New spacetimes for rotating dust in (2 + 1)-dimensional general relativity

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Multiparameter solutions to the Einstein equations in $2 + 1$ dimensions are presented, with stress-energy given by a rotating dust with negative cosmological constant. The matter density is uniform in the corotating frame, and the ratio of the density to the vacuum energy may be freely chosen. The rotation profile of the dust is controlled by two parameters, and the circumference of a circle of a given radius is controlled by two additional parameters. Though locally related to known metrics, the global properties of this class of spacetimes are nontrivial and allow for new and interesting structure, including apparent horizons and closed timelike curves, which can be censored by a certain parameter choice. General members of this class of metrics have two Killing vectors, but parameters can be chosen to enhance the symmetry to four Killing vectors. The causal structure of these geometries, interesting limits, and relationship to the Gödel metric are discussed. An additional solution, with nonuniform dust density in a Gaussian profile and zero cosmological constant, is also presented, and its relation to the uniform-density solutions in a certain limit is discussed.

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I. INTRODUCTION

In the past century of Einstein's theory of gravity, progress in general relativity has been shaped in large part by the discovery of new exact solutions to the field equations. From Schwarzschild's work to the present day, exact solutions provide useful laboratories for investigating the properties of relativistic gravitating systems.

Two exact solutions of particular interest are Gödel's spacetime [1] describing a rotating dust and Bañados, Teitelboim, and Zanelli's (BTZ) black hole [2]. Both of these solutions describe geometries in $2 + 1$ dimensions, with a negative cosmological constant. Gödel's solution demonstrated properties of closed timelike curves (CTCs) in general relativity, while the BTZ geometry showed that black holes can exist in three-dimensional gravity. General relativity in three dimensions provides an interesting arena in which to explore the geometric properties of spacetime, without the complications endemic to a nonzero Weyl tensor [3]. Moreover, $(2 + 1)$-dimensional spacetimes with a negative cosmological constant are of current interest as a result of the AdS/CFT correspondence [4–7], in particular, aspects of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case (see Ref. [8] and references therein for a review, as well as Ref. [9]).

In this paper, a class of axially symmetric spacetime geometries in $2 + 1$ dimensions is presented. The cosmological constant is negative, and the energy-momentum tensor is described by pressureless, rotating dust, as in the Gödel solution. However, this class of geometries is much more general, with multiple free parameters describing the dust density, (spatially dependent) rotation rate, and circumference profile of the spacetime. Depending on the parameters, these spacetimes can exhibit many interesting properties, including CTCs, apparent horizons, spacetime boundaries, enhanced symmetry algebras, and geodesic completeness.

This paper is organized as follows. In Sec. II, the class of solutions is stated and the corotating timelike congruence is examined. Next, in Sec. III the causal structure of the spacetime is investigated, including null congruences, Cauchy horizons, geodesic completeness, and censorship of the CTCs. Particular limiting geometries are discussed in Sec. IV, and the symmetries of the spacetime are found in Sec. V. We examine various special cases of interest in Sec. VI and conclude with the discussion in Sec. VII. In Appendix, another new spacetime geometry is exhibited, with a spinning dust with Gaussian density profile, which in a certain limit reduces to a member of the family of geometries we present in Sec. II.

II. A CLASS OF SPINNING DUST SOLUTIONS

Consider the following stationary metric in $2 + 1$ dimensions:

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$$ds^2 = -dt^2 - w(r)dtd\phi + D(r)d\phi^2 + \frac{dr^2}{N(r)}, \quad (1)$$

where

$$w(r) = 2L\left[j_1\left(\frac{L}{r}\right) + j_2\right],$$
$$D(r) = -j_1^2 ML^2\left(\frac{L}{r}\right)^2 + 2c_1L^2\left(\frac{L}{r}\right) + c_2L^2,$$
$$N(r) = 4(1-M)\left(\frac{r}{L}\right)^2 + 8\left(\frac{j_1j_2+c_1}{j_1^2}\right)\left(\frac{r}{L}\right)^3 + 4\left(\frac{j_2^2+c_2}{j_1^2}\right)\left(\frac{r}{L}\right)^4. \quad (2)$$

Here, j_1, j_2, c_1, c_2, and M are unitless constants and $-\infty < t < \infty$, $0 \le r < \infty$, and $0 \le \phi < 2\pi$.[1] This metric describes a rotating, pressureless dust solution with density $4M/L^2$ and with negative cosmological constant $\Lambda = -1/L^2$. Defining $u^a = \partial_t = (1,0,0)$ for the reference frame of the dust,[2] the Einstein equations are satisfied,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{4M}{L^2}u_a u_b. \quad (3)$$

The metric (1) has several interesting properties. Frame dragging is evidenced by the nonzero $dtd\phi$ component of the metric. However, the dust in this solution does not rotate at a constant rate, but instead at a rate dictated by $w(r)$. In particular, if j_1 and j_2 have opposite signs, then the dust rotates clockwise for small r and counterclockwise for large r, or vice versa, and has vanishing rotation at some radius $r_0 = -\frac{j_1}{j_2}L$. We note that $w(r)$, $D(r)$, and $N(r)$ in Eq. (2) are related by the useful identity

$$j_1^2 L^6 N(r) = r^4[4D(r) + w^2(r)]. \quad (4)$$

Let us consider a timelike geodesic congruence corotating with the dust. We choose the initial tangent vector for our congruence of geodesics to be u^a, and since

$$u^a\nabla_a u^b = 0, \quad (5)$$

we obtain a family of timelike geodesics that continue to point along u^a at all times. Since we can show that u^a is a global timelike Killing vector of the geometry (1), the spacetime is stationary. However, it is not static for $j_1 \neq 0$, since there do not exist hypersurfaces everywhere orthogonal to orbits of the timelike Killing vectors, or equivalently by Frobenius's theorem [10], $u_{[a}\nabla_b u_{c]} \neq 0$.

For this timelike congruence, we can define a spatial metric $h_{ab} = g_{ab} + u_a u_b$; the extrinsic curvature is $B_{ab} = \nabla_b u_a$. Then we find that the expansion, measuring the logarithmic derivative of the area element along the congruence, vanishes,

$$\theta = h^{ab}B_{ab} = 0. \quad (6)$$

Similarly, the shear vanishes, implying that the dust is rigidly rotating,[3]

$$\varsigma_{ab} = B_{(ab)} - \frac{1}{2}\theta h_{ab} = 0. \quad (7)$$

However, the vorticity tensor[4] $\Omega_{ab} = B_{[ab]}$ does not vanish,

$$\Omega_{ab}dx^a \wedge dx^b = 2\Omega_{r\phi}dr \wedge d\phi = -\frac{j_1L^2}{r^2}dr \wedge d\phi. \quad (8)$$

Since the vorticity is nonzero, the geodesic congruence formed by u^a is not hypersurface orthogonal [10]. This is another manifestation of the fact that the spacetime, while stationary, is not static.

An extensive review of other dust solutions can be found in Ref. [12]. In more recent work, Refs. [13,14] gave the general $(2+1)$-dimensional solutions for an irrotational dust and a null dust, respectively.[5] As we will see in Sec. VI A, while our family of metrics contains the Gödel universe as a special case, it is in general distinct from Gödel's solution. Our metric is axially symmetric but is not locally rotationally symmetric in the sense of Ref. [11]. It is distinct from the van Stockum [15] (or more generally, Lanczos [16]) solution, which has nonhomogeneous dust density. The traceless Ricci tensor $R_{ab} - \frac{1}{3}Rg_{ab}$ for the metric given in Eqs. (1) and (2) is proportional to $g_{ab} + 3u_a u_b$, so it is of Petrov-Segre type D_t in the notation of Ref. [17]. The metric is related by a local diffeomorphism to a timelike-squashed AdS_3 solution of topologically massive gravity [17,18] and by a different local diffeomorphism to the

[1]As we will show in Sec. VII, having chosen a value of L, which specifies the overall length scale, the coefficients (M, j_1, j_2, c_1, c_2) indeed specify a five-parameter family of distinct geometries; that is, for generic values of these constants, the geometry one obtains is not equivalent to another member of this family under diffeomorphism.

[2]Unless indicated otherwise, we use an axial (t, r, ϕ) coordinate system throughout.

[3]A theorem of Gödel [1,11] states that if a spacetime possesses spacelike homogeneous hypersurfaces and a timelike geodesic congruence with vanishing expansion and shear, then the metric must locally correspond to either the Gödel universe or the Einstein static universe. For the metric Eq. (1), the matter density is homogeneous, $M = \text{const}$, but the metric is not. Hence, Eq. (1) is allowed to differ from the Gödel or Einstein static universe, which it manifestly does in general, since M is independent of Λ.

[4]This object is sometimes called the twist [10]. However, to avoid confusion with the twist one-form we will later define for null congruences, we use this alternative nomenclature.

[5]In contrast, the dust solution described in this paper is timelike and has nonvanishing vorticity.

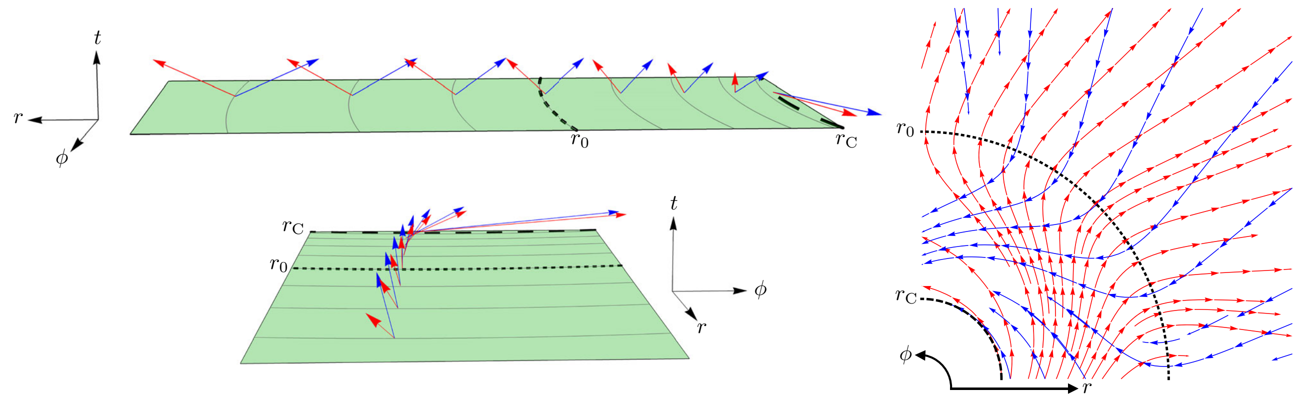

FIG. 1. Illustration of the null congruences orthogonal to circles of fixed r (gray arcs) on some slice of constant t (green surface). The outgoing and ingoing orthogonal null congruences have tangents given in Eq. (9) by k^a and ℓ^a and are illustrated by red and blue arrows, respectively. For this illustration, $j_1 j_2$ was chosen to be negative, so the frame dragging vanishes on the surface at radius r_0 (dotted line) where $w(r_0) = 0$ and goes in the $+\phi$ or $-\phi$ direction inside and outside, respectively. As the Cauchy horizon r_C (dashed line) is approached, the orthogonal null congruences are dragged around the angular direction. For $r < r_C$, the congruence is not defined.

four-parameter family of solutions of Lubo, Rooman, and Spindel [19]. Our class of metrics possesses five unitless parameters, plus a length scale L, and as we will discuss in Sec. VII, the family of globally distinct solutions is five-dimensional; the extra freedom comes from the fact that our metric (1) is an analytic extension of the metric in Ref. [19] and, for various parameter choices, can have different global properties. As we will see in this work, these global properties lead to interesting physical differences in three-dimensional gravity, including considerations of topology, causal structure, horizons, boundaries, geodesic completeness, and singularities.

III. CAUSAL STRUCTURE

To investigate the causal structure of the spacetime, it is useful to first construct some null congruences. We will find apparent horizons, spacetime boundaries, and CTCs, and also discover how to censor them.

A. Null congruences

Consider a surface σ at constant r, for some r chosen such that $N(r)$ and $D(r)$ are positive. There are two future-pointing null geodesic congruences generated by the null vectors with initial tangents k^a and ℓ^a orthogonal to σ,

$$k^a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+\frac{w^2(r)}{4D(r)}}}, \sqrt{N(r)}, \frac{w(r)}{2D(r)\sqrt{1+\frac{w^2(r)}{4D(r)}}}\right),$$
$$\ell^a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+\frac{w^2(r)}{4D(r)}}}, -\sqrt{N(r)}, \frac{w(r)}{2D(r)\sqrt{1+\frac{w^2(r)}{4D(r)}}}\right). \quad (9)$$

Despite the fact that k^ϕ, $\ell^\phi \neq 0$, one can verify that $k_a\phi^a = \ell_a\phi^a = 0$, for $\phi^a = \partial_\phi = (0,0,1)$ giving the unit tangent to σ. We note that $k^2 = \ell^2 = 0$, and we have chosen the relative normalization such that $k\cdot\ell = -1$. Both k and ℓ are future pointing (i.e., $k^t \geq 0$ and $\ell^t \geq 0$), and k is outward pointing while ℓ is inward pointing (i.e., $k^r \geq 0$ and $\ell^r \leq 0$). In Eq. (9), we need $1 + w^2(r)/4D(r) \geq 0$, which is justified by Eq. (4) along with our choice of r such that $N(r)$ and $D(r)$ are both positive. See Fig. 1 for an illustration of the congruences.

From the congruences, we define the induced metric $q_{ab} = g_{ab} + k_a\ell_b + k_b\ell_a$, the null extrinsic curvature $B^{(k)}_{ab} = q^c_a q^d_b \nabla_d k_c$ (and analogously for ℓ), and the null expansion $\theta_k[\sigma] = q^{ab}B^{(k)}_{ab}$ (which measures the logarithmic derivative of the area element along the affine parameter of the null geodesic). After explicit computation, we have

$$\theta_k[\sigma] = -\theta_\ell[\sigma] = \frac{D'(r)\sqrt{N(r)}}{2\sqrt{2}D(r)}. \quad (10)$$

The shear of the congruences vanishes,

$$\varsigma^{(k)}_{ab} = B^{(k)}_{(ab)} - \theta_k q_{ab} = 0,$$
$$\varsigma^{(\ell)}_{ab} = B^{(\ell)}_{(ab)} - \theta_\ell q_{ab} = 0, \quad (11)$$

which is a consequence of the fact that the shear tensor is by definition traceless and that the null congruence is codimension-two (and hence one-dimensional in this three-dimensional spacetime). The vorticity tensors also vanish,

$$\Omega^{(k)}_{ab} = B^{(k)}_{[ab]} = 0,$$
$$\Omega^{(\ell)}_{ab} = B^{(\ell)}_{[ab]} = 0, \quad (12)$$

which is a consequence of the fact that the congruences are hypersurface orthogonal. The twist one-form gauge field (i.e., the Hájíček one-form) is

$$\omega_a = \frac{1}{2} q_{ab} \mathcal{L}_k \ell^b = -q_a{}^b \ell^c \nabla_b k_c$$
$$= \sqrt{\frac{N(r)}{1+\frac{w^2(r)}{4D(r)}}} \frac{w(r)D'(r) - D(r)w'(r)}{8D(r)^2}$$
$$\times (w(r), 0, -2D(r)), \tag{13}$$

where $\mathcal{L}_k$ denotes the Lie derivative along k.

Were we to choose parameters for which we can have $D(r) > 0$ when $N(r) = 0$, the metric in Eq. (1) would have apparent horizons, where θ_k or θ_ℓ vanish, at the zeros of $N(r)$ located at $r = r_\pm$,

$$r_\pm = L\left(\frac{j_1 j_2 + c_1}{j_2^2 + c_2}\right)\left[-1 \pm \sqrt{1 - (1-M)\frac{j_1^2(j_2^2+c_2)}{(j_1 j_2 + c_1)^2}}\right]. \tag{14}$$

Whether these zeros exist (i.e., whether $r_\pm$ is real and positive) in the spacetime depends on the relative signs and magnitudes of j_1, j_2, c_1, c_2, and M.

To guarantee that the angular direction is not timelike at large r, we must take $c_2 \geq 0$. If we choose $c_1 > 0$, then $D(r)$ would reach a maximum at $r = r_{\rm m} = \frac{j_1^2}{c_1} ML$. In this case, the signs of θ_k and θ_ℓ would flip at $r_{\rm m}$. We would have an apparent horizon at $r_{\rm m}$, where the null expansions can vanish. For the rest of this paper, we will consider $c_1 \leq 0$, so that $D'(r) > 0$; this ensures that circles of constant r have circumferences that grow with r asymptotically, as we would intuitively expect. In that case, σ is a normal surface, i.e., a surface for which the outward future-pointing null congruence has positive expansion and the inward future-pointing null congruence has negative expansion.

B. Cauchy horizon and boundary

This spacetime can exhibit Cauchy horizons, defining the boundary of the nonchronological region of the spacetime where $D(r) < 0$. In this region, circles in ϕ at constant t and r are CTCs.[6] The Cauchy horizons are given by the zeros of $D(r)$,

$$r_\pm^{\rm C} = -L\left(\frac{c_1}{c_2}\right)\left(1 \pm \sqrt{1 + \frac{j_1^2 M c_2}{c_1^2}}\right). \tag{15}$$

By the null energy condition, we will take $M \geq 0$, so that the dust has non-negative density. Further, taking $c_2 \geq 0$ and $c_1 \leq 0$ as noted previously, the nonchronological region is given by $r < r_{\rm C}$, where we write $r_{\rm C}$ for $r_+^{\rm C}$, the single Cauchy horizon,

[6]Note that the appearance of CTCs does not run afoul of the theorem in Ref. [20], since the CTCs extend arbitrarily close to $r = 0$, and thus constitute "boundary CTCs" in the sense of Ref. [20].

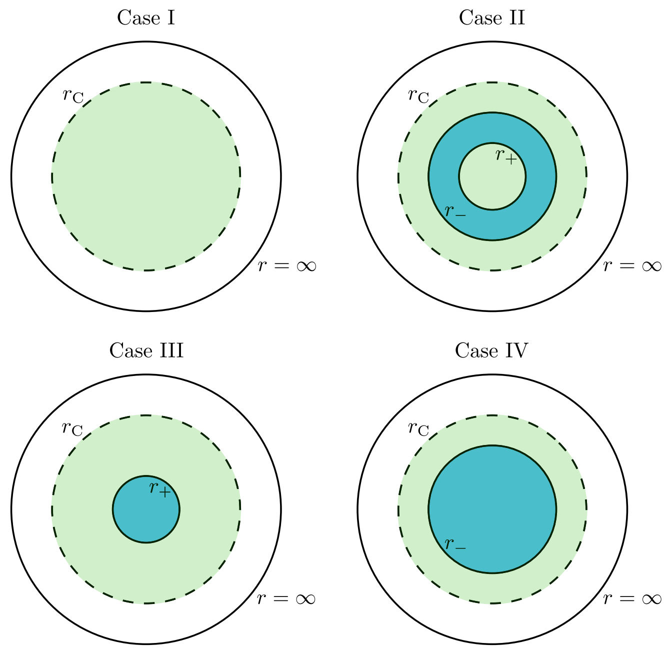

FIG. 2. Cases in Eq. (17) for geometry with metric given in Eqs. (1) and (2). The singularities in the rr component of the metric [the zeros of $N(r)$] are $r_\pm$, while the Cauchy horizon $r_{\rm C}$ is determined by the zero of the $\phi\phi$ component of the metric $D(r)$. In Cases I and II, $0 < M < 1$, while in Cases III and IV, $M \geq 1$. In Cases I and III, $j_1 j_2 + c_1 > 0$, while in Cases II and IV, $j_1 j_2 + c_1 < 0$. If $c_2 \geq 0$ and $c_1 \leq 0$, then $r_{\rm C} \geq r_\pm$.

$$r_{\rm C} = -L\left(\frac{c_1}{c_2}\right)\left(1 + \sqrt{1 + \frac{j_1^2 M c_2}{c_1^2}}\right). \tag{16}$$

We then have the following conditions for the existence of the zeros of $N(r)$ in Eq. (14):

$$\begin{aligned}
&\text{Case I}: 0 < M < 1,\ j_1 j_2 + c_1 > 0 \Rightarrow \text{no zero},\\
&\text{Case II}: 0 < M < 1,\ j_1 j_2 + c_1 < 0 \Rightarrow \text{two zeros at } r_\pm, r_+ < r_-,\\
&\text{Case III}: M \geq 1,\ j_1 j_2 + c_1 > 0 \Rightarrow \text{one zero at } r_+,\\
&\text{Case IV}: M \geq 1,\ j_1 j_2 + c_1 < 0 \Rightarrow \text{one zero at } r_-.
\end{aligned} \tag{17}$$

See Fig. 2 for an illustration. The nonchronological region can extend outside the apparent horizon; i.e., it is possible to have $r_{\rm C} > r_\pm$. Indeed, with our assumptions $c_1 \leq 0$ and $c_2 \geq 0$, one can show that the Cauchy horizon always satisfies $r_{\rm C} \geq r_\pm$, which implies that the congruence in Sec. III A cannot be extended down to $r_\pm$ [since the construction of the congruence required $D(r) > 0$ by Eq. (4)]. Hence, if $c_1 \leq 0$ and $c_2 \geq 0$, the zeros of $N(r)$ are not truly apparent horizons where θ_k or θ_ℓ vanish, since the congruence does not exist there.

In the region where $N(r) < 0$, we can consider a surface μ at constant r. Let us attempt to construct the orthogonal null congruences from μ. Writing one of the tangent vectors to such a congruence as $\bar{k}^a$, we must have $\bar{k}_\phi = 0$ for the congruence to be orthogonal to μ. Thus, we have

$$\bar{k}^2 = (\bar{k}_r)^2 N(r) - \frac{(\bar{k}_t)^2}{1 + \frac{w^2(r)}{4D(r)}}. \tag{18}$$

Since we have $N(r) < 0$ on μ and since Eq. (2) requires $j_1 \neq 0$, Eq. (4) implies that we must have $D(r) < 0$ on μ. By Eq. (4) again, we then have $1 + w^2(r)/4D(r) > 0$ on μ, which by Eq. (18) makes the requirement $\bar{k}^2 = 0$ impossible to satisfy for any choice of $\bar{k}_t$ and $\bar{k}_r$. Hence, the null congruences from μ do not exist; i.e., the region $N(r) < 0$ is of non-Lorentzian causal structure. We therefore exclude this region from the spacetime and take a surface at $r = r_\pm$ to be a boundary of the geometry.

C. Geodesics and completeness

Let us consider the evolution of a timelike geodesic in this spacetime. The geodesic equation,

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0, \tag{19}$$

implies

$$\begin{aligned}
\frac{j_1^2 L^6 N(r)}{r^4} \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} - 2w(r)D'(r)\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + w'(r)\frac{\mathrm{d}r}{\mathrm{d}\tau}\left[2D(r)\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + w(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] &= 0, \\
\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{N'(r)}{2N(r)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{N(r)}{2}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\left[w'(r)\frac{\mathrm{d}t}{\mathrm{d}\tau} - D'(r)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] &= 0, \\
\frac{j_1^2 L^6 N(r)}{r^4}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}r}{\mathrm{d}\tau}\left[(4D'(r) + w(r)w'(r))\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - 2w'(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] &= 0,
\end{aligned} \tag{20}$$

where we used Eq. (4) to write $4D(r) + w^2(r)$ in terms of $N(r)$. Now, near $r = r_\pm$, $N(r)$ is small, so to leading order in this limit, we must have

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{N'(r)}{2N(r)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = 0, \tag{21}$$

where we can approximate $N(r)$ by $N'(r_\pm)(r - r_\pm)$. Thus, near $r_\pm$, we have the solution

$$r(\tau) - r_\pm = r_\mathrm{i} - r_\pm + v_\mathrm{i}\tau + \frac{v_\mathrm{i}^2\tau^2}{4(r_\mathrm{i} - r_\pm)}, \tag{22}$$

where r_i and v_i are constants. Note that r_i can be either larger or smaller than $r_\pm$, so this solution applies to timelike geodesics that originate on either side of $r_\pm$. We find from Eq. (22) that the geodesic can never cross the surface at $r = r_\pm$: $r(\tau)$ has a turning point when $\tau = -2(r_\mathrm{i} - r_\pm)/v_\mathrm{i}$, at $r(\tau) = r_\pm$. That is, the timelike geodesic can just touch the surface and bounce off of it, but cannot pass through; see Fig. 3. Replacing τ by an arbitrary parameter λ respecting the affine connection, the analogous conclusion applies for both null and spacelike geodesics, which also bounce off of this boundary. (Given r_i and v_i, whether the geodesic is timelike, null, or spacelike can be specified by choosing the initial data in the t and ϕ components of the geodesic equation.) This justifies our considering $r_\pm$ to be boundaries of the spacetime. That is, the connected regions of the fixed sign of $N(r)$ (i.e., the three regions $r < r_+$, $r_+ < r < r_-$, and $r > r_-$ in Case II; the two regions $r < r_+$ and $r > r_+$ in Case III; and the two regions $r < r_-$ and $r > r_-$ in Case IV) can be essentially regarded as separate spacetimes. Indeed, if we replace the right-hand side of the geodesic equation with the proper acceleration, we see that any trajectory with nonzero $\mathrm{d}r/d\tau$ at $r = r_\pm$ will necessarily undergo infinite proper acceleration; the barrier at $r = r_\pm$ is insurmountable.

This example suggests that the spacetime in the region where $N(r)$ is positive, bounded by $r_\pm$, is, in fact, geodesically complete. Indeed, we can verify this with the help of the theorem proven in Ref. [21]. In the classification of

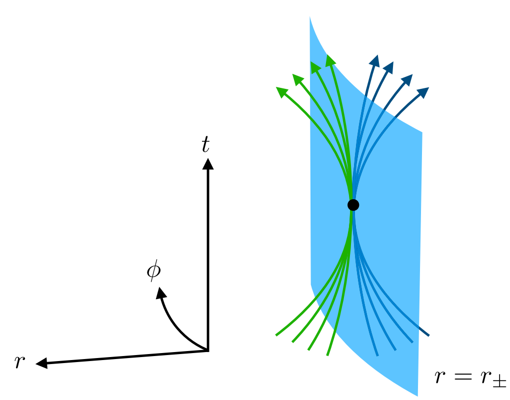

FIG. 3. Geodesics bouncing off of a boundary at $r = r_\pm$. Depicted is a family of geodesics with different initial data r_i and v_i chosen to all intersect the boundary at some fixed spacetime location (black dot). The bounce occurs for geodesics originating on either side of the boundary.

Ref. [21], the family of metrics given in Eq. (1) fits the definition of a Gödel-type spacetime provided $w^2(r)+4D(r)>0$.[7] By Eq. (4), this is true whenever $j_1 \neq 0$ and $N(r) > 0$. Hence, our geometry is within the class of Gödel-type metrics (a class that includes the Gödel universe, some Kerr-Schild metrics, some plane wave metrics, etc.) whenever we are outside the $N(r) = 0$ boundary. Applying Theorem 4.1 of Ref. [21], we observe that since $[1 - D(r) + \sqrt{(1 + D(r))^2 + w^2(r)}]^{-1}$ is bounded from above, it follows that our geometry, with $r_\pm$ as its boundary, is geodesically complete.

D. Censoring CTCs

To avoid $r_C > r_-$ (while keeping $c_1 \leq 0$ and $c_2 \geq 0$), we must take either Case II or Case IV above and further require the tuning of c_1 and c_2 such that

$$2\frac{c_1}{j_1} + j_2 M = \frac{c_2}{j_2} \tag{23}$$

and also $j_1 j_2 < 0$. With the choice (23), the radius of the outer of the $N(r) = 0$ surfaces, which we will call r_H, is located at

$$r_H = r_- = r_C = r_0 = -\frac{j_1}{j_2} L = \left|\frac{j_1}{j_2}\right| L. \tag{24}$$

In this case, the radius r_0 where the frame-dragging reverses direction, the $N(r) = 0$ surface at r_-, and the Cauchy horizon at r_C all coincide. While requiring $c_1 \leq 0$ and $c_2 \geq 0$, it is not possible for the Cauchy horizon to be located a finite distance inside the $N(r) = 0$ surface, i.e., $r_C \not< r_\pm$.

The choice of coefficients in Eq. (23), necessary to censor the CTCs, immediately leads us to an observation about the boundary. Since Eq. (23) implies that $D(r_H) = 0$, one finds from Eq. (10) that $\lim_{r\to r_H} \theta_k$ is not zero but is instead infinite. However, the twist ω_a vanishes as $r \to r_H$ with the choice in Eq. (23). Hence, for generic c_1 and M, the choice in Eq. (23) is incompatible with a smooth, marginally trapped event horizon. We stress that this is not a curvature singularity (neither s.p. nor p.p. type in the sense of Ref. [23]), since all curvature scalars for this metric are regular and, moreover, explicit computation shows that the full Riemann tensor is everywhere finite in a local Lorentz frame.[8] Instead, r_H is a boundary in the Lorentzian structure of the manifold, a singularity in the causal structure in the sense of Ref. [24].

IV. LIMITING GEOMETRY

In this section, we will keep the choice in Eq. (23) that we made in Sec. III D, allowing us to exclude the CTCs from the geometry; as we saw in Sec. III C, we can drop the region $r < r_H$ from the geometry with impunity. We would like to investigate the behavior of the metric for $r \to \infty$ and for $r \to r_H$.

A. Asymptotic geometry

The metric (1) at large r looks like

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - 2d_1 \mathrm{d}t\mathrm{d}\phi + d_2^2 \mathrm{d}\phi^2 + \frac{d_3^4}{r^4}\mathrm{d}r^2, \tag{25}$$

where $d_1 = L j_2$, $d_2^2 = c_2 L^2$, and $d_3^4 = \frac{L^4 j_1^2}{4(j_2^2 + c_2)}$. Defining $r' = d_3^2/r$, $t' = t + d_1 \phi$, and $d_4^2 = d_1^2 + d_2^2$, we have

$$\mathrm{d}s^2 = -(\mathrm{d}t')^2 + d_4^2 \mathrm{d}\phi^2 + (\mathrm{d}r')^2, \tag{26}$$

which is just the geometry of a flat Lorentzian cylinder, $\mathbb{R}^{1,1} \otimes S^1$, of radius d_4. In particular, at constant ϕ, this is simply two-dimensional Minkowski space, and the asymptotic causal structure includes null infinity. However, the time coordinate t' has a "jump" or "winding" property due to its dependence on ϕ, as discussed in Ref. [25].

B. Near-boundary geometry

Let us define a near-boundary coordinate x, where $x^2 = \frac{r - r_H}{r_H}$. For small x, the metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 - 2j_2 L x^2 \mathrm{d}t\mathrm{d}\phi - \frac{2L^3}{r_H}(c_1 + j_1 j_2 M) x^2 \mathrm{d}\phi^2 \\
&\quad - \frac{j_2^2 L r_H \mathrm{d}x^2}{2(c_1 + j_1 j_2 M)} \\
&= -\mathrm{d}t^2 - 2\ell_3 x^2 \mathrm{d}t\mathrm{d}\phi + \ell_1^2 x^2 \mathrm{d}\phi^2 + \ell_2^2 \mathrm{d}x^2,
\end{aligned} \tag{27}$$

where we eliminated c_2 using Eq. (23) and defined $\ell_1^2 = -\frac{2L^3}{r_H}(c_1 + j_1 j_2 M)$, $\ell_2^2 = -\frac{j_2^2 L r_H}{2(c_1 + j_1 j_2 M)}$, and $\ell_3 = j_2 L$. Recalling that $c_1 \leq 0$, $j_1 j_2 < 0$, and $M \geq 0$, we have $c_1 + j_1 j_2 M \leq 0$, and hence the $\mathrm{d}\phi^2$ and $\mathrm{d}x^2$ terms are spacelike. The $\mathrm{d}t\mathrm{d}\phi$ term still induces frame dragging, but let us consider spatial slices at constant t. The spatial metric is just

$$\frac{\ell_1^2}{\ell_2^2}(x')^2 \mathrm{d}\phi^2 + (\mathrm{d}x')^2, \tag{28}$$

where we have defined $x' = \ell_2 x$. This is simply the metric of a two-dimensional plane with a conical defect. Defining $\phi' = \frac{\ell_1}{\ell_2}\phi$ so that the spatial metric is simply $(x')^2(\mathrm{d}\phi')^2 + (\mathrm{d}x')^2$, ϕ' takes the range $0 \leq \phi' < 2\pi\delta$, where

[7] Note that the conclusion of Ref. [22], which exhibited a class of metrics equivalent to the Gödel universe, does not apply to our metric (1), since we will see in Sec. VI A that our class of metrics is strictly larger than the Gödel solution.

[8] That is, $e^\mu{}_a e^\nu{}_b e^\rho{}_c e^\sigma{}_d R_{\mu\nu\rho\sigma}$ is finite, where the dreibein e^μ_a is defined as $g^{\mu\nu} = e^\mu{}_a e^\nu{}_b \eta^{ab}$ for flat metric η.

$$\delta = \frac{\ell_1}{\ell_2} = 2\left|\frac{c_1}{j_1} + j_2 M\right|. \tag{29}$$

We could make this conical defect vanish by enforcing $\ell_1 = \ell_2$, i.e., $\delta = 1$, in which case we could simply take the entire surface at $r = r_{\mathrm{H}}$ to be a single point, with a flat spatial metric, albeit with frame dragging present. Without this condition, we have a conical singularity if we take $r = r_{\mathrm{H}}$ to be identified as a single point. Interpreting the conical defect as a mass m in a flat background [25,26], we have $m = (1-\delta)/4G$. Since δ is manifestly positive, we have $m < 1/4G$, so the universe is not overclosed. When $\delta > 1$, this effective mass is negative. Indeed, m is unbounded from below; as we increase M, δ can grow without limit, and m can become arbitrarily large and negative.

V. SYMMETRIES

Let us now consider the symmetries of our metric in Eq. (1). For a region where $N(r) > 0$ (e.g., $r > \max r_\pm$), the family of geometries described by Eq. (1) has a Lie algebra of Killing vectors v each satisfying the Killing equation $\nabla_{(a} v_{b)} = 0$. The basis of this algebra is given by

$$\begin{aligned}
u^a &= (1,0,0) = \partial_t,\\
\phi^a &= (0,0,1) = \partial_\phi,\\
\chi^a &= \sqrt{N(r)}\left(-\frac{2w^3(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{D(r)}{w^2(r)}\right)}{4D(r)+w^2(r)}\sin\alpha\phi, -2\alpha\cos\alpha\phi, \frac{\frac{\mathrm{d}}{\mathrm{d}r}[4D(r)+w^2(r)]}{4D(r)+w^2(r)}\sin\alpha\phi\right),\\
\psi^a &= \sqrt{N(r)}\left(\frac{2w^3(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{D(r)}{w^2(r)}\right)}{4D(r)+w^2(r)}\cos\alpha\phi, -2\alpha\sin\alpha\phi, -\frac{\frac{\mathrm{d}}{\mathrm{d}r}[4D(r)+w^2(r)]}{4D(r)+w^2(r)}\cos\alpha\phi\right),
\end{aligned} \tag{30}$$

where

$$\alpha = -\frac{2}{j_1}\sqrt{2j_1 j_2 c_1 + c_1^2 + j_1^2[(M-1)c_2 + j_2^2 M]}. \tag{31}$$

The vector u^a is the global timelike Killing field we encountered previously, while the vector ϕ^a is the angular Killing field associated with the axial symmetry of the spacetime. The Killing vectors χ and ψ exist only if α is an integer, so that χ and ψ are single-valued. That is, for general values of the parameters $(M, j_1 j_2, c_1, c_2)$, the symmetry algebra is two-dimensional.

If ψ and χ are well-defined periodic Killing vectors, so that the Lie algebra of symmetries is four-dimensional, then $\alpha = n$, i.e.,

$$2j_1 j_2 c_1 + c_1^2 + j_1^2[(M-1)c_2 + j_2^2 M] = \frac{n^2 j_1^2}{4} \tag{32}$$

for some integer n. In that case, we can reach any spacetime point in the connected region where $N(r) > 0$ using these Killing vectors, by moving in t using u^a, in ϕ using ϕ^a, and in r using χ^a or ψ^a in combination with u^a and ϕ^a. Hence, as in the Gödel metric, the isometry group acts transitively and the spacetime region for $N(r) > 0$ is homogeneous. Thus, if Eq. (32) is satisfied but not Eq. (23), so that CTCs extend to the region where $N(r)$ is positive, the spacetime is everywhere vicious outside the boundary, with CTCs through every point (so the Cauchy horizon disappears). We will discuss the relationship between Eq. (1) and the Gödel metric further in Sec. VI A.

VI. SPECIAL CASES

Let us now compute some special cases and limits of Eq. (1) that are of physical interest, including the choice of parameters that leads to the Gödel metric and the behavior of the geometry in the $\Lambda \to 0$, nonspinning, and $M \to 0$ limits.

A. Relationship to Gödel universe

Choosing

$$\begin{aligned}
M &= \frac{1}{2},\\
j_1 &= -2,\\
j_2 &= 0,\\
c_1 &= 1,\\
c_2 &= 0,
\end{aligned} \tag{33}$$

and defining new unitless radial and time coordinates $r' = \mathrm{arcsinh}\sqrt{L/r}$ and $t' = t/\sqrt{2}L$, our metric in Eq. (1) reduces to the metric of the Gödel universe in $2+1$ dimensions [1,23]:

$$\begin{aligned}
\mathrm{d}s^2 = 2L^2[&-\mathrm{d}t'^2 + \mathrm{d}r'^2 - (\sinh^4 r' - \sinh^2 r')\mathrm{d}\phi^2\\
&+ 2\sqrt{2}\sinh^2 r' \mathrm{d}\phi \mathrm{d}t'].
\end{aligned} \tag{34}$$

Hence, our metric constitutes a generalization of the Gödel metric. Unlike the Gödel universe, in Eq. (1) the general class of metrics (i) corresponds to a matter density $\propto M$ that is a free parameter, rather than being pinned to the cosmological constant, (ii) has two distinct angular momentum parameters j_1 and j_2, allowing the spin of the dust to be nonuniform and even reverse direction at different radii, and (iii) can have apparent horizons for certain choices of the parameters.

One can alternatively view the Gödel solution not as simply describing a dust with negative cosmological constant, but instead as describing arbitrary matter content for which the Einstein tensor in a local Lorentz frame satisfies $G_{ab} \propto \mathrm{diag}(1,1,1)$. For example, describing a perfect fluid with arbitrary pressure, plus a cosmological constant, in order to correspond to the Gödel solution one must satisfy (in $8\pi G = 1$ units) $\rho - p = -2\Lambda$. That is, in the $\Lambda = 0$ case, the Gödel universe corresponds to a stiff fluid with $\rho = p$ [27]. Similarly, one can view the energy-momentum source for our metric in Eq. (1) as describing an arbitrary perfect fluid plus cosmological constant for which the Einstein tensor is $L^2 G_{ab} = \mathrm{diag}(4M-1,1,1)$ in a local Lorentz frame, that is, $\rho + \Lambda = \frac{4M-1}{L^2}$ and $p - \Lambda = \frac{1}{L^2}$. Taking the cosmological constant to vanish, the equation-of-state parameter $p/\rho = \frac{1}{4M-1}$ corresponds to a stiff fluid when $M = \frac{1}{2}$ in accordance with Eq. (33), a cosmological constant when $M = 0$, phantom energy when $0 < M < \frac{1}{4}$, radiation when $M = \frac{3}{4}$, and dust when $M \to \infty$.

It will be illuminating to impose various energy conditions on this combined perfect fluid [28]. Imposing the null energy condition, $R_{ab}k^a k^b \geq 0$ for all null k^a, we require $\rho + p \geq 0$, or equivalently, $M \geq 0$. Similarly, the strong energy condition, $R_{ab}t^a t^b \geq 0$ for all timelike t^a, implies $(D-3)\rho + (D-1)p \geq 0$ in D spacetime dimensions as well as $\rho + p \geq 0$, so in our case it merely stipulates that $M \geq 0$. Imposing the weak energy condition, $G_{ab}t^a t^b \geq 0$ for all timelike t^a, we require $\rho \geq 0$ and $\rho + p \geq 0$, or equivalently, $M \geq \frac{1}{4}$. Finally, imposing the dominant energy condition that $-G^a{}_b t^b$ be causal and future directed for all causal, future-directed t^a, we have $\rho \geq |p|$, so $M \geq \frac{1}{2}$.

Since Eq. (33) does not satisfy Eq. (23), the Gödel metric exhibits the CTCs for which it is known. While the Gödel universe is, in fact, totally vicious—i.e., it has CTCs through every point, since all points are equivalent as a consequence of its larger algebra of Killing vectors—this is not the case in general. That is, since for general choices of coefficients in our metric (1) the tuning in Eq. (32) is not satisfied, homogeneity is broken, and we have no reason to expect CTCs for $r > r_C$. Moreover, if we impose the tuning (32), so that the metric for $r > \max r_\pm$ is homogeneous, we can sequester the CTCs behind the boundary if we simultaneously impose Eq. (23). In that case, homogeneity guarantees that there are no CTCs outside the boundary. The requirements of Eqs. (32) and (23) can simultaneously be satisfied by first imposing Eq. (23) and then requiring $2|c_1 + j_1 j_2 M| = -j_1 n$.

With the choice of parameters (33), the parameter α in Eq. (31) is simply 1, so the Gödel metric has four Killing vectors as expected, with Eq. (32) satisfied with $n = 1$. Note that the converse is not necessarily true, however: satisfying $\alpha = 1$ does not imply that the metric is diffeomorphic to the Gödel universe, since Eq. (32) does not pin the dust density to the cosmological constant (i.e., set M to $1/2$) as is the case in the Gödel metric.

B. $\Lambda \to 0$ limit

For fixed (M, j_1, j_2, c_1, c_2), the $\Lambda \to 0$ (i.e., $L \to \infty$) limit of the metric described in Eqs. (1) and (2) is singular. However, let us define the rescaled parameters

$$\begin{aligned} \iota_1 &= L^3 j_1, \\ \iota_2 &= L j_2, \\ \kappa_1 &= L^3 c_1, \\ \kappa_2 &= L^2 c_2, \\ \mu &= \frac{M}{L^2}, \end{aligned} \tag{35}$$

and then take the $L \to \infty$ limit, holding ι_1, ι_2, κ_1, κ_2, and μ constant. In this case, the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - 2\iota_2 \mathrm{d}t\mathrm{d}\phi + \Delta(r)\mathrm{d}\phi^2 + \frac{\iota_1^2}{4r^4(\Delta(r) + \iota_2^2)}\mathrm{d}r^2, \tag{36}$$

where

$$\Delta(r) = -\frac{\iota_1^2 \mu}{r^2} + \frac{2\kappa_1}{r} + \kappa_2. \tag{37}$$

This metric satisfies Einstein's equation with zero cosmological constant, for a dust with uniform density 4μ in the corotating frame,

$$R_{ab} - \frac{1}{2} R g_{ab} = 4\mu u_a u_b. \tag{38}$$

Defining the winding time coordinate $\bar{t} = t + \iota_2 \phi$ as in Sec. IV A [25], a new periodic coordinate $\varphi = \phi\sqrt{\kappa_2 + \iota_2^2}$, and a radial coordinate $\rho = |\iota_1|/2r\sqrt{\kappa_2 + \iota_2^2}$, the metric (36) becomes

$$\mathrm{d}s^2 = -(\mathrm{d}\bar{t})^2 + f(\rho)\mathrm{d}\varphi^2 + \frac{\mathrm{d}\rho^2}{f(\rho)}, \tag{39}$$

where

$$f(\rho) = 1 - 4\mu\rho^2 + \frac{4\kappa_1}{|\iota_1|\sqrt{\kappa_2 + \iota_2^2}}\rho. \tag{40}$$

We note, remarkably, that if $\kappa_1 = 0$, then the spatial sector of Eq. (39) is simply a Euclideanized two-dimensional de Sitter (dS) space in static slicing; to make the analogy precise, one would require $\kappa_2 + \iota_2^2 = 1/4\mu$, so that the periodicity in φ (the Wick-rotated time coordinate of the dS_2 space) matches the dS length.

C. Nonspinning limit

The geometry described in Eqs. (1) and (2) must always be spinning; there is no way to smoothly take the joint j_1, $j_2 \to 0$ limit while keeping the metric nonsingular. Specifically, the form of $N(r)$ and $D(r)$ dictate that we cannot take $j_1 \to 0$.

However, we can set j_2 to zero. In this case, the $\mathrm{d}t\mathrm{d}\phi$ term in the metric asymptotes to zero for large r. The vorticity tensor (8) for the timelike congruence along ∂_t then satisfies $\lim_{r\to\infty}\Omega_{ab} = 0$, but the twist one-form for the null geodesics described in Sec. III A satisfies $\lim_{r\to\infty}\omega_a = (0, 0, \mp\sqrt{c_2})$, where the $+$ case occurs if and only if j_1 is negative (and vice versa). Thus, even when we send j_2 to zero, information about the spin of the dust is imprinted on the twist of null geodesics at arbitrarily large r, even though the frame-dragging $w(r)$ asymptotically vanishes.

D. Vacuum limit

Let us consider the vacuum limit of the geometry described in Eq. (1). Taking the gas density ($\propto M$) to zero, we find that the Cauchy horizon is located at

$$r_\mathrm{C} = -2\frac{c_1}{c_2}L. \tag{41}$$

For $r > r_\mathrm{C}$, let us define

$$\tilde{r} = L\sqrt{\frac{2c_1 L}{r} + c_2}, \tag{42}$$

so $D(r)\mathrm{d}\phi^2 \to \tilde{r}^2\mathrm{d}\phi^2$. If we continue to impose the condition (23) (which has simply become $c_2/j_2 = 2c_1/j_1$ for $M = 0$) in order to sequester the CTCs behind a boundary, then the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - \frac{j_1\tilde{r}^2}{c_1 L}\mathrm{d}t\mathrm{d}\phi + \tilde{r}^2\mathrm{d}\phi^2 + \left(\frac{4c_1^2}{j_1^2} + \frac{\tilde{r}^2}{L^2}\right)^{-1}\mathrm{d}\tilde{r}^2. \tag{43}$$

The boundary is located at $\tilde{r} = 0$, so the CTC region behind the boundary corresponds to the analytic continuation of $\tilde{r}^2$ to negative values; cf. Ref. [24].

Next, we can rescale the radial coordinate again, define $\hat{r} = |\frac{j_1}{2c_1}|\tilde{r}$, and also define $\hat{\phi} = \frac{2c_1}{j_1}\phi$, $0 \le |\hat{\phi}| < |\frac{4\pi c_1}{j_1}|$, so

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - \frac{2\hat{r}^2}{L}\mathrm{d}t\mathrm{d}\hat{\phi} + \hat{r}^2\mathrm{d}\hat{\phi}^2 + \left(1 + \frac{\hat{r}^2}{L^2}\right)^{-1}\mathrm{d}\hat{r}^2. \tag{44}$$

We can turn Eq. (44) into a metric that locally corresponds to AdS_3 in global coordinates by defining $\bar{\phi} = \hat{\phi} - \frac{t}{L}$, in terms of which we have

$$\mathrm{d}s^2 = -\left(1 + \frac{\hat{r}^2}{L^2}\right)\mathrm{d}t^2 + \left(1 + \frac{\hat{r}^2}{L^2}\right)^{-1}\mathrm{d}\hat{r}^2 + \hat{r}^2\mathrm{d}\bar{\phi}^2. \tag{45}$$

If t is allowed to take values in all of $\mathbb{R}$, then $\bar{\phi}$ spans the real numbers and the metric in Eq. (45) describes a covering space of AdS_3 with decompactified time and angular coordinate. To correspond to AdS_3, we should make t periodic, $0 \le t < 2\pi L$. For consistency, we can then take $2c_1/j_1 = 1$ so that $\hat{\phi} \in [0, 2\pi)$, and we are left with the global AdS_3 geometry. We note that if $2c_1/j_1$ is any integer n, we have $\hat{\phi} \in [0, 2\pi n)$, which (for appropriate periodicity in t) corresponds simply to an n-fold cover of AdS_3, which we may quotient by $\mathbb{Z}_n$ to recover the single copy of the geometry. For any n, Eq. (23) and our $M = 0$ choice together then imply that Eq. (32) is satisfied, enhancing the symmetry algebra (which is expected, since AdS_3, in fact, possesses six Killing vectors).

Hence, we find that in the CTC-free case where we impose Eq. (23), our metric (1) obeys the extension of Birkhoff's theorem to $2+1$ dimensions [29], which states that any solution of the $(2+1)$-dimensional vacuum Einstein equations with a negative cosmological constant that is free of CTCs must correspond to AdS_3, a BTZ geometry [2], or a Coussaert-Henneaux [30] solution.

VII. DISCUSSION

The metric in Eq. (1) contains six free parameters: a length L and five unitless constants (M, j_1, j_2, c_1, c_2). Choosing a value of L effectively sets the length scale of the geometry (relative to the three-dimensional Newton's constant). The remaining five parameters specify a five-dimensional space of geometries that, for general choices of the constants, remain distinct under diffeomorphisms. The parameter M can be measured in the geometry through the value of R_{tt}, while c_2 can be measured by the proper circumference of the spacetime at large r. Moreover, the existence and location of the zeros of $N(r)$, $D(r)$, and $w(r)$ (i.e., $r_\pm$, r_C, and r_0, respectively) allow various combinations of the remaining three parameters to be measured in the spacetime. These locations are all physically meaningful: the rr component of the metric flips sign at $r_\pm$, CTCs appear at r_C, and the frame dragging vanishes at r_0. The relative order and proper distances among these locations and the axis of rotation allow us to conclude that (M, j_1, j_2, c_1, c_2) indeed specifies a five-dimensional family of globally inequivalent geometries.

This paper leaves multiple interesting avenues for further research. We leave for future work the subjects of the dynamical stability of these geometries and their formation from the collapse of a cloud of dust. The question of what other three-dimensional geometries exhibit an analogue of the bouncing boundary discussed in Sec. III C is also a compelling one. Further, it would be interesting to study the causal structure of the $c_1 > 0$ case mentioned in Sec. III A, in which an apparent horizon appears at radius $r_{\rm m}$ and $D(r)$ asymptotically decreases. The appearance of the analytically continued de Sitter metric in Sec. VI B also merits investigation; the double Wick rotation is reminiscent of the "bubble of nothing" solution of Ref. [31]. Finally, an examination of the spatial geodesics of this class of metrics, how they can be embedded within a surrounding vacuum to produce an asymptotically anti–de Sitter geometry, and a subsequent computation of their Ryu-Takayanagi surfaces [32] are well-motivated future directions from a holographic perspective.

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APPENDIX: GAUSSIAN DUST SOLUTION

In this Appendix, we discuss an additional, apparently new solution of the Einstein equations in $2+1$ dimensions. Although not globally a member of the family of metrics presented in Eqs. (1) and (2), we will find that it is related in a certain limit. Consider a dust solution to the Einstein equations with zero cosmological constant,

$$R_{ab} - \frac{1}{2} R g_{ab} = m(r) u_a u_b, \tag{A1}$$

with vector $u^a = \partial_t$ as before. Here, let us take the density $m(r)$ to have a Gaussian profile,

$$m(r) = \frac{1}{a^2} e^{-r^2/a^2}, \tag{A2}$$

for some length parameter a. The metric for the solution is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}\phi + (r^2 - a^2)\mathrm{d}\phi^2 + e^{r^2/a^2}\mathrm{d}r^2. \tag{A3}$$

The solution has a two-dimensional isometry algebra generated by Killing vectors u^a and ϕ^a. Since $u^a \nabla_a u^b = 0$, u^a generates a timelike congruence as before, for which we find that the expansion θ, shear ς, and vorticity Ω all vanish. The solution is clearly distinct from Gödel's universe [1] (since the dust density is not uniform) and van Stockum's solution [15] (since the dust density is finite in the closure of the geometry and goes to zero at the boundary). Such a rotating dust solution, with higher density in the center and exponentially falling density at large distances, may be of astrophysical use if extended to four dimensions. In a sense, this solution may be thought of as corresponding to a $(2+1)$-dimensional galaxy. However, the solution possesses CTCs for $r < a$.

Taking the small-r limit of the rr component of the metric in Eq. (A3), we can write, for $r \ll a$,

$$\mathrm{d}s^2 \simeq -\mathrm{d}\hat{t}^2 + r^2\mathrm{d}\phi^2 + \frac{\mathrm{d}r^2}{1 - \frac{r^2}{a^2}}, \tag{A4}$$

where we define $\hat{t} = t + a\phi$. The metric (A4) is a member of the family of $\Lambda = 0$ solutions discussed in Sec. VI B; if we set the parameters in Eq. (35) to $\mu = 1/4a^2$, $\kappa_1 = \iota_2 = 0$, $\kappa_2 = a^2$, write t in Eq. (36) as $\hat{t}$, and send r in Eq. (36) to $|\iota_1|/2a\sqrt{a^2 - r^2}$, we recover Eq. (A4). Thus, as we would expect, the small-r (and thus nearly constant density) limit of the metric (A3) corresponds to a member of the family of uniform-density rotating dust solutions in Eq. (1).

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