Relativistic effects in the semileptonic B_c decays to charmonium with the Bethe-Salpeter method

Zi-Kan Geng,[†] Tianhong Wang,[*] Yue Jiang,[‡] Geng Li,[§] Xiao-Ze Tan,[∥] and Guo-Li Wang[¶]

Department of Physics, Harbin Institute of Technology, Harbin 150001, China

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Relativistic effects are important in the rigorous study of heavy quarks. In this paper, we study the relativistic corrections of semileptonic B_c decays to charmonium with the instantaneous Bethe-Salpeter method. Within the Bethe-Salpeter framework, we use two methods to study the relativistic effects. One of them is to expand the transition amplitude in powers of $\vec{q}$ which is the relative momentum between the quark and antiquark, and the other is to expand the amplitude base on the wave functions. In the level of decay width, the results show that, for the transition of $B_c \to \eta_c$, the relativistic correction is about 22%; for $B_c \to J/\psi$, it is about 19%; the relativistic effects of $1P$ final states are about 14%–46% larger than those of $1S$ final states; for $2S$ final states, they are about 19%–28% larger than those of $1S$ final states; for $3S$ final states, they are about 12%–13% larger than those of $2S$ final states; for $2P$ final states, they are about 10%–14% larger than those of $1P$ final states; for $3P$ final states, they are about 7%–12% larger than those of $2P$ final states. We conclude that the relativistic corrections of the B_c decays to the orbitally or radially excited charmonium (2S, 3S, 1P, 2P, 3P) are quite large.

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I. INTRODUCTION

When studying the properties of heavy-light mesons, we always pay attention to the relativistic effect of a light quark but ignore that of a heavy quark. However, the careful investigation of the relativistic corrections to the heavy quark is important, especially the charm quark. For example, recently, the authors of Ref. [1] have calculated the relativistic corrections to the form factors of the B_c decays to S-wave charmonium by the nonrelativistic QCD (NRQCD) approach, with the heavy quark relative velocities $\vec{v}^2_{J/\psi} = 0.267$ and $\vec{v}^2_{B_c} = 0.186$. They pointed out that the relativistic corrections can bring about additional 15%–27% contributions. The authors of Ref. [2] have calculated the $\mathcal{O}(v^2)$ corrections to twist-2 light cone distribution amplitudes (LCDAs) of S-wave B_c mesons. They pointed out that the relativistic corrections are sizable and comparable with the next-to-leading order radiative corrections. Another famous example, the leading order NRQCD predictions [3,4] of the production $e^+ + e^- \to J/\psi + \eta_c$ are at least 5 times smaller than the experimental measurements [5,6]. Later, people found that the relativistic corrections increase the results to 2 times as much as the nonrelativistic predictions [7,8]. Therefore, the relativistic effect of heavy quark are important and need to be studied carefully.

The B_c^+ meson consists of a c quark and $\bar{b}$ antiquark, and carries two different flavors. It only decays via weak interactions; thus the B_c meson has attracted a lot of attentions both in theories and experiments [9]. Recently the cross section of the B_c meson is expected to reach the level of μb via the proton-nucleus and the nucleus-nucleus collision modes at the Large Hadron Collider [10]. The LHCb experiment can produce and reconstruct a large number of the B_c meson events, and it provides a solid platform to study the properties of the B_c meson precisely.

A great deal of work has been done on various B_c decays under different approaches, such as the NRQCD approach [1,11–13], the perturbative QCD approach (PQCD) [14–17], the relativistic quark model (RQM) [18–21], light cone sum rules (LCSR) [22], the nonrelativistic constituent quark model (NCQM) [23] and QCD sum rules (QCDSR) [24–27].

In a previous study, according to the numerical wave function which is the solution of the instantaneous Bethe-Salpeter (BS) equation (also called Salpeter equation) [28,29], we have qualitatively pointed out that the relativistic correction of an excited state is larger than that of the corresponding ground state. The relativistic correction can

[*]Corresponding author.
thwang@hit.edu.cn
[†]zikangeng@hit.edu.cn
[‡]jiangure@hit.edu.cn
[§]karlisle@hit.edu.cn
[∥]xz.tan@hit.edu.cn
[¶]gl_wang@hit.edu.cn

not be ignored, so a relativistic model is needed to deal with the problems including excited states [30]. There are also some investigations of the relativistic effects from the "nontraditional" orbital angular momentum components in the BS framework [31–35]. We could extend these works to the double-heavy mesons, and our conclusions are similar to them that such relativistic effects are important and non-negligible. In this paper, we give a quantitative study of this topic and choose the semileptonic B_c decays to charmonium by using the instantaneous BS method. This method has a comparatively solid foundation because both the BS equation and the Mandelstam formula [36] are established on a relativistic quantum field theory. We have solved the full Salpeter equations for different $J^{P(C)}$ states [37–39] and deduced the transition amplitude formula by performing the instantaneous approach to the Mandelstam formula. In these processes, the corresponding quark and antiquark are charm and bottom quarks which both are heavy. The instantaneous approximation is reasonable, and we can provide a relatively rigorous relativistic calculation.

The paper is organized as follows. In Sec. II, we give the useful formulas for the B_c decays to charmonium. In Sec. III, we give the relativistic wave function in the instantaneous BS method. In Sec. IV, we give a method to separate the relativistic corrections. In Sec. V, we give another method to calculate the relativistic corrections. In Sec. VI, we give the numerical results and discussions. We summarize and conclude in Sec. VII, and put the wave functions and Salpeter equation in the Appendix.

II. FORM FACTORS AND SEMILEPTONIC DECAY WIDTH

For the $B_c^+ \to (c\bar{c})\ell^+\nu_\ell$ processes shown in Fig. 1, the transition amplitude element reads

$$T = \frac{G_F}{\sqrt{2}} V_{cb}\bar{u}_{\nu_\ell}\gamma^\mu(1-\gamma_5)v_\ell\langle (c\bar{c})(P_f)|J_\mu|B_c^+(P)\rangle, \qquad (2.1)$$

where $(c\bar{c})$ denotes charmonium, V_{cb} is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element, $J_\mu \equiv V_\mu - A_\mu$ is the charged current responsible for the decays, P and P_f are the momenta of the initial B_c^+ and the final charmonium, respectively.

Taking a η_c meson as an example, the hadronic transition element can be written as the overlapping integral over the initial and final relativistic BS wave functions within Mandelstam formalism. We would not solve the full BS equation, but the instantaneous one, namely, the full Salpeter equation. We perform the instantaneous approximation to the transition element [40] and write it as

$$\langle \eta_c|\bar{b}\gamma^\mu(1-\gamma^5)c|B_c^+\rangle = \int \frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\bar{\varphi}^{++}_{P_f}(\vec{q}\,')\frac{\not{P}}{M}\varphi^{++}_P(\vec{q})\gamma^\mu(1-\gamma^5)\right], \qquad (2.2)$$

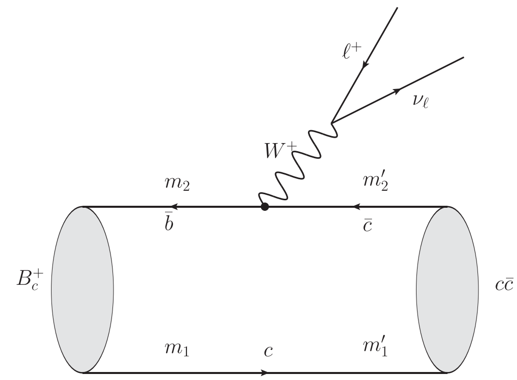


FIG. 1. Feynman diagram corresponding to the semileptonic decays $B_c^+ \to (c\bar{c})\ell^+\nu_\ell$.

where φ^{++}_P denotes the positive energy component of the instantaneous BS wave function of the initial state, $\bar{\varphi}^{++}_{P_f} \equiv \gamma^0\varphi^{++}_{P_f}\gamma^0$ is the Dirac conjugate of the positive energy component of the final state, m_1' and m_2' are the masses of quark and antiquark in the final state, respectively, and $\vec{q}\,' = \vec{q} - \frac{m_1'}{m_1'+m_2'}\vec{P}_f$ is the relative momentum between them. In this paper, we keep only the positive energy component φ^{++} of the relativistic wave functions, because the contributions from other components are much smaller than the 1% in transition of $B_c \to (c\bar{c})$ [41].

For $B_c^+ \to P\ell^+\nu_\ell$ (here P denotes η_c or χ_{c0}), the hadronic matrix element can be written as

$$\begin{aligned}\langle P|\bar{b}\gamma^\mu(1-\gamma^5)c|B_c^+\rangle &= \int \frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\bar{\varphi}^{++}_{P_f}(\vec{q}\,')\frac{\not{P}}{M}\varphi^{++}_P(\vec{q})\gamma^\mu(1-\gamma^5)\right]\\ &= S_+(P+P_f)^\mu + S_-(P-P_f)^\mu,\end{aligned} \qquad (2.3)$$

where S_+ and S_- are the form factors.

For $B_c^+ \to V\ell^+\nu_\ell$ (here V denotes J/ψ, h_c or χ_{c1}), the hadronic matrix element can be written as

$$\begin{aligned}\langle V|\bar{b}\gamma^\mu(1-\gamma^5)c|B_c^+\rangle &= \int \frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\bar{\varphi}^{++}_{P_f}(\vec{q}\,')\frac{\not{P}}{M}\varphi^{++}_P(\vec{q})\gamma^\mu(1-\gamma^5)\right]\\ &= (t_1P^\mu + t_2P_f^\mu)\frac{\epsilon\cdot P}{M} + t_3(M+M_f)\epsilon^\mu\\ &\quad + \frac{2t_4}{M+M_f}i\varepsilon^{\mu\nu\sigma\delta}\epsilon_\nu P_\sigma P_{f\delta},\end{aligned} \qquad (2.4)$$

where ϵ_ν is the polarization vector of the final vector meson, and t_1, t_2, t_3 and t_4 are the form factors.

The summation formulas for polarization of the final vector meson used in this paper are

$$\epsilon_\mu^{(\lambda)}(P_f)P_f^\mu = 0,$$
$$\sum_\lambda \epsilon_\mu^{(\lambda)}(P_f)\epsilon_\nu^{\dagger(\lambda)}(P_f) = -g_{\mu\nu} + \frac{P_{f\mu}P_{f\nu}}{M_f^2}. \quad (2.5)$$

Finally, the semileptonic decay width can be expressed as

$$\Gamma = \frac{1}{8M(2\pi)^3}\int\frac{|\vec{P}_\ell|}{E_\ell}\mathrm{d}|\vec{P}_\ell|\int\frac{|\vec{P}_f|}{E_f}\mathrm{d}|\vec{P}_f|\sum_\lambda|T|^2, \quad (2.6)$$

where $\vec{P}_\ell$ is the three-dimensional momentum of the final lepton, and $\vec{P}_f$ is the three-dimensional momentum of the final meson.

III. RELATIVISTIC WAVE FUNCTION

Usually, the nonrelativistic wave function for a pseudoscalar is written as [42]

$$\Psi_P(\vec{q}) = (\not{P}+M)\gamma_5 f(\vec{q}), \quad (3.1)$$

where M and P are the mass and momentum of the meson, respectively, $\vec{q}$ is the relative momentum between the quark and antiquark in the meson, and the radial wave function $f(\vec{q})$ can be obtained numerically by solving the Schrodinger equation.

But in our method, we solve the full Salpeter equation. The form of the wave function is relativistic and depends on the $J^{P(C)}$ quantum number of the corresponding meson. For a pseudoscalar, the relativistic wave function can be written as the four items constructed by P, $q_\perp$ and γ-matrices [43–47],

$$\varphi_{0^-}(q_\perp) = M\left[\frac{\not{P}}{M}f_1(q_\perp) + f_2(q_\perp) + \frac{\not{q}_\perp}{M}f_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}f_4(q_\perp)\right]\gamma_5, \quad (3.2)$$

where $q = p_1 - \alpha_1 P = \alpha_2 P - p_2$ is the relative momentum between quark (with momentum p_1 and mass m_1) and antiquark (momentum p_2 and mass m_2), $\alpha_1 = \frac{m_1}{m_1+m_2}$, $\alpha_2 = \frac{m_2}{m_1+m_2}$; $q_\perp = q - \frac{P\cdot q}{M^2}P$, in the rest frame of the meson, $q_\perp = (0, \vec{q})$.

All the items in the wave function Eq. (3.2) have the quantum number of 0^-. This wave function is a general relativistic form for a pseudoscalar with the instantaneous approximation. If we set the items with f_3 and f_4 to zero, and set $f_1 = f_2$, the relativistic wave function is reduced to the Schrodinger wave function Eq. (3.1).

Taking into account the last two equations in Eq. (A.14), obtain the relations

$$f_3(q_\perp) = \frac{M(\omega_2-\omega_1)}{(m_1\omega_2+m_2\omega_1)}f_1,$$
$$f_4(q_\perp) = -\frac{M(\omega_1+\omega_2)}{(m_1\omega_2+m_2\omega_1)}f_2, \quad (3.3)$$

where the quark energy $\omega_i = \sqrt{m_i^2 - q_\perp^2} = \sqrt{m_i^2+\vec{q}^2}$ $(i = 1, 2)$. The wave function corresponding to the positive energy projection has the form

$$\varphi_{0^-}^{++}(q_\perp) = \left[A_1(q_\perp) + \frac{\not{P}}{M}A_2(q_\perp) + \frac{\not{q}_\perp}{M}A_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}A_4(q_\perp)\right]\gamma^5, \quad (3.4)$$

where

$$A_1 = \frac{M}{2}\left[\frac{\omega_1+\omega_2}{m_1+m_2}f_1 + f_2\right], \quad A_3 = -\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}A_1,$$
$$A_2 = \frac{M}{2}\left[f_1 + \frac{m_1+m_2}{\omega_1+\omega_2}f_2\right], \quad A_4 = -\frac{M(m_1+m_2)}{m_1\omega_2+m_2\omega_1}A_1. \quad (3.5)$$

The normalization condition reads

$$\int\frac{\mathrm{d}\vec{q}}{(2\pi)^3}4f_1f_2M^2\left\{\frac{m_1+m_2}{\omega_1+\omega_2} + \frac{\omega_1+\omega_2}{m_1+m_2} + \frac{2\vec{q}^2(m_1\omega_1+m_2\omega_2)}{(m_2\omega_1+m_1\omega_2)^2}\right\} = 2M. \quad (3.6)$$

By solving the full Salpeter equation, the numerical values of wave functions f_1, f_2, f_3 and f_4 are obtained. The positive energy component Eq. (3.4) is brought into the Mandelstam formula Eq. (2.2). After the trace and integral are finished, the form factors S_+ and S_- can be calculated numerically. Finally, the decay width of the semileptonic decay $B_c^+ \to \eta_c\ell^+\nu_\ell$ can be obtained within the relativistic BS method. In this paper, besides the wave function for 0^- state, we also need the wave functions for the states of 1^{--} (J/ψ), 1^{+-} (h_c), 0^{++} (χ_{c0}), 1^{++} (χ_{c1}), etc., and we give them in the Appendix.

IV. METHOD I OF SEPARATING THE RELATIVISTIC CORRECTIONS

In this part, how to obtain the relativistic corrections is shown. The transition element is obtained by overlapping integral over the Schrodinger wave functions of the initial and final states in a nonrelativistic model. The main difference between the relativistic and nonrelativistic models comes from the wave functions. If we set the items with $\vec{q}$ (or $q_\perp$) in Eq. (3.4) to zero and let $f_1 = f_2$, the relativistic wave function is reduced to the nonrelativistic one Eq. (3.1).

To see the relativistic corrections at each expansive order, we expand the amplitude in powers of $\vec{q}$. There are some reasons: (i) the quantity $\vec{q}$, which represents the relative momentum between the quark and the antiquark, is a kind of measure of the relativistic effect of a meson; (ii) when $|\vec{q}|$ is small, the ratio $|\vec{q}|/M$ or $|\vec{q}|/m_i$ ($i = 1, 2$) is small and can be dealt as the power to expand the amplitude; when $|\vec{q}|$ is large, its contribution will be suppressed by the wave function $f_i(\vec{q})$, especially for the ground state (η_c and J/ψ); (iii) it has been investigated in NRQCD effective theory that the decay rates can be ordered in powers of the quark relative velocity v [48]. The relative momentum $\vec{q}$ is related to relative velocity $\vec{v}$ and also can be used to perform the Taylor expansion of the amplitude, see Refs. [1,13].

In the transition amplitude, the wave function of the final state is dependent on $\vec{q}'$. We use the relation $\vec{q}' = \vec{q} - \frac{m_1'}{m_1'+m_2'}\vec{P}_f$ during a numerical calculation. But $\vec{q}'$ is treated as an independent variable to maintain covariance when we expand the amplitude. In other words, we perform the Taylor expansion of the amplitude Eq. (2.2) (before doing the integrate over $\vec{q}$) in powers of relative momentum $\vec{q}$ and $\vec{q}'$, where $\vec{q}$ and $\vec{q}'$ are the relative momenta between quark and antiquark in the initial meson and final meson, respectively. The relativistic corrections to form factors can be given at each expansive order. According to the expansion in the transition amplitude, we can obtain the expansion in the level of decay width straightforwardly. We expand the amplitude (or form factors) to the third order of $\vec{q}$, but expand the decay width to the sixth order of $\vec{q}$. This difference results from the cross items in $|T|^2$, similarly as Eq. (5.4).

In some nonrelativistic methods, both the wave functions and the amplitude are nonrelativistic. If we set the items with $\vec{q}$ (or $q_\perp$) in Eq. (3.4) to zero and set $f_1 = f_2$ as mentioned above and modify the normalization condition Eq. (3.6) as

$$\int \frac{d\vec{q}}{(2\pi)^3} 4 f_1 f_2 M^2 \times 2 = 2M, \tag{4.1}$$

the nonrelativistic wave function can be obtained. Taking them into the leading order expansion of the amplitude, we can estimate the decay widths obtained by the nonrelativistic (NR) methods.

V. METHOD II OF CALCULATING THE RELATIVISTIC CORRECTIONS

Though the behavior of a wave function $f_i(\vec{q})$ will suppress the contribution of large $|\vec{q}|$, there is still the problem of rapidity of convergence. This problem will be shown by numerical results later. We would like to provide another method to give the relativistic corrections.

The first two items in Eq. (3.2) are close to the nonrelativistic wave function Eq. (3.1) because of $f_1 \simeq f_2$ numerically. Based on that, we can treat them as the leading order and the last two items as the relativistic corrections. Similarly, the positive energy wave function Eq. (3.4) can be decomposed into two components,

$$\varphi_{0^-}^{++}(q_\perp) = \varphi_0^{++}(q_\perp) + \varphi_1^{++}(q_\perp), \tag{5.1}$$

where $\varphi_0^{++}(q_\perp) = [A_1(q_\perp) + \frac{\not{P}}{M} A_2(q_\perp)]\gamma^5$ is treated as the nonrelativistic (NR) wave function, and $\varphi_1^{++}(q_\perp) = [\frac{\not{q}_\perp}{M} A_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2} A_4(q_\perp)]\gamma^5$ is treated as the relativistic corrections (RC) of the wave function.

If we use the approximate formula $\omega_i = m_i + \vec{q}^2/2m_i$ ($i = 1, 2$) (which is valid in small $|\vec{q}|$, but the contribution from large $|\vec{q}|$ will be suppressed by the wave functions f_i) and set $f_1 = f_2$, then $\varphi_0^{++}(q_\perp) = M f_1[(1 + \frac{\vec{q}^2}{4m_1 m_2}) + \frac{\not{P}}{M}(1 - \frac{\vec{q}^2}{4m_1 m_2})]\gamma^5$. The difference between φ_0^{++} and the NR wave function Eq. (3.1) is left to the second order of $\vec{q}$. Therefore φ_0^{++} is approximated equivalent to the NR wave function. For other states, we can reach the same conclusions.

The hadronic transition element can be decomposed into three components,

$$\begin{aligned}
\langle \eta_c | \bar{b}\gamma^\mu (1-\gamma^5) c | B_c^+ \rangle &= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\bar{\varphi}_{P_f}^{++}(\vec{q}') \frac{\not{P}}{M} \varphi_P^{++}(\vec{q}) \gamma^\mu (1-\gamma^5) \right] \\
&= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[(\bar{\varphi}_0'^{++} + \bar{\varphi}_1'^{++}) \frac{\not{P}}{M} (\varphi_0^{++} + \varphi_1^{++}) \gamma^\mu (1-\gamma^5) \right] \\
&= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[(\bar{\varphi}_0'^{++}) \frac{\not{P}}{M} (\varphi_0^{++}) \gamma^\mu (1-\gamma^5) \right] + \Leftrightarrow \text{the leading order (LO)} \\
&\quad + \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[(\bar{\varphi}_1'^{++}) \frac{\not{P}}{M} (\varphi_0^{++}) \gamma^\mu (1-\gamma^5) + (\bar{\varphi}_0'^{++}) \frac{\not{P}}{M} (\varphi_1^{++}) \gamma^\mu (1-\gamma^5) \right] \\
&\quad + \Leftrightarrow \text{the first order of relativistic correction} (\text{1st}\, RC) \\
&\quad + \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[(\bar{\varphi}_1'^{++}) \frac{\not{P}}{M} (\varphi_1^{++}) \gamma^\mu (1-\gamma^5) \right] \\
&\quad \Leftrightarrow \text{the second order of relativistic correction} (\text{2nd}\, RC).
\end{aligned} \tag{5.2}$$

In the transition amplitude T, the leptonic component $\bar{u}_{\nu_\ell}\gamma^\mu(1-\gamma_5)v_\ell$ is independent of the relative momentum $\vec{q}$. The transition amplitude T can also be decomposed into three components,

$$T = T_0 + T_1 + T_2, \tag{5.3}$$

where T_0 denotes the leading order (LO), T_1 denotes the first order relativistic correction (1st RC), and T_2 denotes the second order relativistic correction (2nd RC). However, the decay width is related to the square of transition amplitude module, as shown in Eq. (2.6). The square of transition amplitude module can be decomposed into five components,

$$\begin{aligned} |T|^2 &= (T_0+T_1+T_2)(T_0^*+T_1^*+T_2^*) \\ &= |T_0|^2 + \Leftrightarrow LO \\ &\quad + T_0T_1^* + T_0^*T_1 + \Leftrightarrow \text{1st}\, RC \\ &\quad + |T_1|^2 + (T_0T_2^* + T_0^*T_2) + \Leftrightarrow \text{2nd}\, RC \\ &\quad + T_1T_2^* + T_1^*T_2 + \Leftrightarrow \text{3rd}\, RC \\ &\quad + |T_2|^2 \Leftrightarrow \text{4th}\, RC. \end{aligned} \tag{5.4}$$

Taking an component into the phase-space integral Eq. (2.6), we can calculate the corresponding order of the decay width. In summary, we separate the positive energy wave function of the initial and final meson into two components, the NR wave function and the RC one. Then we compute each expansive orders of transition amplitude (or form factors), and finally we use them to obtain each expansive orders of decay width.

VI. RESULTS AND DISCUSSIONS

The parameters used in this paper are $\Gamma_{B_c} = 1.298 \times 10^{-12}$ GeV, $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, $m_b = 4.96$ GeV, $m_c = 1.62$ GeV, $V_{cb} = 40.5 \times 10^{-3}$, $M_{h_c(2P)} = 3.887$ GeV, $M_{\chi_{c0}(2P)} = 3.862$ GeV, $M_{\chi_{c1}(2P)} = 3.872$ GeV, $M_{\eta_c(3S)} = 3.949$ GeV, $M_{\psi(3S)} = 4.039$ GeV, $M_{h_c(3P)} = 4.242$ GeV, $M_{\chi_{c0}(3P)} = 4.140$ GeV, $M_{\chi_{c1}(3P)} = 4.229$ GeV.

By solving the corresponding full Salpeter equations, we obtain the numerical results of a wave function for different $J^{P(C)}$ states and show them in Figs. 2–4. In some nonrelativistic approaches, there is only one wave function. According to our results, the dominate two wave functions are almost equivalent, for example, $f_1 \simeq f_2$ for 0^- state, $g_5 \simeq -g_6$ for 1^- state, and $\phi_1 \simeq \phi_2$ for 0^+ state, etc. Therefore these nonrelativistic approaches are reasonable in some cases.

With these relativistic wave functions, the form factors and the corresponding relativistic corrections are obtained. With the nonrelativistic wave functions [see Eq. (4.1)] and the leading order amplitude, the results obtained by those NR approaches are estimated. For each process in method I, we only show one of the form factors which makes the main contribution to the decay width. The corresponding results of the B_c decays to $1S$-wave and $1P$-wave charmonium are shown in Fig. 5, where the green solid lines are the results of relativistic BS method (BSE) without expansion, the red solid lines are the leading order (nonrelativistic) contributions to the form factors, dash lines are the first order relativistic corrections (q^2), dot-dash lines are the second order relativistic corrections (q^3), and dot lines are the third order relativistic corrections (q^3); $t \equiv (P - P_f)^2$, and t_m is the maximum of t. The LO contributions in form factors are dominant. The first relativistic corrections for the $1S$ final state cases are negligible. In the cases of $1P$ final states, the first relativistic corrections provide sizable contributions, especially for χ_{c0} and χ_{c1}. The ratio of the first relativistic corrections to LO in $B_c \to h_c$ is around 15%. But for χ_{c0} or χ_{c1} final states, this ratio can reach up to 80%. Other higher order corrections are less than 10% as much as the leading order and are negligible. Therefore we conclude that the relativistic corrections in the B_c decays to P-wave charmonium have large contributions, even though both the initial state and final state are the double heavy mesons.

The form factors of the B_c decays to radially excited charmonium are plotted in Figs. 6 and 7. The first relativistic corrections are comparable to the leading order for the B_c decays to $2S$ or $3S$ charmonium. The first relativistic corrections are larger than the leading order for the B_c decays to $2P$ or $3P$ charmonium. The second order contribution and the third one may also appear reversal. We conclude that compared with the ground states, the relativistic corrections of the B_c decays to corresponding excited charmonium are much larger.

Then we calculate the decay widths of the semileptonic decay $B_c^+ \to (c\bar{c}) + e^+ + \bar{\nu}_e$. To see the relativistic effects, we give each order expansion of the decay fractions and their sums in two methods. The results from method I are shown in Table I, and the results from method II are shown in Table II. In method I, the amplitude is expanded in powers of $\vec{q}$ before the integral, and $\vec{q}$ relates directly to the relative velocity $\vec{v}$ between quark and antiquark in the meson, $\vec{q} = \frac{m_1 m_2}{m_1+m_2}\vec{v}$. The average relative velocity is not very large in a double-heavy meson, so one can make such a kind of expansion. $|\vec{q}|$ is actually not a fixed quantity, whose range is from zero to infinity. We argue when $|\vec{q}|$ is large, the contribution is suppressed by the values of wave functions. But from the diagrams of wave functions in Figs. 2–4,

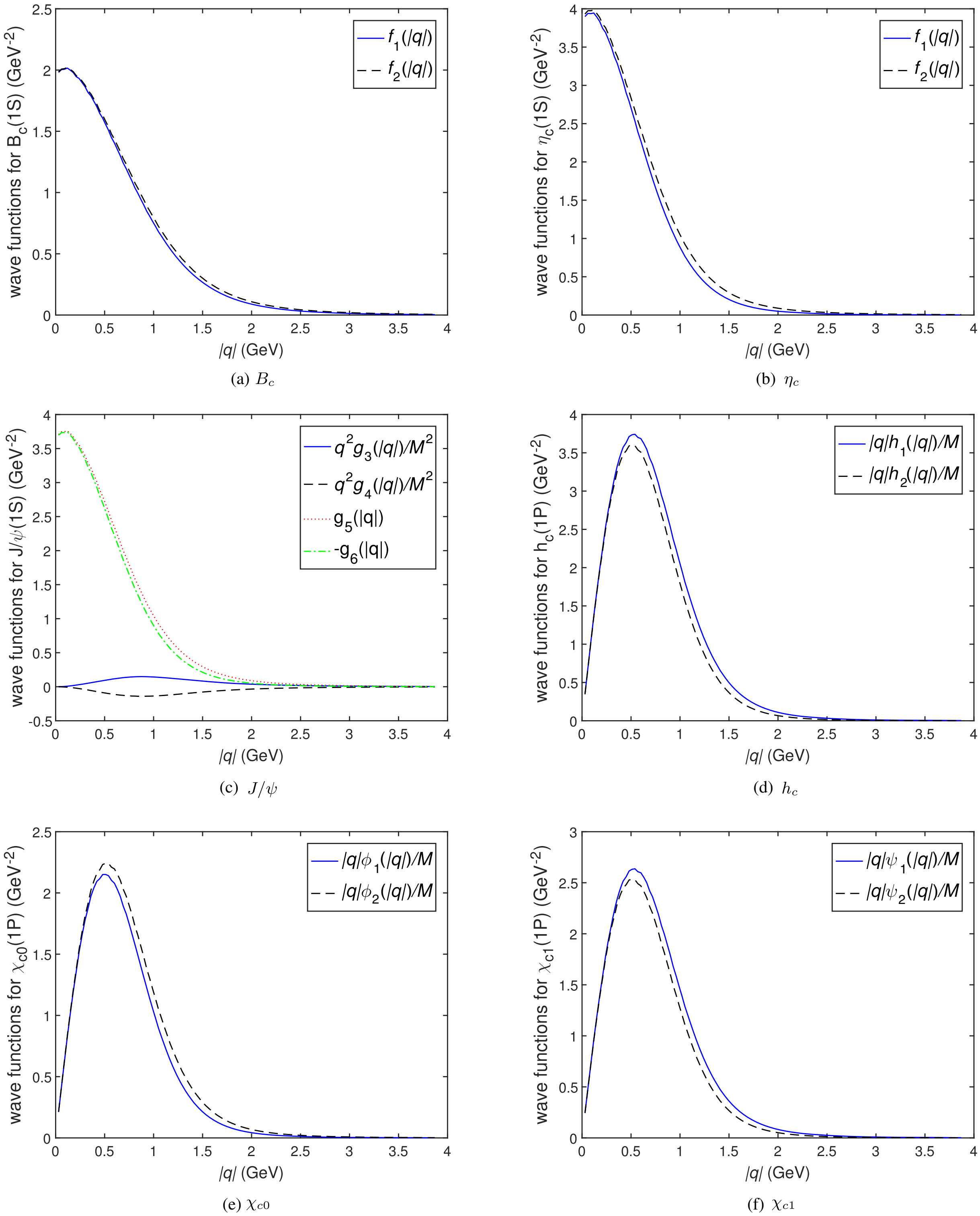


FIG. 2. The wave functions for the ground state mesons.

these arguments may be not very good especially for highly excited states. It may lead to the problem of convergence. For $1S$ wave (η_c and J/ψ) as the final states, the leading order ($\vec{q}^0$) makes dominant contribution. For the channels $2S$ or $3S$ as the final states, the leading order makes the largest contribution. Meanwhile, the first and second order relativistic corrections, $\vec{q}^1$ and $\vec{q}^2$ items, provide comparable contributions

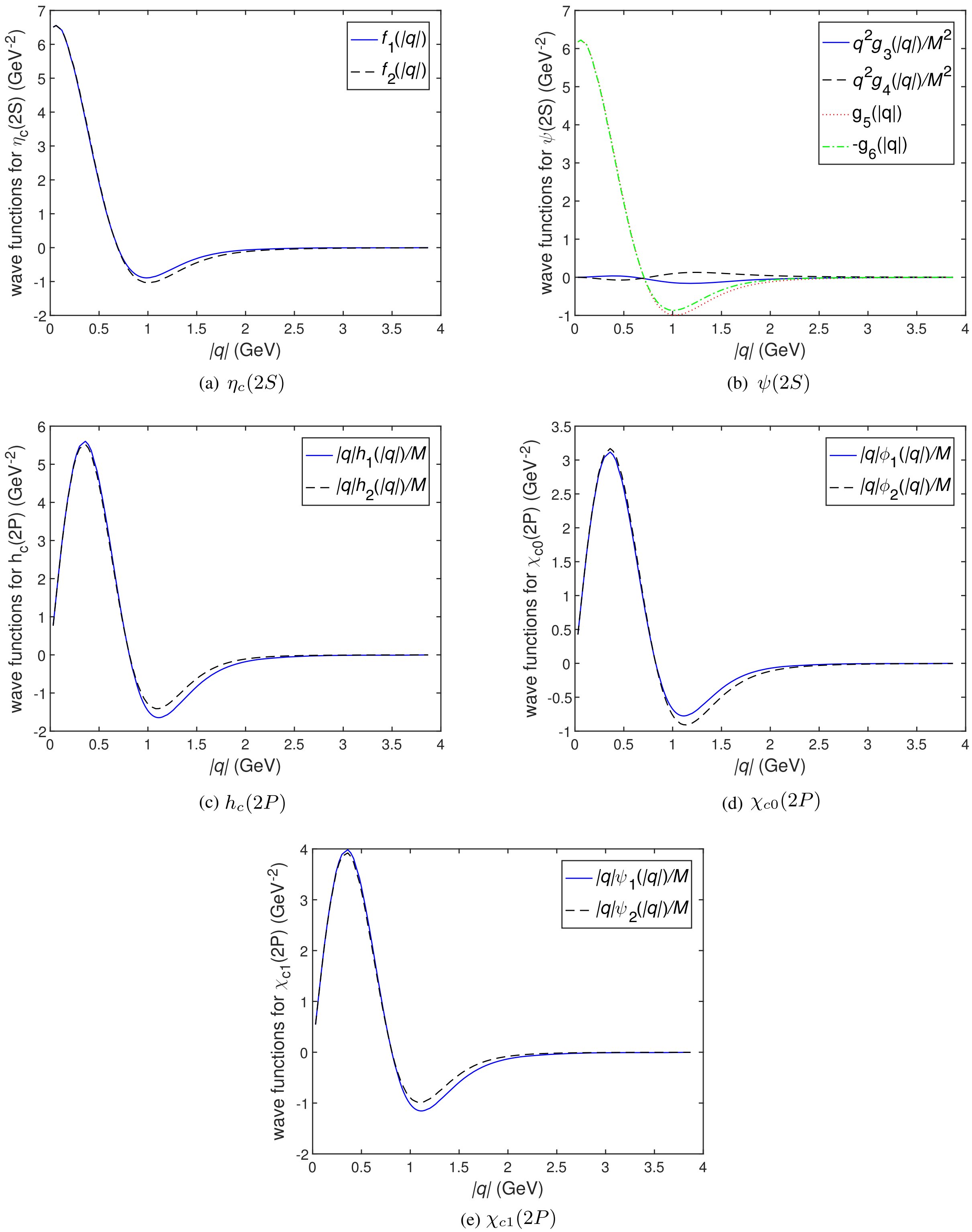
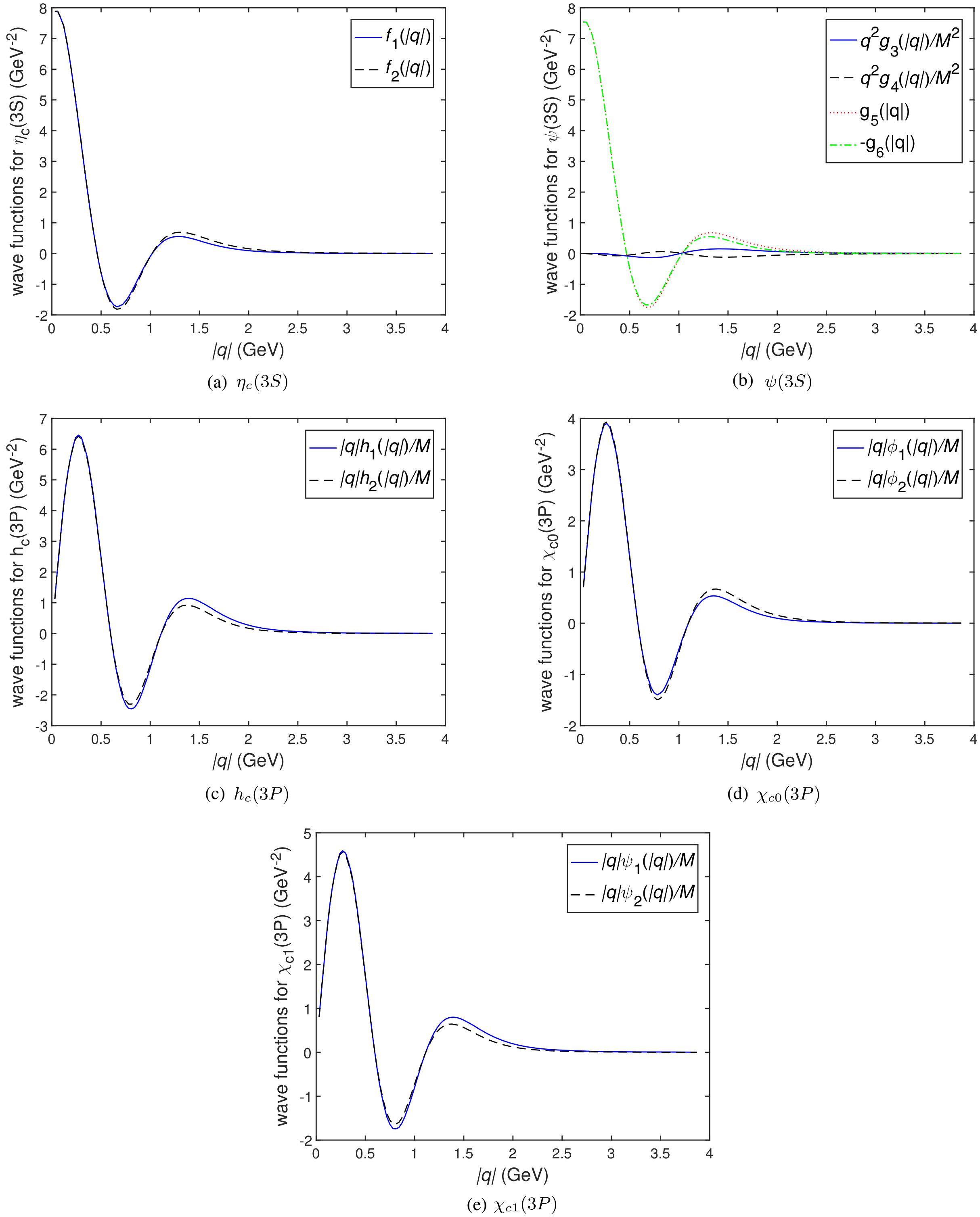


FIG. 3. The wave functions for the radially excited state mesons.

to the leading order. For some of P wave final states, the leading order ($\vec{q}^2$) which is nonrelativistic contributions, may be not dominant. Instead, the first order relativistic corrections make the dominant contributions. For both the orbitally excited states and the radially ones as the final states, the relativistic effects are huge.

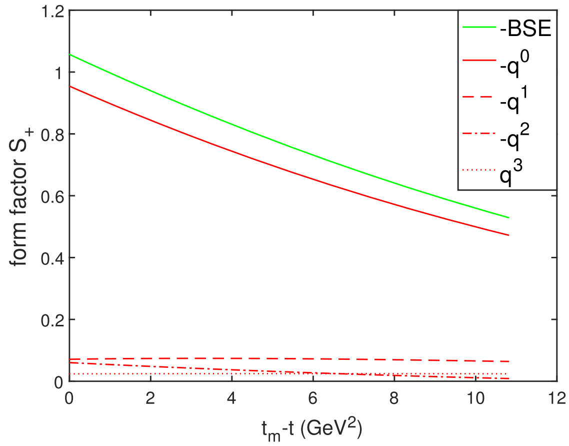
In Table III, the comparisons of the branch ratios obtained by different ways are given, where **sum** column



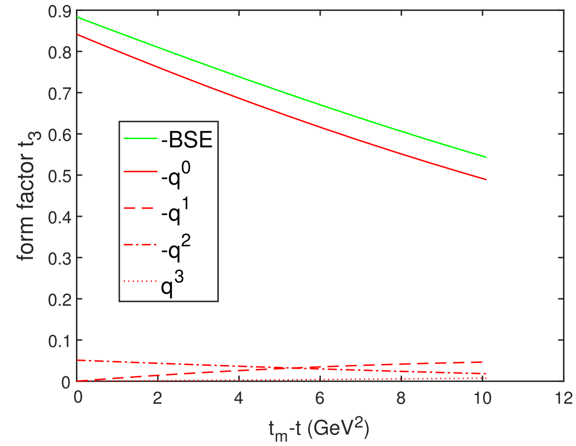
(a) $\eta_c(3S)$ (b) $\psi(3S)$ (c) $h_c(3P)$ (d) $\chi_{c0}(3P)$ (e) $\chi_{c1}(3P)$

FIG. 4. The wave functions for the radially excited state mesons.

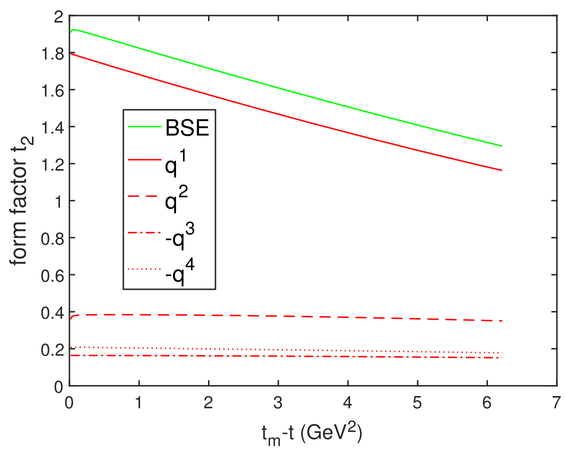
means the sum of all of expansion orders; **BS** column means the results by the BS method without expansion; **NR** means the result by the nonrelativistic wave function and the leading order expansion of the amplitude. For the B_c decays to the $1S$ charmonium, the nonrelativistic results (NR) are very close to those by the BS method. But for the orbitally or radially excited charmonium as the final state, the NR results are closer to the leading orders ($\vec{q}^0$) rather



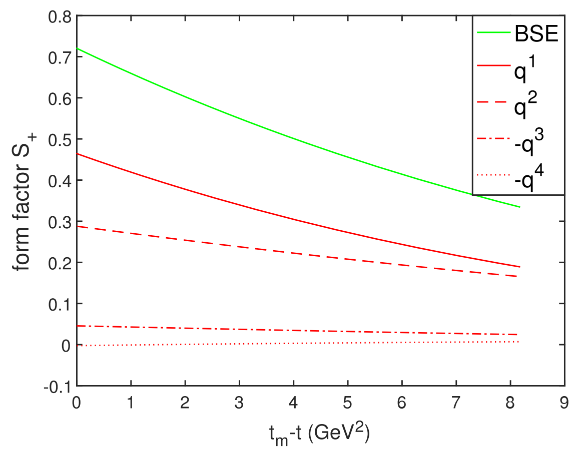
(a) $B_c^+ \to \eta_c e^+ \nu_e$



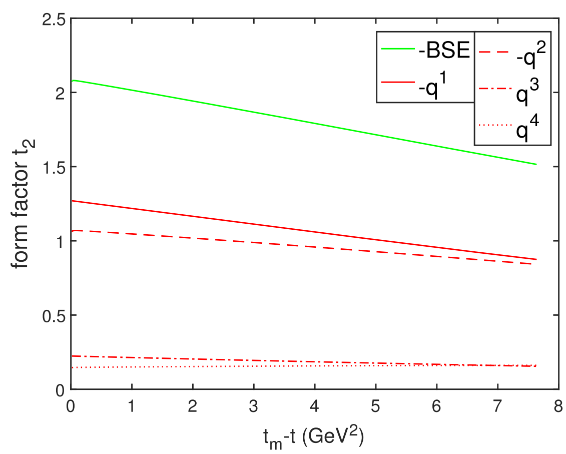
(b) $B_c^+ \to J/\psi e^+ \nu_e$



(c) $B_c^+ \to h_c e^+ \nu_e$



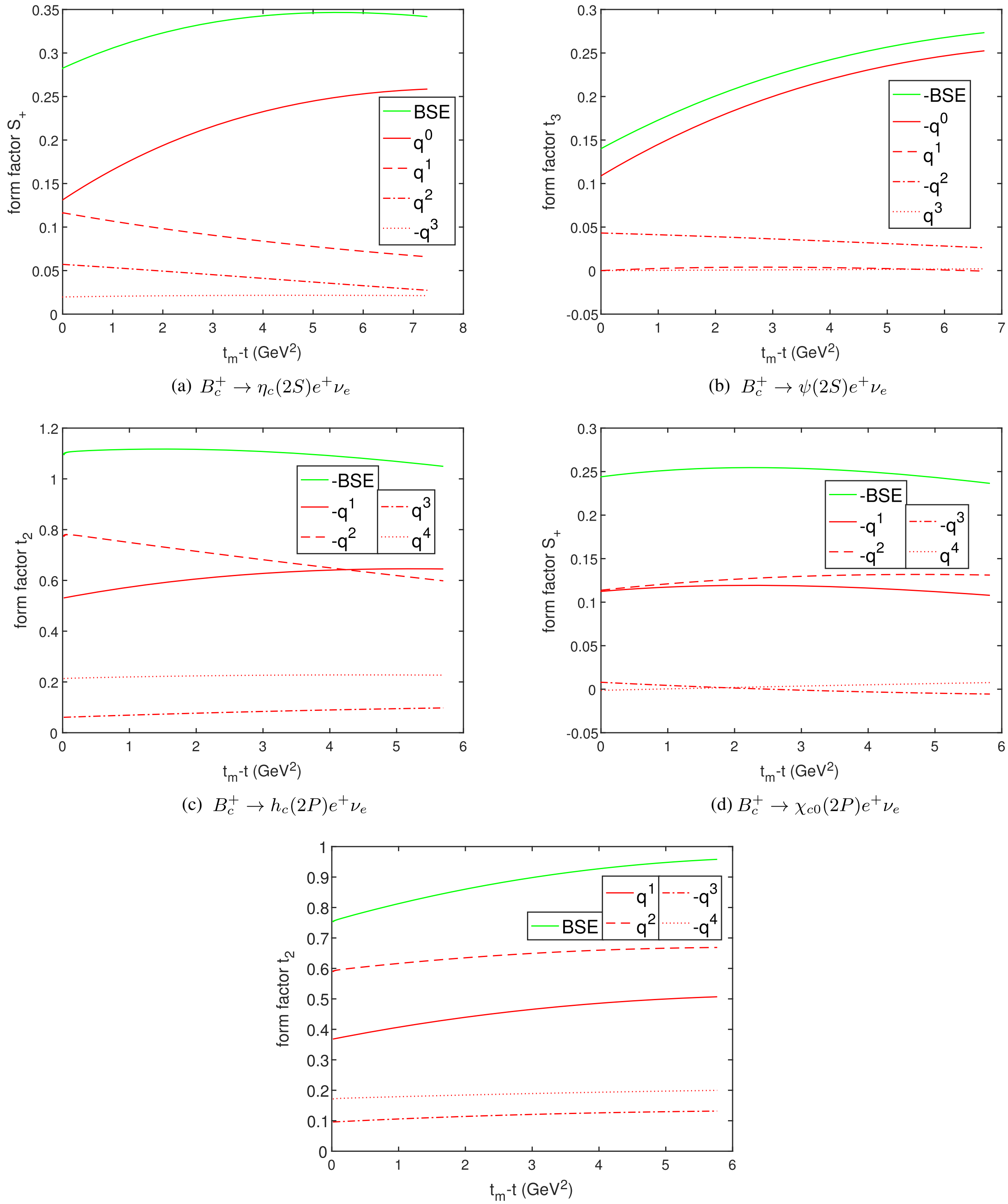
(d) $B_c^+ \to \chi_{c0} e^+ \nu_e$



(e) $B_c^+ \to \chi_{c1} e^+ \nu_e$

FIG. 5. Form factors of the B_c decays to $1S$ and $1P$ charmonium.

than the BS results. Therefore, we conclude that the nonrelativistic approach may be a good approximation for the ground heavy meson, but it is unreasonable for the excited state. The sum of all of expansion orders are not exactly equal to the BS results, which means the convergence rate is not fast enough. In the last column, we show the differences between them and conclude that high order $\vec{q}^n$ $(n > 6)$ contributions are still important for the



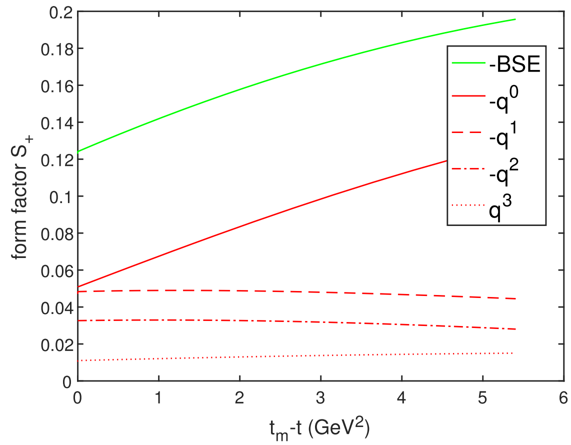
(a) $B_c^+ \to \eta_c(2S)e^+\nu_e$ (b) $B_c^+ \to \psi(2S)e^+\nu_e$

(c) $B_c^+ \to h_c(2P)e^+\nu_e$ (d) $B_c^+ \to \chi_{c0}(2P)e^+\nu_e$

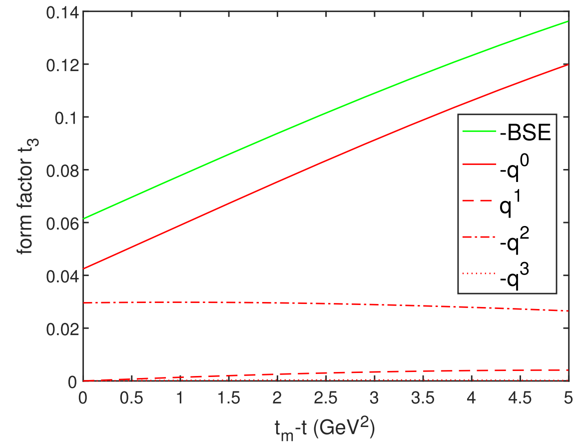
(e) $B_c^+ \to \chi_{c1}(2P)e^+\nu_e$

FIG. 6. Form factors of the B_c decays to $2S$ and $2P$ charmonium.

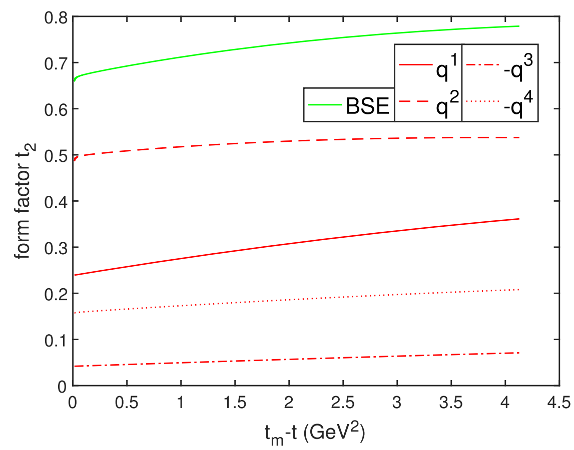
radially excited states. We provide a supplementary method II to see the relativistic effects. The details are given in Sec. V, and numerical results are shown in Table II. The main difference between the two methods is the treatment of $\vec{q}^2$ in ω_1 and ω_2. The amplitude is expanded in powers of $|\vec{q}|/m_i$ or $|\vec{q}|/M$ in method I, which suffers from the problem of convergence. In the method II, the relativistic correction is based on the wave functions. It has no problem



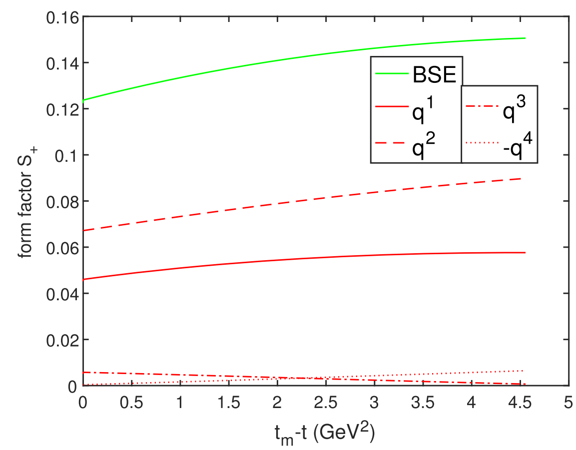
(a) $B_c^+ \to \eta_c(3S)e^+\nu_e$



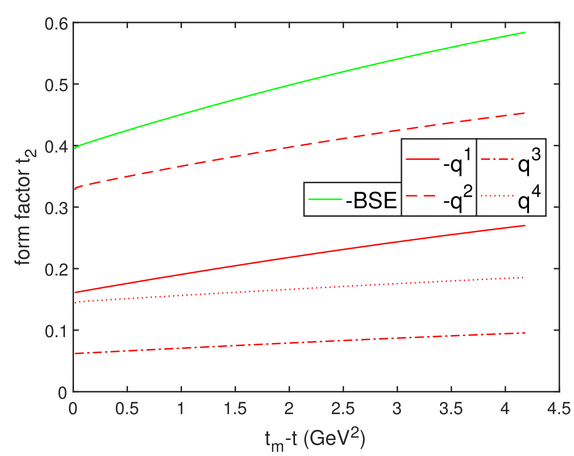
(b) $B_c^+ \to \psi(3S)e^+\nu_e$



(c) $B_c^+ \to h_c(3P)e^+\nu_e$



(d) $B_c^+ \to \chi_{c0}(3P)e^+\nu_e$



(e) $B_c^+ \to \chi_{c1}(3P)e^+\nu_e$

FIG. 7. Form factors of the B_c decays to $3S$ and $3P$ charmonium.

with convergence but suffers from the overlapping problem.

For the highly excited charmonium as final states, the first order relativistic corrections may be larger than the leading order. This special behavior can be understood. In the transition of $B_c \to \chi_{c0}$, for example, the ratio of the first order expansion over LO, $T_1/T_0 \approx 80\%$ (Fig. 5) in the level of form factor. As a rough estimate, see Eq. (5.4),

TABLE I. The branch ratios of $B_c^+ \to (c\bar{c}) + e^+ + \nu_e$ in method I according to the power $\vec{q}^n$ (in 10^{-4}).

Mode	$\vec{q}^0$	$\vec{q}^1$	$\vec{q}^2$	$\vec{q}^3$	$\vec{q}^4$	$\vec{q}^5$	$\vec{q}^6$
η_c	47.4	11.0	4.10	−3.49	−0.390	−0.135	0.0811
J/ψ	157	20.2	18.8	0.517	0.150	−0.00270	0.0203
$\eta_c(2S)$	3.66	2.29	1.43	−0.294	−0.122	−0.0951	0.0284
$\psi(2S)$	6.88	1.87	2.74	0.362	0.185	0.0247	0.00849
$\eta_c(3S)$	0.408	0.339	0.293	−0.0100	−0.0133	−0.0287	0.00675
$\psi(3S)$	0.652	0.241	0.510	0.108	0.0809	0.0138	0.00282
Mode	$\vec{q}^2$	$\vec{q}^3$	$\vec{q}^4$	$\vec{q}^5$	$\vec{q}^6$	$\vec{q}^7$	$\vec{q}^8$
h_c	15.4	11.0	4.20	−0.420	−0.142	0.00354	0.0100
χ_{c0}	5.13	7.90	1.85	−1.12	−0.0765	0.0224	0.00224
χ_{c1}	7.82	1.50	2.30	0.218	−0.480	−0.0189	0.0278
$h_c(2P)$	1.75	1.86	1.11	−0.0110	−0.0810	−0.0168	0.00575
$\chi_{c0}(2P)$	0.508	1.16	0.635	−0.0776	−0.0531	0.00158	0.00121
$\chi_{c1}(2P)$	0.666	0.104	0.444	0.0265	−0.157	−0.00197	0.0159
$h_c(3P)$	0.173	0.249	0.207	0.0205	−0.0136	−0.00470	0.00103
$\chi_{c0}(3P)$	0.0731	0.220	0.170	−0.00436	−0.0183	−0.000377	0.000541
$\chi_{c1}(3P)$	0.0580	0.00784	0.0758	0.00580	−0.0379	−0.000542	0.00539

$(T_0T_1^* + T_0^*T_1)/T_0^2 \approx 160\%$ at most, so the large contribution of the first order expansion is reasonable. For the process of $B_c \to \chi_{c1}$, the ratio of the first order expansion form factor over the LO form factor is as high as 80%. However, the first order expansion of decay width is much smaller than LO decay width. It means there is cancellation when calculating $T_0T_1^* + T_0^*T_1$. Therefore it is not enough to just provide the relativistic corrections of form factors. To investigate the relativistic effects of the decay processes, the relativistic corrections should be given not only in the level of form factors, but also in the level of decay widths.

To see the whole relativistic effects in the level of decay widths, we define the ratios $\frac{\mathrm{BS-LO}}{\mathrm{BS}}$ as the relativistic

TABLE II. The branch ratios of $B_c^+ \to (c\bar{c}) + e^+ + \nu_e$ in method II (in 10^{-4}) according to Eq. (5.4).

Mode	LO	First	Second	Third	Fourth	Total(BS)
η_c	44.1	8.24	7.32	0.650	0.279	60.7
J/ψ	158	18.2	15.2	1.94	0.219	193
$\eta_c(2S)$	3.24	1.81	1.71	0.420	0.166	7.34
$\psi(2S)$	6.96	1.66	2.00	0.353	0.108	11.1
$\eta_c(3S)$	0.355	0.272	0.311	0.101	0.0475	1.09
$\psi(3S)$	0.651	0.201	0.365	0.0834	0.0388	1.34
h_c	14.7	10.2	5.21	0.688	0.0822	30.9
χ_{c0}	5.20	7.88	2.03	−0.736	0.0453	14.4
χ_{c1}	7.75	1.35	2.31	0.307	0.0292	11.8
$h_c(2P)$	1.61	1.67	1.27	0.288	0.0448	4.88
$\chi_{c0}(2P)$	0.523	1.16	0.664	0.0211	0.000490	2.37
$\chi_{c1}(2P)$	0.650	0.0804	0.406	0.0353	0.00421	1.18
$h_c(3P)$	0.159	0.228	0.222	0.0614	0.0111	0.682
$\chi_{c0}(3P)$	0.0760	0.221	0.175	0.0208	0.000717	0.493
$\chi_{c1}(3P)$	0.0562	0.00589	0.0615	0.00427	0.000765	0.129

TABLE III. Comparisons of the branch ratios of $B_c^+ \to (c\bar{c}) + e^+ + \nu_e$ obtained by different ways, where $\vec{q}^0$ means the leading order result, **sum** means the sum of all of expansion orders, **BS** means the result by BS method without expansion, and **NR** means the result by the nonrelativistic wave function and the leading order expansion of the amplitude (in 10^{-4} except the last column).

Mode	$\vec{q}^0$	Sum	BS	NR	$\frac{\mathrm{BS-sum}}{\mathrm{BS}}$
η_c	47.4	58.6	60.7	56.7	3.4%
J/ψ	157	197	193	188	−1.8%
$\eta_c(2S)$	3.66	6.90	7.34	4.48	6.0%
$\psi(2S)$	6.88	12.1	11.1	8.40	−8.8%
$\eta_c(3S)$	0.408	0.995	1.09	0.509	8.7%
$\psi(3S)$	0.652	1.61	1.34	0.806	−20%
Mode	$\vec{q}^2$	Sum	BS	NR	$\frac{\mathrm{BS-sum}}{\mathrm{BS}}$
h_c	15.4	30.0	30.9	18.8	2.9%
χ_{c0}	5.13	13.7	14.4	6.28	4.8%
χ_{c1}	7.82	11.4	11.8	9.60	2.8%
$h_c(2P)$	1.75	4.62	4.88	2.18	5.3%
$\chi_{c0}(2P)$	0.508	2.17	2.37	0.633	8.4%
$\chi_{c1}(2P)$	0.666	1.10	1.18	0.853	7.2%
$h_c(3P)$	0.173	0.633	0.682	0.220	7.1%
$\chi_{c0}(3P)$	0.0731	0.440	0.493	0.0923	11%
$\chi_{c1}(3P)$	0.0580	0.114	0.129	0.0735	11%

TABLE IV. The relativistic effects of $B_c^+ \to (c\bar{c})e^+\nu_e$: $\frac{\mathrm{BS-LO}}{\mathrm{BS}}$ from the two methods (in %).

Method	η_c	J/ψ	$\eta_c(2S)$	$\psi(2S)$	$\eta_c(3S)$	$\psi(3S)$
I	21.9	18.8	50.2	38.0	62.5	51.3
II	27.3	18.5	55.8	37.2	67.3	51.5

TABLE V. The relativistic effects of $B_c^+ \to (c\bar{c})e^+\nu_e$: $\frac{\mathrm{BS-LO}}{\mathrm{BS}}$ from the two methods (in %).

Method	h_c	χ_{c0}	χ_{c1}	$h_c(2P)$	$\chi_{c0}(2P)$	$\chi_{c1}(2P)$	$h_c(3P)$	$\chi_{c0}(3P)$	$\chi_{c1}(3P)$
I	50.2	64.4	33.7	64.1	78.5	43.3	74.6	85.2	54.9
II	52.5	63.9	34.0	67.0	77.9	44.7	76.7	84.6	56.3

TABLE VI. The branch ratios of $B_c^+ \to (c\bar{c}) + \tau^+ + \nu_\tau$ in method I according to the power $\vec{q}^n$ (in 10^{-5}).

Mode	$\vec{q}^0$	$\vec{q}^1$	$\vec{q}^2$	$\vec{q}^3$	$\vec{q}^4$	$\vec{q}^5$	$\vec{q}^6$
η_c	158	19.0	20.0	−5.16	0.109	−0.333	0.0832
J/ψ	405	33.7	48.4	1.20	0.842	0.00264	0.0264
$\eta_c(2S)$	2.01	0.870	1.35	0.0262	0.116	−0.0543	0.00792
$\psi(2S)$	2.99	0.490	1.66	0.121	0.183	0.00833	0.00288
$\eta_c(3S)$	0.0430	0.0187	0.0534	0.00462	0.0120	−0.00245	0.000284
$\psi(3S)$	0.0493	0.00698	0.0571	0.00450	0.0149	0.000764	0.000151
Mode	$\vec{q}^2$	$\vec{q}^3$	$\vec{q}^4$	$\vec{q}^5$	$\vec{q}^6$	$\vec{q}^7$	$\vec{q}^8$
h_c	9.00	6.28	5.61	0.310	0.0620	0.00222	0.00155
χ_{c0}	4.12	7.74	5.65	−0.886	−0.0110	0.00549	0.000419
χ_{c1}	6.84	0.915	3.14	0.226	−0.522	−0.0139	0.0232
$h_c(2P)$	0.141	0.163	0.292	0.0292	0.00428	−0.000102	0.000126
$\chi_{c0}(2P) \times 10^{-2}$	4.98	15.5	26.7	0.272	−0.397	−0.00241	0.00344
$\chi_{c1}(2P)$	0.114	0.0124	0.157	0.00599	−0.0520	−0.000127	0.00480
$h_c(3P) \times 10^{-3}$	0.444	0.779	4.01	0.270	−0.000403	0.00639	0.00111
$\chi_{c0}(3P) \times 10^{-3}$	0.721	3.36	12.5	0.278	−0.406	−0.00608	0.00385
$\chi_{c1}(3P) \times 10^{-3}$	0.443	0.0467	2.36	0.0544	−1.22	−0.00180	0.165

effects and show them in Tables IV and V. Except for the η_c case, the two methods give consistent results of the relativistic effects. The method I suffers from the problem of convergence of $|\vec{q}|$ expansion, but it dose not affect the leading order; the method II suffers from the overlapping problem because ω_i $(i = 1,\ 2)$ has no expansion on $|\vec{q}|$. But in leading order, if we use the approximate formula $\omega_i = m_i + \vec{q}^2/2m_i$, the difference between the two methods is left to the $\vec{q}^2$ order. Therefore both methods give accurate estimate of the relativistic corrections. For the ground states, the relativistic effects are about 20%; for the excited states, we

TABLE VII. The decay fractions of the $B_c^+ \to (c\bar{c}) + \tau^+ + \nu_\tau$ in method II according to Eq. (5.4) (in 10^{-5}).

Mode	LO	First	Second	Third	Fourth	Total(BS)
η_c	152	14.6	24.5	1.18	0.953	193
J/ψ	406	30.5	40.9	3.57	0.507	481
$\eta_c(2S)$	1.89	0.712	1.33	0.234	0.199	4.36
$\psi(2S)$	2.96	0.422	1.30	0.128	0.0991	4.91
$\eta_c(3S)$	0.0417	0.0159	0.0473	0.00865	0.0115	0.125
$\psi(3S)$	0.0464	0.00548	0.0418	0.00351	0.00707	0.104
h_c	8.85	6.16	5.99	0.523	0.0664	21.6
χ_{c0}	4.15	7.78	5.86	−0.657	0.0296	17.2
χ_{c1}	6.82	0.850	3.01	0.233	0.0177	10.9
$h_c(2P)$	0.136	0.157	0.302	0.0374	0.00715	0.640
$\chi_{c0}(2P)$	0.0509	0.157	0.278	0.0125	0.000336	0.497
$\chi_{c1}(2P)$	0.112	0.0104	0.131	0.00457	0.000899	0.260
$h_c(3P)$	0.000446	0.000779	0.00408	0.000274	0.0000527	0.00563
$\chi_{c0}(3P)$	0.000748	0.00343	0.0127	0.000568	0.0000205	0.0175
$\chi_{c1}(3P)$	0.000432	0.0000405	0.00169	0.0000267	0.00000797	0.00220

TABLE VIII. Comparisons of the branch ratios of $B_c^+ \to (c\bar{c}) + \tau^+ + \nu_\tau$ obtained by different ways, where $\vec{q}^0$ means the leading order result, **sum** means the summed values of expanded items, **BS** means the result by BS method without expansion, and **NR** means the result by the nonrelativistic wave function and the leading order expansion of the amplitude (in 10^{-5} except the last column).

Mode	$\vec{q}^0$	Sum	BS	NR	$\frac{\text{BS}-\text{sum}}{\text{BS}}$
η_c	158	191	193	189	1.0%
J/ψ	405	489	481	485	−1.6%
$\eta_c(2S)$	2.01	4.33	4.36	2.46	0.7%
$\psi(2S)$	2.99	5.45	4.91	3.65	−11%
$\eta_c(3S)$	0.0430	0.130	0.125	0.0536	−4.0%
$\psi(3S)$	0.0495	0.134	0.104	0.0609	−29%
Mode	$\vec{q}^2$	Sum	BS	NR	$\frac{\text{BS}-\text{sum}}{\text{BS}}$
h_c	9.00	21.3	21.6	11.0	1.4%
χ_{c0}	4.12	16.6	17.2	5.04	3.1%
χ_{c1}	6.86	10.6	10.9	8.39	2.9%
$h_c(2P)$	0.141	0.628	0.640	0.175	1.9%
$\chi_{c0}(2P)$	0.0498	0.470	0.497	0.0621	5.3%
$\chi_{c1}(2P)$	0.114	0.241	0.260	0.153	7.0%
$h_c(3P)$	0.000444	0.00551	0.00563	0.000562	2.1%
$\chi_{c0}(3P)$	0.000721	0.0164	0.0175	0.000911	6.1%
$\chi_{c1}(3P)$	0.000443	0.00185	0.00220	0.000562	16%

TABLE IX. The relativistic effects of $B_c^+ \to (c\bar{c})\tau^+\nu_\tau$: $\frac{\text{BS}-\text{LO}}{\text{BS}}$ from the two methods (in %).

Method	η_c	J/ψ	$\eta_c(2S)$	$\psi(2S)$	$\eta_c(3S)$	$\psi(3S)$
I	18.4	15.8	53.8	39.1	65.6	52.6
II	21.3	15.7	56.7	39.7	66.6	55.3

TABLE X. The relativistic effects of $B_c^+ \to (c\bar{c})\tau^+\nu_\tau$: $\frac{\text{BS}-\text{LO}}{\text{BS}}$ from the two methods (in %).

Method	h_c	χ_{c0}	χ_{c1}	h_c (2P)	χ_{c0} (2P)	χ_{c1} (2P)	h_c (3P)	χ_{c0} (3P)	χ_{c1} (3P)
I	58.3	76.0	37.2	78.0	90.0	56.2	92.1	95.9	79.8
II	59.0	75.8	37.6	78.8	89.8	56.8	92.1	95.7	80.3

reach much larger relativistic effects. The smallest one is χ_{c1} case, which is 33%. The largest one is $\chi_{c0}(3P)$ case, which is 85%.

We also calculate the processes $B_c^+ \to (c\bar{c})\tau^+\nu_\tau$, as Tables VI–IX and X shown.

VII. CONCLUSION

In this paper, we choose the instantaneous BS method to calculate the semileptonic B_c decays to charmonium, whose final states include 1S, 2S, 3S, 1P, 2P and 3P. We focus on the relativistic effects. Two methods are provided to see the relativistic corrections. In the first method, the amplitude is expanded in powers of $\vec{q}$ which is the relative momentum between the quark and the antiquark. We find this widely used method suffers from the problem of convergence though the quark and antiquark are heavy, and the high order $\vec{q}^n$ items still have sizable contributions. The other method is based on the relativistic wave functions. In this method there is no problem of convergence, while it suffers from the overlapping problem. Both methods give accurate and consistent leading order (nonrelativistic) contributions. In another words, both methods provide accurate relativistic corrections. We find that for the semileptonic B_c decays, the relativistic effects are about 20% when final states are 1S charmonium (η_c and J/ψ), but the relativistic effects are much higher for the excited final states. First of all, the nP final state has larger relativistic corrections than the corresponding nS state; secondly, the $(n+1)S$ state has larger relativistic corrections than the corresponding nS state; thirdly, the $(n+1)P$ state has larger relativistic corrections than the corresponding nP state. Therefore, we conclude that even though the higher excited state has a higher mass, its relativistic effect is larger. A relativistic method is needed to deal with a problem including a excited state, though the corresponding quark and antiquark are heavy.

ACKNOWLEDGMENTS

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APPENDIX A: EQUATION AND SOLUTION FOR HEAVY MESONS

BS equation for a quark-antiquark bound state generally is written as [49]

$$(\not{p}_1 - m_1)\chi_P(q)(\not{p}_2 + m_2) = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4} V(P,k,q)\chi_P(k), \tag{A1}$$

where p_1, p_2; m_1, m_2 are the momenta and masses of the quark and antiquark, respectively, $\chi_P(q)$ is the BS wave function with the total momentum P and relative momentum q, $V(P,k,q)$ is the kernel between the quark-antiquark in the bound state. P and q are defined as

$$\vec{p}_1 = \alpha_1\vec{P} + \vec{q}, \qquad \alpha_1 = \frac{m_1}{m_1+m_2},$$
$$\vec{p}_2 = \alpha_2\vec{P} - \vec{q}, \qquad \alpha_2 = \frac{m_2}{m_1+m_2}. \tag{A2}$$

The instantaneous kernel has the following form:

$$V(P,k,q) \sim V(|k-q|), \tag{A3}$$

especially when the two constituents of meson are very heavy. The kernel we used contains a linear scalar interaction for color-confinement, a vector interaction for one-gluon exchange and a constant V_0 which as a "zero-point", i.e.,

$$I(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3}\frac{\alpha_s(r)}{r}, \tag{A4}$$

where λ is the "string constant", $\alpha_s(r)$ is the running coupling constant. In order to avoid the infrared divergence, a factor $e^{-\alpha r}$ is introduced, i.e.,

$$\begin{aligned} V_s(r) &= \frac{\lambda}{\alpha}(1 - e^{-\alpha r}), \\ V_v(r) &= -\frac{4}{3}\frac{\alpha_s(r)}{r} e^{-\alpha r}. \end{aligned} \tag{A5}$$

In momentum space the kernel reads

$$I(\vec{q}) = V_s(\vec{q}) + \gamma_0 \otimes \gamma^0 V_v(\vec{q}), \tag{A6}$$

where

$$\begin{aligned} V_s(\vec{q}) &= -\left(\frac{\lambda}{\alpha} + V_0\right)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{q}^2 + \alpha^2)^2}, \\ V_v(\vec{q}) &= -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{q})}{\vec{q}^2 + \alpha^2}, \\ \alpha_s(\vec{q}) &= \frac{12\pi}{27}\frac{1}{\ln(a + \vec{q}^2/\Lambda^2_{\mathrm{QCD}})}. \end{aligned} \tag{A7}$$

The fitted parameters are $a = e = 2.7183$, $\alpha = 0.06$ GeV, $\lambda = 0.21$ GeV^2, $\Lambda_{\mathrm{QCD}} = 0.27$ GeV; V_0 is fixed by fitting the mass of the ground state.

We divide the relative momentum q into two parts, $q_{P_\parallel}$ and $q_{P_\perp}$, a parallel part and an orthogonal one to P, respectively,

$$q^\mu = q^\mu_{P_\parallel} + q^\mu_{P_\perp}, \tag{A8}$$

where $q^\mu_{P_\parallel} \equiv (P\cdot q/M^2)P^\mu$, $q^\mu_{P_\perp} \equiv q^\mu - q^\mu_{P_\parallel}$, and M is the mass of the relevant meson. Correspondingly, we have two Lorentz-invariant variables

$$q_P = \frac{P\cdot q}{M}, \qquad q_{P_T} = \sqrt{q_P^2 - q^2} = \sqrt{-q^2_{P_\perp}}. \tag{A9}$$

If we introduce two notations as below

$$\begin{aligned} \eta(q^\mu_{P_\perp}) &\equiv \int \frac{k^2_{P_T}\mathrm{d}k_{P_T}\mathrm{d}s}{(2\pi)^2} V(k_{P_\perp}, s, q_{P_\perp})\varphi(k^\mu_{P_\perp}), \\ \varphi(q^\mu_{P_\perp}) &\equiv \mathrm{i}\int \frac{\mathrm{d}q_P}{2\pi}\chi_P(q^\mu_{P_\parallel}, q^\mu_{P_\perp}), \end{aligned} \tag{A10}$$

then the BS equation can take the form as follows:

$$\chi_P(q^\mu_{P_\parallel}, q^\mu_{P_\perp}) = S_1(p_1^\mu)\eta(q^\mu_{P_\perp})S_2(p_2^\mu). \tag{A11}$$

The propagators of the relevant particles with masses m_1 and m_2 can be decomposed as

$$\begin{aligned} S_i(p_i^\mu) = &\frac{\Lambda^+_{i_P}(q^\mu_{P_\perp})}{J(i)q_P + \alpha_i M - \omega_{i_P} + \mathrm{i}\varepsilon} \\ &+ \frac{\Lambda^-_{i_P}(q^\mu_{P_\perp})}{J(i)q_P + \alpha_i M + \omega_{i_P} - \mathrm{i}\varepsilon}, \end{aligned} \tag{A12}$$

with

$$\begin{aligned} \omega_{i_P} &= \sqrt{m_i^2 + q^2_{P_T}}, \\ \Lambda^\pm_{i_P}(q^\mu_{P_\perp}) &= \frac{1}{2\omega_{i_P}}\left[\frac{\not{P}}{M}\omega_{i_P} \pm J(i)(\not{q}_{P_\perp} + m_i)\right], \end{aligned} \tag{A13}$$

where $i = 1, 2$ for quark and antiquark, respectively, and $J(i) = (-1)^{i+1}$.

Then the instantaneous Bethe-Salpeter equation can be decomposed into the coupled equations

$$\begin{aligned} (M - \omega_{1p} - \omega_{2p})\varphi^{++}(q_{P_\perp}) &= \Lambda_1^+(P_{1p_\perp})\eta(q_{P_\perp})\Lambda_2^+(P_{2p_\perp}), \\ (M + \omega_{1p} + \omega_{2p})\varphi^{--}(q_{P_\perp}) &= -\Lambda_1^-(P_{1p_\perp})\eta(q_{P_\perp})\Lambda_2^-(P_{2p_\perp}), \\ \varphi^{+-}(q_{P_\perp}) = 0, &\qquad \varphi^{-+}(q_{P_\perp}) = 0. \end{aligned} \tag{A14}$$

The instantaneous Bethe-Salpeter wave function for 1^- states mesons has the general form [44–47,50],

$$\begin{aligned} &\varphi_{1^-}(q_\perp) \\ &= (q_\perp\cdot\epsilon)\left[g_1(q_\perp) + \frac{\not{P}}{M}g_2(q_\perp) + \frac{\not{q}_\perp}{M}g_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}g_4(q_\perp)\right] \\ &\quad + M\not{\epsilon}\left[g_5(q_\perp) + \frac{\not{P}}{M}g_6(q_\perp) + \frac{\not{q}_\perp}{M}g_7(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}g_8(q_\perp)\right], \end{aligned} \tag{A15}$$

with

$$\begin{aligned} g_1(q_\perp) &= \frac{q_\perp^2 g_3(\omega_1 + \omega_2) + 2M^2 g_5\omega_2}{M(m_1\omega_2 + m_2\omega_1)}, \\ g_2(q_\perp) &= \frac{q_\perp^2 g_4(\omega_1 + \omega_2) + 2M^2 g_6\omega_2}{M(m_1\omega_2 + m_2\omega_1)}, \\ g_7(q_\perp) &= \frac{M(\omega_1 - \omega_2)}{m_1\omega_2 + m_2\omega_1}g_5, \\ g_8(q_\perp) &= \frac{M(\omega_1 + \omega_2)}{m_1\omega_2 + m_2\omega_1}g_6. \end{aligned} \tag{A16}$$

The wave function corresponding to the positive projection has the form

$$\varphi^{++}_{1^{-}}(q_\perp) = (q_\perp\cdot\epsilon)\left[B_1(q_\perp) + \frac{\not{P}}{M}B_2(q_\perp) + \frac{\not{q}_\perp}{M}B_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}B_4(q_\perp)\right] + M\not{\epsilon}\left[B_5(q_\perp) + \frac{\not{P}}{M}B_6(q_\perp) + \frac{\not{q}_\perp}{M}B_7(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}B_8(q_\perp)\right], \tag{A17}$$

where

$$\begin{aligned}
B_1 &= \frac{1}{2M(m_1\omega_2 + m_2\omega_1)}[(\omega_1+\omega_2)q_\perp^2 g_3 + (m_1+m_2)q_\perp^2 g_4 + 2M^2\omega_2 g_5 - 2M^2 m_2 g_6],\\
B_2 &= \frac{1}{2M(m_1\omega_2 + m_2\omega_1)}[(m_1-m_2)q_\perp^2 g_3 + (\omega_1-\omega_2)q_\perp^2 g_4 + 2M^2\omega_2 g_6 - 2M^2 m_2 g_5],\\
B_3 &= \frac{1}{2}\left[g_3 + \frac{m_1+m_2}{\omega_1+\omega_2}g_4 - \frac{2M^2}{m_1\omega_2+m_2\omega_1}g_6\right],\\
B_4 &= \frac{1}{2}\left[\frac{\omega_1+\omega_2}{m_1+m_2}g_3 + g_4 - \frac{2M^2}{m_1\omega_2+m_2\omega_1}g_5\right],\\
B_5 &= \frac{1}{2}\left[g_5 - \frac{\omega_1+\omega_2}{m_1+m_2}g_6\right], \qquad B_6 = \frac{1}{2}\left[-\frac{m_1+m_2}{\omega_1+\omega_2}g_5 + g_6\right],\\
B_7 &= \frac{M}{2}\frac{\omega_1-\omega_2}{m_1\omega_2+m_2\omega_1}\left[g_5 - \frac{\omega_1+\omega_2}{m_1+m_2}g_6\right],\\
B_8 &= \frac{M}{2}\frac{m_1+m_2}{m_1\omega_2+m_2\omega_1}\left[-g_5 + \frac{\omega_1+\omega_2}{m_1+m_2}g_6\right].
\end{aligned} \tag{A18}$$

If the masses of the quark and antiquark are equal, the normalization condition reads as

$$\int\frac{d\vec{q}}{(2\pi)^3}\frac{16\omega_1\omega_2}{3}\left\{3g_5g_6\frac{M^2}{2m_1\omega_2} + \frac{\vec{q}^2}{2m_1\omega_2}\left[g_4g_5 - g_3\left(g_4\frac{\vec{q}^2}{M^2} + g_6\right)\right]\right\} = 2M. \tag{A19}$$

The instantaneous Bethe-Salpeter wave function for 1^{+-} states mesons has the form [39,44–47],

$$\varphi_{1^{+-}}(q_\perp) = (q_\perp\cdot\epsilon)\left[h_1(q_\perp) + \frac{\not{P}}{M}h_2(q_\perp) + \frac{\not{q}_\perp}{M}h_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}h_4(q_\perp)\right]\gamma_5, \tag{A20}$$

with

$$\begin{aligned}
h_3(q_\perp) &= -\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}h_1,\\
h_4(q_\perp) &= -\frac{M(\omega_1+\omega_2)}{m_1\omega_2+m_2\omega_1}h_2.
\end{aligned} \tag{A21}$$

The wave function corresponding to the positive projection has the form,

$$\varphi^{++}_{1^{+-}}(q_\perp) = q_\perp\cdot\epsilon\left[C_1(q_\perp) + \frac{\not{P}}{M}C_2(q_\perp) + \frac{\not{q}_\perp}{M}C_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}C_4(q_\perp)\right]\gamma^5, \tag{A22}$$

where

$$C_1 = \frac{1}{2}\left[h_1 + \frac{\omega_1+\omega_2}{m_1+m_2}h_2\right],$$
$$C_2 = \frac{1}{2}\left[\frac{m_1+m_2}{\omega_1+\omega_2}h_1 + h_2\right],$$
$$C_3 = -\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}C_1,$$
$$C_4 = -\frac{M(m_1+m_2)}{m_1\omega_2+m_2\omega_1}C_1. \tag{A23}$$

If the masses of the quark and antiquark are equal, the normalization condition reads as

$$\int \frac{d\vec{q}}{(2\pi)^3}\frac{4h_1h_2\omega_1\vec{q}^2}{3m_1} = M. \tag{A24}$$

The instantaneous Bethe-Salpeter wave function for 0^+ states mesons has the form [39,44–47],

$$\varphi_{0^+}(q_\perp) = M\left[\frac{\not{q}_\perp}{M}\phi_1(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}\phi_2(q_\perp) + \phi_3(q_\perp) + \frac{\not{P}}{M}\phi_4(q_\perp)\right], \tag{A25}$$

with

$$\phi_3(q_\perp) = \frac{q_\perp^2(\omega_1+\omega_2)}{M(m_1\omega_2+m_2\omega_1)}\phi_1,$$
$$\phi_4(q_\perp) = \frac{q_\perp^2(\omega_1-\omega_2)}{M(m_1\omega_2+m_2\omega_1)}\phi_2. \tag{A26}$$

The wave function corresponding to the positive projection has the form

$$\varphi_{0^+}^{++}(q_\perp) = D_1(q_\perp) + \frac{\not{P}}{M}D_2(q_\perp) + \frac{\not{q}_\perp}{M}D_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}D_4(q_\perp), \tag{A27}$$

where

$$D_1 = \frac{(\omega_1+\omega_2)q_\perp^2}{2(m_1\omega_2+m_2\omega_1)}\left[\phi_1 + \frac{m_1+m_2}{\omega_1+\omega_2}\phi_2\right],$$
$$D_2 = \frac{(m_1-m_2)q_\perp^2}{2(m_1\omega_2+m_2\omega_1)}\left[\phi_1 + \frac{m_1+m_2}{\omega_1+\omega_2}\phi_2\right],$$
$$D_3 = \frac{M}{2}\left[\phi_1 + \frac{m_1+m_2}{\omega_1+\omega_2}\phi_2\right],$$
$$D_4 = \frac{M}{2}\left[\frac{\omega_1+\omega_2}{m_1+m_2}\phi_1 + \phi_2\right]. \tag{A28}$$

If the masses of the quark and antiquark are equal, the normalization condition reads as

$$\int \frac{d\vec{q}}{(2\pi)^3}\frac{4\phi_1\phi_2\omega_1\vec{q}^2}{m_1} = M. \tag{A29}$$

The instantaneous Bethe-Salpeter wave function for 1^{++} states mesons has the form [39,44–47],

$$\varphi_{1^{++}}(q_\perp) = i\varepsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}q_\perp^\alpha\epsilon^\beta\gamma^\mu\left[\psi_1(q_\perp) + \frac{\not{P}}{M}\psi_2(q_\perp) + \frac{\not{q}_\perp}{M}\psi_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}\psi_4(q_\perp)\right], \tag{A30}$$

with

$$\psi_3(q_\perp) = -\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}\psi_1,$$
$$\psi_4(q_\perp) = -\frac{M(\omega_1+\omega_2)}{m_1\omega_2+m_2\omega_1}\psi_2. \tag{A31}$$

The wave function corresponding to the positive projection has the form,

$$\varphi_{1^{++}}^{++}(q_\perp) = i\varepsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}q_\perp^\alpha\epsilon^\beta\gamma^\mu\left[F_1(q_\perp) + \frac{\not{P}}{M}F_2(q_\perp) + \frac{\not{q}_\perp}{M}F_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}F_4(q_\perp)\right], \tag{A32}$$

where

$$F_1 = \frac{1}{2}\left[\psi_1 + \frac{\omega_1+\omega_2}{m_1+m_2}\psi_2\right],$$
$$F_2 = -\frac{1}{2}\left[\frac{m_1+m_2}{\omega_1+\omega_2}\psi_1 + \psi_2\right],$$
$$F_3 = \frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}F_1,$$
$$F_4 = -\frac{M(m_1+m_2)}{m_1\omega_2+m_2\omega_1}F_1. \tag{A33}$$

If the masses of the quark and antiquark are equal, the normalization condition reads as

$$\int \frac{d\vec{q}}{(2\pi)^3}\frac{8\psi_1\psi_2\omega_1\vec{q}^2}{3m_1} = M. \tag{A34}$$

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