Baryonic handles: Skyrmions as open vortex strings on a domain wall

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(Received 10 September 2018; published 5 December 2018)

We consider the BEC Skyrme model, which is based on a Skyrme-type model with a potential motivated by Bose-Einstein condensates (BECs), and, in particular, we study the Skyrmions in proximity of a domain wall that inhabits the theory. The theory turns out to have a rich flora of Skyrmion solutions that manifest themselves as twisted vortex rings or vortons in the bulk and vortex handles attached to the domain wall. The latter are linked open vortex strings. We further study the interaction between the domain wall and the Skyrmion and between the vortex handles themselves as well as between the vortex handle and the vortex ring in the bulk. We find that the domain wall provides a large binding energy for the solitons and it is energetically preferred to stay as close to the domain wall as possible; other configurations sticking into the bulk are metastable. We find that the most stable 2-Skyrmion is a torus-shaped braided string junction ending on the domain wall, which is produced by a collision of two vortex handles on the wall, but there is also a metastable configuration that is a doubly twisted vortex handle produced by a collision of a vortex handle on the wall and a vortex ring from the bulk.

DOI: 10.1103/PhysRevD.98.125002

I. INTRODUCTION

Skyrmions are topological textures that are the minimizers in energy of a map from 3-space to SU(2) isospin space [1,2] and provide a low-energy effective description of baryons in large-N QCD [3,4]. In condensed matter physics, Skyrmions usually refer to two-dimensional solitons living in a magnetization vector of a suitable host material [5–8]. Nonetheless, considerable strides are being made in condensed matter physics to realize a three-dimensional Skyrmion in a two-component Bose-Einstein condensate (BEC) [9–19] (see Ref. [20] for a nice review). A potential that is motivated by two-component BECs was considered in Refs. [21–23], and it was shown that the Skyrmion is deformed into a twisted vortex ring (or vorton) due to the explicit breaking of the SU(2) symmetry that is normally possessed by Skyrmions (i.e., isospin symmetry).

A convenient parametrization of the Skyrme field (chiral Lagrangian field) is to write the O(4) vector field as a two-component complex scalar field, $\phi_{1,2}$, with the constraint $|\phi_1|^2 + |\phi_2|^2 = 1$. The BEC-inspired potential takes the form $V = M^2|\phi_1|^2|\phi_2|^2$, and thus there are two different vacua; either ϕ_1 or ϕ_2 must vanish in the vacuum. In the vacuum in which one component vanishes, the vacuum manifold S^1 is parametrized by the phase of the other component. For instance, in the vacuum in which ϕ_2 vanishes, $|\phi_1| = 1$ and vice versa. Hence, there exists a global vortex having logarithmically divergent energy [21–23]. In the vortex core, the other component is confined; for instance, in the core of a vortex having a winding in the ϕ_1 component, the ϕ_2 component is confined, which yields a U(1) modulus. This is a global analog of the superconducting cosmic string [24]. These properties are also shared by a model with the potential $V = M^2|\phi_1|^2$ [25]. A Skyrmion in the BEC-Skyrme model takes the form of a twisted vortex ring or a vorton, i.e., a vortex ring along which the phase of the confined component winds from 0 to 2π. If it is twisted B times (i.e., it winds from 0 to $2\pi B$), then it carries baryon number B. This was actually first found in two-component BECs [9–19] and is shared by this model as well as the above-mentioned model with potential $V = M^2|\phi_1|^2$ [25].

The BEC-Skyrme model also admits a domain wall interpolating between the two vacua. Vortices can terminate on the domain wall, just like in two-component BECs [19,26–28] (see also Ref. [29]). This configuration resembles a D-brane in string theory, thereby called a D-brane soliton [30–33]. Vortex end points are also called boojums, since they resemble those in helium-3 superfluids [34,35]. One question is what happens if the two ends of a vortex terminate on the same domain wall. A Skyrme model with

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the potential term $V = M^2(\Re(\phi_1))^2$ also admits a domain wall, and Skyrmions are absorbed into the wall becoming lumps or baby Skyrmions on the wall [36–38]. Thus, a further question is what happens to the Skyrmions in the presence of a domain wall in the BEC-Skyrme model. We will answer this question with the results of this paper.

In this paper, we will study all the physics in the proximity of the above-mentioned domain wall. If we start in the bulk and place a Skyrmion, it is a twisted vortex ring, as mentioned above. If the distance to the domain wall is not too large, an attractive force will pull the Skyrmion into the domain wall, and it will be absorbed. The vortex ring is then connected to the vacuum on the other side of the domain wall, and it thus manifests itself as a handle sitting on the domain wall, sticking into the bulk of the side it came from. This description sounds asymmetric, and, in fact, we shall discover in this paper that the true energy minimizer symmetrizes itself to become symmetric under the exchange of ϕ_1 and ϕ_2; it becomes a link of two vortex handles. This is in stark contrast to the above-mentioned model with the potential $V = M^2(\Re(\phi_1))^2$, in which case a Skyrmion absorbed into the wall becomes a baby-Skyrmion and no complicated structure appears [36–38]. The attraction between the vortex ring and the domain wall is also a result of a large reduction of the static energy by absorption into the domain wall. We perform an explicit calculation and find that there is a large binding energy for the vortex ring to gain at the cost of transforming itself into a handle sitting on the domain wall.

We shall also study the interactions between two vortex handles on the domain wall. Since, as we already spoiled, the handle configuration will turn out to be symmetric, it does not matter on which side of the domain wall the handle sits. We find that there is an attractive channel and a repulsive channel between the two vortex handles. If we place the handles in the attractive channel, they will combine themselves into a braided string junction of a toroidal shape—a 2-Skyrmion absorbed into the domain wall.

We also investigate the interaction between the vortex handle and the vortex ring in the bulk and find that they always attract each other (or quickly rotate themselves into the attractive channel). Next, the vortex ring goes through a string reconnection mechanism to transform the configuration into a doubly twisted vortex handle, which is the analog of a vortex ring that is twisted two times (hence baryon number 2) being absorbed into the domain wall.

Finally, we compare the energies of the two Skyrmion configurations with baryon number 2 and find that the braided string junction has less energy than the doubly twisted vortex handle.

This paper is organized as follows. In Sec. II, we will briefly review the BEC Skyrme model and its symmetry and vacuum properties. Section III A reviews the domain wall. In Sec. III B, we construct the boojum for the first time in the BEC-Skyrme model; it is a semi-infinite string attached to the domain wall. In Sec. IV A, the vortex handle is constructed. In Sec. IV B, the interaction between the domain wall and the vortex ring is studied. In Sec. IV C, we add a twist to the modulus of the vortex handle, producing a 2-Skyrmion. In Sec. IV D, we study the interactions between two vortex handles and find a new 2-Skyrmion: the braided string junction of toroidal shape. Section IV E considers the interaction between the vortex handle on a wall and the vortex ring in the bulk, which reproduces the doubly twisted vortex handle found in Sec. IV C. In Sec. IV F, the energies of the two 2-Skyrmions are compared. Section IV G considers the construction of higher-charged configurations. Finally, we conclude with a discussion in Sec. V.

II. BEC SKYRME MODEL

The model that we will consider in this paper is the generalized Skyrme model, consisting of the kinetic term, the Skyrme term [1,2], the BPS Skyrme term [39,40],

$$\mathcal{L} = \mathcal{L}_2 + c_4\mathcal{L}_4 + c_6\mathcal{L}_6 - V, \tag{1}$$

$$\mathcal{L}_2 = -\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi, \tag{2}$$

$$\begin{aligned}\mathcal{L}_4 &= \frac{1}{8}(\partial_\mu\phi^\dagger\partial_\nu\phi)(\partial^{[\mu}\phi^\dagger\partial^{\nu]}\phi)\\ &\quad+\frac{1}{8}(\partial_\mu\phi^\dagger\sigma^2\partial_\nu\phi)(\partial^{[\mu}\phi^\dagger\sigma^2\partial^{\nu]}\phi)\\ &= -\frac{1}{4}(\partial_\mu\phi^\dagger\partial^\mu\phi)^2 + \frac{1}{16}(\partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi)^2,\end{aligned} \tag{3}$$

$$\mathcal{L}_6 = \frac{1}{4}(\epsilon^{\mu\nu\rho\sigma}\phi^\dagger\partial_\nu\phi\partial_\rho\phi^\dagger\partial_\sigma\phi)^2, \tag{4}$$

and finally the BEC-inspired potential [21–23]

$$V = \frac{1}{8}M^2[1-(\phi^\dagger\sigma^3\phi)^2] = \frac{1}{2}M^2|\phi_1|^2|\phi_2|^2, \tag{5}$$

where σ^a are the Pauli matrices. The vector $\phi \equiv (\phi_1(x), \phi_2(x))^{\mathrm{T}}$ is a complex 2-vector field; the spacetime indices μ, ν, ρ, σ run over 0 through 3; the flat Minkowski metric is taken to be of the mostly positive signature; and, finally, the nonlinear sigma model constraint is imposed as $\phi^\dagger\phi = |\phi_1|^2 + |\phi_2|^2 = 1$. The relation of the complex vector field ϕ, or equivalently the two complex fields $\phi_{1,2}$, to the usual chiral Lagrangian field used in the Skyrme model is given by

$$U = (\phi \quad -i\sigma^2\bar{\phi}) = \begin{pmatrix}\phi_1 & -\bar{\phi}_2\\ \phi_2 & \bar{\phi}_1\end{pmatrix}, \tag{6}$$

and thus the nonlinear sigma model constraint reads $\det U = |\phi_1|^2 + |\phi_2|^2 = 1$.

The target space (i.e., the vacuum manifold) for $M = 0$ is $\mathcal{M} \simeq \mathrm{O}(4)/\mathrm{O}(3) \simeq \mathrm{SU}(2) \simeq S^3$, and thus the one-point compactified space $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ supports topological solitons (Skyrmions) characterized by

$$\pi_3(\mathcal{M}) = \mathbb{Z}. \tag{7}$$

The topological degree of the map ϕ is $B \in \pi_3(S^3)$ and can be calculated as

$$B = \frac{1}{4\pi^2} \int d^3x \epsilon^{ijk} \phi^\dagger \partial_i \phi \partial_j \phi^\dagger \partial_k \phi. \tag{8}$$

Once we turn on a nonvanishing potential $M > 0$, the Skyrmions survive, but the vacuum of the theory and the physics of the solitons change.

The two vacua of the model with nonvanishing potential V in (5) are

$$\begin{aligned} \odot &: \phi = (e^{i\alpha}, 0)^{\mathrm{T}}, \\ \otimes &: \phi = (0, e^{i\beta})^{\mathrm{T}}, \end{aligned} \tag{9}$$

which by the nonlinear sigma-model constraint yield the other component to be at its maximum.

The symmetry of the model with the potential V in Eq. (5) is explicitly broken from O(4) down to

$$G = \mathrm{U}(1) \times \mathrm{O}(2) \simeq \mathrm{U}(1)_0 \times [\mathrm{U}(1)_3 \rtimes (\mathbb{Z}_2)_{1,2}], \tag{10}$$

where the group is defined by the symmetries

$$\mathrm{U}(1)_0\colon \ \phi \to e^{i\alpha}\phi, \tag{11}$$

$$\mathrm{U}(1)_3\colon \ \phi \to e^{i\beta\sigma^3}\phi, \tag{12}$$

$$(\mathbb{Z}_2)_{1,2}\colon \ e^{i\pi\sigma^{1,2}/2}\phi, \tag{13}$$

and $\mathrm{U}(1)_3$ acts on $\mathbb{Z}_2$ in such a way that they define a semidirect product denoted by $\rtimes$. The unbroken symmetry groups in the vacua (9) are thus

$$H_\odot = \mathrm{U}(1)_{0-3}\colon \ \phi \to e^{i\alpha}e^{-i\alpha\sigma^3}\phi, \tag{14}$$

$$H_\otimes = \mathrm{U}(1)_{0+3}\colon \ \phi \to e^{i\alpha}e^{+i\alpha\sigma^3}\phi. \tag{15}$$

The target space (vacuum manifold) is thus given by the coset group

$$\begin{aligned} \mathcal{M} \simeq G/H &= \frac{\mathrm{U}(1)_0 \times [\mathrm{U}(1)_3 \rtimes (\mathbb{Z}_2)_{1,2}]}{\mathrm{U}(1)_{0\pm3}} \\ &\simeq \mathrm{SO}(2)_{0\mp3} \rtimes (\mathbb{Z}_2)_{1,2} = \mathrm{O}(2), \end{aligned} \tag{16}$$

and the nontrivial homotopy groups of this manifold read

$$\pi_0(\mathcal{M}) = \mathbb{Z}_2, \qquad \pi_1(\mathcal{M}) = \mathbb{Z}. \tag{17}$$

The theory thus supports both domain walls and vortices in addition to the Skyrmions.

Although we have included both the Skyrme term, $\mathcal{L}_4$, and the BPS Skyrme term, $\mathcal{L}_6$, in the model, we will only use either of the terms as follows:

$$\begin{aligned} 2+4 \text{ model}&\colon \ c_4 = 1, \quad c_6 = 0, \\ 2+6 \text{ model}&\colon \ c_4 = 0, \quad c_6 = 1. \end{aligned} \tag{18}$$

It turns out that the two models give qualitatively the same results, so we will only show some of the results for both models in the next section.

In Ref. [23], we studied the domain wall, the vortices, and the Skyrmions in one vacuum (the $H_\otimes$ vacuum).

In this paper, we study the theory in the presence of the domain wall of Ref. [23] with Skyrmions as closed or open vortex strings in proximity of the domain wall.

III. VORTICES AND THE DOMAIN WALL

We will now consider that the three-dimensional space has a domain wall separating two phases with vacua $\otimes$ and $\odot$, respectively. Without loss of generality, we will consider the vortices in the $\otimes$ phase; the results apply to the $\odot$ phase by interchanging the complex scalar fields, $\phi_1 \leftrightarrow \phi_2$.

A. Domain wall

As we will place everything in this paper in the presence of the domain wall, we will give a short review of the domain wall solution [21–23] here. Since the domain wall is a codimension-1 soliton, only the potential (5) and the kinetic term, $\mathcal{L}_2$, contribute to its energy,

$$2E = |\partial_z \phi_1|^2 + |\partial_z \phi_2|^2 + M^2 |\phi_1|^2 |\phi_2|^2. \tag{19}$$

The solution is thus

$$\phi = \frac{1}{\sqrt{1 + e^{\pm 2M(z-z_0)}}} \begin{pmatrix} e^{i\chi} \\ e^{\pm M(z-z_0)+i\vartheta} \end{pmatrix}, \tag{20}$$

with χ and ϑ being constant phase parameters. We will choose the upper sign throughout this paper, and hence the $\otimes$ vacuum is always at $z > 0$ (up) and the $\odot$ vacuum is at $z < 0$ (down). Furthermore, we will set $z_0 = 0$ from now on. This is the translational modulus of the domain wall, and we can always adjust our coordinate system such that the domain wall is at $z = 0$.

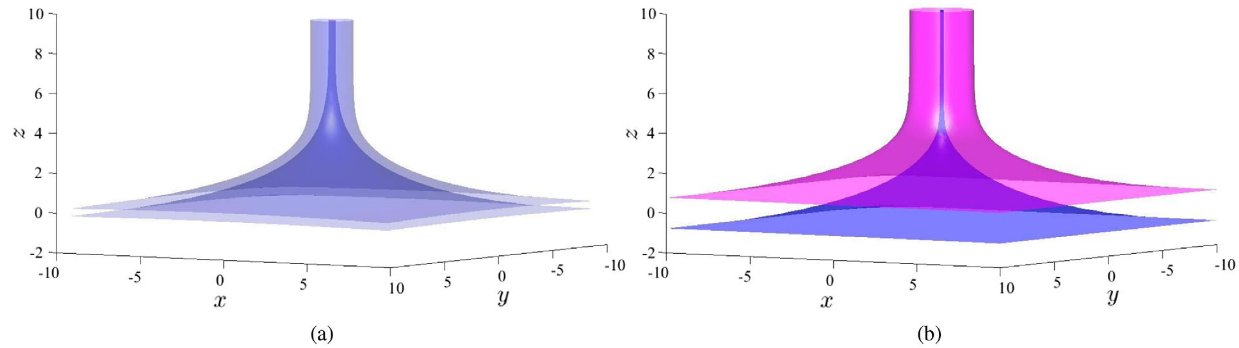

FIG. 1. The vortex string ending on the domain wall from the ⊗ phase in the 2 + 4 model: (a) energy isosurface(s), the maximum of the energy is between the two surfaces; (b) isosurfaces of $0.1|\phi_1|$ (magenta) and $0.1|\phi_2|$ (blue), showing the surfaces close to the vacua. The typical logarithmic bending of the domain wall characteristic of a boojum is clearly visible. The parameters $c_4 = 1$ and $M = 3$ were used for this calculation.

B. Boojum or D-brane soliton

We will now consider attaching a single (infinitely long) open vortex string to the domain wall from the side of the ⊗ phase. The energy of such a system is thus divergent for three different reasons: the domain wall carries an infinite energy; the semi-infinite vortex string has infinite energy; and, finally, the fact that it is a global vortex implies that the energy in the direction transverse to the string in the ⊗ phase diverges logarithmically due to the winding contribution to the kinetic term. This solution serves mostly as an illustration.

It will be convenient to parametrize the fields as follows:

$$\phi = \begin{pmatrix} e^{i\chi}\cos f \\ e^{i\vartheta}\sin f \end{pmatrix}. \tag{21}$$

The vortex is a solution that winds in ϕ_2 (since we are in the ⊗ phase), which thus means that ϑ is a winding phase. The "profile function" of the (global) vortex is $\sin f$, and due to the winding phase, it must vanish at the vortex center; we thus choose $f = 0$ there. Asymptotically, f tends to its vacuum value in the ⊗ phase, which is $f = \frac{\pi}{2}$.

An initial condition for the numerical calculation of the boojum can thus be constructed by combining the domain wall solution and the vortex ansatz,

$$\phi = (\cos f \quad e^{i\vartheta}\sin f), \tag{22}$$

$$\sin f = \begin{cases} \frac{e^{Mz}}{1+e^{2Mz}}, & z \leq 0, \\ \frac{e^{Mz}}{1+e^{2Mz}}\mathcal{F}(\rho), & z > 0, \end{cases} \tag{23}$$

$$\vartheta = \begin{cases} 0, & z \leq 0, \\ \theta, & z > 0, \end{cases} \tag{24}$$

where $\mathcal{F}(\rho)$ is a suitable guess for the vortex profile function obeying $\mathcal{F}(0) = 0$ and $\mathcal{F}(\infty) = 1$.

Figure 1 shows the numerical solution for the boojum. A well-known characteristic of the boojum junction of the vortex string and the domain wall is that the domain wall bends logarithmically due to the junction.

IV. SKYRMIONS AS A VORTEX HANDLE ON A DOMAIN WALL

Without loss of generality, we will again consider Skyrmions in the ⊗ phase; the results apply to the ⊙ phase by interchanging the complex scalar fields, $\phi_1 \leftrightarrow \phi_2$. We will consider the Skyrmions to be closed vortex strings (of ϕ_2) in the ⊗ phase and open vortex strings attached to the domain wall, from the side of the domain wall with the ⊗ phase. This soliton picture closely resembles the physics in string theory. In the supplemental material [41], we provide various videos of the simulations carried out in this section.

A. Vortex handle—Open string loop

We will now consider a single Skyrmion ($B = 1$) as an open vortex string loop emanating from the domain wall (at $z = 0$) into the ⊗ phase and returning back to the domain wall.

The vortex *a priori* does not bear baryon charge (π_3 charge density). We will now explain how it comes about. Using the parametrization (21) of ϕ, the vortex position implies $f = 0$ in the (x, y) plane at $z = 0$, and the phase ϑ winds locally around the vortex zero. The string extends into the ⊗ phase and eventually returns to the domain wall, providing another zero in f in a position different from the starting point. Since the phase winds around the vortex all the way into the bulk phase and on the way back, the vortex that comes back to the domain wall looks to the domain wall like an antivortex. The solution so far can be made with two components of the ϕ field equal, and hence it is clear that the baryon charge vanishes. The way this soliton becomes a Skyrmion is by turning on a twisting of its U(1) modulus, which lives in the string world volume.

More precisely, we will let the phase field χ wind from 0 to π on the way out and away from the domain wall and from π to 2π on the way back to the domain wall. This extra "winding" will make the string handle cover the 3-sphere (the target space) and hence comprise a unit baryon charge.

The above description is quite idealistic; however, the vortex string will have a tension that will tend to shorten the string as much as possible. For this reason, the unit charge Skyrmion or the 1-Skyrmion as a single vortex string handle attached to the domain wall will be quite short.

There is a simple way to extend the above construction to a B-Skyrmion, i.e., by twisting the phase function χ not one time but B times. More precisely, we can twist the function χ from 0 to πB on the way into the bulk and from πB to $2\pi B$ on the way back to the domain wall. This can produce a vortex handle with baryon charge or topological degree B. We will see shortly that the configuration becomes more complicated for higher twists than what we described here.

We are now ready to present the numerical solution of the single vortex string handle attached to the domain wall. The numerical solution is shown in Fig. 2. Figure 2(a) shows the energy isosurfaces in transparent blue and the baryon charge isosurfaces with an unconventional color scheme that we will describe shortly. Figure 2(b) shows the vacua or, more precisely, the values $0.1|\phi_1|$ in magenta and $0.1|\phi_2|$ in blue. The domain wall approaches the vacuum exponentially, so the true vacuum value is only reached asymptotically. The vortex, on the other hand, approaches the $\otimes$ vacuum linearly, so the true vortex zero is quite close to the tubelike surface surrounding it. Figures 2(c) and (d) show the vortices of ϕ_2 and ϕ_1, respectively, in the (x, y) plane at $z = 0$ (in the middle of the domain wall). The arrows are unit vectors pointing in the direction given by $\arg(x + iy) = \arg(\phi_2)$. The length of the arrows is normalized to the unit cell of the lattice showing the configuration. Although it is possible to identify the vortex as opposed to the antivortex from the arrows alone, we have overlaid a color scheme that is based on the following empirical expression:

$$Q = -\frac{\Re(\partial_{[x}\phi_2)\Im(\partial_{y]}\phi_2)}{2\pi(|\phi_2|^2 + \epsilon)}. \tag{25}$$

Large positive values of Q are plotted with red, and large negative values are plotted with blue; green is zero, and other colors are interpolation values between red and blue. The brackets around the spatial indices indicate that the indices are antisymmetrized, and, finally, ϵ is an *ad hoc* small number that is regularizing the quantity at the vortex cores. Q is used solely for the intent of clarifying which vortices are vortices and which are antivortices.

We will now explain the color scheme utilized for coloring in the baryon charge isosurface that shows the Skyrmion configuration in Fig. 2(a). The color scheme is based on the parametrization (21) as

$$\begin{cases} 0 \le \vartheta \le \frac{\pi}{10}, & \text{white} \\ \frac{\pi}{10} < \vartheta < \frac{9\pi}{10}, & \text{color} \\ \frac{9\pi}{10} \le \vartheta \le \frac{11\pi}{10}, & \text{black} \\ \frac{11\pi}{10} < \vartheta < \frac{19\pi}{10}, & \text{color} \\ \frac{19\pi}{10} \le \vartheta \le 2\pi, & \text{white.} \end{cases} \tag{26}$$

The color is defined as a map from χ to the color circle (the hue), such that $\chi = 0$ is red, $\chi = 2\pi/3$ is green, $\chi = 4\pi/3$ is blue, $\chi = \pi/3$ is yellow, $\chi = \pi$ is cyan, and $\chi = 5\pi/3$ is magenta.

There is a surprise, which is not obvious from the construction we described above; it turns out that there is a dual string, meaning a string in the ϕ_1 field in addition to the string in the ϕ_2 field that we pictured so far. The nature of the Skyrmion, covering the entire 3-sphere (target space), implies that the closed string in the ($\otimes$) bulk will have a dual string (of ϕ_1) piercing through the Skyrmion. When the Skyrmion is attached to the domain wall, the vortex ring (of ϕ_2) becomes a handle, but the dual string piercing the Skyrmion becomes a dual handle. Therefore, the 1-Skyrmion is actually symmetric between the two phases. Had we described everything in terms of strings in the $\odot$ phase, the vortex would be a string in the ϕ_1 field, and it also would become a handle attached to the domain wall, albeit from the other side; see Fig. 2(b). The vortex zero in ϕ_2 is depicted by a blue isosurface, and the vortex zero in ϕ_1 is shown with a magenta isosurface. The two vortices (blue and magenta) link each other once (if we include their respective vacua).

Figure 3 shows the same 1-Skyrmion configuration as in Fig. 2, but in the $2 + 6$ model instead of in the $2 + 4$ model. The $2 + 4$ model is given by the Lagrangian (1) with $c_4 = 1$, $c_6 = 0$, and the $2 + 6$ model has $c_4 = 0$, $c_6 = 1$.

B. Interactions between wall and closed vortex string

In this section, we will start from a 1-Skyrmion that can stably exist in the bulk [23] and put it near the domain wall. The 1-Skyrmion in the $\otimes$ bulk exists as a vortex ring (closed vortex string) in the ϕ_2 field (blue). If we were to place it in the $\odot$ phase instead, the 1-Skyrmion would be a vortex ring in the ϕ_1 field (magenta). If the 1-Skyrmion is placed far way from the domain wall, the force between them is exponentially suppressed, and it will take a long time before an attraction will accelerate the 1-Skyrmion toward the domain wall. Therefore, we will place the 1-Skyrmion in the bulk in near proximity to the domain wall, and the attraction happens quite rapidly.

The simulation is made not with a relativistic kinetic term but with the relaxation method, which is dissipative. If the dynamics was made with a real relativistic kinetic term, the interaction would be oscillating many times and only come to a final fixed point when all the excess energy has been radiated away. Instead, we will evolve the dynamics of

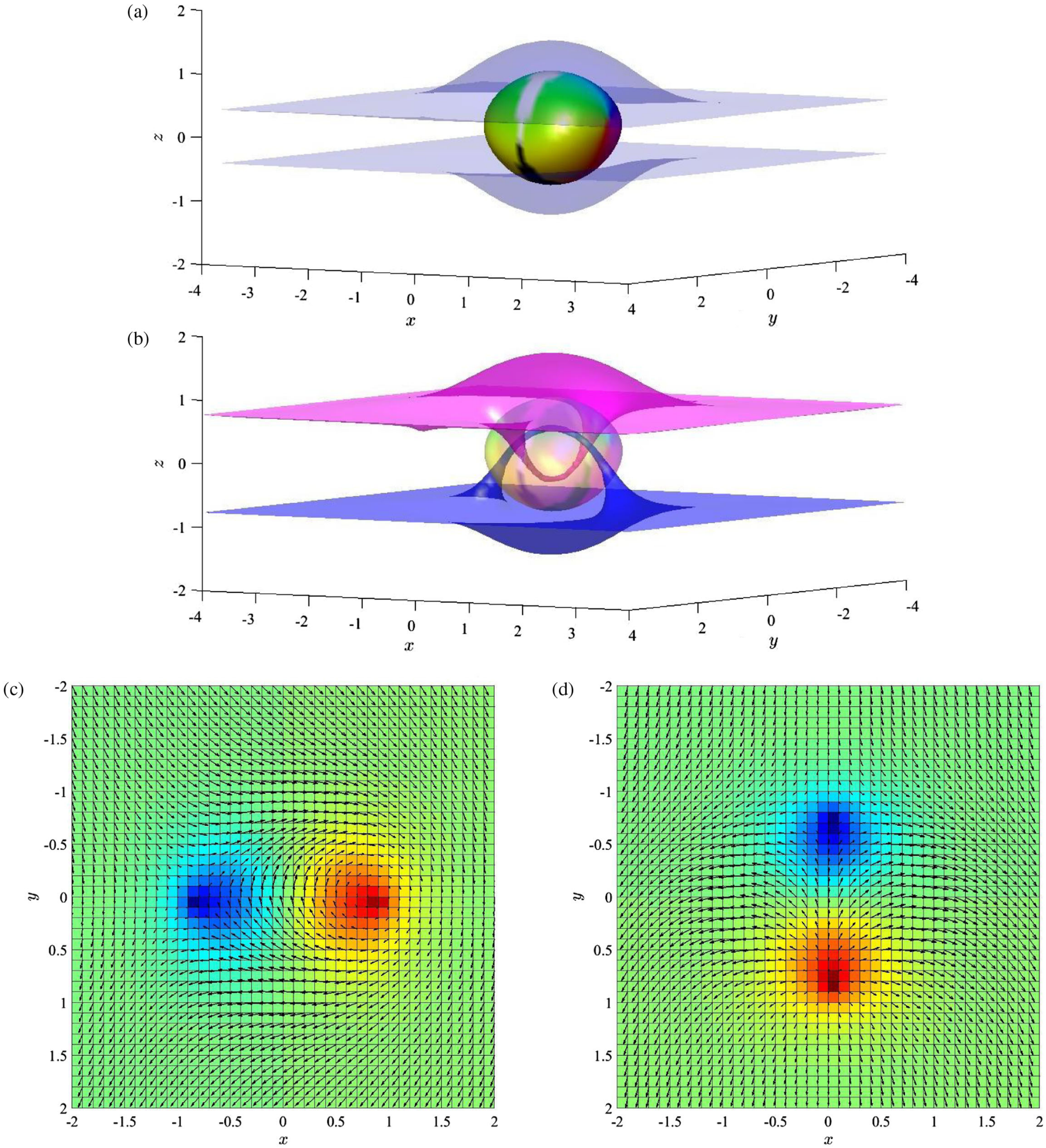

FIG. 2. Skyrmion as a vortex handle with a single twist on the domain wall in the $2+4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pair of ϕ_1. In this figure, we have taken $M=3$.

the simulation with a first-order kinetic term, which is dissipative as mentioned already. This means that the energy is not conserved and the configuration will quickly approach the fixed point losing the excess energy to the dissipative term in the evolution.

The numerical calculation is shown in Fig. 4. We have placed the 1-Skyrmion near to the domain wall in the ⊗ phase with an orientation such that the plane of the vortex ring is perpendicular to the domain wall. This happens to be the optimal orientation for attraction between the

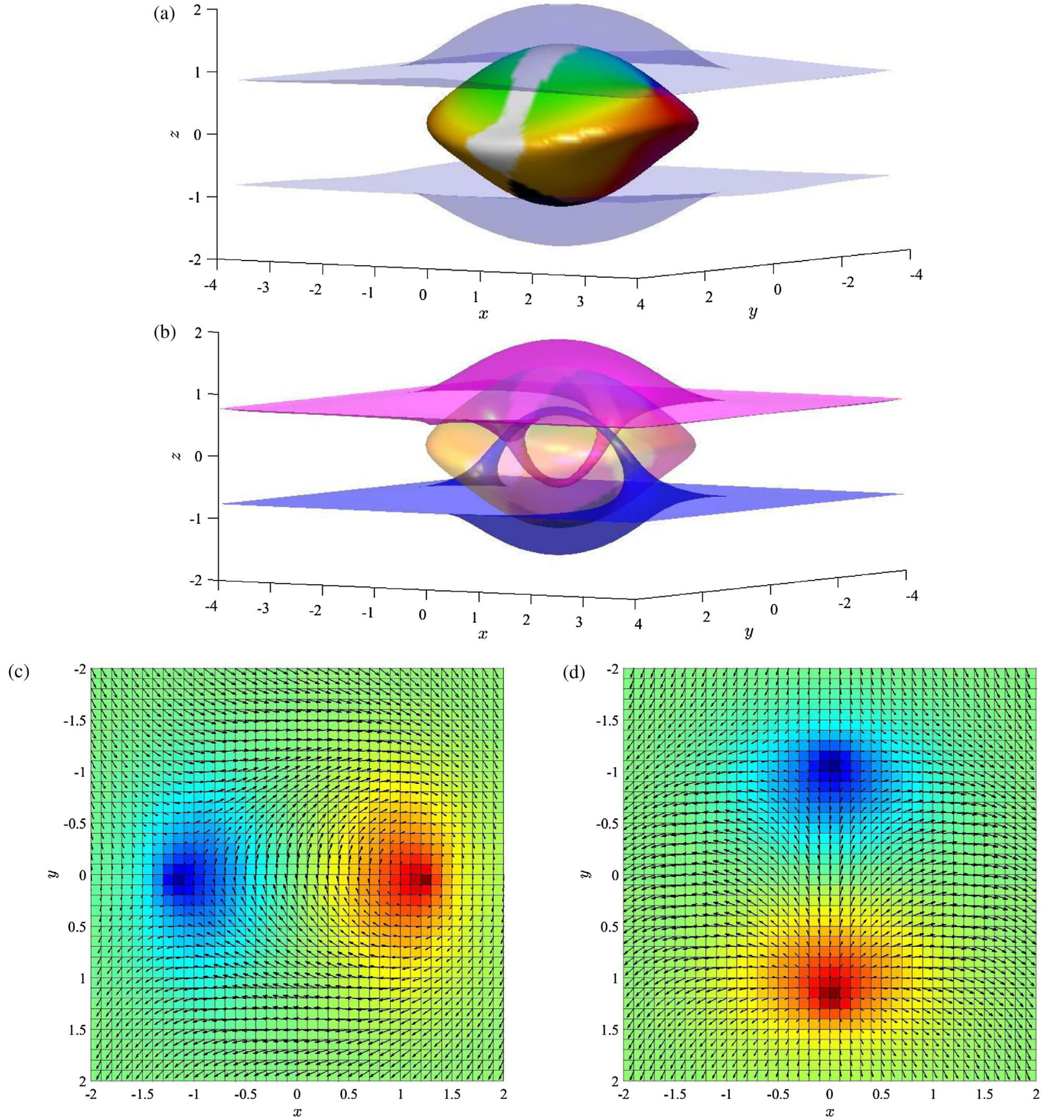

FIG. 3. The Skyrmion as a vortex handle with a single twist on the domain wall in the $2+6$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pair of ϕ_1. In this figure, we have taken $M = 3$.

1-Skyrmion and the domain wall. The four columns in Fig. 4 depict the energy/baryon charge, the vacua, the (anti)vortices of ϕ_2 on the domain wall, and the (anti)vortices of ϕ_1 on the domain wall, respectively. The rows are snapshots in imaginary time corresponding to the evolution of the relaxation time. The attraction between the 1-Skyrmion and the domain wall happens quickly, and the first thing that happens is that the vortex ring in ϕ_2 (blue) is attracted and partially absorbed into the vacuum on the other side of the domain wall (the ⊙ phase); see rows 2 and 3. This is the creation of the vortex handle, and because it is only partially absorbed, the remaining part

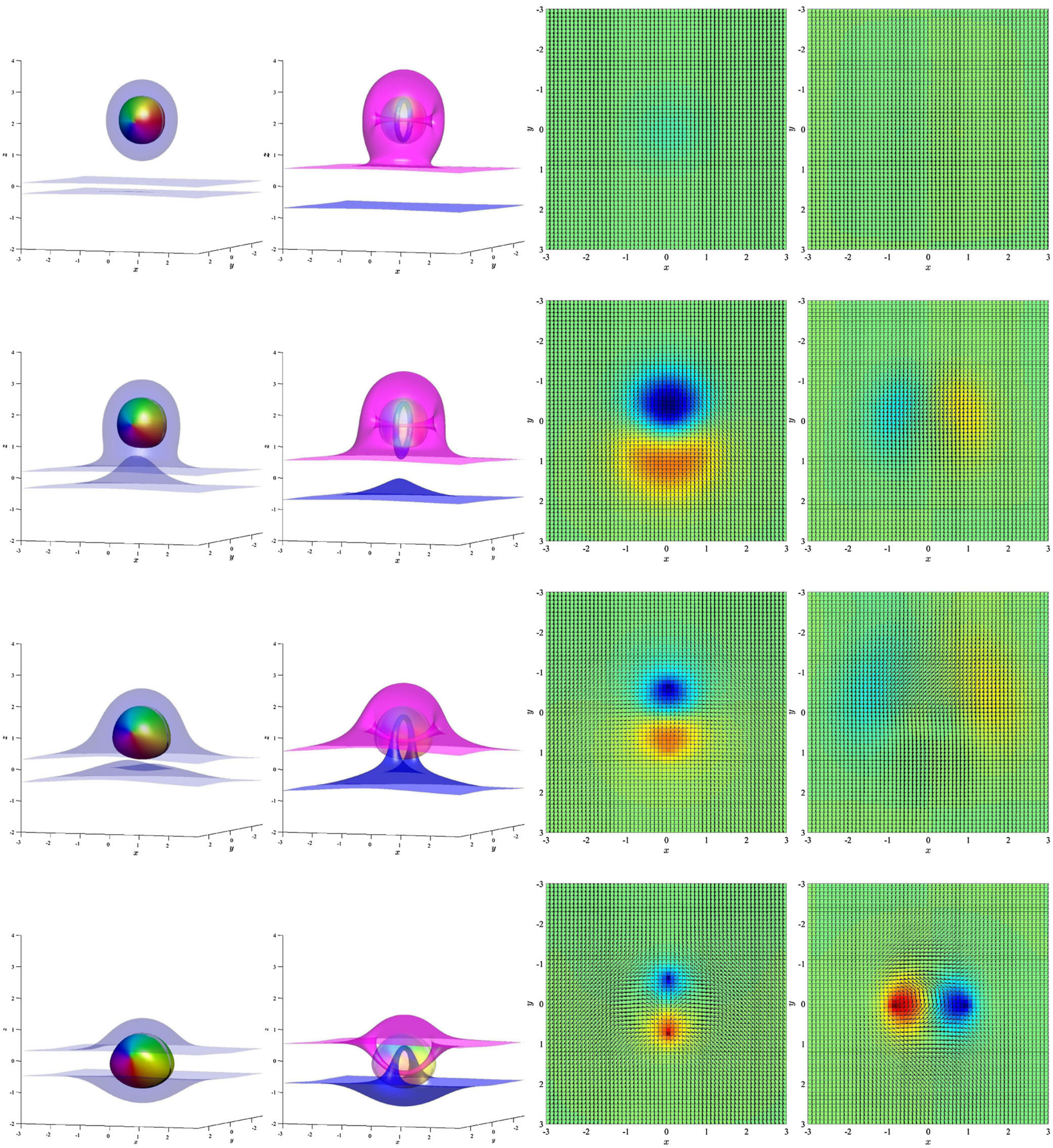

FIG. 4. Skyrmion as a bulk vortex ring in the ⊗ phase interacting with the domain wall in the $2+4$ model. The four columns display (1) the energy isosurfaces and baryon charge isosurface, (2) the zeros of ϕ_2 (blue) and ϕ_1 (magenta), (3) the vortex (red) and antivortex (blue) pair of ϕ_2, and (4) the vortex (red) and antivortex (blue) pair of ϕ_1. The rows correspond to the (imaginary) relaxation time of the simulation. The rows are not equidistant in relaxation time. In this figure, we have taken $M=3$.

is a handle of the ϕ_2 vortex sticking out into the ⊗ bulk (positive-z direction).

From this simulation, we can also see the origin of the dual handle, i.e., the vortex handle that exists in the ϕ_1 field. In the ⊗ bulk, it was just a string that pierced through the 1-Skyrmion, making it into a torus as claimed in Ref. [23]. This piercing string connects the vacua on both sides of the torus, and, energywise, we find it more intuitive to think about the 1-Skyrmion as being a wrapped-up vortex ring in the ϕ_2 field. However, once the vortex ring in the ϕ_2 field is absorbed into the wall and becomes a vortex handle, the dual string (in the ϕ_1 field) is drawn down into the domain wall and becomes a dual handle. In fact, the configuration is symmetric if we flip the $z \to -z$ and make a 90 deg rotation with a flip as $x \to y$ and $y \to x$. The second flip is necessary for keeping the baryon charge positive (otherwise, it would become an anti-Skyrmion).

Now that we have understood the 1-Skyrmion absorbed into the domain wall from two different perspectives, let us consider the 2-Skyrmions in the next section.

C. Handle with double twist

In this section, we will take the approach of adding a twist as described briefly in Sec. IVA. The string in ϕ_2 that emanates from the domain wall into the $\otimes$ phase is twisted in the χ field from 0 to 2π, and on the way back to the domain wall, it further twists to 4π, yielding a total twist of $4\pi = 2\pi B$; hence, $B = 2$. As mentioned in Sec. IVA, there is unavoidably a dual string in ϕ_1 in the Skyrmion, and we will see that it is related to the number of twists. More precisely, once the twist is higher than 1, the dual string is either multiple wound or there are several dual strings.

The numerical calculations for the $2+4$ model and the $2+6$ model are shown in Figs. 5 and 6, respectively. All the subpanels are in the same format as presented in Fig. 2 of Sec. IVA. We can see that the handle, being the vortex string in ϕ_2 (blue), is elongated and the dual string in ϕ_1 (magenta) has become two separate dual strings, instead of one. The linking number between the blue and magenta strings (including the vacuum) is 2; see Figs. 5(b) and 6(b). It is clear that there are two vortex antivortex pairs in the field ϕ_1 in Figs. 5(d) and 6(d). We can also see that the doubly twisted handle is much bigger in the $2+6$ model than in the $2+4$ model with the normalization of the terms used in the Lagrangian (1). From the doubly twisted handle, it is clear why we would like to associate the handle to the vortex in ϕ_2 (blue); however, we could just as well take the opposite point of view and say that it is two separate handles in ϕ_1 (magenta) that are closely bound together because they are linked with the same vortex in ϕ_2 (blue).

D. Interactions between two handles and the braided string junction terminating on a wall as a $B=2$ Skyrmion

The way that we created a 2-Skyrmion on the domain wall (sticking a bit out into the $\otimes$ bulk) in the previous section was by adding an extra twist to the 1-Skyrmion, topping the baryon number up to 2. This created an asymmetric configuration in which the ϕ_2 vortex (blue) is linked with two individual ϕ_1 vortices (magenta).

In this section, we study the interactions of two individual 1-Skyrmions, both already absorbed into the domain wall. Under normal circumstances, this is also a way of creating multi-Skyrmions. The recipe is to find the attractive channel between the two 1-Skyrmions and let them combine into the (probably) optimal 2-Skyrmion. It turns out that the theory is much more complex when we have a domain wall, as we shall see shortly.

As mentioned above, it is well known that there is an attractive channel and a repulsive channel for Skyrmions, depending on their mutual orientations in field space (target space). In this case, in which we consider the interaction in the world volume of the domain wall, this will provide us with a new interpretation of what is going on, at least in this flavor of the Skyrme-like model (note that the potential is different and the Skyrmion has been transformed due to the absorption by the domain wall). Usual Skyrmions can be oriented in any direction in SO(3) or equivalently SU(2), but the Skyrmion handle absorbed into the domain wall can only be rotated in the plane, which reduces the possibilities of spatial rotation to U(1). Recall also that the vacuum manifold is $O(2) \sim U(1) \times \mathbb{Z}_2$, and so we really only have the rotations in the plane to play with.[1] First, we will orient the two 1-Skyrmions in the attractive channel and see what happens next.

The numerical result is shown in Fig. 7. The columns display the energy and baryon charge density isosurfaces, the vacua/vortex zeros, the (anti)vortices in the field ϕ_2 on the domain wall, and the (anti)vortices in the field ϕ_1 on the domain wall, respectively. The rows show the evolution in (imaginary) relaxation time. For a simplified picture of the interactions that govern the physics on the domain wall, we can look at the vortex-antivortex pairs of the two fields $\phi_{2,1}$ in the third and fourth columns of Fig. 7. What happens for the interaction of the two vortex handles in the attractive channel is that they meet with one vortex-vortex pair (in ϕ_2) colliding in the domain wall (plane) and make a 90-degree scattering into two new vortices; see the third column of Fig. 7. Since the vortices are global vortices, we would expect two vortices to mutually repel each other and the vortex to be attracted to the antivortex. A simplistic explanation for the attraction is that in the field ϕ_2 there are two facing vortices that want to repel each other, but in the field ϕ_1, there are 2 vortex-antivortex pairs that both attract each other. The two attractive forces win over the single repulsive force, and the net force is attractive.

The resulting interaction attracts the two vortex handles, and they combine to form a torus in the plane of the domain wall. The torus configuration in the standard Skyrme model is well known, of course. It is, nevertheless, interesting to look at the Skyrmion-Skyrmion interactions in terms of the vortices in the two fields $\phi_{1,2}$. In the second column of Fig. 7, we can see the vortex lines (ϕ_1 is magenta, and ϕ_2 is blue). To see the direction of the vortex, i.e., whether it is a vortex or an antivortex in the (x, y) plane, we have to refer

[1] It is possible to flip two of the spatial directions to get a different orientation, but that will not be essential here.

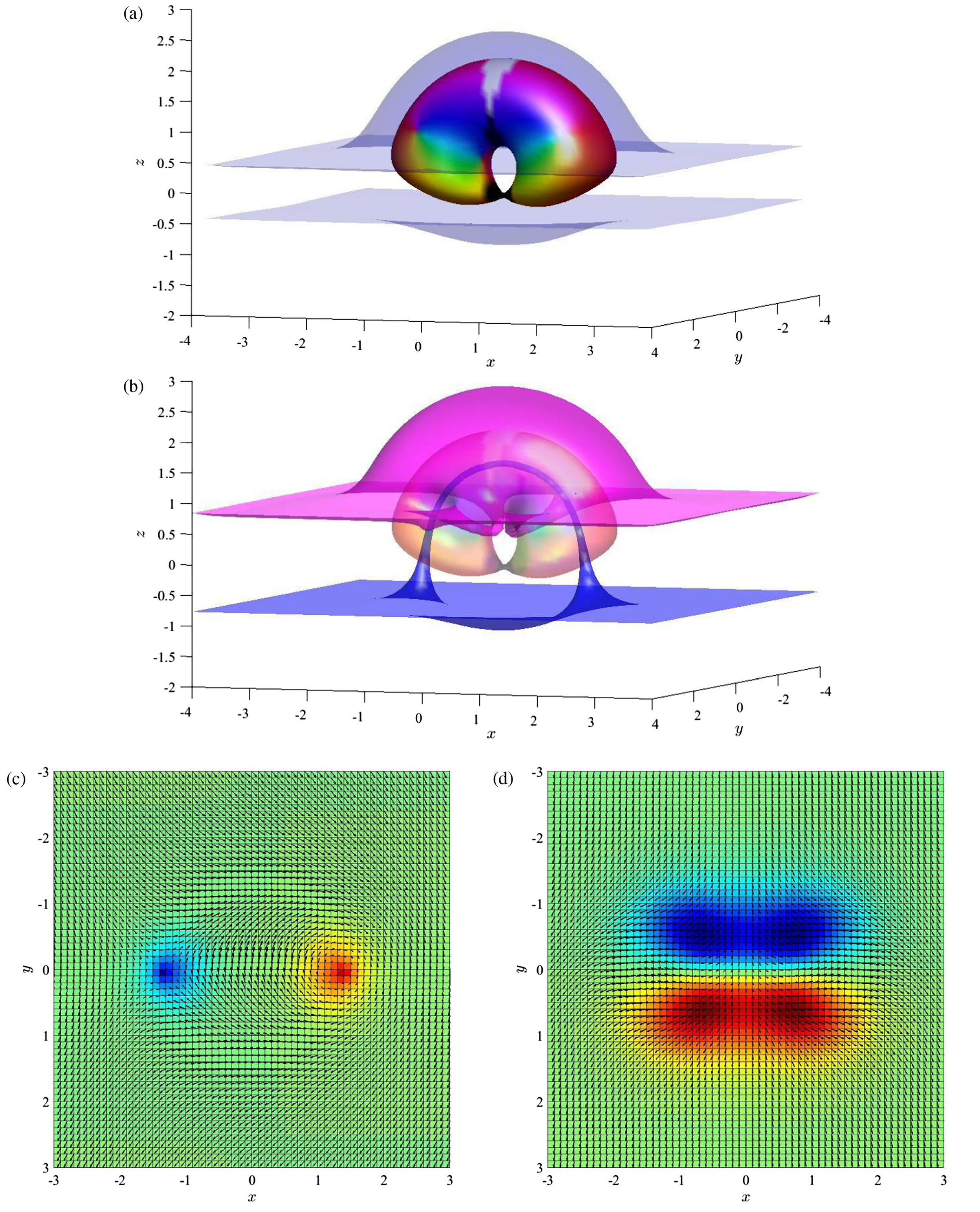

FIG. 5. 2-Skyrmion as a vortex handle with double twist on the domain wall in the $2 + 4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pairs of ϕ_1. In this figure, we have taken $M = 3$.

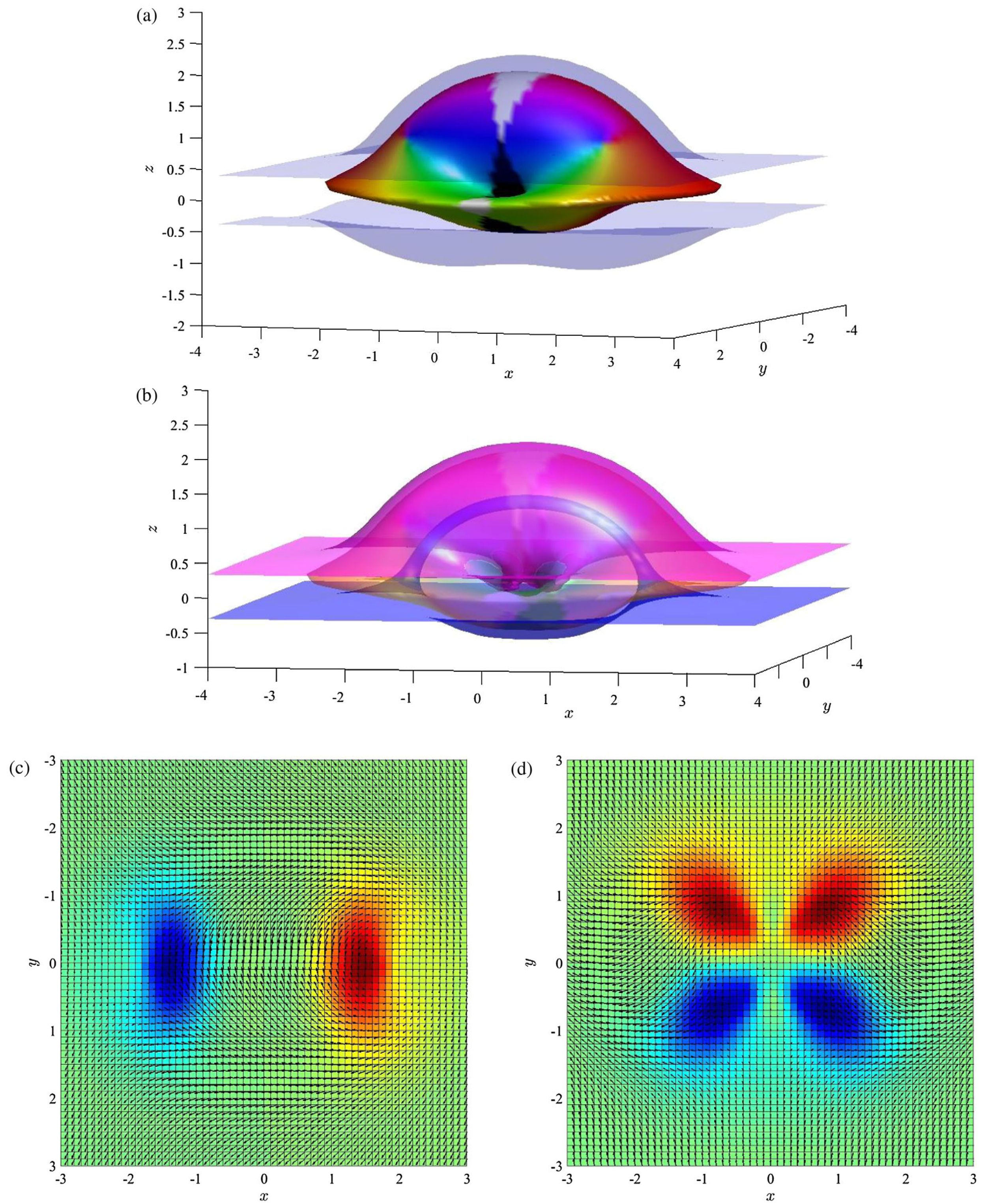

FIG. 6. 2-Skyrmion as a vortex handle with double twist on the domain wall in the $2+6$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pairs of ϕ_1. In this figure, we have taken $M=7$.

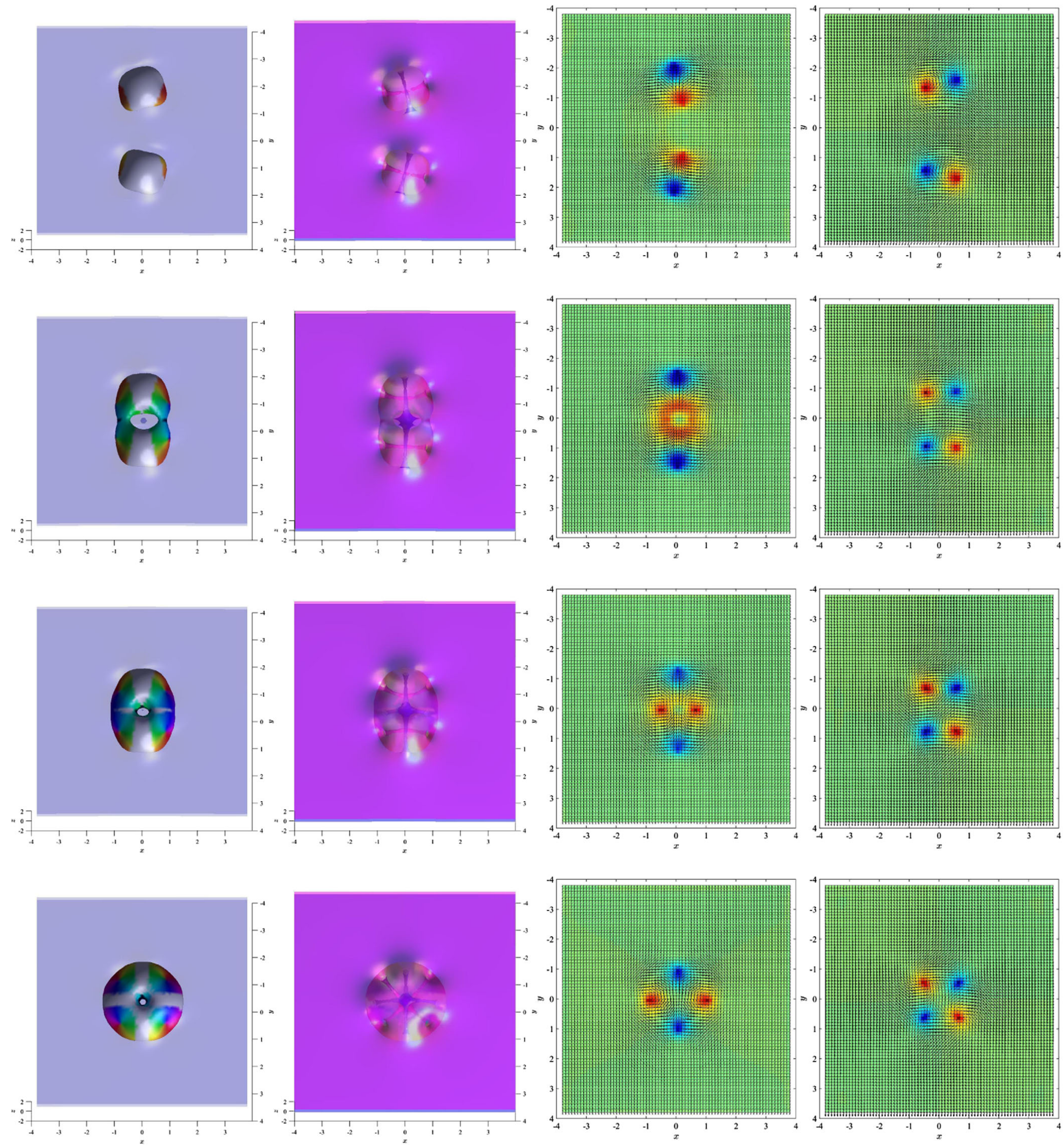

FIG. 7. Skyrmion-Skyrmion interaction in the domain wall: the vortex handle interacts with another vortex handle in the attractive channel in the $2 + 4$ model. The four columns display (1) the energy isosurfaces and baryon charge isosurface, (2) the zeros of ϕ_2 (blue) and ϕ_1 (magenta), (3) the vortex (red) and antivortex (blue) pairs of ϕ_2, and (4) the vortex (red) and antivortex (blue) pairs of ϕ_1. The rows correspond to the (imaginary) relaxation time of the simulation. The rows are not equidistant in relaxation time. In this figure, we have taken $M = 7$.

to columns 3 and 4 of Fig. 7. The interaction in the full three-dimensional picture is somewhat more complicated than in the simplistic picture of the interactions seen only in the plane of the domain wall. The colliding vortex-vortex pair still occurs, of course, but what happens more precisely is that after the vortex-vortex collision in the plane has happened a vortex string junction that is left in the ⊗ bulk (out of the plane of the domain wall) has been made. After the creation of the string junction in the field ϕ_2, there are still just two independent strings in ϕ_1 (magenta), but the Skyrmion configuration is quite oval, and once it relaxes into a more symmetric (toroidal) shape, the two vortex strings in ϕ_1 also form the same string junction, but sitting in the $\odot$ bulk and from a top view, the positions of the (anti)vortices on the domain wall are rotated by 45 deg with respect to the ϕ_2 ones; see Fig. 7. It is a bit difficult to see the three-dimensional structure of the configuration in the second column of Fig. 7, so we have duplicated these images, but from a different view point in Fig. 8. Now, we can better see that the configuration has two string junctions with four vortices emanating (two vortices and two antivortices) and the two junctions are thus braiding their four fingers.

The above example illustrates well what happens in the attractive channel. We will now consider the case shown in Fig. 9, in which we have rotated both the Skyrmion handles in such a way that there are two repulsive interactions coming from a vortex-vortex pair and an

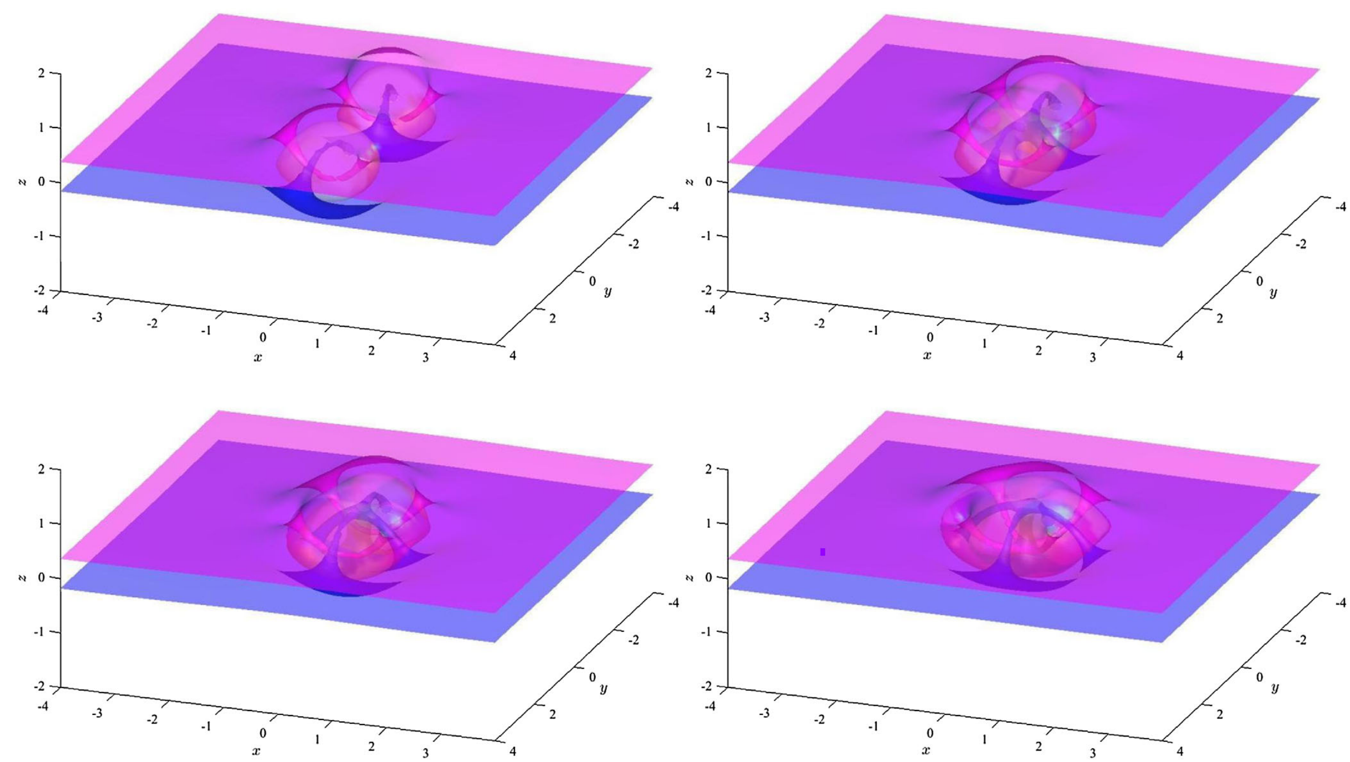

FIG. 8. Same as the second column of Fig. 7, but with a titled view point. The time evolution of the simulation has the following order: upper left, upper right, lower left, and lower right.

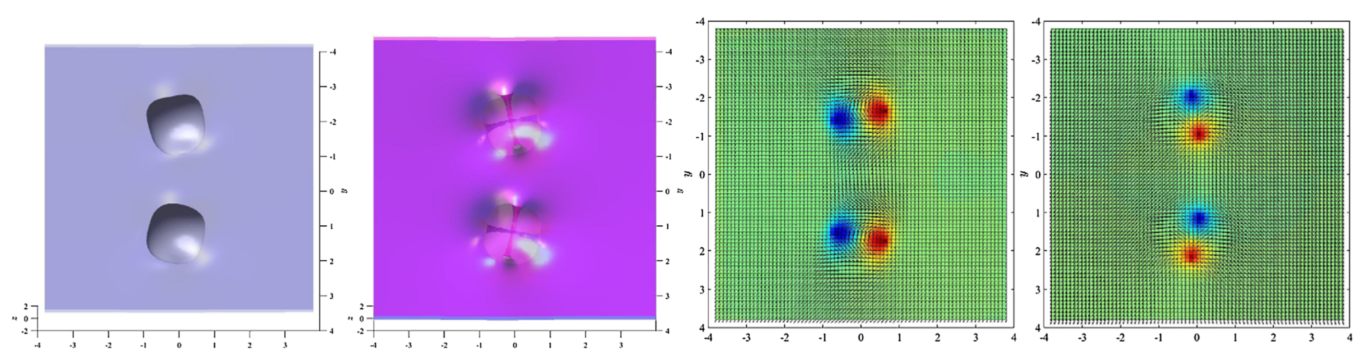

FIG. 9. Skyrmion-Skyrmion interaction in the domain wall: the vortex handle interacts with another vortex handle in the repulsive channel in the $2 + 4$ model. The four columns display (1) the energy isosurfaces and baryon charge isosurface, (2) the zeros of ϕ_2 (blue) and ϕ_1 (magenta), (3) the vortex (red) and antivortex (blue) pairs of ϕ_2, and (4) the vortex (red) and antivortex (blue) pairs of ϕ_1. In this figure, we have taken $M = 7$.

antivortex-antivortex pair in the ϕ_2 field, which dominates over the vortex-antivortex attraction in ϕ_1. We have only shown a single snapshot of the configuration, because what happens next is that they both run away from each other.

E. Interactions between handle and closed vortex string

In this section, we will consider a different kind of interaction, namely, between the vortex handle on the domain wall and the vortex ring in the $(\otimes)$ bulk.

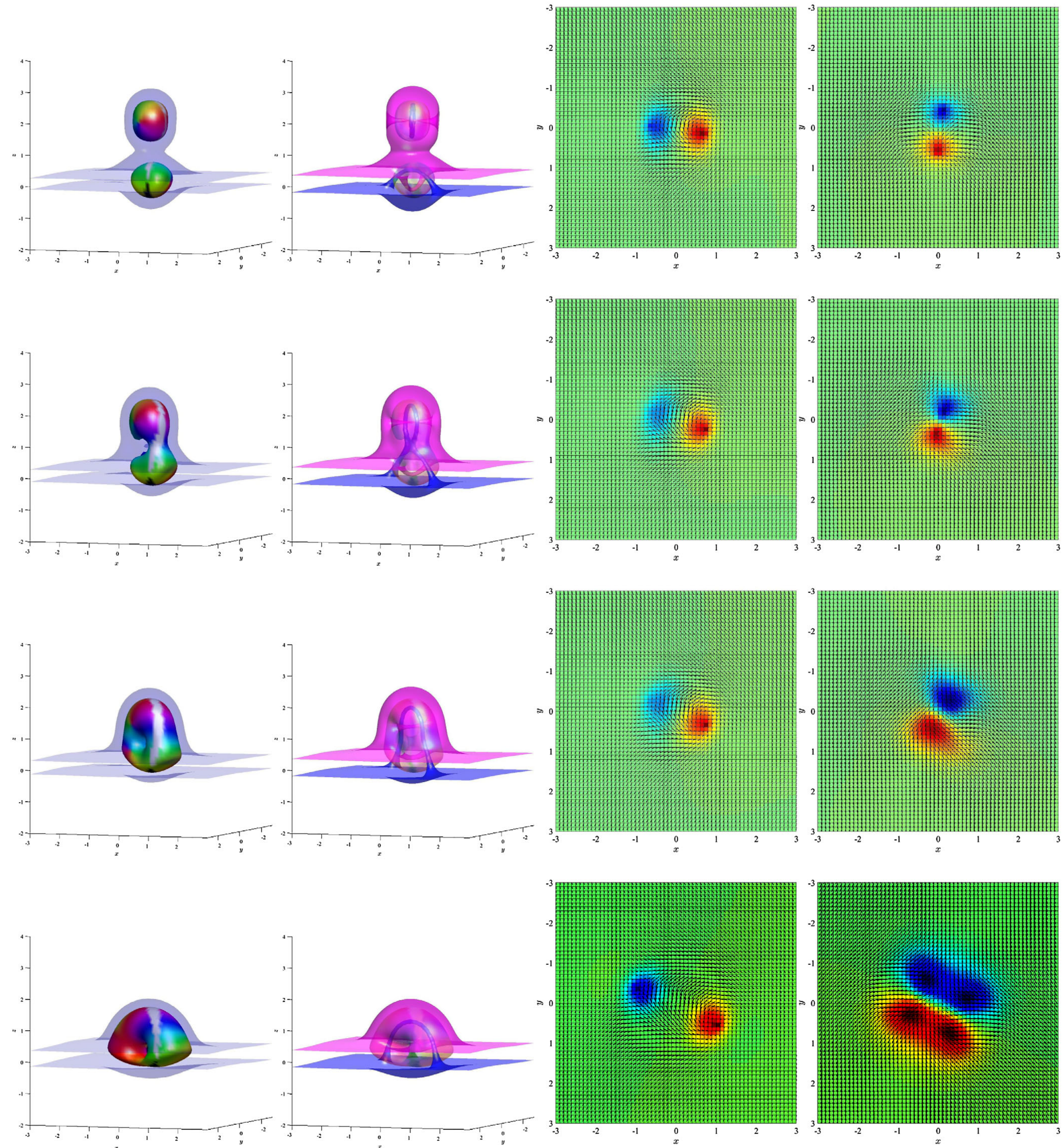

FIG. 10. Bulk Skyrmion-domain wall Skymion interaction: a Skyrmion vortex ring in the $\otimes$ bulk interacts with a vortex handle in the $2+4$ model. The four columns display (1) the energy isosurfaces and baryon charge isosurface, (2) the zeros of ϕ_2 (blue) and ϕ_1 (magenta), (3) the vortex (red) and antivortex (blue) pair of ϕ_2, and (4) the vortex (red) and antivortex (blue) pair of ϕ_1. The rows correspond to the (imaginary) relaxation time of the simulation. The rows are not equidistant in relaxation time. In this figure, we have taken $M=7$.

TABLE I. Energies of various configurations in the $2+4$ model.

Type	B	M	E
Handle	1	3	69.70
Ring	1	3	94.59
Handle	1	7	84.73
Ring	1	7	112.4
Braided string junction	2	3	132.8
Doubly twisted handle	2	3	144.3
Braided string junction	2	7	163.3
Doubly twisted handle	2	7	173.6

The numerical calculation is shown in Fig. 10. Similarly to the matrix in the figures in the previous section, we show the configuration at four (imaginary) time steps as four rows in the figure. The initial orientation of either the vortex handle on the domain wall or the vortex ring in the (⊗) bulk is not too important, because the vortex ring in the bulk will rotate into the attractive channel as chosen as the initial configuration (first row) in Fig. 10. The vortex ring (in ϕ_2, blue) in the ⊗ bulk is initially oriented perpendicularly to the plane of the domain wall. The first thing that happens in the interaction is that the vortex in ϕ_2 (blue) is pulled down toward the vortex handle on the domain wall, and it *reconnects* such that the string becomes a longer handle; see rows 1–3 of Fig. 10. The beauty of the reconnection is that it automatically produces a vortex handle with a double twist in the χ field [see the parametrization (21)]. This can be seen from the fact that the vortex handle in ϕ_2 encloses (links with) two dual strings in ϕ_1; see the third row of Fig. 10. The final phase in the relaxation just minimizes the energy by making the doubly twisted vortex handle shorter and more compact; see the fourth row of Fig. 10.

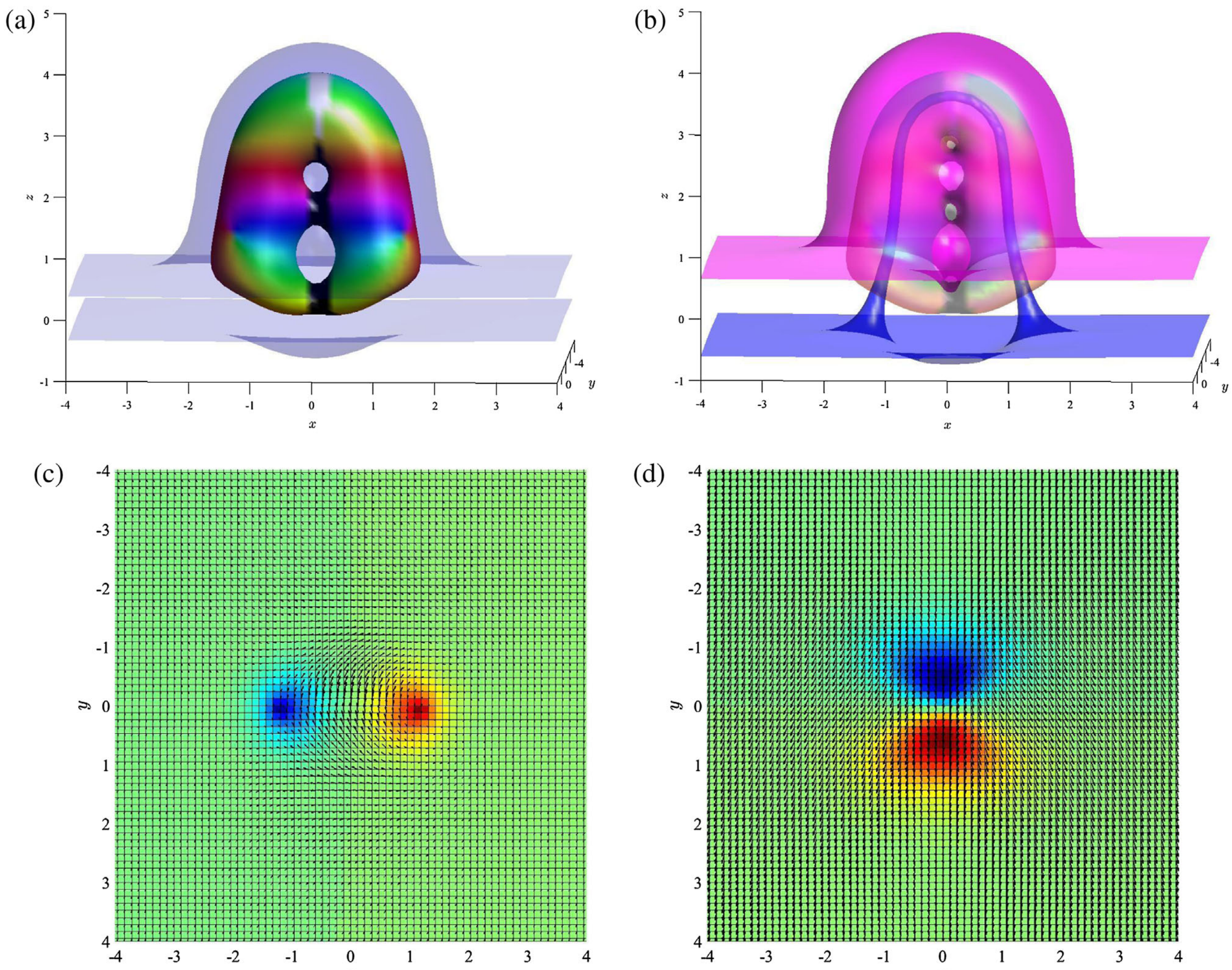

FIG. 11. 3-Skyrmion as a vortex handle with three twists on the domain wall in the $2+4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pair of ϕ_1. In this figure, we have taken $M=3$.

F. Energy comparison of the two $B = 2$ Skyrmions

It is interesting to see that the interaction between the two handles in the plane of the domain wall created a different 2-Skyrmion (a braided string junction torus) compared to the interaction of the vortex handle and the vortex ring in the bulk, which created the doubly twisted vortex handle that we constructed in Sec. IV C. To know which configuration is the stable one, we will compare their energies numerically. As the domain wall has an infinite energy in the infinite space, we will subtract off the domain wall energy and just calculate the energy of the Skyrmion configurations, with zero being the empty domain wall.

The results of the numerical calculations for the energies are shown in Table I. In particular, we first calculate the masses of the vortex handle for $M = 3$, 7 compared to the vortex ring and see that there is a strong binding energy for the Skyrmion to be gained by getting absorbed into the domain wall. Next, we compare the braided string junction of Sec. IV D with the doubly twisted handles of Secs. IV C and IV E. The conclusion drawn from the energy measurements is clear; the braided string junction of toroidal shape has the far lowest energy of the two different 2-Skyrmions and thus is the stable one.

G. Higher-charged handles

As we have seen in the previous sections, the interaction dynamics is quite intricate, and the number of ways of combining handles and rings with various numbers of twists is overwhelmingly large. Therefore, a complete study of higher-charged Skyrmions in this theory is beyond the scope of the paper. However, let us mention that there are in principle three different possibilities: multi-Skyrmions as multisolitons in the domain wall (as, e.g., the braided string junction as a 2-Skyrmion); multi-Skyrmions as higher-twisted handles sticking into the bulk; and, finally, a hybrid of the previous two options.

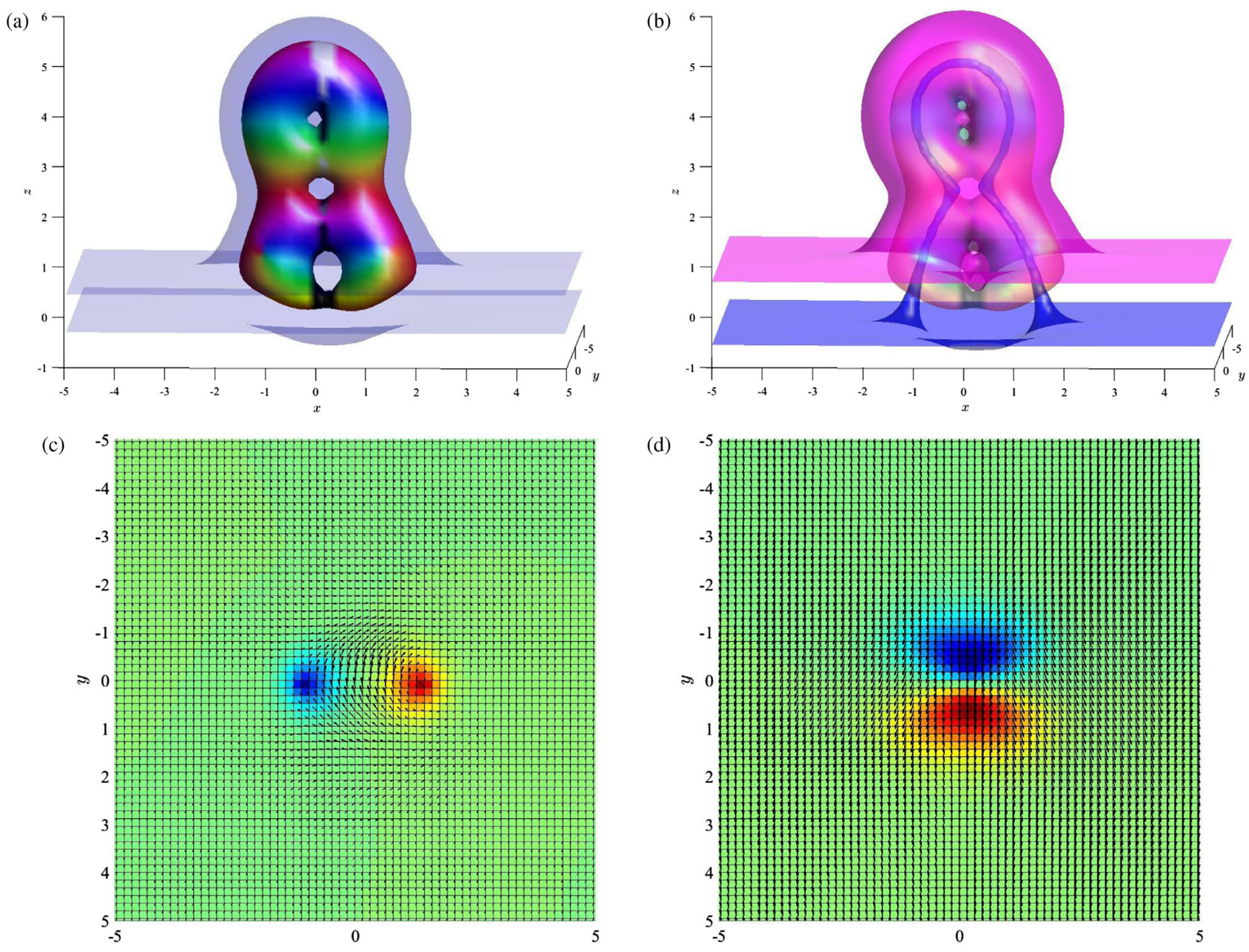

FIG. 12. 4-Skyrmion as a vortex handle with four twists on the domain wall in the $2 + 4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pair of ϕ_2. (d) The vortex (red) and antivortex (blue) pair of ϕ_1. In this figure, we have taken $M = 3$.

In this section, we only explore one possibility as the initial condition but then use the numerical calculations and relax them to the nearest metastable configuration; that is, we consider a single open vortex handle with various twists as the initial guess.

The first numerical result is for $B = 3$, i.e., a single vortex in ϕ_2 that is twisted three times yielding a 3-Skyrmion; see Fig. 11. It turns out to be metastable, and it has exactly one vortex handle in the field ϕ_2, and it is linked with three individual dual strings of ϕ_1.

Next, we try to add a twist to the previous initial guess in order to create a 4-Skyrmion as a vortex handle on the domain wall. The result is shown in Fig. 12, and it also turns out to be metastable. A curiosity is that the four dual string are located near the top "hole" of the Skyrmion and the bottom hole, but not near the middle hole, which seems to appear due to a stereo cusp in the ϕ_2 vortex handle.

We now add two more twists to the initial guess and try to search for a 6-Skyrmion, but this time, the metastability of the configuration was not found (or the guess was too far away from such a solution). The numerical result is shown in Fig. 13, in which the vortex handle has collapsed into a single handle and a 5-Skyrmion that consists of a double string junction in the field ϕ_1 with complicated twists around it (dual strings).

The last attempt at looking for a higher-charged handle is to add another twist to the former guess, yielding a total of

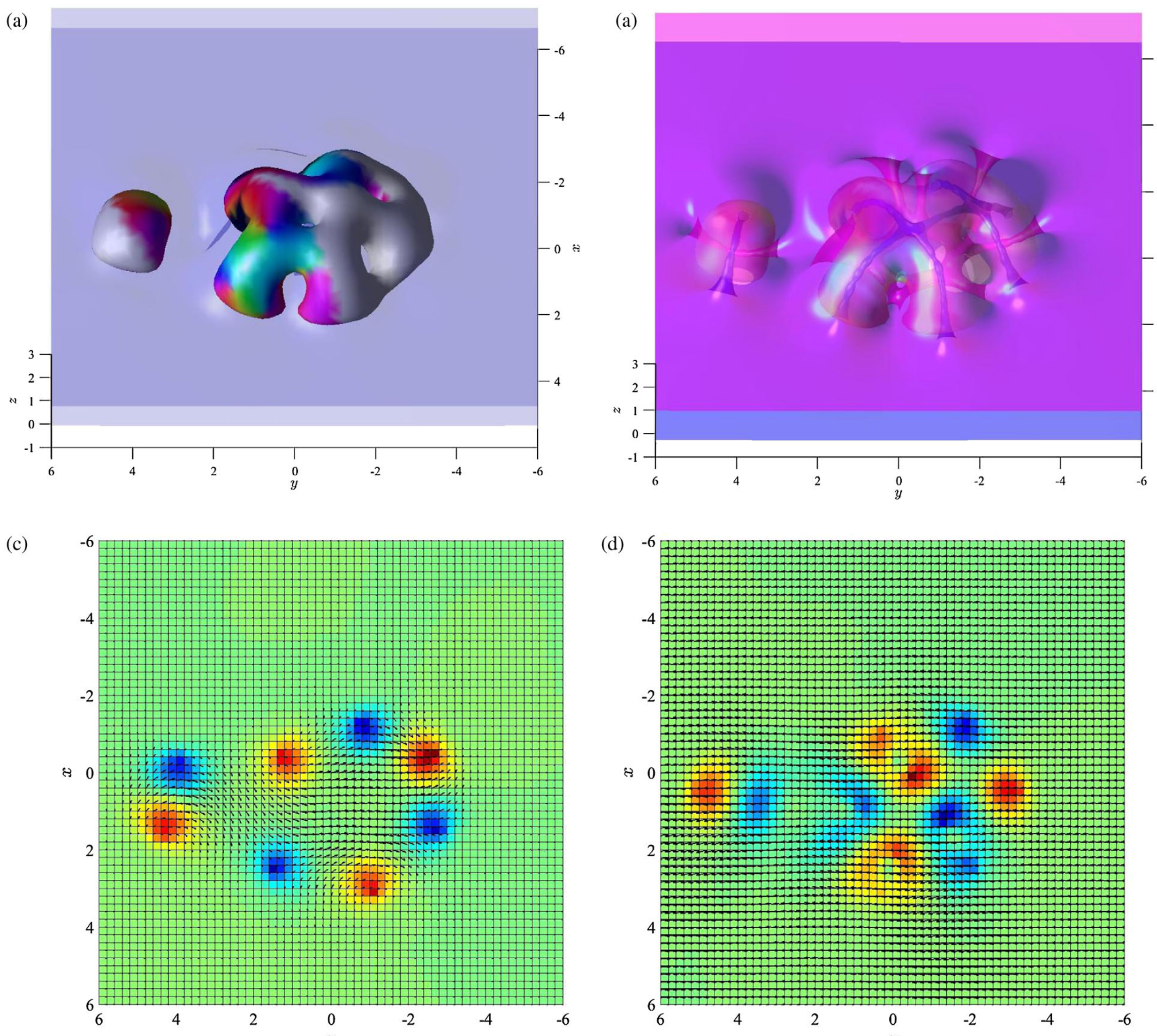

FIG. 13. 5-Skyrmion as a double vortex junction with twists on the domain wall next to a 1-Skyrmion as a handle, in the $2 + 4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pairs of ϕ_2. (d) The vortex (red) and antivortex (blue) pairs of ϕ_1. In this figure, we have taken $M = 3$.

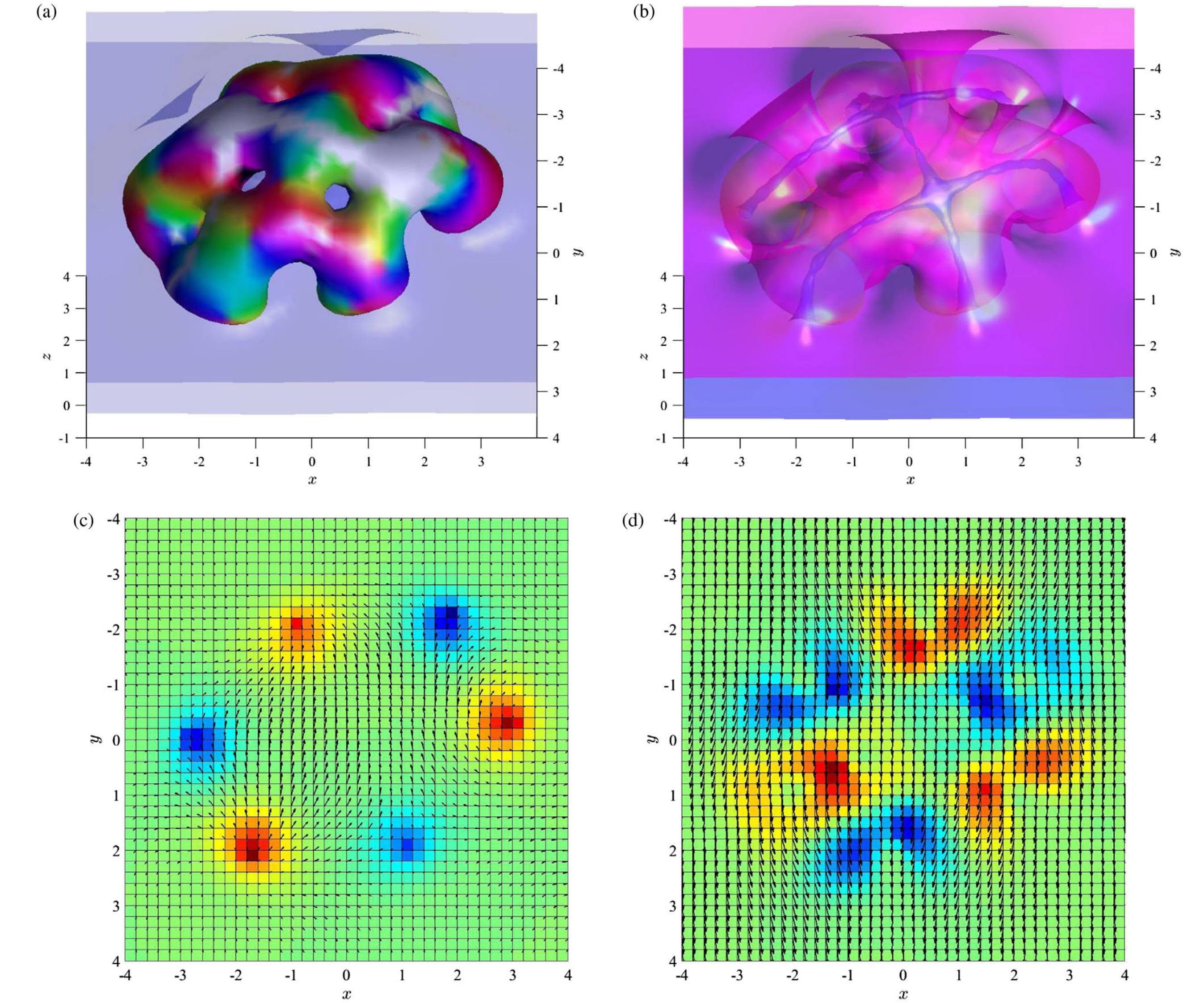

FIG. 14. 7-Skyrmion as a double vortex junction with twists on the domain wall in the $2+4$ model. (a) The energy density is shown with blue transparent isosurfaces illustrating the domain wall, and the baryon charge density is shown with an isosurface with the color scheme described in the text. (b) The blue isosurface (bottom) represents the zeros of ϕ_2, and the magenta isosurface (top) is the zeros of ϕ_1. The baryon charge is added transparently. (c) The vortex (red) and antivortex (blue) pairs of ϕ_2. (d) The vortex (red) and antivortex (blue) pairs of ϕ_1. In this figure, we have taken $M = 3$.

seven twists. The numerical result is shown in Fig. 14, and it also collapsed from a vortex handle into a double string junction, intricately braided with dual strings, yielding a 7-Skyrmion.

The true minimizers of the energy for $B \geq 3$ may not be the solutions found here.

V. DISCUSSION AND OUTLOOK

In this paper, we have considered a BEC-inspired potential in a sextic version of the Skyrme model, which due to the potential possesses vortex strings, and in particular we have studied the setting in the presence of a domain wall—which is also possessed by the theory with the BEC-inspired potential. The vortex strings contain a U(1) modulus, which, once twisted by 2π, yields a unit baryon charge. The first intuitive picture is that we wound the vortex string once to get a 1-Skyrmion in the bulk, and when it was placed in the vicinity of the domain wall, we found that attractive forces absorbed the Skyrmion and it became a vortex handle sitting on the domain wall. It turned out that it inevitably also had a dual string, i.e., a vortex in the complementary component of the two-component complex scalar field ϕ, and the configuration was made of two linked open strings ending on the domain wall. We further studied the interactions between two handles and found that when oriented in the attractive channel they attract and combine as a torus-shaped braided string junction. Another way to create a 2-Skyrmion is to place a vortex ring in the vicinity of a vortex handle sitting on the domain wall, which also attract each other and create a doubly twisted vortex handle. The former has the lowest energy, nevertheless.

An obvious direction for further studies is to consider higher-charged Skyrmions, which may yield a large number of metastable configurations; in particular, we have not necessarily found the true energy minimizers for $B \geq 3$. This would require a large search for configurations based on many different initial guesses. We will leave this for future work.

An interesting observation is that all the Skyrmions of baryon number B seem to be composed of vortex zeros of ϕ_1 and of ϕ_2 that link each other B times. It could be interesting to study this fact further and investigate whether this is always the case.

An important development would be to see how many of the results in this model can be carried over to BECs and under what circumstances.

ACKNOWLEDGMENTS

This work is supported by the Ministry of Education, Culture, Sports, Science (MEXT)-Supported Program for the Strategic Research Foundation at Private Universities "Topological Science" (Grant No. S1511006) and by a Grant-in-Aid for Scientific Research on Innovative Areas "Topological Materials Science" (KAKENHI Grant No. 15H05855) from MEXT, Japan. The work of M. N. is also supported in part by the Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (KAKENHI Grants No. 16H03984 and No. 18H01217). The TSC computer of the "Topological Science" project at Keio University was used for the numerical calculations.

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[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.98.125002 for videos of the simulations shown in Fig. 4 [ringM3c4a2_comp.mp4 and ringM3c4a2_vortex_lines_comp.mp4], in Fig. 7 [intM7c4_rot_comp.mp4 and intM7c4_rot_vortex_lines_comp.mp4], and in Fig. 10 [ringintM7c4a2_s4_comp.mp4 and ringintM7c4a2_s4_vortex_lines_comp.mp4]. Additional simulations corresponding to different initial conditions for Figs. 4 and 10 are shown in [ringM3c4a2_rot_comp.mp4 and ringM3c4a2_rot_vortex_lines_comp.mp4] and [ringintM7c4a2_s4_rot_comp.mp4 and ringintM7c4a2_s4_rot_vortex_lines_comp.mp4], respectively.