Entropy maximization in the emergent gravity paradigm

P. B. Krishna[*] and Titus K. Mathew[†]

Department of Physics, Cochin University of Science and Technology, Kochi, Kerala 682022, India

The accelerated expansion of the Universe can be interpreted as a quest for satisfying holographic equipartition. It can be expressed by a simple law, $\Delta V = \Delta t (N_{\text{surf}} - N_{\text{bulk}})$ which leads to the standard Friedmann equation. This novel idea suggested by Padmanabhan in the context of general relativity has been generalized by Cai and Yang *et al.* to Gauss-Bonnet and Lovelock gravities for a spatially flat universe in different methods. We investigate the consistency of these generalizations with the constraints imposed by the maximum entropy principle. Interestingly, both these generalizations imply entropy maximization even if their basic assumptions are different. Further, we analyze the consistency of Verlinde's emergent gravity with the maximum entropy principle in the cosmological context. In particular, we consider the generalization suggested by Shu and Gong, in which an energy flux through the horizon is assumed, in addition. Even though the conceptual formulations are different, these two emergent perspectives of gravity describes a universe which behaves as an ordinary macroscopic system. Our results provide further support to the emergent gravity paradigm.

I. INTRODUCTION

The deep connection between gravitational dynamics and thermodynamics motivates the emergent interpretation of gravity. Such a connection was realized after the discovery of black hole thermodynamics by Bekenstein and Hawking [1–4]. A great step in this field was put forward by Jacobson. He obtained Einstein's field equations from the fundamental Clausius relation on a local Rindler causal horizon [5]. Following this, various schemes for relating gravity and thermodynamics were discussed for a variety of gravity theories [6,7]. Later, in [8] Padmanabhan showed that Newton's law of gravity can be derived by combining the equipartition law of energy for the horizon degrees of freedom (d.o.f.) and thermodynamic relation $S = E/2T$, where S and T are the entropy and temperature of the horizon and E is the active gravitational mass.

Verlinde introduced gravity as an entropic force caused due to the changes in entropy associated with the positions of material bodies [9]. He derived Newton's law of gravitation and Einstein's field equations using the holographic principle and equipartition law of energy. Using this idea, the authors of [10,11] derived Friedmann equations through different methods. In [12], Miao and Wang discussed the implications of holographic dark energy in the entropic force frame work. In a recent work, Verlinde explains the possibility of a common origin for dark matter

and dark energy [13]. This entropic force formalism also attracted a lot of investigations [14–19].

Most of these studies treat the gravitational field as an emergent phenomenon assuming the spacetime background as pre-existing. Recently, a more elegant way to view gravity as an emergent phenomenon was suggested by Padmanabhan where spacetime itself is considered as an emergent structure [20]. However it is conceptually difficult to think of time as being emerged from some pregeometric variables. Also it is hard to imagine the space around finite gravitating systems as emergent. But, Padmanabhan argued that these difficulties disappear in the cosmological context, when we choose the time variable as the proper time of the geodesic observers to whom the cosmic microwave background radiation appears homogeneous and isotropic. Thus the spatial expansion of the universe can be described as the emergence of cosmic space with the progress of cosmic time. He successfully derived the Friedmann equation of a flat Friedmann-Lamaitre-Robertson-Walker (FLRW) universe in general relativity, using this new idea.

Cai generalized Padmanabhan's proposal to a higher $n + 1$ dimensional spacetime. By properly modifying the d.o.f. and the volume increase, he also obtained the Friedmann equation of a flat FLRW universe in Gauss-Bonnet and Lovelock gravity [21]. With some modifications, this procedure was extended by Sheykhi to derive the dynamical equations of the universe with a spatial curvature [22]. Following this, Ali arrived at the Friedmann equations by considering a general form of entropy [23]. Another generalization was suggested by Yang *et al.* for the flat universe [24] which is further extended by

[*]krishnapb@cusat.ac.in
[†]titus@cusat.ac.in

Wen-Yuan Ai *et al.* for the nonflat case [25]. Instead of modifying the d.o.f., in [24,25] the change in Hubble volume is assumed to be proportional to a more general function $f(\Delta N, N_{\text{surf}})$, where $\Delta N = N_{\text{surf}} - N_{\text{bulk}}$. The authors of [26], extended Padmanabhan's proposal to a non flat universe by using the appropriate invariant volume instead of the volume of the sphere in a flat space. In Ref. [27], Padmanabhan's conjuncture and its modified versions were obtained from Friedmann equations. This paper also discusses the difficulties in generalizing Padmanabhan's proposal to a nonflat universe. In [28], Friedmann equations are extracted in the brane world scenarios. This emergent paradigm has been further explored by Padmanabhan and his collaborators from a variety of perspectives [29–31]. For recent investigations on this novel idea, see [32–34].

As is well known, every macroscopic system evolves to a state of thermodynamic equilibrium consistent with their constraints [35]. The entropy of such systems should attain a certain maximum value in the long run. In [36], it is shown that our universe with a Hubble expansion history behaves as an ordinary macroscopic system. In a previous work, we have proved the equivalency of holographic equipartition law and the maximum entropy principle in the context of general relativity [37]. In this paper we study the consistency of the generalized holographic equipartition in [21,24] with the maximum entropy principle for a spatially flat FLRW universe. It is also our great interest to see whether Verlinde's hypothesis implies entropy maximization. We analyze Verlinde's proposal in the cosmological context following the method in [11], suggested by Shu and Gong and will check its consistency with the maximum entropy principle.

The paper is organized as follows. In the upcoming section, we obtain the constraints imposed by the generalized second law (GSL) and the maximum entropy principle for a spatially flat FLRW universe in $n + 1$ dimensional Einstein's gravity, Gauss-Bonnet gravity and Lovelock gravity. We will also prove the consistency of an asymptotically de Sitter universe with these constraints obtained. In Sec. III, we analyze whether the modified equipartition law in [21,24] ensures the maximization of entropy. In Sec. IV, we will check the consistency of Verlinde's entropic force formalism with the maximization entropy principle in the cosmological context. Section V compares Padmanabhan's proposal with Verlinde's hypothesis and Sec. VI is devoted to our conclusions.

II. MAXIMIZATION OF ENTROPY

An ordinary, isolated macroscopic system spontaneously evolves to an equilibrium state of maximum entropy consistent with their constraints. This implies,

$$\dot{S} \geq 0, \text{always} \tag{1}$$

$$\ddot{S} < 0 \text{ at least in long run}, \tag{2}$$

where "S" denotes the total entropy of the universe and dots indicate the derivatives with respect to a relevant variable like cosmic time. Based on the Hubble expansion history, Pavan and Radicella have shown that our universe behaves as an ordinary macroscopic system that proceeds to a maximum entropy state [36]. In this section we extend this procedure to $n + 1$ dimensional Einstein's gravity, Gauss Bonnet gravity and Lovelock gravity for a spatially flat universe.

The total entropy of the universe, S can be approximated as the horizon entropy since the entropy contribution from all other components is negligibly small [38]. Using Bekenstein result, the horizon entropy can be expressed as [1,3],

$$S = \frac{A_{n+1}}{4L_p^{n-1}}, \tag{3}$$

where $A = n\Omega_n/H^{n-1}$, for $n \geq 3$, with Ω_n being the volume of the unit n-sphere. Here L_p represents the Plank length and H is the Hubble parameter. The rate of change of entropy with respect to the cosmic time is,

$$\dot{S} = -\frac{n(n-1)\Omega_n}{4L_p^{n-1}} \frac{\dot{H}}{H^n}. \tag{4}$$

Since the Hubble parameter H is always positive for an expanding universe, the horizon entropy will not decrease, if $\dot{H} \leq 0$. The measurements on the Hubble parameter [39,40] and the numerical simulations [41,42] have confirmed that, $\dot{H} < 0$. Hence the entropy of the universe will never decrease. Even though this implies a possible thermodynamic evolution, whether it attains a state of equilibrium or not, is determined by the constraint on the second derivative of the entropy. The system approaches an equilibrium state, if the entropy corresponding to it is maximum. A maximum entropy state is characterized by the inequality, $\ddot{S} < 0$ to be satisfied at least in the long run. From Eq. (4), we have the second derivative of entropy,

$$\ddot{S} = \frac{n(n-1)\Omega_n}{4L_p^{n-1}} \left[\left(\frac{n\dot{H}^2}{H^{n+1}} \right) - \left(\frac{\ddot{H}}{H^n} \right) \right]. \tag{5}$$

Then, one can immediately find the constraint for entropy maximization,

$$n\left(\frac{\dot{H}^2}{H^{n+1}} \right) < \left(\frac{\ddot{H}}{H^n} \right), \tag{6}$$

in the asymptotic limit. As per the observational data, we have, $\ddot{H} > 0$ and $\dot{H} \rightarrow 0$ in the asymptotic limit [39–42] and thus the above inequality holds true for an expanding universe. This shows the consistency of a $n + 1$ dimensional flat FLRW universe in Einstein's gravity with the maximum entropy principle.

So far we have discussed the conditions for entropy maximization in Einstein's gravity, where the horizon entropy follows the area law. But in Gauss-Bonnet and Lovelock gravity, horizon entropy takes more complex form. Hence it is worth noting, the conditions for entropy maximization in those gravity theories.

The Gauss-Bonnet gravity is a natural extension of Einstein's gravity such that, the Gauss-Bonnet action consist of second order terms, in addition. The entropy-area law of the spherically symmetric and static black hole in Gauss-Bonnet gravity is assumed to be satisfied for the FRW universe also. Following this, the entropy of the Hubble horizon can be expressed as [43,44],

$$S = \frac{A}{4L_p^{n-1}}\left[1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{H^{-2}}\right]. \qquad (7)$$

Here $A = n\Omega_n/H^{n-1}$ and $\tilde{\alpha} = (n-2)(n-3)\alpha$, where α is the Gauss Bonnet coefficient which is positive [45]. Then, the rate of change of entropy can be calculated from Eq. (7) as,

$$\dot{S} = -\frac{n(n-1)\Omega_n}{4L_p^{n-1}H^n}(1 + 2\tilde{\alpha}H^2)\dot{H}, \qquad (8)$$

where $n \geq 4$. Since $\dot{H} \leq 0$, the generalized second law will be satisfied. When $t \rightarrow \infty$, $\dot{H} \rightarrow 0$ and $\dot{S} \rightarrow 0$ indicating a state of equilibrium in the final stage.

Taking the time derivative of Eq. (8), we find

$$\ddot{S} = \frac{n(n-1)\Omega_n}{4L_p^{n-1}H^{n+1}}[\dot{H}^2[n + (2n-4)\tilde{\alpha}H^2] - [H\ddot{H}(1 + 2\tilde{\alpha}H^2)]]. \qquad (9)$$

Since $\ddot{H}$ is always positive as per the observational data, the entropy maximization demands,

$$|\dot{H}^2[n + (2n-4)\tilde{\alpha}H^2]| < |H\ddot{H}(1 + 2\tilde{\alpha}H^2)| \qquad (10)$$

in the final stage. Since $\dot{H} \rightarrow 0$, in the asymptotic limit the above inequality holds and the entropy of the apparent horizon will never grow unbounded.

We will now move to the more general Lovelock gravity. Lovelock gravity [46] is a generalization of Gauss-bonnet gravity, such that the Lagrangian consists of dimensionally extended Euler densities. The entropy of the horizon in Lovelock gravity is assumed to be of the same form as the entropy of black hole and is given by,

$$S = \frac{A}{4L_p^{n-1}}\sum_{i=1}^{m}\frac{i(n-1)}{(n-2i+1)}\hat{c}_iH^{2i-2} \qquad (11)$$

where $m = n/2$ and the coefficients $\hat{c}_i$ are given by $\hat{c}_0 = \frac{c_0}{n(n-1)}$, $\hat{c}_1 = 1$, $\hat{c}_i = c_i\prod_{j=3}^{m}(n+1-j)i > 1$. One can

obtain the rate of change of horizon entropy in Lovelock gravity from Eq. (11) as,

$$\dot{S} = -\frac{n(n-1)\Omega_n\dot{H}}{4L_p^{n-1}H^{n+2}}\sum_{i=1}^{m}i\hat{c}_iH^{2i}. \qquad (12)$$

Since $\dot{H} \leq 0$ for an asymptotically de Sitter universe, the above equation ensures its consistency with the generalized second law. Also when $t \rightarrow \infty$, $\dot{H} \rightarrow 0$, indicating a state of equilibrium in the final stage. Now we will consider the second derivative of entropy by differentiating Eq. (12),

$$\ddot{S} = \frac{n(n-1)\Omega_n}{4L_p^{n-1}}\sum_{i=1}^{m}i\hat{c}_iH^{2i}\left[(n+2-2i)\frac{\dot{H}^2}{H^{n+3}} - \frac{\ddot{H}}{H^{n+2}}\right]. \qquad (13)$$

Since $\ddot{H}$ is always positive, the horizon entropy tends to some maximum value if,

$$\left|\sum_{i=1}^{m}i\hat{c}_iH^{2i}(n+2-2i)\frac{\dot{H}^2}{H^{n+2}}\right| < \left|\sum_{i=1}^{m}i\hat{c}_iH^{2i}\frac{\ddot{H}}{H^{n+2}}\right| \qquad (14)$$

in the long run. Since $\dot{H} \rightarrow 0$, when $t \rightarrow \infty$, the above inequality holds true in the last stage of evolution indicating the entropy maximization.

Here we have discussed the constraints imposed by the GSL and the maximum entropy principle for a flat FLRW universe in $n+1$ dimensional Einstein, Gauss Bonnet and Lovelock gravity. These constraints are generally satisfied by an asymptotically de Sitter universe. This entropy maximization in the final de Sitter epoch has been already emphasized in several previous studies [47–49] from different perspectives. It is worth mentioning that in Ref. [50], the rise of complexity content and the validity of GSL demands an asymptotically de Sitter universe with equation of state parameter, $\omega \geq -1$. On the other hand, in [51,52], the authors discuss the thermodynamic motivation for the existence of dark energy. The dark energy component also seems unavoidable for the attainment of equilibrium in the braneworld scenario [53]. Since the attainment of equilibrium demands an asymptotically de Sitter universe, our results also point in the same direction.

III. HOLOGRAPHIC EQUIPARTITION AND ENTROPY MAXIMIZATION

Our aim here is to see whether the generalized holographic equipartition in [21,24] leads to the maximization of entropy. In [37], it is shown that the holographic equipartition law suggested by Padmanabhan for a flat universe effectively implies entropy maximization in the context of Einstein's gravity. In this section we extend this procedure for a spatially flat FLRW universe in $n+1$ Einstein's gravity, Gauss Bonnet gravity and Lovelock gravity.

A. Friedmann equations from emergence of space

Here we will briefly review the necessary background to which our work is closely related. We start with Padmanabhan's idea of holographic equipartition which is followed by a brief discussion on the generalized holographic equipartition law in [21,24].

Padmanabhan observed that, a pure de Sitter universe with a Hubble constant H obeys holographic principle in the form,

$$N_{\text{surf}} = N_{\text{bulk}}. \tag{15}$$

Here N_{surf} is the number of d.o.f. on the Hubble horizon with radius H^{-1} and is given by,

$$N_{\text{surf}} = \frac{4\pi}{L_p^2 H^2}. \tag{16}$$

where L_p^2 is the Plank area, attributed to one d.o.f. N_{bulk} denotes the effective number of d.o.f. residing in the region enclosed by the horizon and is given by,

$$N_{\text{bulk}} = \frac{|E|}{\frac{1}{2}k_B T}. \tag{17}$$

where $|E| = |\rho + 3p|V$, the Komar energy contained inside the Hubble volume, $V = \frac{4\pi}{3H^3}$; k_B, the Boltzmann constant and $T = \frac{H}{2\pi}$, the Gibbon's Hawking temperature. From the above equations, one could reach at the condition, $|E| = \frac{1}{2}N_{\text{surf}}k_B T$. This equality can be called as holographic equipartition since it relates the d.o.f. in the bulk region of space, determined by the equipartition condition to the d.o.f. on its boundary surface.

Even though the present universe is not exactly de Sitter, many of the astronomical observations indicate that it is proceeding to a pure de Sitter state. Based on these facts, it is suggested that the accelerated expansion of the universe can be explained as a quest for satisfying holographic equipartition. It can be expressed by a simple law,

$$\frac{dV}{dt} = L_p^2 (N_{\text{surf}} - N_{\text{bulk}}). \tag{18}$$

where V is again the Hubble volume and t is the cosmic time in Planck units ($k_B = c = \hbar = 1$). Substituting for each term in (18) and by integrating with the help of continuity equation one gets the standard Friedmann equation.

We will now turn our attention to the generalized holographic equipartition law suggested by Cai [21]. For a spatially flat FLRW universe in $n + 1$ dimensional space time, the surface d.o.f. can be defined as,

$$N_{\text{surf}} = \frac{\alpha A}{L_p^{n-1}} \tag{19}$$

where $A = n\Omega_n / H^{n-1}$, and

$$N_{\text{bulk}} = \frac{-4\pi\Omega_n}{H^{n+1}} \frac{(n-2)\rho + np}{n-2}. \tag{20}$$

Also the equipartition law given in (18) is modified for an $(n + 1)$ dimensional space as,

$$\alpha \frac{dV}{dt} = L_p^{n-1} (N_{\text{surf}} - N_{\text{bulk}}). \tag{21}$$

where, $V = \Omega_n / H^n$, the volume of the n-sphere. Substitution of the d.o.f. in Eqs. (19) and (20) in the holographic equipartition law given in (21) gives the Friedmann equation of the flat FLRW universe in $n + 1$ dimensional space time. Now we will briefly describe the extension of the above method to Gauss-Bonnet gravity and more general Lovelock gravity.

In Gauss-Bonnet gravity, the effective area corresponding to the horizon entropy can be defined from Eq. (7) as,

$$\tilde{A} = A\left[1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{H^{-2}}\right]. \tag{22}$$

The corresponding increase in effective volume is given by,

$$\frac{d\tilde{V}}{dt} = -\frac{n\Omega_n}{H^{n+1}}(1 + 2\tilde{\alpha}H^2)\dot{H}. \tag{23}$$

From the above expression, the d.o.f. at the apparent horizon can be assumed as,

$$N_{\text{surf}} = \alpha \frac{n\Omega_n}{H^{n+1}L_p^{n-1}}(H^2 + \tilde{\alpha}H^4). \tag{24}$$

The d.o.f. in the bulk is still given by (20). Then, from Eq. (21) the Friedmann equation of the flat FLRW universe in Gauss-Bonnet gravity can be derived.

In Lovelock gravity, the increase in effective volume can be calculated from (11) as,

$$\frac{d\tilde{V}}{dt} = -\frac{n\Omega_n}{H^{n+3}}\left(\sum_{i=1}^m i\hat{c}_i H^{2i}\right)\dot{H}. \tag{25}$$

From the above equation the surface d.o.f. can be defined as,

$$N_{\text{surf}} = \alpha \frac{n\Omega_n}{H^{n+1}L_p^{n-1}}\sum_{i=1}^m \hat{c}_i H^{2i}. \tag{26}$$

As described earlier from (20), (25), (26), and (21) one can arrive at the Friedmann equation of a flat FLRW universe in Love1ock gravity.

Inspired by this work, Yang *et al.* proposed another generalization of Padmanabhan's holographic equipartition law [24]. In this case, the surface d.o.f. on the Hubble surface is assumed to be proportional to the area of the surface regardless of the gravity theory while the bulk d.o.f. obeys the equipartition law of energy. Here the emergence of cosmic space is described by a general form of a dynamical equation,

$$\frac{dV}{dt} = L_p^{n-1} f(\Delta N, N_{\text{surf}}),$$ (27)

where, $V = \Omega_n/H^n$, irrespective of the gravity theory and $\Delta N = N_{\text{surf}} - N_{\text{bulk}}$. The authors have used the relations in (19) and (20) for defining N_{surf} and N_{bulk} respectively in an $n + 1$ dimensional space time. If $f(\Delta N, N_{\text{surf}})$ is chosen to be the most simplest form $f(\Delta N) = \frac{\Delta N}{\alpha}$, as in [8,21], one could arrive at the Friedmann equations for a flat universe in Einstein's gravity. In order to derive the Friedmann equation in Gauss-Bonnet gravity one has to choose,

$$f(\Delta N, N_{\text{surf}}) = \frac{\Delta N/\alpha + \tilde{\alpha} K (N_{\text{surf}}/\alpha)^{1+\frac{2}{1-n}}}{1 + 2\tilde{\alpha} K (N_{\text{surf}}/\alpha)^{\frac{2}{1-n}}},$$ (28)

where $K = (n\Omega_n/L_p^{n-1})^{\frac{2}{n-1}}$ and $\tilde{\alpha}$ is a parameter with length dimension 2. If $f(\Delta N, N_{\text{surf}})$ is assumed to be a more general function,

$$f(\Delta N, N_{\text{surf}}) = \frac{\Delta N/\alpha + \sum_{i=2}^{m} \tilde{c}_i K_i (N_{\text{surf}}/\alpha)^{1+\frac{2i-2}{1-n}}}{1 + \sum_{i=2}^{m} i \tilde{c}_i K_i (N_{\text{surf}}/\alpha)^{\frac{2i-2}{1-n}}},$$ (29)

where $K_i = (n\Omega_n/L_p^{n-1})^{\frac{2i-2}{n-1}}$, $m = [n/2]$ and $\tilde{c}_i$ are some coefficients with the length dimension $(2i - 2)$; one can arrive at the Friedmann equation of the $n + 1$ dimensional spatially flat FLRW universe in Lovelock gravity.

B. Holographic equipartition and Entropy maximization: Analysis of Cai's proposal

Let us consider a spatially flat $n + 1$ dimensional universe in Einstein's gravity. The time derivative of the cosmic volume, $V = \Omega_n/H^n$, can be calculated as,

$$\frac{dV}{dt} = -n\Omega_n \frac{\dot{H}}{H^{n+1}}.$$ (30)

Now recalling Eq. (4), we can write

$$\frac{dV}{dt} = \frac{4L_p^{n-1}}{H(n-1)} \dot{S}.$$ (31)

Then, holographic equipartition law in Eq. (21), can be written as,

$$\dot{S} = \frac{(n-2)H}{2}(N_{\text{surf}} - N_{\text{bulk}}).$$ (32)

For an expanding universe, we have $\frac{dV}{dt} \geq 0$, which demands

$$N_{\text{surf}} - N_{\text{bulk}} \geq 0.$$ (33)

This in turn ensures the non-negativity of the r.h.s. of Eq. (32). Consequently the generalized second law in the form $\dot{S} \geq 0$ will be satisfied.

Taking the differential of (32), we get

$$\ddot{S} = \frac{(n-2)\dot{H}}{2}(N_{\text{surf}} - N_{\text{bulk}}) + \frac{(n-2)H}{2}$$
$$\times \frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}).$$ (34)

We have in the asymptotic limit, $N_{\text{bulk}} \to N_{\text{surf}}$ and the first term in the above expression vanishes. As per the holographic equipartition law, the evolution of the universe can be explained as a tendency to equalize the d.o.f. on the horizon and that in the bulk. In other words, one can say that the universe is trying to minimize the holographic discrepancy with the progress of time. Since N_{bulk} cannot exceed N_{surf}, we have "$N_{\text{surf}} - N_{\text{bulk}}$," always positive and tending to zero in the asymptotic limit. This in turn implies that

$$\frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}) < 0$$ (35)

which guarantees the nonpositivity of $\ddot{S}$ in the long run, ensuring the consistency with the maximum entropy principle. Also, substituting (19) and (20) in (34) we get the condition for $\ddot{S} < 0$ as,

$$n\left(\frac{\dot{H}^2}{H^{n+1}}\right) < \left(\frac{\ddot{H}}{H^n}\right),$$ (36)

This is nothing but the constraint in Eq. (6) we obtained in the last section for the entropy maximization which is satisfied by an asymptotically de Sitter universe.

Next, we will investigate whether the holographic equipartition law proposed in the context of Gauss-Bonnet gravity implies entropy maximization. Combining (23) and (8), one can relate the rate of change of effective volume within the apparent horizon to the rate of change of entropy as,

$$\frac{d\tilde{V}}{dt} = \frac{4L_p^{n-1}}{H(n-1)} \dot{S}.$$ (37)

Note that the above equation is in the same form of Eq. (31) with V and $\dot{S}$ are replaced with the corresponding expressions

in Gauss-Bonnet gravity. Hence, the holographic equipartition law in Eq. (21) can be rewritten as,

$$\dot{S} = \frac{(n-2)H}{2}(N_{\text{surf}} - N_{\text{bulk}}). \qquad (38)$$

just as in the case before, where N_{surf} and N_{bulk} are given by Eq. (24) and Eq. (20) respectively. For an expanding universe, we have $\frac{dV}{dt} \geq 0$, which demands $N_{\text{surf}} - N_{\text{bulk}} \geq 0$. Hence the above equation guarantees the consistency of the universe with the generalized second law. Now we have the second derivative of entropy,

$$\ddot{S} = \frac{(n-2)\dot{H}}{2}(N_{\text{surf}} - N_{\text{bulk}}) + \frac{(n-2)H}{2}$$
$$\times \frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}) \qquad (39)$$

as we have seen earlier. Here also, the first term vanishes in the asymptotic limit and the entropy tends to a certain maximum value if, $\frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}) < 0$.

Since, the universe is trying to decrease the holographic discrepancy with the progress of time, the above inequality holds and the entropy gets saturated in the asymptotic limit. Substituting the Eqs. (24) and (20) in (39), we get the constraint for the nonpositivity of $\ddot{S}$,

$$|\dot{H}^2[n + (2n-4)\tilde{\alpha}H^2]| < |H\ddot{H}(1 + 2\tilde{\alpha}H^2)| \qquad (40)$$

which is same as the inequality in (10) that we obtained earlier for the entropy maximization in Gauss-Bonnet gravity.

Let us now generalize this procedure to Lovelock gravity. Recalling the Eqs. (21), (25), and (12), the holographic equipartition law in Lovelock gravity can be expressed as,

$$\dot{S} = \frac{(n-2)H}{2}(N_{\text{surf}} - N_{\text{bulk}}), \qquad (41)$$

where N_{surf} and N_{bulk} are given by Eq. (26) and Eq. (20) respectively. Note that the above relation takes the same form as in the Einstein's and Gauss-Bonnet gravity. We have $N_{\text{surf}} - N_{\text{bulk}} \geq 0$ for an expanding universe which guarantees the consistency with the generalized second law, $\dot{S} \geq 0$. The second derivative of entropy takes the form,

$$\ddot{S} = \frac{(n-2)\dot{H}}{2}(N_{\text{surf}} - N_{\text{bulk}}) + \frac{(n-2)H}{2}$$
$$\times \frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}). \qquad (42)$$

Here, $\ddot{S}$ will be negative in the asymptotic limit, if $\frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}) < 0$, just as in the previous case. As the holographic discrepancy is a decreasing function of time, the

above inequality will be satisfied in the long run. Substituting (26) and (20) in Eq. (42), we obtained the constraint for entropy maximization,

$$\left|\sum_{i=1}^{m} i\hat{c}_i H^{2i}(n + 2 - 2i)\frac{\dot{H}^2}{H^{n+2}}\right| < \left|\sum_{i=1}^{m} i\hat{c}_i H^{2i}\frac{\ddot{H}}{H^{n+2}}\right|. \qquad (43)$$

As expected, this is same as the constraint (14) we have obtained for the entropy maximization in Lovelock gravity in the previous section. Thus, the validity of the holographic equipartition law ensures the validity of maximum entropy principle. The tendency for satisfying the holographic equipartition can be explained as a tendency for maximizing entropy in Einstein, Gauss Bonnet and Lovelock gravity theories for a flat FLRW universe. Since the law of emergence has the same form in Cai's proposal irrespective of the gravity theories, the Eq. (32), (38), (41) and their derivatives take the same form, although the definitions of each term in those equations are different. The above discussions strengthens the deep connection between the emergence of space and the entropy maximization.

C. Holographic equipartition and entropy maximization: Analysis of Yang *et al.*'s proposal

Even though Cai obtained Friedmann equations for a flat universe in Gauss-Bonnet and Lovelock gravity, his work was criticized for using effective volume for the volume change and plain ordinary volume for defining the bulk d.o.f. In order to overcome this discrepancy Yang *et al.* used the plain ordinary volume for defining both the rate of emergence and the bulk d.o.f. [24]. In the context of general relativity, the generalized holographic law given in Eq. (27) is not different from Cai's proposal. Then, as we have seen earlier the holographic discrepancy, ΔN vanishes in the long run ensuring the consistency with the generalized second law and the maximum entropy principle. On the other hand, in Gauss-Bonnet and Lovelock gravity,"ΔN" is generally nonvanishing. Even in the final de Sitter state, the holographic discrepancy will not be zero. In this case, the emergence of cosmic space could not be explained as a tendency for equalizing the d.o.f. Hence it is worth investigating whether this generalization fulfills the generalized second law and the maximum entropy principle.

Now, from Eqs. (8), (27), and (28), the rate of change of entropy with respect to the cosmic time can be expressed as,

$$\dot{S} = \frac{(n-1)H}{4}(1 + 2\tilde{\alpha}H^2)f(\Delta N, N_{\text{surf}}). \qquad (44)$$

Since $dV/dt \geq 0$, for an expanding universe, Eq. (27) guarantees the non-negativity of $f(\Delta N, N_{\text{surf}})$ and thus ensures the consistency with the generalized second law. Differentiating the above equation, we get

$$\ddot{S} = \frac{(n-1)}{4} \frac{d}{dt} (H(1 + 2\tilde{\alpha}H^2)) f(\Delta N, N_{\text{surf}})$$
$$+ \frac{(n-1)}{4} H(1 + 2\tilde{\alpha}H^2) \frac{d}{dt} (f(\Delta N, N_{\text{surf}})). \quad (45)$$

In order to fulfill the maximum entropy principle, the right-hand side (r.h.s.) of the above equation must be negative in the long run. In the asymptotic limit, when $t \to \infty$, the rate of emergence of space, dV/dt will tend to zero. Hence as per equation (27), $f(\Delta N, N_{\text{surf}}) \to 0$ and the first term in the above expression vanishes. Since the rate of emergence is always positive, tending to zero in the long run, from Eq. (27), we get

$$\frac{d}{dt} (f(\Delta N, N_{\text{surf}})) < 0 \quad (46)$$

in the final stage. This guarantees the nonpositivity of "$\ddot{S}$" in the last stage of evolution and thus ensures the consistency with the maximum entropy principle.

In Lovelock gravity, from Eqs. (12), (27), and (29), the change in entropy can be obtained as,

$$\dot{S} = \frac{(n-1)}{4} \sum_{i=1}^{m} i\hat{c}_i H^{2i-1} f(\Delta N, N_{\text{surf}}). \quad (47)$$

and its derivative,

$$\ddot{S} = \frac{(n-1)}{4} \frac{d}{dt} \left(\sum_{i=1}^{m} i\hat{c}_i H^{2i-1} \right) f(\Delta N, N_{\text{surf}})$$
$$+ \frac{(n-1)}{4} \sum_{i=1}^{m} i\hat{c}_i H^{2i-1} \frac{d}{dt} (f(\Delta N, N_{\text{surf}})). \quad (48)$$

As per the earlier arguments, the above equations guarantee the validity of the generalized second law and the maximum entropy principle. What is striking in our result is that even if the law governing the emergence of space in [24] does not lead to the condition $N_{\text{surf}} = N_{\text{bulk}}$, it guarantees the validity of the generalized second law and the maximum entropy principle. In short, a flat FLRW universe that obeys the generalized holographic equipartition law in [21,24], behaves as an ordinary macroscopic system in the context of Einstein, Gauss-Bonnet, and Lovelock gravity. In the light of above discussions, the achievement of holographic equipartition could be interpreted as the attainment of maximum entropy in a spatially flat universe.

IV. ENTROPY MAXIMIZATION IN VERLINDE'S EMERGENT GRAVITY

In [9], Verlinde introduced the treatment of gravity as an emergent phenomenon. He interprets gravity as an entropic force experienced by a material body when it approaches a holographic screen. We, here, consider some of the arguments in [9] and analyze the consistency of this proposal with the generalized second law and the maximum entropy principle in the cosmological context.

In [9], the number of bits of information on a holographic screen of area A is assumed as,

$$N = \frac{Ac^3}{G\hbar}. \quad (49)$$

Then, from the equipartition law, the total energy of the system can be calculated as,

$$E = \frac{1}{2} N k_B T \quad (50)$$

where T is the temperature on the screen. The energy E in the above expression is assumed to be equal to Mc^2 where M represents the mass that would emerge on the part of space enclosed by the screen. Verlinde arrive at Newton's law of gravitation and Einstein's field equations from these postulates.

By generalizing this proposal to dynamic spacetimes Shu and Gong [11] and Cai et al. [10] derive Friedmann equations using different methods. We assume the Hubble horizon as the boundary of the universe, as earlier, and hence follow the method of [11]. Apart from the equipartition law and the holographic principle the authors of [11] assume an energy flux through the horizon. If there is an energy flux through the horizon, the energy ε enclosed by it will increase in course of time. Consequently, as per the equipartition law, the temperature and the number of bits on the screen changes. If the universe is assumed to be flat, the radius of the apparent horizon will be equal to the Hubble radius, r_H. Then, the change in total energy in an infinitesimal interval of time dt can be expressed as,

$$d\varepsilon = \frac{1}{2} N_H dT_H + \frac{1}{2} T_H dN_H \quad (51)$$

where $N_H = \frac{4\pi r_H^2}{L_p^2}$ and $T_H = \frac{\hbar}{2\pi r_H}$, the Hawking temperature. Here the increase in the number of bits on the Hubble sphere,

$$dN_H = \frac{8\pi r_H}{L_p^2} dr_H \quad (52)$$

and the change in Hawking temperature is,

$$dT_H = -\frac{\hbar}{2\pi r_H^2} dr_H \quad (53)$$

as in [11]. This change in energy, $d\varepsilon$ will be equal to the energy flow through the horizon within a time interval dt which is given by,

$$-dE = d\varepsilon = 4\pi r_H^3 (\rho + p) H dt. \quad (54)$$

Combining Eqs. (51), (52), and (53), we arrive at,

$$\frac{d\varepsilon}{dt} = -\frac{\dot{H}}{L_p^2 H^2}. \tag{55}$$

From the definition of horizon entropy, $S = A/4$ the above relation can be expressed as,

$$T\frac{dS}{dt} = -\frac{\dot{H}}{L_p^2 H^2}. \tag{56}$$

With the help of Friedmann equation, the energy flux through the horizon can be defined from equation (54) as,

$$\frac{d\varepsilon}{dt} = \frac{4\pi^2(1+\omega)}{H^2}\rho. \tag{57}$$

If the universe is assumed to be asymptotically de Sitter, the equation of state, $\omega \to -1$, when $t \to \infty$. In consequence, as per the above equation $\frac{d\varepsilon}{dt} \to 0$. Also for $\omega \geq -1$, $\frac{d\varepsilon}{dt} \geq 0$. Here the universe is trying to minimize the energy flux through the horizon with the progress of cosmic time. In other words, the evolution of the universe can be interpreted as a tendency for minimizing the flux through the horizon. From Eqs. (55) and (56), the rate of change of entropy can be expressed as,

$$\dot{S} = \frac{2\pi}{H}\frac{d\varepsilon}{dt}. \tag{58}$$

Since $\frac{d\varepsilon}{dt} \geq 0$ for $\omega \geq -1$, the law of emergence is consistent with the GSL.

Now, we will check the convexity condition for the maximization of entropy. Differentiating the above equation once again with respect to time, we get

$$\ddot{S} = \frac{8\pi^2}{H^3}(1+\omega)\dot{\rho} - \frac{24\pi^2\dot{H}}{H^4}(1+\omega)\rho + \frac{8\pi^2}{H^3}\dot{\omega}\rho. \tag{59}$$

In an asymptotically de Sitter universe, $\omega \to -1$, as $t \to \infty$ and the first two terms in the above expression vanishes. Since $\dot{\omega}$ is always negative the total entropy will never grow unbounded. Thus, Verlinde's proposal which is generalized in [11] is in agreement with the generalized second law and the maximum entropy principle.

V. PADMANABHAN'S PROPOSAL VS VERLINDE'S HYPOTHESIS

Padmanabhan describes the evolution of the universe as a quest for decreasing the holographic discrepancy [20]. But, based on the arguments in [11], one can interpret the cosmic evolution as a tendency for reducing the energy flux through the horizon. Although the basic assumptions are different, the approaches in [20,11] leads to the same results in the context of cosmology. Both these proposals assure the validity of GSL and the maximum entropy principle.

Moreover, both of them describe a universe that proceeds to a pure de Sitter state and thus demand the presence of dark energy which is not too different from the cosmological constant. The authors of [15], have pointed out a possible connection between Verlinde's and Padmanabhan's arguments by proposing a generalized entropy.

It is worth mentioning that, the authors of [11], assume the conventional first law of black hole dynamics, $TdS = dE$, where dE is the energy flux through the horizon. This energy flux, dE is getting reduced and finally vanishes, ensuring the consistency with the maximum entropy principle. Meanwhile, Padmanabhan's holographic equipartition law could be expressed in the form, $TdS = dE_G + PdV$, where we have an extra term PdV. Here, $E_G = \frac{c^4}{G}\left(\frac{A_H}{16\pi}\right)^{\frac{1}{2}}$, is the energy associated with the horizon of area A_H and PdV is the work function of the matter source. For detailed discussion see [54,55]. When the universe evolves to the final de Sitter state with a constant Hubble parameter, both dE_G and PdV will vanish resulting in the maximization of entropy. But, one could easily reach at the maximum entropy principle directly from the holographic equipartition law [37].

We wish to emphasize that the approach in [11] is slightly different from Verlinde's original proposal, as the authors assume an energy flux through the horizon in addition. Now, following Verlinde's proposal in [9], the total energy can be defined as,

$$E = \frac{1}{2}Nk_BT = Mc^2 \tag{60}$$

where $N = \frac{Ac^3}{G\hbar}$, the d.o.f. on the holographic screen. In an FLRW universe "M" is usually taken as the Komar mass $|\rho + 3p|V$ [10,16], instead of the total mass ρV. Hence, assuming the Hubble horizon as the boundary of the universe the above equation can be written as,

$$N = \frac{|E|}{\frac{1}{2}k_BT}, \tag{61}$$

where $|E| = \frac{4\pi\rho|1+3\omega|}{3H^3}$, the Komar energy inside the Hubble volume. Assuming a thermal equilibrium between the horizon and the fluid inside the horizon, we take $T = H/2\pi$, the Gibbons-Hawking temperature. Also, in Ref. [56], it is argued that even though the radiation can not reach thermal equilibrium with the horizon, nonrelativistic matter may and dark energy might. With the help of Friedmann equation, the Eq. (61) can be expressed as,

$$\frac{4\pi}{L_p^2 H^2} = |1+3\omega|\frac{2\pi c^3}{G\hbar H^2}. \tag{62}$$

In the final de Sitter state $\omega \to -1$ and the r.h.s. of the above equation becomes equal to the l.h.s. Thus, in a pure de Sitter universe, the d.o.f. on the horizon can be defined as,

$$N_{\text{surf}} = \frac{|E|}{\frac{1}{2}k_B T}. \tag{63}$$

This is nothing but Padmanabhan's holographic equipartition condition, since it relates the d.o.f. on the surface to the d.o.f. determined by the equipartition condition.

But, it has to be noted that Eq. (62) holds true only in the final de Sitter state. Generally we have $\omega \geq -1$, throughout the evolution for a universe that tends to a final de Sitter state. For instance, $\omega = \frac{1}{3}$ in the radiation dominated phase, $\omega = 0$ for the matter dominated phase and $\omega = -1$, for the dark energy dominated phase. Hence Eq. (62) should be rewritten as the inequality,

$$\frac{4\pi}{L_p^2 H^2} \geq |1 + 3\omega| \frac{2\pi c^3}{G\hbar H^2}. \tag{64}$$

From the definitions of N_{surf} and N_{bulk}, the above inequality can be expressed as,

$$N_{\text{surf}} \geq N_{\text{bulk}}. \tag{65}$$

In the final de Sitter state N_{bulk} approaches N_{surf} and the holographic equipartition is achieved. Conversely, starting from Padmanabhan's argument $N_{\text{surf}} \geq N_{\text{bulk}}$ one can deduce the inequality,

$$N_{\text{surf}} \geq \frac{|E|}{\frac{1}{2}k_B T}, \tag{66}$$

which reduces to $E = \frac{1}{2}Nk_B T = Mc^2$, in the asymptotic limit. Hence if we assume the Hubble horizon as the boundary of the universe Verlinde's assumption in Eq. (60) can be obtained as a limiting case of Padmanabhan's relation in (65). However, we wish to highlight the fact that the approaches in [9,20] are conceptually different and there exist no equivalency between them [57].

VI. CONCLUSION

In this paper, we investigate the consistency of the generalized holographic equipartition with the maximum entropy principle. In particular, we have considered the generalizations in [21,24], where the authors extended Padmanabhan's proposal to Gauss-Bonnet and Lovelock gravities for a spatially flat universe. We have also analyzed the consistency of Verlinde's entropic force formalism with the maximum entropy principle in the cosmological context.

In [36], Pavon and Radicella have shown that our universe behaves as an ordinary macroscopic system that proceeds to a maximum entropy state. But, their results are restricted to the $3 + 1$ dimensional Einstein's gravity. Hence we first extended the procedure in [36] to $n + 1$ dimensional Einstein, Gauss-Bonnet, and Lovelock gravities

and obtained the constraints for the maximization of entropy. These constraints are generally satisfied by an expanding universe that proceeds to a final de Sitter epoch.

One of our main aim was to check whether the generalized holographic equipartition in [21,24] imply entropy maximization. Following the modified holographic equipartition suggested by Cai in [21], we have found that the condition $\dot{S} \geq 0$ implies $N_{\text{surf}} - N_{\text{bulk}} \geq 0$ and $\ddot{S} < 0$ (in the long run) leads to $\frac{d}{dt}(N_{\text{surf}} - N_{\text{bulk}}) < 0$ in Einstein, Gauss-Bonnet and Lovelock gravity theories. We showed that these conditions are compatible with the respective constraints that we have obtained for entropy maximization in each gravity theory. Thus, an asymptotically de Sitter universe which evolves to minimize the holographic descrepancy $N_{\text{surf}} - N_{\text{bulk}}$ proceeds to a maximum entropy state. On the other hand, following [24], we found that $\dot{S} \geq 0$ leads to $f(\Delta N, N_{\text{surf}}) \to 0$ and $\ddot{S} < 0$ (in the long run) leads to $\frac{d}{dt}(f(\Delta N, N_{\text{surf}})) < 0$ in all these gravity theories. Since the rate of emergence is always positive tending to zero in the final stage, $f(\Delta N, N_{\text{surf}})$ satisfies the above conditions, ensuring the entropy maximization. What is remarkable here is that, even if the law of emergence does not guarantees the condition $N_{\text{surf}} = N_{\text{bulk}}$, it ensures the consistency with the GSL and the maximum entropy principle. The above results provide a thermodynamic basis for the law of emergence beyond Einstein's gravity.

It may be noted that Verlinde's entropic force formalism has been generalized to the cosmological context in different methods. We have considered one of such generalizations in [11], where the authors assume an energy flow, $d\varepsilon$ through the horizon. Following this approach, we found that this energy flow through the horizon is getting reduced in course of time. In this case, the generalized second law, $\dot{S} \geq 0$ implies the condition $\frac{d\varepsilon}{dt} \geq 0$. This energy flow through the horizon will eventually stop in the final de Sitter state of maximum entropy. Hence it can be argued that the universe is trying maximize its entropy by reducing the energy flow through the horizon.

Finally, we made a comparison between Padmanabhan's proposal and Verlinde's hypothesis. According to Padmanabhan, the evolution of the universe can be interpreted as a tendency for decreasing the holographic descrepancy. But, following the argument in [11], one can interpret the cosmic evolution as a tendency for reducing the energy flux through the horizon. However both these proposals ensures the consistency with the GSL and the maximum entropy principle. Moreover, both of them demands an asymptotically de Sitter universe which in turn implies the presence of dark energy which is not too different from the cosmological constant. We have already mentioned that the approach in [11] is slightly different from Verlinde's original proposal, as the authors assume an energy flux through the horizon in addition. It should also be noted that, one of the basic assumption of Verlinde,

$\frac{1}{2}Nk_BT = Mc^2$ will hold true only in a pure de Sitter universe. This relation could in general be expressed as the inequality $N \geq \frac{Mc^2}{\frac{1}{2}k_BT}$ which has the same form of Padmanabhan's assumption $N_{\text{surf}} \geq N_{\text{bulk}}$.

Although the conceptual formulations in [11,20] are different, both these emergent perspectives of gravity describes a universe that behave as an ordinary macroscopic system. In other words, in both these perspectives, the cosmic evolution could be explained as a tendency for maximizing entropy. Our approach gives a thermodynamic basis and thus provides further support to the emergent gravity paradigm.

[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[2] J. D. Bekenstein, Phys. Rev. D **9**, 3292 (1974).
[3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[4] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[5] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
[6] C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006).
[7] T. Padmanabhan, Rep. Prog. Phys. **73**, 046901 (2010).
[8] T. Padmanabhan, Mod. Phys. Lett. A **25**, 1129 (2010).
[9] E. P. Verlinde, J. High Energy Phys. 04 (2011) 029.
[10] R.-G. Cai, L.-M. Cao, and N. Ohta, Phys. Rev. D **81**, 061501 (2010).
[11] F.-W. Shu and Y. Gong, Int. J. Mod. Phys. D **20**, 553 (2011).
[12] L. Miao and Y. Wang, Phys. Lett. B **687**, 243 (2010).
[13] E. P. Verlinde, SciPost Phys. **2**, 016 (2017).
[14] G. Changjun, Phys. Rev. D **81**, 087306 (2010).
[15] H. Moradpour, Int. J. Theor. Phys. **55**, 4176 (2016).
[16] H. Moradpour, R. C. Nunes, E. M. C. Abreu, and J. Ananias Neto, Mod. Phys. Lett. A **32**, 1750078 (2017).
[17] A. Sheykhi, Phys. Rev. D **81**, 104011 (2010).
[18] A. Sheykhi and S. H. Hendi, Phys. Rev. D **84**, 044023 (2011).
[19] R. B. Mann and J. R. Mureika, Phys. Lett. B **703**, 167 (2011).
[20] T. Padmanabhan, arXiv:1206.4916.
[21] R.-G. Cai, J. High Energy Phys. 11 (2012) 016.
[22] A. Sheykhi, Phys. Rev. D **87**, 061501 (2013).
[23] A. F. Ali, Phys. Lett. B **732**, 335 (2014).
[24] K. Yang, Y.-X. Liu, and Y.-Q. Wang, Phys. Rev. D **86**, 104013 (2012).
[25] W.-Y. Ai, H. Chen, X.-R. Hu, and J.-B. Deng, Gen. Relativ. Gravit. **46**, 1680 (2014).
[26] M. Eune and W. Kim, Phys. Rev. D **88**, 067303 (2013).
[27] E. Chang-Young and D. Lee, J. High Energy Phys. 04 (2014) 125.
[28] A. Sheykhi, M. Dehghani, and S. Hosseini, J. Cosmol. Astropart. Phys. 04 (2013) 038.
[29] S. Chakraborty and T. Padmanabhan, Phys. Rev. D **90**, 084021 (2014).
[30] S. Chakraborty and T. Padmanabhan, Phys. Rev. D **90**, 124017 (2014).
[31] S. Chakraborty, J. High Energy Phys. 08 (2015) 029.
[32] N. Komatsu, Phys. Rev. D **96**, 103507 (2017).
[33] W. Zhang and X.-M. Kuang, Adv. High Energy Phys. **2018**, 6758078 (2018).
[34] M. Hashemi, S. Jalalzadeh, and S. Farahani, Gen. Relativ. Gravit. **47**, 139 (2015).
[35] H. B. Callen, *Thermodynamics* (Wiley, New York, 1960).
[36] D. Pavon and N. Radicella, Gen. Relativ. Gravit. **45**, 63 (2013).
[37] P. B. Krishna and T. K. Mathew, Phys. Rev. D **96**, 063513 (2017).
[38] C. A. Egan and C. H. Lineweaver, Astrophys. J. **710**, 1825 (2010).
[39] J. Simon, L. Verde, and R. Jimenez, Phys. Rev. D **71**, 123001 (2005).
[40] D. Stern, R. Jimenez, L. Verde, M. Kamionkowski, and S. A. Stanford, J. Cosmol. Astropart. Phys. 02 (2010) 008.
[41] S. M. Crawford, A. L. Ratsimbazafy, C. M. Cress, E. A. Olivier, S.-L. Blyth, and K. van der Heyden, Mon. Not. R. Astron. Soc. **406**, 2569 (2010).
[42] J. C. Carvalho and J. S. Alcaniz, Mon. Not. R. Astron. Soc. **418**, 1873 (2011).
[43] R.-G. Cai, Phys. Rev. D **65**, 084014 (2002).
[44] R.-G. Cai and Q. Guo, Phys. Rev. D **69**, 104025 (2004).
[45] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
[46] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[47] S. M. Carroll and A. Chatwin-Davis, Phys. Rev. D **97**, 046012 (2018).
[48] A. Albrecht, J. Phys. Conf. Ser. **174**, 012006 (2009).
[49] L. Dyson, M. Kleban, and L. Susskind, J. High Energy Phys. 10 (2002) 011.
[50] H. Moradpour and N. Riazi, Int. J. Theor. Phys. **55**, 268 (2016).
[51] N. Radicella and D. Pavon, Gen. Relativ. Gravit. **44**, 685 (2012).
[52] H. Moradpour, A. Sheykhi, N. Riazi, and B. Wang, Adv. High Energy Phys. **2014**, 718583 (2014).

[53] N. Radicella and D. Pavon, Phys. Lett. B **704**, 260 (2011).
[54] T. Padmanabhan, *Gravity and Spacetime: An Emergent Perspective, Handbook of Spacetime* (Springer, New York, 2013).
[55] D. Kothawala, Phys. Rev. D **83**, 024026 (2011).
[56] J. P. Mimoso and D. Pavon, Phys. Rev. D **94**, 103507 (2016).
[57] T. Padmanabhan, arXiv:1602.01474.