Deflection of light and time delay in closed Einstein-Straus solution

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We investigate strong lensing by a spherically symmetric mass distribution in the framework of the Einstein-Straus solution with a positive cosmological constant and concentrate on the case of a spatially closed Universe ($k = +1$). We develop a method based on integration of differential equations in order to make possible the computation of light deflection and time delay. By applying our results to the lensed quasar sloan digital sky survey (SDSS) J1004 + 4112, we find that the bending of light and the time delay depend on wether the present value of the scale factor a_0 is or is not much smaller than a value close to 9.1×10^{26} m. Beyond this value, the results are almost the same as for the spatially flat Universe. Moreover, it turns out that a positive cosmological constant attenuates the light bending in agreement with Rindler and Ishak's finding.

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I. INTRODUCTION

On the strength of Rindler and Ishak's finding [1], and several subsequent works [2–6], it is already well established that a positive cosmological constant reduces the bending of light near an isolated spherical mass in Kottler space-time.

Many authors have considered the simplest case of a spatially flat universe (null curvature $k = 0$) to investigate the same problem in a more realistic model, namely, the Einstein-Straus solution [7,8] in the presence of a cosmological constant [9]. In such a model, it is assumed that the bending of light by a lens happens only inside a Kottler vacuole (Schücking sphere) embedded in an expanding Friedmann universe. Within the same framework, Ishak *et al.* [10], Schücker [11], Kantowski *et al.* [12], and Schücker [13] have proven that the effect of the cosmological constant on light bending is only diminished without however being dropped, contrary to what has been argued in Refs. [14–16]. In Refs. [17–19], the authors have gone a step further and investigated the cosmological constant's effect on time delay, in which case the computation of the photon's travel time outside the vacuole is particularly simple using some properties of Euclidean geometry.

Even though the current observations predict that the Universe is very close to flat, it would nevertheless be interesting to treat the bending of light and the time delay in the Einstein-Straus solution considering the case of a spatially closed Universe (positive curvature $k = +1$) and discuss the possible impact of positive curvature on light bending and time delay. In such a case, the photon outside the vacuole no longer travels in a straight line. For this reason, the argumentation will be based only on the integration of differential equations.

We will use the same units as in Ref. [11]: astroseconds (as), astrometers (am) and astrograms (ag),

$$
\begin{aligned}
1 \text{ as} &= 4.34 \times 10^{17} \text{ s} = 13.8 \text{ Gyr},\\
1 \text{ am} &= 1.30 \times 10^{26} \text{ m} = 4221 \text{ Mpc},\\
1 \text{ ag} &= 6.99 \cdot 10^{51} \text{ kg} = 3.52 \times 10^{21}\ M_\odot,
\end{aligned} \tag{1}
$$

where $M_\odot$ is the solar mass. In these units,

$$
c = 1 \text{ am}\,\text{as}^{-1}, \qquad 8\pi G = 1 \text{ am}^3\,\text{as}^{-2}\,\text{ag}^{-1}, \qquad H_0 = 1 \text{ as}^{-1}, \tag{2}
$$

where H_0 is the Hubble constant.

II. MATCHING OF KOTTLER'S AND CLOSED FRIEDMANN'S SOLUTIONS WITH A COSMOLOGICAL CONSTANT

To construct the closed Einstein-Straus metric, we shall start by matching between the Kottler and closed Friedmann metrics on the Schücking sphere (Kottler vacuole) by calculating the Jacobian of the transformation between the Schwarzschild and the Friedmann coordinates. Let us denote by (T, r, θ, φ) the Schwarzschild coordinates and by $(t, \chi, \theta, \varphi)$ the Friedmann coordinates. The Kottler metric

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$$ds^2 = B(r)dT^2 - B(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$B(r) \coloneqq 1 - \frac{M}{4\pi r} - \frac{\Lambda}{3}r^2 \quad (3)$$

reigns inside a vacuole of radius $r_{\text{Schü}}(T)$ centered around a spherical mass distribution (the lens) with $r \leq r_{\text{Schü}}$. The Friedmann metric in the case of a spatial positive curvature, $k = +1$,

$$ds^2 = dt^2 - a(t)^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (4)$$

describes the space-time geometry outside the vacuole, $\chi \geq \chi_{\text{Schü}}$, where the scale factor $a(t)$ is determined by integration of the Friedmann equation

$$\frac{da}{dt} = f(a), \qquad f(a) \coloneqq \sqrt{\frac{A}{a} + \frac{\Lambda}{3}a^2 - k}, \quad (5)$$

where the constant A results from the energy conservation law

$$A \coloneqq \rho a^3/3 = \rho_{\text{dust}0} a_0^3/3, \quad (6)$$

for a nonrelativistic matter-dominated universe with the present dust density $\rho_{\text{dust}0}$,

$$\rho_{\text{dust}0} = 3 - \Lambda + 3\Omega_{k0}, \qquad \Omega_{k0} \coloneqq k a_0^{-2}, \quad (7)$$

computed using the fact that the Hubble constant $H_0 = a_t(0)/a_0$ in the system of units (1) is 1 as^{-1} (the lower index t denotes differentiation with respect to time). The parameters Ω_{k0} and a_0 represent, respectively, the curvature density of space at the present time and the scale factor at the present time. The two solutions are connected at the boundary of the Schücking sphere by requiring

$$r_{\text{Schü}}(T) \coloneqq a(t)\sin\chi_{\text{Schü}}. \quad (8)$$

The mass M is expressed in terms of the Schücking radius as

$$M = \frac{4\pi}{3} r_{\text{Schü}}^3 \rho = 4\pi A \sin^3\chi_{\text{Schü}}, \quad (9)$$

which can be inverted to give

$$\chi_{\text{Schü}} = \arcsin\left[\left(\frac{M}{4\pi A}\right)^{1/3}\right]. \quad (10)$$

So, we have on the Schücking sphere

$$B_{\text{Schü}} \coloneqq B(r_{\text{Schü}}) = 1 - \left(\frac{A}{a} + \frac{\Lambda}{3}a^2\right)\sin\chi_{\text{Schü}}^2. \quad (11)$$

To connect the two solutions, proceeding in a way analogous to what has been done by Balbinot *et al.* [9] and Schücker [11], we will pass from Schwarzschild coordinates (T, r) and Friedmann coordinates (t, χ) to the new coordinate system (b, r). In this new coordinate system, the Kottler metric can be rewritten as

$$ds^2 = B\Psi(b)^2 db^2 - \frac{1}{B}dr^2 - r^2 d\Omega^2, \quad (12)$$

where the function $\Psi(b)$ is defined by

$$\Psi(b) \coloneqq \frac{dT}{db}. \quad (13)$$

We now rewrite first the Friedmann metric in the coordinate system (a, χ),

$$ds^2 = \frac{da^2}{f(a)^2} - a^2 d\chi^2 - a^2\sin^2\chi d\Omega^2, \quad (14)$$

and convert the factor $a^2\sin^2\chi$ in r^2 under a second coordinate transformation $(a, \chi) \to (b, r)$,

$$a \coloneqq \Phi(b, r), \qquad \sin\chi \coloneqq r/\Phi(b, r), \quad (15)$$

with the boundary condition that at the Schücking radius old and new time coordinates coincide,

$$a = b = \Phi(b, b\sin\chi_{\text{Schü}}). \quad (16)$$

The Friedmann metric then becomes

$$ds^2 = \Phi_b^2\left(\frac{1}{C_1^2} - \frac{r^2}{\Phi^2 - r^2}\right)db^2 - \left[\frac{(\Phi - r\Phi_r)^2}{\Phi^2 - r^2} - \frac{\Phi_r^2}{C_1^2}\right]dr^2$$
$$+ 2\Phi_b\left[\frac{r(\Phi - r\Phi_r)}{\Phi^2 - r^2} + \frac{\Phi_r}{C_1^2}\right]db dr - r^2 d\Omega^2, \quad (17)$$

with $\Phi_b = \partial\Phi/\partial b$, $\Phi_r = \partial\Phi/\partial r$ and

$$C_1 \coloneqq \sqrt{\frac{A}{\Phi} + \frac{\Lambda}{3}\Phi^2 - 1}. \quad (18)$$

Since the metric should be diagonal, this implies the absence of mixed terms, i.e., $g_{br}^F = 0$. Then,

$$\Phi_r = -\frac{rC_1^2}{\Phi B_1}, \qquad B_1 \coloneqq 1 - \left(\frac{A}{\Phi^3} + \frac{\Lambda}{3}\right)r^2. \quad (19)$$

Therefore, the Friedmann metric can be put in the form

$$ds^2 = \frac{\Phi_b^2}{1 - r^2/\Phi^2}\frac{B_1}{C_1^2}db^2 - \frac{1}{B_1}dr^2 - r^2 d\Omega^2. \quad (20)$$

Differentiating the boundary condition (16) with respect to b, we obtain at $\chi = \chi_{\text{Schü}}$

$$\Phi_b|_{\text{Schü}} := \Phi_b(b, b\sin\chi_{\text{Schü}}) = 1 - \Phi_r|_{\text{Schü}}\sin\chi_{\text{Schü}} = \frac{\cos^2\chi_{\text{Schü}}}{B_{\text{Schü}}}. \quad (21)$$

The matching of the two solutions continuously on the Schücking sphere in this coordinate system (b, r, θ, φ) results in

$$g^F_{bb}|_{\text{Schü}} = g^K_{bb}|_{\text{Schü}}, \qquad g^F_{rr}|_{\text{Schü}} = g^K_{rr}|_{\text{Schü}}. \quad (22)$$

We can demonstrate that

$$B_1|_{\text{Schü}} = B_{\text{Schü}}, \qquad C_1|_{\text{Schü}} = \frac{C_{\text{Schü}}}{\tan\chi_{\text{Schü}}}, \quad (23)$$

with $C_{\text{Schü}}$ defined by

$$C_{\text{Schü}} := \sqrt{1 - \frac{B_{\text{Schü}}}{\cos^2\chi_{\text{Schü}}}}. \quad (24)$$

The relations (22) yield

$$\Psi(b) = \frac{\sin\chi_{\text{Schü}}}{B_{\text{Schü}}(b)C_{\text{Schü}}(b)}. \quad (25)$$

Repeated use of the chain rule then gives

$$\frac{\partial t}{\partial T} = \frac{\partial t}{\partial a}\frac{\partial \Phi}{\partial b}\frac{\partial b}{\partial T}, \qquad \frac{\partial t}{\partial r} = \frac{\partial t}{\partial a}\frac{\partial \Phi}{\partial r},$$
$$\frac{\partial \chi}{\partial T} = \frac{\partial \chi}{\partial b}\frac{\partial b}{\partial T}, \qquad \frac{\partial \chi}{\partial r} = \frac{1}{\cos\chi}\left(\frac{1}{\Phi} - \frac{r}{\Phi^2}\frac{\partial \Phi}{\partial r}\right). \quad (26)$$

Hence, the Jacobian of the coordinate transformation $(T, r) \to (t, \chi)$ at the Schücking radius $\chi = \chi_{\text{Schü}}$ is given by

$$\left.\frac{\partial t}{\partial T}\right|_{\text{Schü}} = \cos\chi_{\text{Schü}}, \qquad \left.\frac{\partial t}{\partial r}\right|_{\text{Schü}} = -\frac{C_{\text{Schü}}\cos\chi_{\text{Schü}}}{B_{\text{Schü}}},$$
$$\left.\frac{\partial \chi}{\partial T}\right|_{\text{Schü}} = -\frac{C_{\text{Schü}}\cos\chi_{\text{Schü}}}{a}, \qquad \left.\frac{\partial \chi}{\partial r}\right|_{\text{Schü}} = \frac{\cos\chi_{\text{Schü}}}{aB_{\text{Schü}}}, \quad (27)$$

and the Jacobian of the inverse coordinate transformation $(t, \chi) \to (T, r)$ is given by

$$\left.\frac{\partial T}{\partial t}\right|_{\text{Schü}} = \frac{\cos\chi_{\text{Schü}}}{B_{\text{Schü}}}, \qquad \left.\frac{\partial T}{\partial \chi}\right|_{\text{Schü}} = \frac{aC_{\text{Schü}}\cos\chi_{\text{Schü}}}{B_{\text{Schü}}},$$
$$\left.\frac{\partial r}{\partial t}\right|_{\text{Schü}} = C_{\text{Schü}}\cos\chi_{\text{Schü}}, \qquad \left.\frac{\partial r}{\partial \chi}\right|_{\text{Schü}} = a\cos\chi_{\text{Schü}}. \quad (28)$$

To compare the Schwarzschild coordinate time T with that of Friedmann t on the Schücking sphere $\chi = \chi_{\text{Schü}}$, we consider the parametrized curve, $T = p$, $r = b\sin\chi_{\text{Schü}}$ $(\theta = \pi/2,\ \varphi = 0)$. Its 4-velocity is given by

$$\frac{dT}{dp} = 1,$$
$$\frac{dr}{dp} = \frac{dr}{db}\left.\frac{db}{dT}\right|_{\text{Schü}}\frac{dT}{dp} = B_{\text{Schü}}C_{\text{Schü}} \quad (29)$$

in Schwarzschild coordinates and

$$\frac{dt}{dp} = \left.\frac{\partial t}{\partial T}\right|_{\text{Schü}}\frac{dT}{dp} + \left.\frac{\partial t}{\partial r}\right|_{\text{Schü}}\frac{dr}{dp} = \frac{B_{\text{Schü}}}{\cos\chi_{\text{Schü}}}$$
$$\frac{d\chi}{dp} = \left.\frac{\partial \chi}{\partial T}\right|_{\text{Schü}}\frac{dT}{dp} + \left.\frac{\partial \chi}{\partial r}\right|_{\text{Schü}}\frac{dr}{dp} = 0, \quad (30)$$

in Friedmann coordinates. Finally, we deduce the relation

$$\left.\frac{dT}{dt}\right|_{\text{Schü}} = \frac{dT}{dp}\frac{dp}{dt} = \frac{\cos\chi_{\text{Schü}}}{B_{\text{Schü}}}, \quad (31)$$

which allows us to pass from the Schwarzschild coordinate time to the Friedmann coordinate time and vice versa.

III. EQUATIONS OF NULL GEODESIC MOTION

Since these are the final conditions on Earth that are known, it seems more convenient to consider two photons C and D emitted by a source S (quasar) at different times, $t_S \neq t'_S$, follow different trajectories and are received on Earth E simultaneously at $t_E = t'_E = 0$ with the angles α and α'. As shown in Fig. 1, these two photons, during their propagation in curved Friedmann's space-time, penetrate the Schücking sphere, respectively at $t_{\text{SchüS}}$ and $t'_{\text{SchüS}}$, where they get deflected by a lens L (galaxy cluster) respectively at minimum distances r_P and r'_P (perilens), which are much larger than the Schwarzschild radius $(r_{\text{Schw}} := 2GM = M/4\pi)$, and then leave the Schücking sphere respectively at $t_{\text{SchüE}}$ and $t'_{\text{SchüE}}$.

In the presence of a cosmological constant and a nonzero spatial curvature, exact analytical solutions of the Friedmann equation (5) for the cosmic time $t(a)$ can be obtained in terms of elliptic integrals of the second and third kinds, the inverse function for the scale factor $a(t)$ of which is not known. Nevertheless, one can alternatively proceed to a numerical resolution. We will therefore

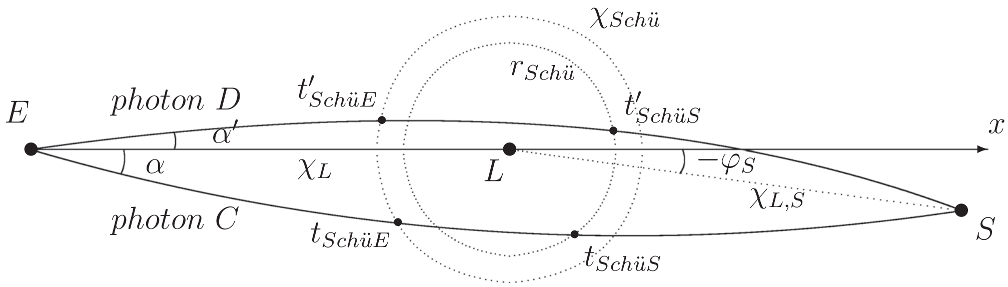

FIG. 1. Two photons emitted by a source S, bent inside the Schücking sphere and finally received at Earth E. Outside the Schücking sphere, these two photons do not travel in straight lines.

determine the scale factor $a(t)$ by numerical integration of the Friedmann equation with final condition $a(t=0)=a_0$. To compare our results with those obtained previously in the flat case, we use the same experimentally measured value of cosmological constant $\Lambda = 0.77\cdot 3\ \mathrm{am}^{-2} \pm 20\%$ or in cm^{-2} units $\Lambda = 1.36\times 10^{-56}\ \mathrm{cm}^{-2}\pm 20\%$ as in Refs. [5,11,18] and dust density $\rho_{\mathrm{dust}0} = 3 - \Lambda + 3a_0^{-2}$.

To determine the Earth-lens geodesic distance χ_L and the Earth-source geodesic distance χ_S, we will also need to solve numerically the radial null geodesic

$$d\chi = -\frac{dt}{a(t)}, \tag{32}$$

which ensures that $\chi(t)$ is a decreasing function of time. But the right-hand side of this equation depends on the solution $a(t)$ of the Friedmann equation, which is not known. We can get around this problem by introducing the function $f(a)$ through the Friedmann equation $dt = da/f(a)$ and the redshift formula $1+z = a_0/a$,

$$\chi(z) = \int_{\frac{a_0}{1+z}}^{a_0} \frac{da}{af(a)}, \tag{33}$$

where we have taken the origin $\chi(z=0)=0$ at the position of the Earth; thereby, χ_L and χ_S will be respectively calculated for any given value of z_L and z_S. Since these are the final conditions at the arrival on Earth which are given, it is more appropriate to proceed backward in time. We will first determine $t'_{\mathrm{SchüE}}$ and $t_{\mathrm{SchüE}}$ and then $t'_{\mathrm{SchüS}}$ and $t_{\mathrm{SchüS}}$ to calculate the angle φ_S and finally t'_S and t_S for the computation of the time delay.

A. Null geodesics between the Schücking sphere and the Earth

In this region, the photon trajectory is governed by the closed Friedmann metric (4). The corresponding nonzero Christoffel symbols in the equatorial plane $\theta = \pi/2$ are

$$\Gamma^t_{\chi\chi} = aa_t, \qquad \Gamma^t_{\varphi\varphi} = aa_t\sin^2\chi, \qquad \Gamma^\chi_{\varphi\varphi} = -\sin\chi\cos\chi,$$
$$\Gamma^\chi_{t\chi} = a_t/a, \qquad \Gamma^\varphi_{t\varphi} = a_t/a, \qquad \Gamma^\varphi_{\chi\varphi} = \cot\chi. \tag{34}$$

Then, the geodesic equations read

$$\ddot{t} + aa_t(\dot{\chi}^2 + \sin^2\chi\dot{\varphi}^2) = 0, \tag{35}$$

$$\ddot{\chi} + 2a^{-1}a_t\dot{t}\dot{\chi} - \sin\chi\cos\chi\dot{\varphi}^2 = 0, \tag{36}$$

$$\ddot{\varphi} + 2(a^{-1}a_t\dot{t} + \cot\chi\dot{\chi})\dot{\varphi} = 0, \tag{37}$$

where $\dot{} := d/dp$, with p an affine parameter other than s because the space-time interval for light is zero $ds = 0$. It should be noted that the second equation might not be needed for our purposes.

The upper photon would arrive at Earth with final conditions ($p=0$)

$$t=0, \qquad \chi=\chi_L, \qquad \varphi=\pi, \qquad \dot{t}=1,$$
$$\dot{\chi} = \frac{1}{a_0\sqrt{1+\cos^2\chi_L\tan^2\alpha'}}, \qquad \dot{\varphi} = \frac{\cot\chi_L\tan\alpha'}{a_0\sqrt{1+\cos^2\chi_L\tan^2\alpha'}}, \tag{38}$$

where we use the fact that the physical angle α' coincides with the coordinate angle $\arctan|\tan\chi_L\dot{\varphi}/\dot{\chi}|$. These final conditions make it possible to integrate the geodesic equations, which yield

$$\dot{t} = \frac{a_0}{a(t)}, \qquad \dot{\varphi} = \frac{a_0\chi'_P}{a^2\sin^2\chi}, \qquad \varphi = \pi - \arcsin\frac{\chi'_P\cot\chi}{\sqrt{1-\chi'^2_P}} + \beta', \tag{39}$$

where the constants χ'_P (perilens) and β' are defined by

$$\chi'_P := \frac{\cos\chi_L\sin\chi_L\tan\alpha'}{\sqrt{1+\cos^2\chi_L\tan^2\alpha'}}, \qquad \beta' := \arcsin\frac{\chi'_P\cot\chi_L}{\sqrt{1-\chi'^2_P}}. \tag{40}$$

In Refs. [11,18], the distance traveled by the photon between its exit point from the vacuole and its arrival point on Earth, noted by $\chi'_{E,\mathrm{SchüE}}$ (or $\chi_{E,\mathrm{SchüE}}$), is computed by means of some properties of Euclidean geometry that are no longer valid in a curved Friedmann space-time in which the geodesic distances are not straight lines. The same remark must be made regarding the distance traveled by the photon between the source and its entry point into the vacuole ($\chi'_{S,\mathrm{SchüS}}$ and $\chi_{S,\mathrm{SchüS}}$). For this reason, we have developed a method based only on Friedmann differential equations, which remains valid in either flat or curved space-time.

Differentiating the solution φ (39) with respect to χ and substituting into the closed Friedmann metric (4) for a photon, we can easily get an equation linking the variable χ with time

$$\int \frac{dt}{a(t)} = \int \frac{d\chi}{g'(\chi)}, \qquad g'(\chi) \coloneqq \sqrt{1-\chi_P'^2/\sin^2\chi}, \tag{41}$$

where we have taken into account that χ between the vacuole and the Earth increases with time. The right-hand side of this equation could be easily evaluated between $\chi_{\mathrm{Schü}}$ and χ_L to give precisely the geodesic distance $\chi'_{E,\mathrm{SchüE}}$ in terms of the arcsin function, but we do not know the expression of scale factor in terms of time in the left-hand side, so we shall first have to integrate with respect to the scale factor by inserting the function $f(a)$ via $dt = da/f(a)$. We obtain

$$\int_{a'_{\mathrm{SchüE}}}^{a_0} \frac{da}{af(a)} = \arcsin\frac{\cos\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P'^2}} - \arcsin\frac{\cos\chi_L}{\sqrt{1-\chi_P'^2}}, \tag{42}$$

from which we easily deduce numerically the value of $a'_{\mathrm{SchüE}}$, where $\chi_{\mathrm{Schü}}$ and χ_L are respectively given by Eqs. (10) and (33). The value of $t'_{\mathrm{SchüE}}$, at which the upper photon emerges from the Schücking sphere, is then calculated by numerical integration of the Friedmann equation, i.e.,

$$t'_{\mathrm{SchüE}} = \int_{a_0}^{a'_{\mathrm{SchüE}}} \frac{da}{f(a)}. \tag{43}$$

We point out that an equation linking the variable φ with time may be established by the same manner as before and serve to deduce the value of $t'_{\mathrm{SchüE}}$ using the polar angle at the exit point from the vacuole

$$\varphi'_{\mathrm{SchüE}} = \pi - \arcsin\frac{\chi'_P \cot\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P'^2}} + \beta'. \tag{44}$$

Similar formulas apply in the case of the lower-trajectory photon, with π replaced by $-\pi$ and α' replaced by $-\alpha$. The final conditions upon arrival on Earth are

$$t=0, \qquad \chi=\chi_L, \qquad \varphi=-\pi, \qquad \dot t = 1,$$
$$\dot\chi = \frac{1}{a_0\sqrt{1+\cos^2\chi_L\tan^2\alpha}}, \qquad \dot\varphi = \frac{-\cot\chi_L\tan\alpha}{a_0\sqrt{1+\cos^2\chi_L\tan^2\alpha}}. \tag{45}$$

The integration of geodesic equations thus yields

$$\dot t = \frac{a_0}{a(t)}, \qquad \dot\varphi = \frac{-a_0\chi_P}{a^2\sin^2\chi},$$
$$\varphi = -\pi + \arcsin\frac{\chi_P\cot\chi}{\sqrt{1-\chi_P^2}} - \beta, \tag{46}$$

with

$$\chi_P \coloneqq \frac{\cos\chi_L\sin\chi_L\tan\alpha}{\sqrt{1+\cos^2\chi_L\tan^2\alpha}}, \qquad \beta \coloneqq \arcsin\frac{\chi_P\cot\chi_L}{\sqrt{1-\chi_P^2}}. \tag{47}$$

It follows that the polar angle at which the lower-trajectory photon emerges from the Schücking sphere is

$$\varphi_{\mathrm{SchüE}} = -\pi + \arcsin\frac{\chi_P\cot\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P^2}} - \beta. \tag{48}$$

The value of $t_{\mathrm{SchüE}}$ at which the lower photon emerges from the Schücking sphere is obtained by means of a formula similar to Eq. (43), i.e.,

$$t_{\mathrm{SchüE}} = \int_{a_0}^{a_{\mathrm{SchüE}}} \frac{da}{f(a)}, \tag{49}$$

in terms of $a_{\mathrm{SchüE}}$, which in turn is calculated by means of a formula similar to Eq. (42), i.e.,

$$\int_{a_{\mathrm{SchüE}}}^{a_0} \frac{da}{af(a)} = \arcsin\frac{\cos\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P^2}} - \arcsin\frac{\cos\chi_L}{\sqrt{1-\chi_P^2}}. \tag{50}$$

However, as in the flat case [18], one can proceed otherwise, without further numerical computation to obtain the time $t_{\mathrm{SchüE}}$ for the lower-trajectory photon. The method consists of determining $t_{\mathrm{SchüE}}$ by difference with $t'_{\mathrm{SchüE}}$ through an approximate analytical expression. Combining Eqs. (42) and (50), one gets, after evaluating the integrals with respect to time,

$$\int_{t_{\mathrm{SchüE}}}^{t'_{\mathrm{SchüE}}} \frac{dt}{a(t)} = \arcsin\frac{\cos\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P^2}} - \arcsin\frac{\cos\chi_L}{\sqrt{1-\chi_P^2}} - \arcsin\frac{\cos\chi_{\mathrm{Schü}}}{\sqrt{1-\chi_P'^2}} + \arcsin\frac{\cos\chi_L}{\sqrt{1-\chi_P'^2}}. \tag{51}$$

To approximate the left-hand side of this equation, one may use the fact that the scale factor $a(t)$ varies significantly only on cosmological timescales. Then,

$$\mathrm{lhs} \simeq \frac{t'_{\mathrm{SchüE}} - t_{\mathrm{SchüE}}}{a'_{\mathrm{SchüE}}}. \tag{52}$$

A plausible approximation could be done for the right-hand side in the limit of a small perilens, $\chi_P \ll 1$ and $\chi'_P \ll 1$ (of order 10^{-6} in our case), which is motivated by the observed small α and α' of the order of a few arc seconds ($\sim 10^{-5}$). One gets up to second order in the physical angles α and α', on account of Eqs. (40) and (47),

$$\mathrm{rhs} \simeq \frac{1}{2}(\cot\chi_{\mathrm{Schü}} - \cot\chi_L)(\chi_P^2 - \chi_P'^2) \tag{53}$$

$$\simeq \frac{1}{2}\sin(\chi_L - \chi_{\mathrm{Schü}})\cos^2\chi_L \frac{\sin\chi_L}{\sin\chi_{\mathrm{Schü}}}(\alpha^2 - \alpha'^2). \tag{54}$$

Equating the two sides, we get

$$t'_{\mathrm{SchüE}} - t_{\mathrm{SchüE}} \simeq \frac{1}{2} a'_{\mathrm{SchüE}} \sin(\chi_L - \chi_{\mathrm{Schü}})\cos^2\chi_L \times \frac{\sin\chi_L}{\sin\chi_{\mathrm{Schü}}}(\alpha^2 - \alpha'^2). \tag{55}$$

Knowing the value of $t'_{\mathrm{SchüE}}$ from Eq. (43), one can deduce the value of $t_{\mathrm{SchüE}}$ from this approximate expression. This is clearly positive for $\alpha > \alpha'$, meaning that the lower photon leaves the vacuole before the upper one.

The two 4-velocities of the upper and lower photons at points of exit from the vacuole are respectively

$$\dot{t}'_{\mathrm{SchüE}} = \frac{a_0}{a'_{\mathrm{SchüE}}}, \qquad \dot{\chi}'_{\mathrm{SchüE}} = \frac{a_0 g'_{\mathrm{Schü}}}{a'^2_{\mathrm{SchüE}}}, \qquad \dot{\varphi}'_{\mathrm{SchüE}} = \frac{a_0 \chi'_P}{r'^2_{\mathrm{SchüE}}} \tag{56}$$

and

$$\dot{t}_{\mathrm{SchüE}} = \frac{a_0}{a_{\mathrm{SchüE}}}, \qquad \dot{\chi}_{\mathrm{SchüE}} = \frac{a_0 g_{\mathrm{Schü}}}{a^2_{\mathrm{SchüE}}}, \qquad \dot{\varphi}_{\mathrm{SchüE}} = \frac{-a_0 \chi_P}{r^2_{\mathrm{SchüE}}}, \tag{57}$$

using Eq. (39), in Friedmann coordinates, where the value of $a_{\mathrm{SchüE}}$ is calculated by numerical integration of the Friedmann equation (49), with $g'_{\mathrm{Schü}} \coloneqq g'(\chi_{\mathrm{Schü}})$, $g_{\mathrm{Schü}} \coloneqq g(\chi_{\mathrm{Schü}})$ (41), and $r'_{\mathrm{SchüE}} = a'_{\mathrm{SchüE}}\sin\chi_{\mathrm{Schü}}$, $r_{\mathrm{SchüE}} = a_{\mathrm{SchüE}}\sin\chi_{\mathrm{Schü}}$, using the matching condition (8). These two 4-velocities can be translated, thanks to the inverse Jacobian (28), into Schwarzschild coordinates,

$$\dot{T}'_{\mathrm{SchüE}} = \frac{a_0\cos\chi_{\mathrm{Schü}}}{a'_{\mathrm{SchüE}} B'_{\mathrm{SchüE}}}(1 + C'_{\mathrm{SchüE}} g'_{\mathrm{Schü}}),$$
$$\dot{r}'_{\mathrm{SchüE}} = \frac{a_0\cos\chi_{\mathrm{Schü}}}{a'_{\mathrm{SchüE}}}(C'_{\mathrm{SchüE}} + g'_{\mathrm{Schü}}) \tag{58}$$

for the upper photon and

$$\dot{T}_{\mathrm{SchüE}} = \frac{a_0\cos\chi_{\mathrm{Schü}}}{a'_{\mathrm{SchüE}} B'_{\mathrm{SchüE}}}(1 + C_{\mathrm{SchüE}} g_{\mathrm{Schü}}),$$
$$\dot{r}_{\mathrm{SchüE}} = \frac{a_0\cos\chi_{\mathrm{Schü}}}{a_{\mathrm{SchüE}}}(C_{\mathrm{SchüE}} + g_{\mathrm{Schü}}), \tag{59}$$

for the lower photon, where $B'_{\mathrm{SchüE}}$ and $B_{\mathrm{SchüE}}$ have been defined in Eq. (11) with $C'_{\mathrm{SchüE}} = \sqrt{1 - B'_{\mathrm{SchüE}}/\cos^2\chi_{\mathrm{Schü}}}$ and $C_{\mathrm{SchüE}} = \sqrt{1 - B_{\mathrm{SchüE}}/\cos^2\chi_{\mathrm{Schü}}}$.

Let γ'_K and γ_K denote respectively the smaller coordinate angles between the unoriented direction of the upper-trajectory photon and the direction toward the lens and between the unoriented direction of the lower-trajectory photon and the direction toward the lens. We have

$$\gamma'_K \coloneqq \arctan\left| r'_{\mathrm{SchüE}} \frac{\dot{\varphi}'_{\mathrm{SchüE}}}{\dot{r}'_{\mathrm{SchüE}}} \right| = \arctan \frac{\chi'_P (C'_{\mathrm{SchüE}} + g'_{\mathrm{Schü}})^{-1}}{\cos\chi_{\mathrm{Schü}}\sin\chi_{\mathrm{Schü}}} \tag{60}$$

and

$$\gamma_K \coloneqq \arctan\left| r_{\mathrm{SchüE}} \frac{\dot{\varphi}_{\mathrm{SchüE}}}{\dot{r}_{\mathrm{SchüE}}} \right| = \arctan \frac{\chi_P (C_{\mathrm{SchüE}} + g_{\mathrm{Schü}})^{-1}}{\cos\chi_{\mathrm{Schü}}\sin\chi_{\mathrm{Schü}}}. \tag{61}$$

B. Null geodesics inside the Kottler vacuole

In this region where the Kottler metric prevails, we shall not go deeper into the details leading to the same results already discussed in the flat case [11,18], and what we will have to do is to first calculate the scale factors $a'_{\mathrm{SchüS}}$ and $a_{\mathrm{SchüS}}$ and then their corresponding times $t'_{\mathrm{SchüS}}$ and $t_{\mathrm{SchüS}}$, at which the two photons enter inside the vacuole, by integrating the Friedman equation with final conditions at the exit points from the vacuole.

The partially integrated geodesic equations for the upper photon in Kottler space-time (3) are

$$\dot{T} = \frac{1}{B(r)}, \qquad \dot{r} = \pm\left(1 - \frac{r'^2_P}{r^2}\frac{B(r)}{B(r'_P)}\right)^{1/2},$$
$$\dot{\varphi} = \frac{r'_P}{r^2\sqrt{B(r'_P)}}, \tag{62}$$

where r'_P is the perilens. The travel time of the upper photon inside the vacuole from the entry point to the exit point can be obtained by using the relation (31) between the Schwarzschild time T and the Friedmann time t,

$$T'_{\mathrm{SchüE}} - T'_{\mathrm{SchüS}} = \cos\chi_{\mathrm{Schü}} \int_{t'_{\mathrm{SchüS}}}^{t'_{\mathrm{SchüE}}} \frac{dt}{B_{\mathrm{Schü}}(t)} \tag{63}$$

$$= \cos\chi_{\mathrm{Schü}} \int_{a'_{\mathrm{SchüS}}}^{a'_{\mathrm{SchüE}}} \frac{da}{B_{\mathrm{Schü}}(a) f(a)}, \tag{64}$$

owing to $dt = da/f(a)$. Another expression for this travel time can be obtained by making use of the well-known equation

$$dT = \pm \frac{dr}{v'(r)}, \qquad v'(r) \coloneqq B(r)\sqrt{1 - \frac{r_P'^2}{r^2}\frac{B(r)}{B(r_P')}}, \tag{65}$$

which follows immediately by eliminating the affine parameter between $\dot{T}$ and $\dot{r}$ in Eq. (62). The perilens r'_P is given approximately by [11]

$$r'_P \simeq r'_{\mathrm{Sch\ddot{u}E}} \sin\gamma'_{KE} - M/8\pi, \tag{66}$$

which is obtained by developing the smaller coordinate angle γ'_K (60), using $\dot{r}$ and $\dot{\varphi}$ of Eq. (62), to first order in the ratio of the Schwarzschild radius to the perilens $M/4\pi r'_P$, which in our case is of order 10^{-5}. The integral of the right-hand side of Eq. (65) can be split in two integrals according to the fact that r decreases with time from $r'_{\mathrm{Sch\ddot{u}S}}$ to r'_P while it increases from r'_P to $r'_{\mathrm{Sch\ddot{u}E}}$, i.e.,

$$T'_{\mathrm{Sch\ddot{u}E}} - T'_{\mathrm{Sch\ddot{u}S}} = \int_{r'_P}^{r'_{\mathrm{Sch\ddot{u}E}}} \frac{dr}{v'(r)} + \int_{r'_P}^{r'_{\mathrm{Sch\ddot{u}S}}} \frac{dr}{v'(r)}, \tag{67}$$

where $r'_{\mathrm{Sch\ddot{u}E}}$ and $r'_{\mathrm{Sch\ddot{u}S}}$ are related respectively to $a'_{\mathrm{Sch\ddot{u}E}}$ and $a'_{\mathrm{Sch\ddot{u}S}}$ by the matching condition (8). It follows from Eqs. (64) and (67) that

$$\int_{r'_P}^{a'_{\mathrm{Sch\ddot{u}E}}\sin\chi_{\mathrm{Sch\ddot{u}}}} \frac{dr}{v'(r)} + \int_{r'_P}^{a'_{\mathrm{Sch\ddot{u}S}}\sin\chi_{\mathrm{Sch\ddot{u}}}} \frac{dr}{v'(r)} = \cos\chi_{\mathrm{Sch\ddot{u}}} \int_{a'_{\mathrm{Sch\ddot{u}S}}}^{a'_{\mathrm{Sch\ddot{u}E}}} \frac{da}{B_{\mathrm{Sch\ddot{u}}}(a) f(a)}, \tag{68}$$

which enables us to obtain $a'_{\mathrm{Sch\ddot{u}S}}$ by numerical integration. Then, we have to integrate numerically the Friedmann equation to obtain $t'_{\mathrm{Sch\ddot{u}S}}$, i.e.,

$$t'_{\mathrm{Sch\ddot{u}S}} = \int_{a_0}^{a'_{\mathrm{Sch\ddot{u}S}}} \frac{da}{f(a)}. \tag{69}$$

Following the same reasoning as for the upper trajectory, we get for the lower trajectory similar formulas, i.e.,

$$t_{\mathrm{Sch\ddot{u}S}} = \int_{a_0}^{a_{\mathrm{Sch\ddot{u}S}}} \frac{da}{f(a)}, \tag{70}$$

with $a_{\mathrm{Sch\ddot{u}S}}$ calculated by numerically solving the equation

$$\int_{r_P}^{a_{\mathrm{Sch\ddot{u}E}}\sin\chi_{\mathrm{Sch\ddot{u}}}} \frac{dr}{v(r)} + \int_{r_P}^{a_{\mathrm{Sch\ddot{u}S}}\sin\chi_{\mathrm{Sch\ddot{u}}}} \frac{dr}{v(r)} = \cos\chi_{\mathrm{Sch\ddot{u}}} \int_{a_{\mathrm{Sch\ddot{u}S}}}^{a_{\mathrm{Sch\ddot{u}E}}} \frac{da}{B_{\mathrm{Sch\ddot{u}}}(a) f(a)}, \tag{71}$$

where r_P is also expressed by a formula similar to Eq. (66),

$$r_P \simeq r_{\mathrm{Sch\ddot{u}E}} \sin\gamma_{KE} - M/8\pi. \tag{72}$$

We can avoid numerically computing the time $t_{\mathrm{Sch\ddot{u}S}}$ for the lower-trajectory photon following the same approximation method described in Ref. [18]. It could be determined by difference with $t'_{\mathrm{Sch\ddot{u}S}}$ via an approximate analytical expression. First, thanks to the relation (31) between the Schwarzschild time T and the Friedmann time t, we may express the difference in the travel times of both photons inside the vacuole as

$$\begin{aligned} T'_{\mathrm{Sch\ddot{u}E}} - T'_{\mathrm{Sch\ddot{u}S}} - (T_{\mathrm{Sch\ddot{u}E}} - T_{\mathrm{Sch\ddot{u}S}}) &= \cos\chi_{\mathrm{Sch\ddot{u}}} \int_{t_{\mathrm{Sch\ddot{u}E}}}^{t'_{\mathrm{Sch\ddot{u}E}}} \frac{dt}{B_{\mathrm{Sch\ddot{u}}}(t)} \\ &\quad - \cos\chi_{\mathrm{Sch\ddot{u}}} \int_{t_{\mathrm{Sch\ddot{u}S}}}^{t'_{\mathrm{Sch\ddot{u}S}}} \frac{dt}{B_{\mathrm{Sch\ddot{u}}}(t)} \end{aligned} \tag{73}$$

$$\begin{aligned} &\simeq \cos\chi_{\mathrm{Sch\ddot{u}}} \frac{t'_{\mathrm{Sch\ddot{u}E}} - t_{\mathrm{Sch\ddot{u}E}}}{B'_{\mathrm{Sch\ddot{u}E}}} \\ &\quad - \cos\chi_{\mathrm{Sch\ddot{u}}} \frac{t'_{\mathrm{Sch\ddot{u}S}} - t_{\mathrm{Sch\ddot{u}S}}}{B'_{\mathrm{Sch\ddot{u}S}}}, \end{aligned} \tag{74}$$

where we have used the fact that $B_{\mathrm{Sch\ddot{u}}}$ vary appreciably only on cosmological time intervals with $B'_{\mathrm{Sch\ddot{u}E}} \coloneqq B_{\mathrm{Sch\ddot{u}}}(t'_{\mathrm{Sch\ddot{u}E}})$ and $B'_{\mathrm{Sch\ddot{u}E}} \coloneqq B_{\mathrm{Sch\ddot{u}}}(t'_{\mathrm{Sch\ddot{u}E}})$. Second, we can make use of Eq. (65) to write this difference in travel times as

$$\begin{aligned} T'_{\mathrm{Sch\ddot{u}E}} - T'_{\mathrm{Sch\ddot{u}S}} - (T_{\mathrm{Sch\ddot{u}E}} - T_{\mathrm{Sch\ddot{u}S}}) &= \Delta T_K + \int_{r_{\mathrm{Sch\ddot{u}E}}}^{r'_{\mathrm{Sch\ddot{u}E}}} \frac{dr}{v(r)} + \int_{r_{\mathrm{Sch\ddot{u}S}}}^{r'_{\mathrm{Sch\ddot{u}S}}} \frac{dr}{v(r)}, \end{aligned} \tag{75}$$

where we have broken up the integrals to produce the following expression:

$$\begin{aligned} \Delta T_K = \int_{r'_P}^{r'_{\mathrm{Sch\ddot{u}E}}} \frac{dr}{v'(r)} + \int_{r'_P}^{r'_{\mathrm{Sch\ddot{u}S}}} \frac{dr}{v'(r)} \\ - \left(\int_{r_P}^{r'_{\mathrm{Sch\ddot{u}E}}} \frac{dr}{v(r)} + \int_{r_P}^{r'_{\mathrm{Sch\ddot{u}S}}} \frac{dr}{v(r)} \right). \end{aligned} \tag{76}$$

We may interpret this as the difference in the travel times between the upper photon and an imaginary lower photon that starts from the same point as the upper photon $r'_{\mathrm{Sch\ddot{u}S}}$, deflected by the lens at the perilens r_P, and finally arrives at the same point as the upper photon $r'_{\mathrm{Sch\ddot{u}E}}$. This latter expression, Eq. (76), is almost identical to an expression already involved in the calculation of time delay in the Kottler solution [20], just with r_T replaced by $r'_{\mathrm{Sch\ddot{u}E}}$, r_S replaced by $r'_{\mathrm{Sch\ddot{u}S}}$, r'_0 replaced by r'_P, and r_0 replaced by r_P. The result is

$$\Delta T_K \simeq \frac{r_P^2 - r_P'^2}{2}\left(\frac{1}{r'_{\text{SchüE}}} + \frac{1}{r'_{\text{SchüS}}}\right) + \frac{M}{2\pi}\ln\frac{r_P}{r'_P} - 3\left(\frac{M}{4\pi r'_P}\right)^2 \frac{r_P^2 - r_P'^2}{8r_P^2} \times \sqrt{\frac{3}{\Lambda}}\left[\operatorname{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r'_{\text{SchüE}}\right) + \operatorname{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r'_{\text{SchüS}}\right)\right]. \tag{77}$$

Furthermore, since we deal with smaller length and timescales than cosmological ones,

$$\int_{r_{\text{SchüE}}}^{r'_{\text{SchüE}}} \frac{dr}{v(r)} \simeq \frac{r'_{\text{SchüE}} - r_{\text{SchüE}}}{v'_{\text{SchüE}}}, \quad \int_{r_{\text{SchüS}}}^{r'_{\text{SchüS}}} \frac{dr}{v'(r)} \simeq \frac{r'_{\text{SchüS}} - r_{\text{SchüS}}}{v'_{\text{SchüS}}}, \tag{78}$$

and, using the Friedmann equation,

$$r'_{\text{SchüE}} - r_{\text{SchüE}} = (a'_{\text{SchüE}} - a_{\text{SchüE}})\sin\chi_{\text{Schü}} \simeq f'_{\text{SchüE}}\sin\chi_{\text{Schü}}(t'_{\text{SchüE}} - t_{\text{SchüE}}), \tag{79}$$

$$r'_{\text{SchüS}} - r_{\text{SchüS}} = (a'_{\text{SchüS}} - a_{\text{SchüS}})\sin\chi_{\text{Schü}} \simeq f'_{\text{SchüS}}\sin\chi_{\text{Schü}}(t'_{\text{SchüS}} - t_{\text{SchüS}}). \tag{80}$$

with $v'_{\text{SchüE}} \coloneqq v(r'_{\text{SchüE}})$, $v'_{\text{SchüS}} \coloneqq v(r'_{\text{SchüS}})$, $f'_{\text{SchüE}} \coloneqq f(a'_{\text{SchüE}})$, and $f'_{\text{SchüS}} \coloneqq f(a'_{\text{SchüS}})$. Using this together with Eqs. (77), (75), and (74), we therefore get

$$t_{\text{SchüS}} - t'_{\text{SchüS}} \simeq \frac{\Delta T_K + \left(\frac{f'_{\text{SchüE}}\sin\chi_{\text{Schü}}}{v'_{\text{SchüE}}} - \frac{\cos\chi_{\text{Schü}}}{B'_{\text{SchüE}}}\right)(t'_{\text{SchüE}} - t_{\text{SchüE}})}{\left(\frac{f'_{\text{SchüS}}\sin\chi_{\text{Schü}}}{v'_{\text{SchüS}}} + \frac{\cos\chi_{\text{Schü}}}{B'_{\text{SchüS}}}\right)}. \tag{81}$$

Then, the knowledge of $t'_{\text{SchüS}}$ allows one to deduce $t_{\text{SchüS}}$, where $t'_{\text{SchüE}} - t_{\text{SchüE}}$ is given by Eq. (55). It should be noted that the lower photon enters the vacuole after the upper one, even though it leaves the vacoule before. Admittedly, this is caused by the fact that the upper photon more strongly undergoes the gravitational effect since it passes closest to the lens ($r'_P < r_P$) as shown in Fig. 1.

We should also compute the angles $\varphi'_{\text{SchüS}}$ and $\varphi_{\text{SchüS}}$ as well as the 4-velocities at points of entry into the vacuole, needed for the next section. It would be necessary to make use of the well-known equation

$$d\varphi = \pm\frac{dr}{u'(r)}, \quad u'(r) \coloneqq r\sqrt{\frac{r^2}{r_P'^2} - 1}\sqrt{1 - \frac{M}{4\pi r'_P}\left(\frac{r'_P}{r} + \frac{r}{r + r'_P}\right)}, \tag{82}$$

which follows from Eq. (62), in which the cosmological constant Λ incidentally disappeared. Because the angle φ increases when the upper photon approaches the lens as well as when it moves away, the integral of the right-hand side of Eq. (82) along the trajectory may thus be split up as

$$\varphi'_{\text{SchüE}} - \varphi'_{\text{SchüS}} = \int_{r'_P}^{r'_{\text{SchüE}}} \frac{dr}{u'(r)} + \int_{r'_P}^{r'_{\text{SchüS}}} \frac{dr}{u'(r)} \tag{83}$$

and gives, to first order in the ratio $M/4\pi r'_P$,

$$\varphi'_{\text{SchüS}} \simeq \varphi'_{\text{SchüE}} - \pi + \arcsin\frac{r'_P}{r'_{\text{SchüE}}} + \arcsin\frac{r'_P}{r'_{\text{SchüS}}} - \frac{M}{8\pi r'_P}\left(\sqrt{1 - \frac{r_P'^2}{r_{\text{SchüE}}'^2}} + \sqrt{1 - \frac{r_P'^2}{r_{\text{SchüS}}'^2}} + \sqrt{\frac{r'_{\text{SchüE}} - r'_P}{r'_{\text{SchüE}} + r'_P}} + \sqrt{\frac{r'_{\text{SchüS}} - r'_P}{r'_{\text{SchüS}} + r'_P}}\right), \tag{84}$$

where $\varphi'_{\text{SchüE}}$ is given by Eq. (44). In the same manner, we obtain for the lower trajectory

$$\varphi_{\text{SchüS}} \simeq \varphi_{\text{SchüE}} + \pi - \arcsin\frac{r_P}{r_{\text{SchüE}}} - \arcsin\frac{r_P}{r_{\text{SchüS}}} + \frac{M}{8\pi r_P}\left(\sqrt{1 - \frac{r_P^2}{r_{\text{SchüE}}^2}} + \sqrt{1 - \frac{r_P^2}{r_{\text{SchüS}}^2}} + \sqrt{\frac{r_{\text{SchüE}} - r_P}{r_{\text{SchüE}} + r_P}} + \sqrt{\frac{r_{\text{SchüS}} - r_P}{r_{\text{SchüS}} + r_P}}\right), \tag{85}$$

where $\varphi_{\text{SchüE}}$ is given by Eq. (48). On account of Eq. (62), the two 4-velocities of the upper and lower photons at points of entry into the vacuole are respectively

$$\dot{T}'_{\text{SchüS}} = \frac{1}{B'_{\text{SchüS}}}, \qquad \dot{r}'_{\text{SchüS}} = -\sqrt{1 - \frac{r_P'^2 B'_{\text{SchüS}}}{r_{\text{SchüS}}'^2 B(r'_P)}}, \qquad \dot{\varphi}'_{\text{SchüS}} = \frac{r'_P}{r_{\text{SchüS}}'^2\sqrt{B(r'_P)}} \tag{86}$$

and

$$\dot{T}_{\rm SchüS} = \frac{1}{B_{\rm SchüS}}, \qquad \dot{r}_{\rm SchüS} = -\sqrt{1 - \frac{r_P^2 B_{\rm SchüS}}{r_{\rm SchüS}^2 B(r_P)}}, \qquad \dot{\varphi}_{\rm SchüS} = \frac{-r_P}{r_{\rm SchüS}^2 \sqrt{B(r_P)}}, \tag{87}$$

in Schwarzschild coordinates, where we have taken into account that r is decreasing with time since both photons approach the lens while φ is increasing with time for the upper photon and decreasing with time for the lower photon. These two 4-velocities can be translated, thanks to the Jacobian (27), into Friedmann coordinates,

$$\dot{t}'_{\rm SchüS} = \frac{\cos\chi_{\rm Schü}}{B'_{\rm SchüS}}\left(1 + C'_{\rm SchüS}\sqrt{1 - \frac{r_P'^2 B'_{\rm SchüS}}{r_{\rm SchüS}'^2 B(r'_P)}}\right), \tag{88}$$

$$\dot{\chi}'_{\rm SchüS} = \frac{-\cos\chi_{\rm Schü}}{a'_{\rm SchüS} B'_{\rm SchüS}}\left(C'_{\rm SchüS} + \sqrt{1 - \frac{r_P'^2 B'_{\rm SchüS}}{r_{\rm SchüS}'^2 B(r'_P)}}\right) \tag{89}$$

for the upper trajectory and

$$\dot{t}_{\rm SchüS} = \frac{\cos\chi_{\rm Schü}}{B_{\rm SchüS}}\left(1 + C_{\rm SchüS}\sqrt{1 - \frac{r_P^2 B_{\rm SchüS}}{r_{\rm SchüS}^2 B(r_P)}}\right), \tag{90}$$

$$\dot{\chi}_{\rm SchüS} = \frac{-\cos\chi_{\rm Schü}}{a_{\rm SchüS} B_{\rm SchüS}}\left(C_{\rm SchüS} + \sqrt{1 - \frac{r_P^2 B_{\rm SchüS}}{r_{\rm SchüS}^2 B(r_P)}}\right) \tag{91}$$

for the lower trajectory.

C. Null geodesics between the source and the Schücking sphere

In this region where the Friedmann metric prevails, we shall follow the same procedure as described in Sec. III A. The integration of the two geodesic equations for the upper photon, Eqs. (35) and (37), with the final conditions at the point of entry into the vacuole,

$$t = t'_{\rm SchüS}, \qquad \chi = \chi_{\rm Schü}, \qquad \varphi = \varphi'_{\rm SchüS}, \qquad \dot{t} = \dot{t}'_{\rm SchüS}, \qquad \dot{\chi} = \dot{\chi}'_{\rm SchüS} \qquad \dot{\varphi} = \dot{\varphi}'_{\rm SchüS}, \tag{92}$$

gives

$$\dot{t} = \frac{E'}{a(t)}, \qquad \dot{\varphi} = \frac{J'}{a^2 \sin^2\chi}, \qquad \varphi = \varphi'_{\rm SchüS} + \arcsin\frac{(J'/E')\cot\chi}{\sqrt{1-(J'/E')^2}} - \gamma', \tag{93}$$

where the constants E', J', and γ' are defined by

$$E' \coloneqq \frac{a'_{\rm SchüS}\cos\chi_{\rm Schü}}{B'_{\rm SchüS}}\left(1 + C'_{\rm SchüS}\sqrt{1 - \frac{r_P'^2 B'_{\rm SchüS}}{r_{\rm SchüS}'^2 B(r'_P)}}\right), \qquad J' \coloneqq \frac{r'_P}{\sqrt{B(r'_P)}}, \qquad \gamma' \coloneqq \arcsin\frac{(J'/E')\cot\chi_{\rm Schü}}{\sqrt{1-(J'/E')^2}}. \tag{94}$$

We then deduce the expression of the angle φ'_S,

$$\varphi'_S = \varphi'_{\rm SchüS} + \arcsin\frac{(J'/E')\cot\chi_{L,S}}{\sqrt{1-(J'/E')^2}} - \gamma', \tag{95}$$

where the geodesic distance $\chi_{L,S}$ between the lens and the source is approximated by

$$\chi_{L,S} \simeq \chi_S - \chi_L, \tag{96}$$

owing to the fact that the deflection angle $-\varphi_S$ is of the order of a few arc seconds ($\sim 10^{-5}$). For the lower photon, we just have to replace J' by $-J$ and then obtain the solution of the geodesic equations. We only give here the expression of the angle φ_S,

$$\varphi_S = \varphi_{\rm SchüS} - \arcsin\frac{(J/E)\cot\chi_{L,S}}{\sqrt{1-(J/E)^2}} + \gamma, \tag{97}$$

where $\varphi_{\rm SchüS}$ is given by Eq. (85) and

$$E \coloneqq \frac{a_{\rm SchüS}\cos\chi_{\rm Schü}}{B_{\rm SchüS}}\left(1 + C_{\rm SchüS}\sqrt{1 - \frac{r_P^2 B_{\rm SchüS}}{r_{\rm SchüS}^2 B(r_P)}}\right), \qquad J \coloneqq \frac{r_P}{\sqrt{B(r_P)}}, \qquad \gamma \coloneqq \arcsin\frac{(J/E)\cot\chi_{\rm Schü}}{\sqrt{1-(J/E)^2}}. \tag{98}$$

The angles φ'_S and φ_S must be equal due to the fact that both photons are emitted by the same source. Among all the parameters involved in the calculation of these angles, their equality could only be ensured by a fixed value of the mass; i.e., we have to vary M in order to get $\varphi_S = \varphi'_S$. Once the correct mass is determined, one should be able to compute the time delay between both photons $t_S - t'_S$ (difference between their total travel times). To achieve this, we first need to compute the distance traveled by the upper photon between the source and its entry point into the vacuole ($\chi'_{S,{\rm SchüS}}$). Differentiating the solution φ (93) with respect to χ and substituting into the closed Friedmann metric (4), one easily get an equation similar to Eq. (41), by using J'/E' instead of χ'_P,

$$\int \frac{dt}{a(t)} = -\int \frac{d\chi}{h'(\chi)}, \qquad h'(\chi) = \sqrt{1-(J'/E')^2/\sin^2\chi}, \tag{99}$$

where the negative sign indicates that χ between the source and the vacuole is decreasing with time. Obviously, the evaluation of the right-hand side of the last equation between $\chi_{L,S}$ and $\chi_{\text{Schü}}$ gives the geodesic distance $\chi'_{S,\text{SchüS}}$,

$$\int_{a'_S}^{a'_{\text{SchüS}}} \frac{da}{af(a)} = \arcsin \frac{\cos\chi_{\text{Schü}}}{\sqrt{1-(J'/E')^2}} - \arcsin \frac{\cos\chi_{L,S}}{\sqrt{1-(J'/E')^2}}, \quad (100)$$

where we have inserted the function $f(a)$ in the left-hand side via $dt = da/f(a)$. Numerically solving this equation, one can deduce $a'_{\text{SchüS}}$. Then, if one is interested in t'_S, it suffices to use the Friedmann equation, i.e.,

$$t'_S = \int_{a_0}^{a'_S} \frac{da}{f(a)}. \quad (101)$$

Likewise, the emission time t_S of the lower photon is obtained by numerical integration of the Friedmann equation, i.e.,

$$t_S = \int_{a_0}^{a_S} \frac{da}{f(a)}, \quad (102)$$

where a_S is calculated by numerically solving the equation

$$\int_{a_S}^{a_{\text{SchüS}}} \frac{da}{af(a)} = \arcsin \frac{\cos\chi_{\text{Schü}}}{\sqrt{1-(J/E)^2}} - \arcsin \frac{\cos\chi_{L,S}}{\sqrt{1-(J/E)^2}}. \quad (103)$$

However, one can proceed, as in the flat case [18], in a different manner directly computing the time delay through an approximate analytical expression. Combining Eqs. (100) and (103), one gets, after evaluating the integrals with respect to time,

$$\int_{t'_{\text{SchüS}}}^{t_{\text{SchüS}}} \frac{dt}{a(t)} - \int_{t'_S}^{t_S} \frac{dt}{a(t)} = \arcsin \frac{\cos\chi_{\text{Schü}}}{\sqrt{1-(J/E)^2}} - \arcsin \frac{\cos\chi_{L,S}}{\sqrt{1-(J/E)^2}} - \arcsin \frac{\cos\chi_{\text{Schü}}}{\sqrt{1-(J'/E')^2}} + \arcsin \frac{\cos\chi_{L,S}}{\sqrt{1-(J'/E')^2}}. \quad (104)$$

Using the fact that the scale factor $a(t)$ varies noticeably only over cosmological timescales, the left-hand side of this equation can be approximated by

$$\text{lhs} \simeq \frac{t_{\text{SchüS}} - t'_{\text{SchüS}}}{a'_{\text{SchüS}}} - \frac{t_S - t'_S}{a'_S}. \quad (105)$$

Expanding the right-hand side to second order in J'/E' and J/E ($\sim 10^{-6}$ in our case), on account of Eqs. (95) and (97), one gets

$$\text{rhs} \simeq \frac{1}{2}(\cot\chi_{\text{Schü}} - \cot\chi_{L,S})\left[\left(\frac{J}{E}\right)^2 - \left(\frac{J'}{E'}\right)^2\right] \quad (106)$$

$$\simeq \frac{1}{2}\frac{(\varphi_{\text{SchüS}} - \varphi_S)^2 - (\varphi'_{\text{SchüS}} - \varphi_S)^2}{\cot\chi_{\text{Schü}} - \cot\chi_{L,S}}, \quad (107)$$

where $\varphi'_{\text{SchüS}} - \varphi_S$ and $|\varphi_{\text{SchüS}} - \varphi_S|$ are of order 10^{-2}. Equating the two sides, one arrives finally at

$$\Delta t \coloneqq t_S - t'_S \simeq a'_S\left[\frac{t_{\text{SchüS}} - t'_{\text{SchüS}}}{a'_{\text{SchüS}}} - \frac{1}{2}\frac{(\varphi_{\text{SchüS}} - \varphi_S)^2 - (\varphi'_{\text{SchüS}} - \varphi_S)^2}{\cot\chi_{\text{Schü}} - \cot\chi_{L,S}}\right]. \quad (108)$$

To sum up, the upper photon, after being the first emitted by the source, also penetrates the first the vacuole, but it leaves it the last in such a way that it reaches the Earth at the same time with the lower photon.

IV. APPLICATION TO THE LENSED QUASAR SDSS J1004+4112

In this last step, we apply our results to the lensed quasar SDSS J1004 + 4112 [21–25] with

$$\alpha' = 5'' \pm 10\%, \qquad \alpha = 10'' \pm 10\%,$$
$$z_L = 0.68, \qquad z_S = 1.734. \quad (109)$$

The Earth-lens and Earth-source distances are calculated by Eq. (33); $\chi_L \coloneqq \chi(z_L)$ and $\chi_S \coloneqq \chi(z_S)$, with $\Lambda = 0.77 \cdot 3\ \text{am}^{-2} \pm 20\%$. As we said before, the mass of the cluster of galaxies M is varied until the equality $\varphi_S = \varphi'_S$ is satisfied. It is worth noting that neither a value of a_0 nor that of the cosmological constant Λ could make φ_Ss coincide. The special case $a_0 = 5$ am is addressed in detail for maximum +, central ±0, and minimum − values of α', α, and Λ. In addition, we have treated the case without cosmological constant $\Lambda = 0$. The results are recorded in Tables I, II, III, and IV.

We can see from Tables I, II, and III that an increasing cosmological constant Λ within its error bar by 20% leads to a decrease of the deflection angle $-\varphi_S$ by about 8%, but this variation only increases the cluster mass M by about 1%. Thus, the bending of light clearly depends on the cosmological constant. Note also the monotonous dependence of the time delay Δt on the cosmological constant.

In contrast to the case of flat Einstein-Straus model [11,18], where the present scale factor can be set to any value without loss of generality, the results in Tables V

TABLE I. Upper limit value of Λ: $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2} + 20\%$ ($a_0 = 5$ am).

$\alpha' \pm 10\%$	$+$	$+$	$+$	± 0	± 0	± 0	$-$	$-$	$-$
$\alpha \pm 10\%$	$+$	± 0	$-$	$+$	± 0	$-$	$+$	± 0	$-$
$M(10^{13} M_\odot)$	**2.14**	1.95	1.75	1.95	1.77	1.59	1.75	1.59	**1.43**
$-\varphi_S('')$	9.99	8.17	**6.35**	10.89	9.08	7.26	**11.80**	9.99	8.17
Δt(yr)	11.52	8.95	**6.58**	12.04	9.52	7.19	**12.44**	9.99	7.71

TABLE II. Central value of Λ: $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2}$ ($a_0 = 5$ am).

$\alpha' \pm 10\%$	$+$	$+$	$+$	± 0	± 0	± 0	$-$	$-$	$-$
$\alpha \pm 10\%$	$+$	± 0	$-$	$+$	± 0	$-$	$+$	± 0	$-$
$M(10^{13} M_\odot)$	**2.16**	1.97	1.77	1.97	1.79	1.61	1.77	1.61	**1.45**
$-\varphi_S('')$	10.95	8.96	**6.97**	11.95	9.96	7.97	**12.95**	10.95	8.96
Δt(yr)	11.47	8.92	**6.56**	11.98	9.48	7.17	**12.36**	9.94	7.68

TABLE III. Lower limit value of Λ: $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2} - 20\%$ ($a_0 = 5$ am).

$\alpha' \pm 10\%$	$+$	$+$	$+$	± 0	± 0	± 0	$-$	$-$	$-$
$\alpha \pm 10\%$	$+$	± 0	$-$	$+$	± 0	$-$	$+$	± 0	$-$
$M(10^{13} M_\odot)$	**2.14**	1.95	1.75	1.95	1.77	1.59	1.75	1.59	**1.43**
$-\varphi_S('')$	11.59	9.48	**7.38**	12.64	10.54	8.43	**13.70**	11.59	9.48
Δt(yr)	11.27	8.77	**6.46**	11.76	9.32	7.05	**12.13**	9.75	7.55

TABLE IV. $\Lambda = 0$ ($a_0 = 5$ am).

$\alpha' \pm 10\%$	$+$	$+$	$+$	± 0	± 0	± 0	$-$	$-$	$-$
$\alpha \pm 10\%$	$+$	± 0	$-$	$+$	± 0	$-$	$+$	± 0	$-$
$M(10^{13} M_\odot)$	**2.00**	1.82	1.64	1.82	1.65	1.49	1.64	1.49	**1.34**
$-\varphi_S('')$	12.99	10.63	**8.27**	14.18	11.81	9.45	**15.36**	12.99	10.63
Δt(yr)	10.36	8.07	**5.95**	10.80	8.57	6.49	**11.12**	8.96	6.94

TABLE V. $\alpha' = 5''$, $\alpha = 10''$, $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2}$.

a_0(am)	0.1	0.3	0.5	0.7	1	2	3	4	5	6
$M(10^{13} M_\odot)$	0.04	0.29	0.62	0.91	1.22	1.62	1.73	1.77	1.79	1.80
$-\varphi_S('')$	4.03	6.64	8.05	8.82	9.38	9.85	9.93	9.95	9.96	9.96
Δt(yr)	0.17	1.30	2.90	4.40	6.08	8.44	9.10	9.36	9.48	9.55

and VI show that the mass, the deflection angle, and the time delay are somehow related to the present scale factor parameter. They decrease significantly as the present scale factor gets smaller and smaller below a limit value close to 7 am ($\Omega_{k0} \simeq 0.02$). This dependence on a_0 comes from the curvature density term in the Friedmann equation that vanishes in the case of flat space ($k = 0$). Nevertheless, above this limit value ($\Omega_{k0} < 0.02$), the effect of a_0 is very small, and the obtained steady values for the mass, the deflection angle, and the time delay are found to be indistinguishable from the flat case.[1] The same is true for all possible values of α', α, and Λ within their error bars.

[1]For accuracy, we correct the rounding error in our previous work [18]: the time delay, with central values of α', α, and Λ, should be 9.71 yr, not 9.72 yr.

TABLE VI. $\alpha' = 5''$, $\alpha = 10''$, $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2}$.

a_0(am)	7	8	9	10	11	12	15	19	30	100
$M(10^{13} M_\odot)$	1.80	1.81	1.81	1.81	1.81	1.82	1.82	1.82	1.82	1.82
$-\varphi_S('')$	9.97	9.97	9.97	9.97	9.97	9.97	9.97	9.97	9.97	9.97
Δt(yr)	9.59	9.62	9.64	9.65	9.66	9.67	9.69	9.70	9.71	9.71

V. CONCLUSION

In this paper, we have computed the bending of light and time delay in the positively curved Einstein-Straus model with a positive cosmological constant. Unlike the case of the flat Einstein-Straus model in which the results remain the same regardless of which value is chosen for the present scale factor [11,18], we show that the bending of light and time delay decrease considerably as the present scale factor becomes smaller than a limit value about 7 am. The same holds true for the mass of the lens. But for larger values of the present scale factor, the results are almost compatible with those obtained in flat Einstein-Straus model. This limit value is an interesting result that could be regarded as a lower bound on the radius of the Universe today so that the spatial curvature, which is a property of the Friedmann universe, does not affect the bending of light nor the time delay.

Furthermore, the results confirm Rindler and Ishak's claim that a positive cosmological constant attenuates the bending of light [1].

Even in the case of a negatively curved Einstein-Straus model, we expect that the present value of the scale factor should also have an effect on the bending of light and time delay since the spatial curvature only changes its sign ($k = -1$), leading to a nonzero curvature density as in the case of the positively curved Einstein-Straus model. This case will be investigated in a forthcoming work. The same problem should also be extended to include the interior Kottler solution, wherein the photons could pass through the mass distribution [26].

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