Magnetic tidal Love numbers clarified

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In this brief paper, we clarify certain aspects related to the magnetic (i.e., odd parity or axial) tidal Love numbers of a star in general relativity. Magnetic tidal deformations of a compact star had been computed in 2009 independently by Damour and Nagar [1] and by Binnington and Poisson [2]. More recently, Landry and Poisson [3] showed that the magnetic tidal Love numbers depend on the assumptions made on the fluid, in particular they are different (and of opposite sign) if the fluid is assumed to be in static equilibrium or if it is irrotational. We show that the zero-frequency limit of the Regge-Wheeler equation forces the fluid to be irrotational. For this reason, the results of Damour and Nagar are equivalent to those of Landry and Poisson for an irrotational fluid, and are expected to be the most appropriate to describe realistic configurations.

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I. INTRODUCTION

The deformability of a self-gravitating object immersed in a tidal field is measured by the tidal Love numbers (TLNs) [4]. The theory of relativistic TLNs in general relativity has been developed in Refs. [1,2,5,6] for nonspinning bodies, and then extended to rotating bodies in [7–11]. This theory has then been applied to compact binary systems, in order to compute the contribution of the tidal deformation to the emitted gravitational waveform [12–18].

For nonspinning objects,[1] the TLNs can be separated into two classes according to the parity of the perturbation induced by the tidal field: induced mass multipole moments are related to the so-called electric (or even-parity or polar) TLNs—which also exist in Newtonian theory [4]—whereas induced current multipole moments are related to the so-called *magnetic* (or odd-parity or axial) TLNs. The current multipole moments are induced by an external magnetic-type tidal field. Since the latter is not a source of the gravitational field in Newton's theory, the magnetic TLNs are a genuine prediction of general relativity, which might possibly be relevant for very compact objects.

Tidal deformability affects the gravitational-wave phase of a binary inspiral at high post-Newtonian order [5], with the magnetic TLNs giving a small contribution relative to the electric ones [15,18,21]. Nonetheless, their characterization is important to develop accurate waveform models and to compare the post-Newtonian predictions with those of numerical simulations [18,22–25].

There is some confusion in the literature related to the magnetic TLNs. These were computed independently in 2009 by Binnington and Poisson [2] (hereafter, BP) and by Damour and Nagar [1] (hereafter, DN) by considering axial perturbations of a perfect-fluid star in general relativity (see also [26] for an earlier study by Favata in the context of post-Newtonian theory). These perturbations can be reduced to a single second-order master equation; however, it has been previously noted that the master equation of BP and that of DN are inequivalent [9] and give rise to different magnetic TLNs. Meanwhile, in 2013 Yagi [21] used the result of DN to compute the effect of the magnetic TLNs in the waveform and to compute some quasi-universal relations [27,28] among TLNs of different parity and different multipole moments. In 2015, Landry and Poisson (hereafter, LP) discovered [3] that the magnetic TLNs depend on the properties of the fluid (see also [20,29]). In particular, they found that the magnetic TLNs for irrotational fluids or for static fluids are different and have the opposite sign. Consequently, the quasi-universal relations involving magnetic TLNs also depend on the fluid properties [18,20,30].

Thus, at the present stage we are left with three different types of magnetic TLNs: those computed by DN, those computed by BP, and those computed by LP for irrotational fluids. The scope of this short note is to clarify certain aspects of the magnetic TLNs and to unveil the relation between the different magnetic TLNs presented in previous

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[1]When the object is spinning, angular momentum gives rise to spin-tidal coupling and to a new class of rotational TLNs [3,9,10,19,20]. In this paper we focus on static objects so we shall not consider the rotational TLNs.

work. As we shall show, the magnetic TLNs computed by DN are actually equivalent [modulo a prefactor given in Eq. (14) below] to those computed by LP for irrotational fluids, whereas the magnetic TLNs computed by BP refer to strictly static configurations.

II. AXIAL PERTURBATIONS OF A PERFECT-FLUID STAR

We consider magnetic (i.e., odd parity or axial) perturbations of Einstein's equations in the Regge-Wheeler gauge [31]. In our analysis the perturbations can be time dependent; we shall analyze the static limit later on. We use geometrical units in which $G = c = 1$.

We consider a (spherically symmetric) background described by an isotropic perfect fluid with stress-energy tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$, where u^μ is the four-velocity of the fluid, and p and ρ are the pressure and the energy density, respectively. The background metric, $g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2$, satisfies the Tolman-Oppenheimer-Volkoff equations,

$$M' = 4\pi r^2 \rho, \qquad \nu' = 2\frac{M + 4\pi r^3 p}{r(r-2M)},$$
$$p' = -(p+\rho)\frac{M + 4\pi r^3 p}{r(r-2M)}, \tag{1}$$

where a prime denotes a derivative with respect to r, and we have defined the radial mass function $M(r)$ such that $e^{-\lambda} = 1–2M/r$. In this background, the unperturbed fluid velocity reads $u^\mu = u^\mu_0 = \{e^{-\nu/2}, 0, 0, 0\}$.

The perturbed metric reads $g_{\mu\nu}(t, r, \vartheta, \varphi) = g^{(0)}_{\mu\nu} + \delta g^{\rm odd}_{\mu\nu}(t, r, \vartheta, \varphi)$, with

$$\delta g^{\rm odd}_{\mu\nu} = \sum_\ell \sum_{m=-l}^{l} \begin{pmatrix} 0 & 0 & h^\ell_0(t,r)S^\ell_\vartheta & h^\ell_0(t,r)S^\ell_\varphi \\ * & 0 & h^\ell_1(t,r)S^\ell_\vartheta & h^\ell_1(t,r)S^\ell_\varphi \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \tag{2}$$

where asterisks represent symmetric components, $Y^\ell = Y^\ell(\vartheta, \varphi)$ are the scalar spherical harmonics, and $(S^\ell_\vartheta, S^\ell_\varphi) \equiv (-\frac{1}{\sin\vartheta} Y^\ell_{,\varphi}, \sin\vartheta Y^\ell_{,\vartheta})$ are the (odd-parity) vector spherical harmonics. Since the background is spherically symmetric, the azimuthal number m is degenerate and the perturbation equations depend only on ℓ. Under parity transformations ($\vartheta \to \pi - \vartheta$, $\varphi \to \varphi + \pi$), the perturbations are multiplied by $(-1)^{\ell+1}$ and therefore are called odd-parity or "axial"; we shall use the two notations indistinctly.

In the axial sector the metric perturbations are not coupled to pressure and density perturbations, but are coupled to axial fluid perturbations. The only nonvanishing odd-parity fluid perturbation is the axial fluid velocity (we follow the notation of Ref. [32] in the nonrotating case):

$$\delta u^\mu = [4\pi e^{-\nu/2} r^2 (\rho + p)]^{-1} \left(0, 0, S^\ell_\vartheta, \frac{S^\ell_\varphi}{\sin^2\theta}\right) U^\ell(t, r), \tag{3}$$

such that $u^\mu = u^\mu_0 + \delta u^\mu$. By linearizing Einstein's equations on the background $g^{(0)}_{\mu\nu}$, one can obtain a system of three differential equations for the axial sector only

$$e^{-\nu}\dot{h}_0 - e^{-\lambda}h_1' - \frac{1}{r^2}(2M - 4\pi(\rho - p)r^3)h_1 = 0, \tag{4}$$

$$e^{-\nu}(\dot{h}_0{}' - \ddot{h}_1) - \frac{2e^{-\nu}}{r}\dot{h}_0 - \frac{(l-1)(l+2)}{r^2}h_1 = 0, \tag{5}$$

$$e^{-\lambda}(h_0'' - \dot{h}_1') - 4\pi(\rho + p)r(h_0' - \dot{h}_1') - \frac{2e^{-\lambda}}{r}\dot{h}_1$$
$$-\frac{1}{r^3}(l(l+1)r - 4M + 8\pi(\rho + p)r^3)h_0 - 4e^\nu U = 0, \tag{6}$$

where for clarity we omitted the multipolar index ℓ from the perturbation variables and used a dot to denote a time derivative.

We immediately see that Eq. (4) can be generically solved for h_0 in terms of h_1, *provided* the perturbations are not strictly *static*, in which case $\dot{h}_0 = 0$ and Eq. (4) becomes a constraint equation for h_1.

More precisely, Eq. (4) can be written as

$$\dot{h}_0 = e^{(\nu-\lambda)/2}(\psi r)', \tag{7}$$

where ψ is defined such that

$$h_1 = e^{(\lambda-\nu)/2}\psi r, \tag{8}$$

and we have used the background equations (1). Below, we consider the static and time-dependent cases separately.

A. Static axial perturbations

For strictly *static* perturbations, $\dot{h}_i = 0$ and $U = 0$. In this case Eq. (5) yields $h_1 = 0$, which also satisfies Eq. (4). On the other hand, Eq. (6) yields a second-order differential equation for h_0:

$$e^{-\lambda}h_0'' - 4\pi r(p+\rho)h_0' - \left(\frac{l(l+1)}{r^2} - \frac{4M}{r^3} + 8\pi(p+\rho)\right)h_0 = 0. \tag{9}$$

This equation is equivalent to that obtained by BP (cf. Eq. (4.29) in Ref. [2]) which indeed studied the axial perturbations of a strictly static fluid.

Although Ref. [2] reported that Eq. (9) is also equivalent to Eq. (31) in DN [1], this is actually not the case, as already noticed in Ref. [9]. We shall elucidate the reason for this discrepancy in the next section.

B. Time-dependent axial perturbations

Let us consider the Fourier transform of the perturbations, i.e., $h_i(t,r) = \int dt h_i(\omega, r) e^{-i\omega t}$, with a slight abuse of notation. In this case Eq. (7) can be solved for h_0 in terms of h_1 and its derivative:

$$h_0(\omega, r) = i \frac{e^{(\nu-\lambda)/2}}{\omega} (\psi r)'. \tag{10}$$

Notice that the above equation does not have a well-defined limit as $\omega \to 0$. Inserting Eq. (10) into Eq. (5) yields

$$e^{(\nu-\lambda)/2}(e^{(\nu-\lambda)/2}\psi')' + \left[\omega^2 - e^{\nu}\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3} + 4\pi(\rho - p)\right)\right]\psi = 0, \tag{11}$$

which is the standard Regge-Wheeler equations for axial perturbations inside the star (see e.g., Ref. [32]). In the limit $\omega \to 0$ this equation coincides with Eq. (31) in DN [1].

We shall now show that the $\omega \to 0$ limit of Eq. (11) is inequivalent to Eq. (9). The underlying reason for this fact can be traced back to the perturbation of the fluid velocity, which for $\omega \neq 0$ is (see, e.g., [32])

$$U = -4\pi(\rho + p)e^{-\nu} h_0. \tag{12}$$

The above equation can be obtained by an appropriate combination of the components of Einstein's equations or, more directly, by the axial component of the stress-energy tensor conservation. Therefore, even when $\omega \to 0$ the fluid velocity is nonvanishing and the configuration is not strictly static. By replacing Eq. (12) into Eq. (6), it is straightforward to obtain an equation for h_0 which, in the limit $\omega \to 0$, reads

$$e^{-\lambda} h_0'' - 4\pi r(p+\rho) h_0' - \left(\frac{l(l+1)}{r^2} - \frac{4M}{r^3} - 8\pi(p+\rho)\right) h_0 = 0. \tag{13}$$

This equation coincides with Eq. (5.6) in LP for an *irrotational* fluid ($\lambda = 1$ in their notation). As noticed in LP, Eq. (13) is actually very similar to Eq. (9) for the static case, the only difference being the opposite sign in front of the $(\rho + p)$ term.

One can easily check that the fluid in this configuration is *irrotational*, i.e., the vorticity vector $\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} u_{\beta;\mu} u_\nu$ identically vanishes [33]. This corresponds to the configuration studied by LP [3]. In our case this condition is enforced by Eq. (12), while in the static case $U = 0$.

The fact that Eqs. (9) and (13) are inequivalent shows that the limit $\omega \to 0$ of the axial sector is discontinuous, i.e., in this limit the Regge-Wheeler equation is not equivalent to Eq. (9) which describes the static case, $\omega = 0 = U$. The latter is an isolated point in the space of the solutions.

III. DISCUSSION

In summary, we showed that the equation describing the magnetic TLNs computed by DN coincide with that computed by LP for an irrotational fluid. This is due to the zero-frequency limit of the Regge-Wheeler equation, which forces the fluid to be irrotational rather than static. This fact also explains why the master equations computed by DN and by BP are inequivalent, because in the former case the fluid is irrotational, whereas in the latter case the fluid is static. To the best of our knowledge, this connection was not pointed out in the past.

In particular, the relation between the magnetic TLNs computed by DN (denoted as j_ℓ) and those computed by LP (denoted as $\tilde{k}_\ell^{\rm mag}$) for an irrotational fluid is (see also Eq. (6) in Ref. [15] for the $\ell = 2$ case)

$$j_\ell = \frac{4(\ell+2)(\ell+1)}{\ell(\ell-1)} \left(\frac{\mathcal{M}}{\mathcal{R}}\right) \tilde{k}_\ell^{\rm mag}. \tag{14}$$

Note that the two definitions differ by a factor of the compactness, $\mathcal{M}/\mathcal{R}$, where $\mathcal{M} = M(\mathcal{R})$ and $\mathcal{R}$ are the stellar mass and radius, respectively.

Yagi [21] used the master equation derived by DN so he actually computed the magnetic TLNs j_ℓ which, as we have just shown, correspond to the case of an irrotational fluid. In particular, the static and irrotational magnetic TLNs satisfy two different approximately-universal relations, as discussed in Refs. [18,20], where some fits for such relations are provided in both cases.

Finally, since the irrotational case is obtained as the zero-frequency limit of the Regge-Wheeler equation, we consider it to be more physical, which is also on the line of recent numerical relativity simulations of binary neutron star mergers [34–37], and therefore we expect it should describe more accurately relevant astrophysical configurations [3,26,38].

It is also worth mentioning that the magnetic TLNs of static and of irrotational fluids are similar in absolute values (and of opposite sign). This implies that in both cases their contribution to the waveform is very small, and might be possibly be relevant only for third-generation gravitational-wave detectors, as recently analyzed in detail [18].

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