is regarded as a universal process in extremal spacetimes in all dimensions.

The self-gravity effect of colliding objects on the BSW collision cannot be neglected since the collision occurs in a near-horizon limit. However, it is very difficult to treat the self-gravity of particles analytically in a Kerr spacetime. Kimura *et al.* considered the collision of two thin shells [51,52] in a Reissner-Nordström spacetime instead of a Kerr spacetime for its simplicity and they calculated analytically the CM energy of the shell collision including their self-gravity [53–55].

Can we treat thin shells in stationary and axisymmetric spacetimes? The analytical treatment of a thin shell in the Kerr spacetime is very difficult [56–58] but the difficulty of the technical problem depends on the dimension of the spacetime [59–62]. In Ref. [59], Mann *et al.* investigated the collapse of a shell in a three-dimensional stationary and axisymmetric spacetime and they showed that the motion of the shell is tractable.

A black hole solution in three dimensions was obtained by Bañados, Teitelboim, and Zanelli (BTZ) [63,64]. The BTZ spacetime has a negative cosmological constant since gravity in three dimensions is weaker than the one in four dimensions. In the BTZ spacetime, particle motions [65], the BSW collision [40–42,45], gravitational perturbations induced by falling particles [66], and thermodynamics of thin shells [67–69] have been investigated.

In this paper, we investigate the collision of two dust thin shells in the BTZ spacetime with an angular momentum and a negative cosmological constant in three dimensions. We use a thin-shell formalism [70–72] for corotating thin shells investigated by Mann *et al.* [59]. We study the collision of two particles with vanishing angular momenta as the counterpart of the shell collision and then we investigate the effects of the self-gravity of the shells on their collision with a high CM energy.

The organization of this paper is as follows. In Sec. II, we investigate a collision of two particles with vanishing angular momenta. We review the thin shell formalism in a corotating frame in Sec. III. In Sec. IV, we investigate the collision of two dust thin shells. Section V is devoted to the discussion and conclusion. In this paper, we use the units in which the speed of light and $8G$ are unity as in Sec. III of Ref. [59], where G is Newton's constant in three dimensions.

II. PARTICLE COLLISION IN THE BTZ SPACETIME

In this section, we review the center-of-mass energy of the collision of two particles in the BTZ spacetime. The metric [63,64] is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2[d\varphi - \Omega(r)dt]^2, \tag{2.1}$$

where

$$f(r) \equiv -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \tag{2.2}$$

$$\Omega(r) \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{J}{2r^2}, \tag{2.3}$$

$$\ell \equiv \sqrt{\frac{1}{-\Lambda}}. \tag{2.4}$$

Here M, J, and $\Omega(r)$ are the mass, the angular momentum, and angular velocity of the spacetime, respectively, and ℓ is the scale of a curvature related to the negative cosmological constant $\Lambda < 0$. The spacetime is a stationary and axisymmetric spacetime with two Killing vectors ∂_t and ∂_φ. Without loss of generality, we assume that the angular momentum J is non-negative. In this paper, we concentrate on the case where M is positive. The spacetime has an event horizon at

$$r = r^H \equiv \ell\sqrt{\frac{M}{2}\left(1 + \sqrt{1 - \frac{J^2}{\ell^2 M^2}}\right)}, \tag{2.5}$$

for $J \le \ell M$ and it is called extremal spacetime for $J = \ell M$. For $J > \ell M$, it has a naked singularity. We discuss the last case in Appendix.

We consider a particle motion with a 3-momentum p^μ and a rest mass m. The conserved energy and angular momentum of the particle are given by

$$E \equiv -g_{\mu\nu}(\partial_t)^\mu p^\nu \quad \text{and} \quad L \equiv g_{\mu\nu}(\partial_\varphi)^\mu p^\nu, \tag{2.6}$$

respectively. From Eq. (2.6) and the condition $p^\mu p_\mu = -m^2$, we obtain the components of the 3-momentum as

$$p^t(r) = \frac{S(r)}{f(r)}, \tag{2.7}$$

$$p^r(r) = \sigma\sqrt{R(r)}, \tag{2.8}$$

$$p^\varphi(r) = \frac{\Omega(r)S(r)}{f(r)} + \frac{L}{r^2}, \tag{2.9}$$

where σ, $S(r)$, and $R(r)$ are defined as

$$\sigma \equiv \mathrm{sgn}(p^r) = \pm 1, \tag{2.10}$$

$$S(r) \equiv E - \Omega(r)L, \tag{2.11}$$

$$R(r) \equiv S^2(r) - \left(m^2 + \frac{L^2}{r^2}\right)f(r), \tag{2.12}$$

[1]The collision of charged particles in a rotating and charged spacetime was also investigated by Hejda and Bičák [50].

respectively. Note that a forward-in-time condition

$$p^t(r) \geq 0, \tag{2.13}$$

must be satisfied for a particle motion.

An energy equation which describes the radial motion of the particle is given by, from Eq. (2.8),

$$\left(\frac{dr}{d\tau}\right)^2 + V(r) = 0, \tag{2.14}$$

where $V(r) \equiv -R(r)$ is the effective potential for the radial motion of the particle and τ is its proper time. Here we have used relations $p^\mu = mu^\mu$ and $u^\mu = dx^\mu/d\tau$, where u^μ is the 3-velocity of the particle. We call a condition $V(r^H) = 0$ critical condition. The critical condition is rewritten as

$$E - \Omega_H L = 0, \tag{2.15}$$

where

$$\Omega_H \equiv \Omega(r^H) = \frac{J}{2(r^H)^2}, \tag{2.16}$$

is the angular velocity of the horizon.

The angular velocity of a particle $\omega(r)$ is defined as

$$\omega(r) \equiv \frac{d\varphi}{dt} = \frac{p^\varphi}{p^t}. \tag{2.17}$$

When a particle has a zero conserved angular momentum $L = 0$, from Eqs. (2.7), (2.9), and (2.17), the angular velocity of the particle coincides with the angular velocity of the spacetime, i.e., $\omega(r) = \Omega(r)$. This means that the particle with $L = 0$ corotates with the background spacetime.

We concentrate on the motion of a particle with $L = 0$ which is the counterpart of a shell corotating with the background spacetime. Using a dimensionless radial coordinate $x \equiv r/\ell$, the effective potential of the particle with the specific energy $e \equiv E/m$ and the position of the event horizon are expressed as

$$V(x) = -e^2 + f(x) = x^2 - M - e^2 + \frac{c}{x^2}, \tag{2.18}$$

and

$$x^H \equiv \frac{r^H}{\ell} = \sqrt{\frac{M + \sqrt{M^2 - 4c}}{2}}, \tag{2.19}$$

respectively, where

$$f(x) = x^2 - M + \frac{c}{x^2}, \qquad c \equiv \frac{J^2}{4\ell^2}. \tag{2.20}$$

As x increases from 0 to infinity, $V(x)$ and $f(x)$ begin with infinity, monotonically decrease to a local minimum at $x = x^m \equiv c^{\frac{1}{4}}$, and monotonically increase to infinity. The particle motion is restricted to a region $x^- \leq x \leq x^+$ where the effective potential $V(x)$ is nonpositive. The boundaries $x^\pm$ are obtained as

$$x^\pm \equiv \sqrt{\frac{M + e^2 \pm \sqrt{(M + e^2)^2 - 4c}}{2}}. \tag{2.21}$$

We notice a relation

$$x^- \leq x^m \leq x^H \leq x^+. \tag{2.22}$$

We obtain $x^m = x^H$ in the extremal case, $x^H = x^+$ in the critical case, and $x^- = x^m = x^H = x^+$ in the extremal and critical case.

We consider a collision of two particles, named particle 1 and 2, with vanishing conserved angular momenta $L_1 = L_2 = 0$ in a region $x^H \leq x \leq x_a^+$ where is seen by an observer who stands at the outside of the horizon x^H. Hereinafter p_a^μ, E_a e_a, m_a, $x_a^\pm$, V_a, and σ_a denote p^μ, E, e, m, $x^\pm$, V, and σ of particle a ($a = 1$ and 2), respectively. From Eq. (2.15), the critical condition for particle a with $L_a = 0$ is given by

$$E_a = 0, \quad \text{i.e.,} \quad e_a = 0. \tag{2.23}$$

The CM energy $E_{\rm cm}(x)$ of the particles is given by

$$\begin{aligned} E_{\rm cm}^2(x) &\equiv -g_{\mu\nu}(m_1 u_1^\mu + m_2 u_2^\mu)(m_1 u_1^\nu + m_2 u_2^\nu) \\ &= m_1^2 + m_2^2 + 2m_1 m_2 \frac{e_1 e_2 - \sigma_1 \sigma_2 \sqrt{V_1(x) V_2(x)}}{f(x)}. \end{aligned} \tag{2.24}$$

Let us consider a rear-end collision, i.e., we choose $\sigma_1 = \sigma_2 = -1$. As x increases from x^H to x_a^+, the CM energy begins with

$$E_{\rm cm}(x^H) = \sqrt{m_1^2 + m_2^2 + m_1 m_2 \left(\frac{e_2}{e_1} + \frac{e_1}{e_2}\right)}, \tag{2.25}$$

and monotonically increases to

$$E_{\rm cm}(x_a^+) = E_{\rm cm}^{\rm max}, \tag{2.26}$$

where

$$E_{\rm cm}^{\rm max} \equiv \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \frac{e_1 e_2}{e_a^2}}. \tag{2.27}$$

We have used l'Hopital's rule to estimate $E_{\rm cm}(x^H)$. We get $E_{\rm cm}(x) = m_1 + m_2$ if the both particles have a same specific energy $e_1 = e_2$. We set particle 1 to be inner than particle 2 in the rest of this section.

We are interested in the collision of the inner particle with a critical limit $e_1 \to 0$ and the outer particle which is not critical in the extremal BTZ spacetime since the collision will correspond with the BSW collision in the extremal Kerr spacetime [3] and in the extremal Reissner-Nordström spacetime [49]. In this case, the collisional point must be $x \to x^H + 0$ because of inequality (2.22) and $x_1^{\pm} \to x^H \pm 0$ and the CM energy $E_{\rm cm}$ diverges there. Notice that a particle with the critical condition has $V_1(x^H) = V_1'(x^H) = 0$ and $V_1''(x^H) > 0$, where a prime denotes a derivative with respect to x, in the extremal BTZ spacetime while one has $V_1(x^H) = V_1'(x^H) = 0$ and $V_1''(x^H) < 0$ in an extremal Kerr black hole spacetime [3] and an extremal Reissner-Nordström spacetime [49]. The positive sign of $V_1''(x^H)$ is caused by the negative cosmological constant and it will not affect on the BSW collision strongly as long as the collision occurs near the horizon.

III. DUST THIN SHELL AND ITS MOTION IN THE BTZ SPACETIME

In this section, as a preparation to study a shell collision, we review the Darmois-Israel junction conditions [70–72] and the motion of a dust thin shell in the BTZ spacetime [59]. Hereinafter, we use x^μ denoting coordinates in every domain for simplicity.

We consider a two-dimensional hypersurface Σ which divides the BTZ spacetime into an interior domain D_1 and an exterior domain D_2. Domain D_A ($A = 1$ and 2) has a mass M_A and an angular momentum J_A, see Fig. 1. For simplicity, we assume that D_1 and D_2 have the same ℓ. We assume that we can take same coordinates y^i on Σ in both the domains.

The projection operator from the three-dimensional BTZ spacetime to Σ is defined as

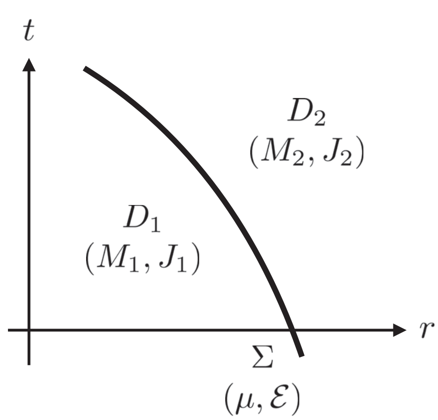

FIG. 1. Schematic picture of the BTZ spacetime divided into two domains D_1 and D_2 by the hypersurface Σ. Domain D_A ($A = 1$ and 2) has a mass M_A and an angular momentum J_A. A thin shell on the hypersurface Σ has a proper mass μ and specific energy $\mathcal{E} \equiv (M_2 - M_1)/\mu$.

$$e^\mu_i \equiv \frac{\partial x^\mu}{\partial y^i}. \tag{3.1}$$

The induced metric on the hypersurface Σ in domain D_A is defined as

$$g^A_{ij} \equiv g^A_{\mu\nu} e^\mu_i e^\nu_j. \tag{3.2}$$

Using a unit vector n^μ normal to the hypersurface Σ, which directed from D_1 to D_2, the extrinsic curvature of the hypersurface Σ in the domain D_A is defined as

$$K^A_{ij} \equiv e^\mu_i e^\nu_j \nabla^A_\mu n_\nu, \tag{3.3}$$

where ∇^A_μ is the covariant derivative within D_A. The first and second junction conditions are given by

$$[g_{ij}] = 0 \quad \text{and} \quad [K_{ij}] = 0, \tag{3.4}$$

respectively. Here the bracket is defined as

$$[\Psi] \equiv \Psi(D_2)|_\Sigma - \Psi(D_1)|_\Sigma, \tag{3.5}$$

where Ψ is any quantity defined on the both sides of Σ.

When we introduce a corotating frame on Σ with an azimuth coordinate

$$d\phi \equiv d\varphi - \frac{J_A}{2\mathcal{R}^2(t)} dt, \tag{3.6}$$

the metric in D_A is given by

$$\begin{aligned} ds_A^2 &= g^A_{\mu\nu} dx^\mu dx^\nu \\ &= -f_A(r) dt^2 + \frac{dr^2}{f_A(r)} \\ &\quad + r^2 \left[d\phi + \frac{J_A}{2} \left(\frac{1}{\mathcal{R}^2(t)} - \frac{1}{r^2} \right) dt \right]^2, \end{aligned} \tag{3.7}$$

where

$$f_A(r) \equiv -M_A + \frac{r^2}{\ell^2} + \frac{J_A^2}{4r^2}. \tag{3.8}$$

From the first junction condition $[g_{ij}] = 0$, the induced metric g^A_{ij} on Σ with

$$t = \mathcal{T}(\tau) \quad \text{and} \quad r = \mathcal{R}(\mathcal{T}(\tau)) = \mathcal{R}(\tau), \tag{3.9}$$

is given by

$$\begin{aligned} ds_\Sigma^2 &= g^A_{ij} dy^i dy^j = -d\tau^2 + \mathcal{R}^2(\tau) d\phi^2 \\ &= \left[-f_A(\mathcal{R}) \dot{\mathcal{T}}^2 + \frac{\dot{\mathcal{R}}^2}{f_A(\mathcal{R})} \right] d\tau^2 + \mathcal{R}^2(\tau) d\phi^2, \end{aligned} \tag{3.10}$$

where a dot denotes a derivative with respect to the proper time τ of an observer on Σ. This implies

$$f_A(\mathcal{R})\dot{\mathcal{T}} = \beta_A, \tag{3.11}$$

where

$$\beta_A \equiv \sqrt{\dot{\mathcal{R}}^2 + f_A(\mathcal{R})}. \tag{3.12}$$

The components of e_i^μ and n_μ are described by

$$e_\tau^\mu = (\dot{\mathcal{T}}, \dot{\mathcal{R}}, 0), \qquad e_\phi^\mu = (0,0,1), \tag{3.13}$$

and

$$n_\mu = (-\dot{\mathcal{R}}, \dot{\mathcal{T}}, 0), \tag{3.14}$$

respectively. We obtain the components of the extrinsic curvature (3.3) and its trace $K^A \equiv g^{ij}K^A_{ij}$ as

$$K^A_{\tau\tau} = -\frac{\dot{\beta}_A}{\dot{\mathcal{R}}}, \qquad K^A_{\phi\phi} = \mathcal{R}\beta_A, \qquad K^A_{\tau\phi} = \frac{J_A}{2\mathcal{R}}, \tag{3.15}$$

and

$$K^A = \frac{\dot{\beta}_A}{\dot{\mathcal{R}}} + \frac{\beta_A}{\mathcal{R}}, \tag{3.16}$$

respectively. We notice that the second junction condition $[K_{ij}] = 0$ is violated unless we consider a trivial case. Therefore, we introduce a thin shell on Σ following equations

$$\pi S_{ij} = -([K_{ij}] - [K]g_{ij}), \tag{3.17}$$

where S_{ij} is the surface stress-energy tensor of the thin shell.

Let us consider a dust thin shell with the surface stress-energy tensor S_{ij} given by

$$S_{ij} = \rho u_i u_j, \tag{3.18}$$

where ρ and $u_i = (-1, 0)$ are the surface energy density and the 2-velocity of the shell, respectively. Using Eqs. (3.10), (3.15), (3.16), and (3.18), the (τ,τ), (ϕ,ϕ), and (τ,ϕ) components of Eq. (3.17) are obtained as

$$[\beta] + \pi\rho\mathcal{R} = 0, \tag{3.19}$$

$$[\dot{\beta}] = 0, \tag{3.20}$$

and

$$[J] = 0, \tag{3.21}$$

respectively. From Eqs. (3.19) and (3.20), we obtain

$$\frac{d}{d\tau}(\pi\rho\mathcal{R}) = 0. \tag{3.22}$$

Therefore, we can define the proper mass μ of the shell

$$\mu \equiv 2\pi\rho\mathcal{R}, \tag{3.23}$$

which is constant along its trajectory. We define the specific energy $\mathcal{E}$ of the shell as

$$\mathcal{E} \equiv \frac{[M]}{\mu}. \tag{3.24}$$

We assume that the proper mass μ and the specific energy $\mathcal{E}$ are positive. This implies that the masses satisfy the relation $M_1 < M_2$. From Eq. (3.21), the angular momenta in all the domains must be the same, i.e.,

$$J \equiv J_1 = J_2. \tag{3.25}$$

Using Eqs. (3.12), (3.19), and (3.23), we obtain the energy equation as

$$\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + V(\mathcal{R}) = 0, \tag{3.26}$$

where $V(\mathcal{R})$ is the effective potential of the shell motion. Using $x \equiv \mathcal{R}/\ell$, the effective potential is expressed as

$$V(x) = x^2 - \langle M\rangle - \mathcal{E}^2 + \frac{c}{x^2} - \frac{\mu^2}{16}, \tag{3.27}$$

where

$$\langle M\rangle \equiv \frac{M_2 + M_1}{2}. \tag{3.28}$$

As x increases from 0 to infinity, $V(x)$ begins with infinity, monotonically decreases to a local minimum at $x = x^m \equiv c^{\frac{1}{4}}$, and monotonically increases to infinity.

The shell motion is restricted to the region $x^- \le x \le x^+$ where the effective potential is nonpositive. Here

$$x^\pm \equiv \sqrt{\frac{b \pm \sqrt{b^2 - 4c}}{2}}, \tag{3.29}$$

have been obtained as the positive solutions of $V(x) = 0$, where

$$b \equiv \frac{\mu^2}{16} + \langle M\rangle + \mathcal{E}^2 = \left(\frac{\mu}{4} - \mathcal{E}\right)^2 + M_2. \tag{3.30}$$

We will call a limit $\mu \to 0$ and $M_1 \to \mathcal{M} \equiv M_2$ with $\mathcal{E} =$ (constant) $\neq 0$ test shell limit. In the test shell limit, the effective potential of the shell (3.27) is obtained as

$$V(x) = x^2 - \mathcal{M} - \mathcal{E}^2 + \frac{c}{x^2}, \tag{3.31}$$

and it takes the same form as the effective potential of a particle (2.18).

IV. DUST THIN SHELL COLLISION IN THE BTZ SPACETIME

In this section, we investigate the collision of two dust thin shells in the BTZ spacetime. We assume that shell 1 and shell 2 are on an inner hypersurface Σ_1 and an outer hypersurface Σ_2, respectively. These two hypersurfaces divide the BTZ spacetime into an interior domain D_1, a middle domain D_2, and an exterior domain D_3. We assume that every domain D_A ($A = 1$, 2, and 3) is the BTZ spacetime with the same ℓ for simplicity. From Eq. (3.21), all the domains have the same angular momenta J. See Fig. 2.

When $J \leq \ell M_A$, where M_A is a mass in D_A, is satisfied, the position of the event horizon is obtained as $x = x_A^H$, where

$$x_A^H \equiv \sqrt{\frac{M_A + \sqrt{M_A^2 - 4c}}{2}}. \tag{4.1}$$

We assume $M_1 < M_2 < M_3$. We consider five cases according to the value of J as shown in Table I: case I (II) for $J < \ell M_1$ ($J = \ell M_1$), case III for $J = \ell M_2$, and case IV (V) for $J = \ell M_3$ ($J > \ell M_3$).

The effective potential of shell a ($a = 1$ and 2) is given by

$$V_a(x) = -Z_a^2 + f(x), \tag{4.2}$$

where

$$Z_1 \equiv \frac{\mu_1}{4} - \mathcal{E}_1, \tag{4.3}$$

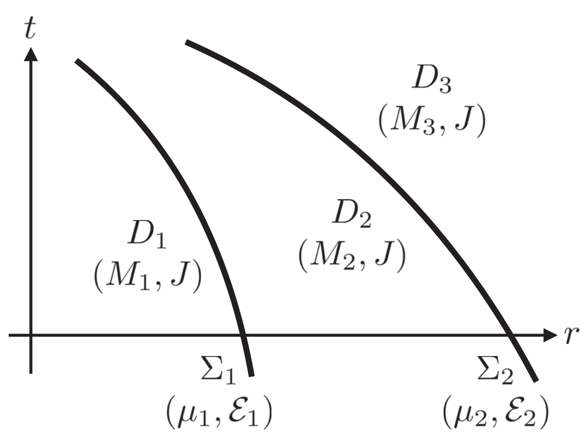

FIG. 2. Schematic picture of the BTZ spacetime divided into three domains D_1, D_2, and D_3 by hypersurfaces Σ_1 and Σ_2. Domain D_A ($A = 1$, 2, and 3) is the BTZ spacetime with a mass M_A and the same angular momentum J. A thin shell a ($a = 1$ and 2) on the hypersurfaces Σ_a has a proper mass μ_a and specific energy $\mathcal{E}_a \equiv (M_{a+1} - M_a)/\mu_a$.

TABLE I. Five cases of the spacetime according to the value of J. Symbols S, E, and O denote the subextremal, extremal, and overspinning spacetimes, respectively. X is defined as the minimum value of $f_2(x)$.

Case	D_1	D_2	D_3	J	X
I	S	S	S	$J < \ell M_1$	$X < 0$
II	E	S	S	$J = \ell M_1$	$X < 0$
III	O	E	S	$J = \ell M_2$	$X = 0$
IV	O	O	E	$J = \ell M_3$	$X > 0$
V	O	O	O	$J > \ell M_3$	$X > 0$

$$Z_2 \equiv \frac{\mu_2}{4} + \mathcal{E}_2, \tag{4.4}$$

$$f(x) \equiv f_2(x) = x^2 - M_2 + \frac{c}{x^2}. \tag{4.5}$$

As x increases from 0 to infinity, $V_a(x)$ begins with infinity, monotonically decreases to a local minimum

$$V_a(x^m) = -Z_a^2 + X, \tag{4.6}$$

where

$$X \equiv f(x^m) = 2\sqrt{c} - M_2, \tag{4.7}$$

at $x = x^m \equiv c^{\frac{1}{4}}$, and monotonically increases to infinity. Note that x^m for shells 1 and 2 are the same.

A motion of shell a is restricted to the region $x_a^- \leq x \leq x_a^+$, where the effective potential $V_a(x)$ is nonpositive. Here $x_a^\pm$ is given by

$$x_a^\pm \equiv \sqrt{\frac{b_a \pm \sqrt{b_a^2 - 4c}}{2}}, \tag{4.8}$$

and

$$b_a \equiv Z_a^2 + M_2. \tag{4.9}$$

In order for such the region to exist, a condition

$$X \leq Z_a^2, \tag{4.10}$$

must be satisfied.

For a comparison with a particle satisfying the critical condition (2.23), one might have an interest in a shell with $V_1(x_2^H) = V_1'(x_2^H) = 0$. We can show easily that

$$V_1(x_2^H) = V_1'(x_2^H) = 0, \tag{4.11}$$

is satisfied if and only if

$$Z_1 = 0, \tag{4.12}$$

and $J = \ell M_2$ are satisfied. Therefore, we call the condition $Z_1 = 0$ critical condition for shell 1.

A. Center-of-mass energy

The CM energy $E_{\rm cm}(x)$ of two shells at a collisional point is given by [53,55]

$$\begin{aligned} E^2_{\rm cm}(x) &\equiv -g_{\mu\nu}(\mu_1 U^\mu_1 + \mu_2 U^\mu_2)(\mu_1 U^\nu_1 + \mu_2 U^\nu_2) \\ &= \mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\left(f(x)\dot{\mathcal{T}}_1\dot{\mathcal{T}}_2 - \frac{\dot{\mathcal{R}}_1\dot{\mathcal{R}}_2}{f(x)}\right), \end{aligned} \tag{4.13}$$

where $g_{\mu\nu}$ is the metric in D_2, $U^\mu_a = (\dot{\mathcal{T}}_a, \dot{\mathcal{R}}_a, 0)$ is the 3-velocity of shell a, where

$$\dot{\mathcal{T}}_a \equiv \frac{\sqrt{f(x) - V_a(x)}}{f(x)}, \tag{4.14}$$

$$\dot{\mathcal{R}}_a \equiv \sigma_a\sqrt{-V_a(x)}, \tag{4.15}$$

and $\sigma_a \equiv \mathrm{sgn}(\dot{\mathcal{R}}_a) = \pm 1$.

We will concentrate on the rear-end collision of two dust thin shells. In this case, we should choose $\sigma_1 = \sigma_2 = -1$ and the CM energy is expresses as

$$E^2_{\rm cm}(x) = \mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\frac{|Z_1|Z_2 - \sqrt{V_1(x)V_2(x)}}{f(x)}. \tag{4.16}$$

We notice $E_{\rm cm}(x) = \mu_1 + \mu_2$ when $|Z_1| = Z_2$ is satisfied. The shells must satisfy a relation $\dot{\mathcal{R}}_1(x) \le \dot{\mathcal{R}}_2(x)$ to collide at the point. It implies $V_1(x) \ge V_2(x)$, $|Z_1| \le Z_2$, or $x_1^+ \le x_2^+$.

1. Cases I, II, and III

In the cases I, II, and III, we concentrate on the region $x_2^H \le x \le x_1^+$ as in particle collisions. As x increases from x_2^H to x_1^+, the CM energy begins with

$$E_{\rm cm}(x_2^H) = \sqrt{\mu_1^2 + \mu_2^2 + \mu_1\mu_2\left(\frac{Z_2}{|Z_1|} + \frac{|Z_1|}{Z_2}\right)}, \tag{4.17}$$

and monotonically increases to

$$E_{\rm cm}(x_1^+) = E_{\rm cm}^{\rm max}, \tag{4.18}$$

where

$$E_{\rm cm}^{\rm max} \equiv \sqrt{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\frac{Z_2}{|Z_1|}}. \tag{4.19}$$

Here we have used l'Hopital's rule to calculate $E_{\rm cm}(x_2^H)$. In a critical limit $Z_1 \to 0$, both $E_{\rm cm}(x_2^H)$ and $E_{\rm cm}(x_1^+)$ are arbitrarily large and x_1^+ coincides with x_2^H. The event horizon, however, moves from x_2^H to x_3^H because of the self-gravity of shell 2. Thus, the arbitrarily large CM energy cannot be seen by an observer outside the event horizon x_3^H.

2. Cases IV and V

In cases IV and V, x_2^H does not exist. Let us consider the region $x_1^- \le x \le x_1^+$. The CM energy monotonically decreases from

$$E_{\rm cm}(x_1^-) = E_{\rm cm}^{\rm max}, \tag{4.20}$$

to

$$E_{\rm cm}(x^m) = \sqrt{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\frac{|Z_1|Z_2 - \sqrt{(Z_1^2 - X)(Z_2^2 - X)}}{X}}, \tag{4.21}$$

as x increases from x_1^- to x^m and it monotonically increases as from x^m to x_1^+ and then it reaches

$$E_{\rm cm}(x_1^+) = E_{\rm cm}^{\rm max}. \tag{4.22}$$

In these cases, a critical shell with $Z_1 = 0$ is forbidden since its effective potential is positive. When $V_1(x^m) = 0$, i.e., $|Z_1| = \sqrt{X}$, shell 1 can be only at $x = x^m = x_1^\pm$. The CM energy there is given by

$$E_{\rm cm}(x = x^m = x_1^\pm) = \sqrt{\mu_1^2 + \mu_2^2 + \frac{2\mu_1\mu_2 Z_2}{\sqrt{X}}}. \tag{4.23}$$

B. Collision at $x \ge x_3^H$

Let us consider a shell collision at $x \ge x_3^H$ for the cases I-IV. An observer at $x \ge x_3^H$ in domain D_3 may see the parts of the products after the collision. A condition $x_3^H \le x_1^+$ must be satisfied for the existence of the inner shell 1. From Eqs. (4.1) and (4.8), the condition $x_3^H \le x_1^+$ is expressed as

$$\mu_2\mathcal{E}_2 \le Z_1^2. \tag{4.24}$$

The finite upperbound of the CM energy is given by, from Eqs. (4.18), (4.22), and (4.24),

$$E_{\rm cm}(x = x_3^H = x_1^+) = \sqrt{\mu_1^2 + \mu_2^2 + \frac{\mu_1\sqrt{\mu_2}(\mu_2 + 4\mathcal{E}_2)}{2\sqrt{\mathcal{E}_2}}}. \tag{4.25}$$

This shows that the self-gravity caused by the colliding shells suppresses the CM energy. For an equal mass $\mu \equiv \mu_1 = \mu_2$, the upperbound of the CM energy is given by

$$E_{\rm cm}(x = x_3^H = x_1^+) = \frac{\mu^{\frac{3}{4}}}{\sqrt{2}\mathcal{E}_2^{\frac{1}{4}}}(2\sqrt{\mathcal{E}_2} + \sqrt{\mu}), \quad (4.26)$$

and it becomes, for a small mass $\mu \ll \mathcal{E}_2$,

$$E_{\rm cm}(x = x_3^H = x_1^+) \simeq 2^{1/2}\mathcal{E}_2^{1/4}\mu^{3/4}. \quad (4.27)$$

C. Test shell limit

Here we consider test shell limits for shells 1 and 2 $\mu_1 \to 0$ and $M_1 \to \mathcal{M}_- \equiv M_2$ with $\mathcal{E}_1 = (\text{constant}) \neq 0$ and $\mu_2 \to 0$ and $M_2 \to \mathcal{M}_+ \equiv M_3$ with $\mathcal{E}_2 = (\text{constant}) \neq 0$, respectively. Notice $x_a^H \to x_{a+1}^H$ in the test shell limit for shell a. From Eqs. (4.18) and (4.19), the CM energy of the shells with the equal mass μ at $x = x_1^+$ in the test shell limits for shells 1 and 2 is obtained as

$$\frac{E_{\rm cm}^2(x_1^+)}{2\mu^2} = 1 + \frac{\frac{\mu}{4} + \mathcal{E}_2}{|\frac{\mu}{4} - \mathcal{E}_1|} \to 1 + \frac{\mathcal{E}_2}{\mathcal{E}_1}. \quad (4.28)$$

We realize that it is corresponds to the CM energy (2.26) and (2.27) of the particle collision at $x = x_1^+$ with an equal mass $m \equiv m_1 = m_2$ given by

$$\frac{E_{\rm cm}^2(x_1^+)}{2m^2} = 1 + \frac{e_2}{e_1}. \quad (4.29)$$

V. DISCUSSION AND CONCLUSION

We have considered the rear-end collision of two dust thin shells in the rotating BTZ spacetime to investigate the effects of the self-gravity of colliding objects on the high energy collision. The shells divide the BTZ spacetime into three domains and the domains are matched by Darmois-Israel's method. From the junction condition, all the domains must have the same angular momenta J. The angular momenta imply that the shells and domains corotate.

We have revealed that there are two effects of the self-gravity of thin shells. First, we have shown that the mass of inner shell affects its critical condition (4.12). Second, the position of the event horizon changed from x_1^H to x_3^H because of the masses of two shells.

We have considered the shell collision in five cases according to the value of J. The cases I ($J < \ell M_1$), II ($J = \ell M_1$), and III ($J = \ell M_2$) would be especially interesting cases because of following reasons. The case I is a usual astrophysical situation as a black hole subextremely rotates and two objects collide near its event horizon. In case II, the black hole extremely rotates initially and the collision of two falling shells corresponds with the BSW collision of two particles with an arbitrary high CM energy in the extremal black hole spacetime. In case III, inner shell 1 can be satisfied the critical condition (4.12) that the effective potential for shell 1 becomes $V_1(x_2^H) = V_1'(x_2^H) = 0$ on the extremal event horizon $x = x_2^H$ as with the BSW process in extremal black hole spacetimes [3,4]. In cases I-III, the CM energy of the shells can be arbitrarily large if inner shell 1 satisfies the critical condition (4.12) and if outer shell 2 does not. However, an observer outside the event horizon x_3^H cannot see the products of the collision with the arbitrary large CM energy because it occurs inside the event horizon x_3^H.

If a shell collision occurs in a region $x \geq x_3^H$, an observer who is outer than the collisional point may see products of the collision. We have obtained the finite upperbound of the CM energy of the collision there. Finally, we have concluded that the self-gravity of colliding objects suppresses its CM energy and the observer can only see the suppressed collision.

We have also found a test shell limit. We have shown that the CM energy and the effective potentials for shells in the test shell limit are very similar to the ones of particles. The test shell limit would help us to understand the effect of the self gravity of the thin shells on the collisions.

We have considered only simple shell collisions on this paper. We hope that our paper stimulates further work on shell collisions and that researchers will investigate more realistic cases in the future.

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APPENDIX: REAR-END COLLISION IN THE OVERSPINNING BTZ SPACETIME

In this Appendix, we consider a rear-end collision of two particles with vanishing conserved momenta $L_1 = L_2 = 0$ in the overspinning BTZ spacetime with $J > \ell M$.

From Eq. (2.18), the effective potential of particle a takes the minimum value at $x = x^m$ as

$$V_a(x^m) = -e_a^2 + X, \quad (A1)$$

where

$$X \equiv f(x^m) = \frac{J}{\ell} - M > 0. \quad (A2)$$

Particle a with the nonpositive effective potential $V_a(x)$ can be within a region $x_a^- \leq x \leq x_a^+$ if a condition

$$X \leq e_a^2, \quad (A3)$$

is satisfied.

The CM energy monotonically decreases from $E_{\rm cm}^{\max}$ (2.27) to

$$E_{\rm cm}(x^m) = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \left[\frac{e_1 e_2 - \sqrt{(e_1^2 - X)(e_2^2 - X)}}{X}\right]}, \tag{A4}$$

as x increases from x_a^- to x^m. It monotonically increases and reaches $E_{\rm cm}^{\max}$ as x increases from x^m to x_a^+.

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